Solutions [and grading guidelines]
for the midterm exam in FYS5120

[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions, energy not conserved etc.)! 'Obvious' small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for explanations demonstrating special insight (but this cannot result in more points than available for a given problem).]

Problem 1

a) The generating functional is defined as **[0.5 pts]**

$$Z[J] \equiv \int \mathcal{D}\phi \, \exp\left[i \int d^4x \, \mathcal{L} + J\phi \right] . \tag{1}$$

By the basic properties of functional derivatives, the correlation function can thus alternatively be written as **[0.5 pts]**

$$\langle \Omega | \phi(x_1) ... \phi(x_n) | \Omega \rangle = \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)} \right) ... \left(-i \frac{\delta}{\delta J(x_n)} \right) Z \Big|_{J=0} \tag{2}$$

With $\mathcal{L} \equiv \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$, the generating functional can be expanded as

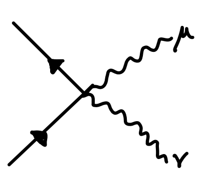
$$Z[J] \equiv \int \mathcal{D}\phi \, \exp\left[i \int d^4x \, \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} + J\phi \right] \tag{3}$$

$$= \int \mathcal{D}\phi \left(1 + i \int d^4x' \, \mathcal{L}_{\text{int}}[\phi(x')] + ... \right) \exp\left[i \int d^4x \, \mathcal{L}_{\text{free}} + J\phi \right] \; \textbf{[1 pt]} \tag{4}$$

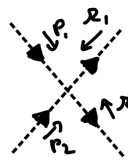
$$= \left(1 + i \int d^4x' \, \mathcal{L}_{\text{int}}[-i \frac{\delta}{\delta J(x')}] + ... \right) \int \mathcal{D}\phi \, \exp\left[i \int d^4x \, \mathcal{L}_{\text{free}} + J\phi \right] \; \textbf{[1 pt]} \tag{5}$$

When plugging this expression into Eq. (2), that equation doesn't change its form. The only thing that remains is thus to evaluate the path integral in the last term, which describes the generating functional for the free theory.

b) Since the mass dimension of $\mathcal{L}$ must be 4, we have $[\lambda] = -1$ in the first and $[\lambda] = -2$ in the second case **[1 pt]**. Assuming all particles to be ingoing, the vertex rules are given by: **[1 pt per correct diagram, including arrow directions, 1 pt per correct rule; -0.5 when not using momentum conservation to simplify the last one]**



$$= -2i\lambda g^{\mu\nu} \gamma^5$$

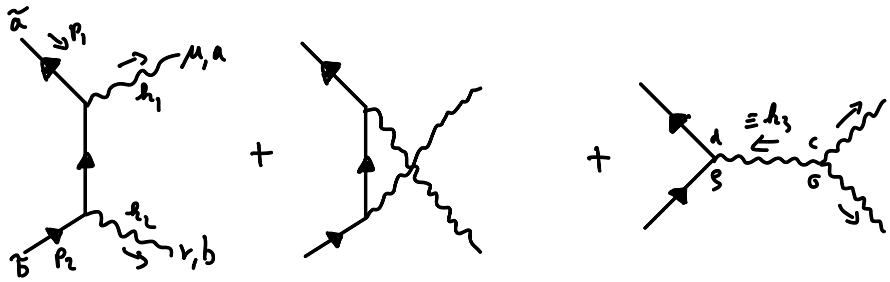


$$= i\lambda \{ p_1 \cdot k_2 + p_1 \cdot k_1 + p_2 \cdot k_1 + p_2 \cdot k_2 \}$$
$$= -2i\lambda (p_1 \cdot p_2 + m^2)$$

Important things to note here are: i) a symmetry factor of 2! for each field appearing twice, ii) the replacement of $\partial_\mu \to -ip_\mu$ for ingoing momenta and iii) assigning particles and antiparticles correctly.

Problem 2

a) From left to right, the following are the contributing t-, u- [1 pt] and s-channel [1 pt] diagrams, respectively: **[0.5 pt subtractions for wrong/missing arrows or momentum assignments not as specified]**



The corresponding contributions to the matrix element are given by **[1 pt for each of the two amplitude – if fully correct, simplified and including color indices.]**

$$i\mathcal{M}^{\mu\nu}_{t+u} = (ig)^2 \bar{v}_{\tilde{a}}(p_1)\left\{\gamma^\mu t^a_{\tilde{a}\tilde{c}} \frac{i\delta_{\tilde{c}\tilde{d}}}{\not{k}_1 - \not{p}_1 - m}\gamma^\nu t^b_{\tilde{d}\tilde{b}} + \gamma^\nu t^b_{\tilde{a}\tilde{c}} \frac{i\delta_{\tilde{c}\tilde{d}}}{\not{p}_2 - \not{k}_1 - m}\gamma^\mu t^a_{\tilde{d}\tilde{b}}\right\} u_{\tilde{b}}(p_2)\,, \tag{6}$$

$$i\mathcal{M}^{\mu\nu}_{s} = ig\bar{v}_{\tilde{a}}(p_1)\gamma^\rho t^d_{\tilde{a}\tilde{b}} u_{\tilde{b}}(p_2) \frac{-i\delta^{dc} g_{\rho\sigma}}{k_3^2} \times$$
$$\times g f^{abc}\left\{g^{\mu\nu}(k_2 - k_1)^\sigma + g^{\nu\sigma}(k_3 - k_2)^\mu + g^{\sigma\mu}(k_1 - k_3)^\nu\right\}. \tag{7}$$

b) We first note that $\epsilon^{+*}_\mu(k_1) = k_1^\mu/(\sqrt{2}\,|\mathbf{k_1}|)$ [1 pt]. From now on suppressing fermion color indices, we thus get:

$$\begin{aligned} i\mathcal{M}^{\mu\nu}_{t+u}\epsilon^{+*}_\mu(k_1) &= -i\frac{g^2}{\sqrt{2}\,|\mathbf{k_1}|}\bar{v}(p_1)\left\{t^a t^b \frac{\gamma^\mu}{\not{k}_1 - \not{p}_1 - m}\gamma^\nu + t^b t^a \gamma^\nu \frac{\gamma^\mu}{\not{p}_2 - \not{k}_1 - m}\right\} u(p_2) k_1^\mu \\ &= -i\frac{g^2}{\sqrt{2}\,|\mathbf{k_1}|}\bar{v}(p_1)\left\{t^a t^b \frac{(-\not{p}_1 - m) + \not{k}_1}{\not{k}_1 - \not{p}_1 - m}\gamma^\nu + t^b t^a \gamma^\nu \frac{(-\not{p}_2 + m) + \not{k}_1}{\not{p}_2 - \not{k}_1 - m}\right\} u(p_2) \\ &= -i\frac{g^2}{\sqrt{2}\,|\mathbf{k_1}|}\bar{v}(p_1)\underbrace{\left[t^a, t^b\right]}_{if^{abc}t^c}\gamma^\nu u(p_2) \equiv C^{abc}\,\bar{v}t^c\gamma^\nu u\,, \end{aligned} \tag{8}$$

i.e. $C^{abc} = g^2 f^{abc}/(\sqrt{2}\,|\mathbf{k_1}|)$ [1.5 pts]. In the second step, the inserted terms in parentheses vanish because of the Dirac equation; in order to cancel the denominator, as in the next step, you may have to use momentum conservation (depending on whether you had expressed it in terms of $\not{k}_1$ or $\not{k}_2$).

For the s-channel, we find instead

$$\begin{aligned}
\mathcal{M}_s^{\mu\nu}\epsilon_\mu^{+*}(k_1) &= C^{abc}\bar{v}(p_1)\gamma_\sigma t^c u(p_2)\frac{1}{k_3^2}k_{1\mu}\Big\{g^{\mu\nu}(k_2-k_1)^\sigma + g^{\nu\rho}(k_3-k_2)^\mu + g^{\sigma\mu}(k_1-k_3)^\nu\Big\} \\
&= \frac{C^{abc}}{k_3^2}\bar{v}(p_1)\gamma_\sigma t^c u(p_2)\Big\{k_1^\nu(k_2-k_1)^\sigma + g^{\nu\rho}(k_3-k_2)\cdot k_1 + k_1^\sigma(k_1-k_3)^\nu\Big\} \\
&= \frac{C^{abc}}{k_3^2}\bar{v}(p_1)\gamma_\sigma t^c u(p_2)\Big\{k_1^\nu k_2^\sigma + g^{\nu\sigma}(k_3-k_2)\cdot k_1 - k_1^\sigma k_3^\nu\Big\} \qquad (9)
\end{aligned}$$

Using momentum conservation, we can simplify as follows:

- $(k_3-k_2)\cdot k_1 = -(k_3-k_2)\cdot(k_3+k_2) = k_2^2 - k_3^2$
- $k_1^\nu k_2^\sigma - k_1^\sigma k_3^\nu = -(k_2+k_3)^\nu k_2^\sigma + (k_2+k_3)^\sigma k_3^\nu = -k_2^\nu k_2^\sigma (+k_3^\sigma k_3^\nu)$

Following the hint, furthermore, we can omit the last term in parenthesis because k_3 dotted into the fermion current vanishes. Overall, this gives the claimed structure: **[1.5 pts for any correct derivation of this form]**

$$\mathcal{M}_s^{\mu\nu}\epsilon_\mu^{+*}(k_1) = \frac{C^{abc}}{k_3^2}\bar{v}(p_1)\gamma_\sigma t^c u(p_2)\Big\{g^{\nu\sigma}(k_2^2-k_3^2) - k_2^\nu k_2^\sigma\Big\}\,. \qquad (10)$$

c) On-shell gauge bosons with momentum k satisfy $k^2 = 0$ and $k\cdot\epsilon^T = 0$ **[0.5 pt each]**. When multiplying Eq. (10) with any of the two $\epsilon_\nu^T(k_2)$, this therefore gives

$$\mathcal{M}_s^{\mu\nu}\epsilon_\mu^{+*}(k_1)\epsilon_\nu^T(k_2) = -C^{abc}\,\bar{v}t^c \not{\epsilon}^T u\,, \qquad (11)$$

i.e. exactly the same, up to a minus sign, as the result from the $t+u$ channel **[1 pt]**. One would generically expect a vanishing amplitude because one of the final states (the one with the '+' polarization) is unphysical **[1 pt]**. Note that $C^{abc}=0$ in the Abelian case, where only the $t+u$ channels contribute; in this case, the amplitude already vanishes in the previous step, problem b), as a direct consequence of the Ward identity **[1 bonus point for noting this]**.

d) Starting from the result as stated in b), we get

$$\begin{aligned}
\mathcal{M}^{\mu\nu}\epsilon_\mu^{+*}(k_1)\epsilon_\nu^{-*}(k_2) &= \frac{C^{abc}}{k_3^2}(\bar{v}t^c\gamma_\sigma u)\Big\{g^{\nu\sigma}k_3^2 + g^{\nu\sigma}(\underbrace{k_2^2}_{0} - k_3^2) - k_2^\nu k_2^\sigma\Big\}\epsilon_\nu^{-*}(k_2) \\
&= -\frac{C^{abc}}{k_3^2}(\bar{v}t^c \not{k}_2 u)\,k_2\cdot\epsilon_\nu^{-*}(k_2)\,.\textbf{[1 pt]} \qquad (12)
\end{aligned}$$

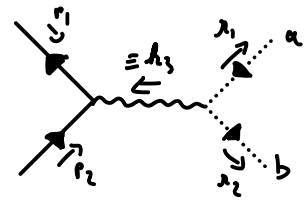
Using again the on-shell condition, $k_2^2 = (k_2^0)^2 - (\mathbf{k}_2)^2 = 0$, we can evaluate

$$k_2\cdot\epsilon_\nu^{-*}(k_2) = \frac{1}{\sqrt{2}\,|\mathbf{k}_2|}(k_2^0,\mathbf{k}_2)\cdot(k_2^0,-\mathbf{k}_2) = \frac{(k_2^0)^2+(\mathbf{k}_2)^2}{\sqrt{2}\,|\mathbf{k}_2|} = \sqrt{2}\,|\mathbf{k}_2|\,, \qquad (13)$$

which does *not* vanish **[1 pt; only 0.5 pt without final simplification]**. Linear combinations of time-like and longitudinal polarizations are unphysical. The amplitude for such on-shell states hence does that not correspond to any real, physical process; therefore, the fact that it formally does not vanish is in principle of no concern **[1 pt]**. Finally, using the explicit form for C^{abcd} derived earlier, we can further simplify this amplitude as

$$i\mathcal{M}_{+-} \equiv i\mathcal{M}^{\mu\nu}\epsilon_\mu^{+*}\epsilon_\nu^{-*} = -g^2 f^{abc} k_3^{-2} \frac{|\mathbf{k_2}|}{|\mathbf{k_1}|} (\bar{v} t^c \not{k}_2 u)\,. \tag{14}$$

e) As indicated in the figure, only the s-channel diagram contributes **[1 pt]**. With the hint, combined with Eq.(4) in the exam, the Feynman rule for the ghost vertex follows from $\mathcal{L} \subset -g f^{abc} \bar{c}^a \partial^\mu A_\mu^b c^c$ as $-g f^{abc} k_2^\mu$ **[1 pt]**.

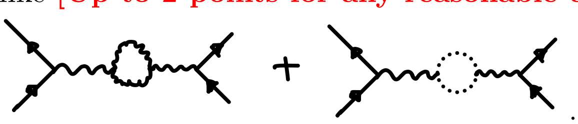


The amplitude thus reads **[1 pt for the final form]**

$$i\mathcal{M} = ig\bar{v}(p_1)\gamma_\mu t^c u(p_2) \frac{-i}{k_3^2}(-g) f^{abc} k_2^\mu = -\frac{g^2}{k_3^2} f^{abc} \bar{v} \not{k}_2 t^c u\,. \tag{15}$$

From the previous results we note that this can be written as $\mathcal{M} = (|\mathbf{k_1}|\,/\,|\mathbf{k_2}|)\mathcal{M}_{+-} = (|\mathbf{k_2}|\,/\,|\mathbf{k_1}|)\mathcal{M}_{-+}$.

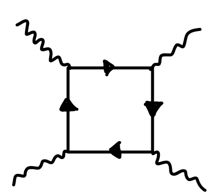
f) Neither ghosts nor gauge bosons with unphysical polarizations can appear in external states, so the results are physically *only* relevant for off-shell momenta; for ghosts, these appear necessarily in loops, adding a factor of (-1) due to their anti-commuting nature. At a technical level, the role of ghost loops is thus to cancel the unphysical degrees of freedom contributing to gauge boson loops, e.g. in the sum of diagrams like **[Up to 2 points for any reasonable explanation]**



Problem 3

a) The Higgs field would still break the (now global) symmetry as $SU(2)_L \times U(1)_Y \to U(1)$, and not affect $SU(3)$ **[1 pt]**. The number of *global* symmetries that are broken is therefore $4 - 1 = 3$, so there would be 3 massless Gauge bosons **[1 pt]**.

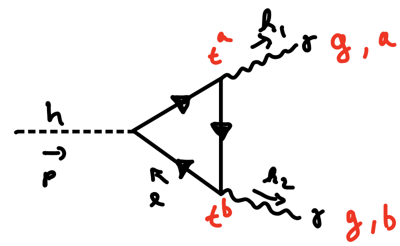
b) Light-by-light scattering is mediated by electron loops, as shown in the figure **[1 pt for any valid diagram]**. The presence of four photon vertices implies that the amplitude scales as e^4, and hence $n = 4$ for the cross section **[1 pt]**.



Exchanging the electron in the loop with a fermion f of charge Q_f introduces an additional factor of Q_f per vertex (since $Q = -1$ for electrons). Since there are no diagrams (at lowest order) with different fermions, the enhancement factor is given by $\sum_f Q_f^4$ for the amplitude, and $(\sum_f Q_f^4)^2$ for the cross section **[1 pt]**. Taking into account that there are three generations of fermions, and that quarks come in three colors, this evaluates in the SM to **[1 pt; 0.5 pt if correct up to either of the factors of three]**.

$$\sum_f Q_f^4 = 3\left[1^4 + 3\left(-\frac{1}{3}\right)^4 + 3\left(\frac{2}{3}\right)^4\right] = \frac{44}{9}\,. \tag{16}$$

c) The Higgs is electrically neutral, so there is no tree-level coupling to photons **[1 pt]**. Since it couples to fermions via the Yukawa coupling $y_f \propto m_f$, the by far largest contribution will come from a top quark loop **[1 pt]**. One of the two diagrams is depicted in the figure, the other is obtained by crossing the final photon legs (or reversing the fermion arrows).



[1 pt for a fully correct diagram, another 1 pt if the other is correctly described or drawn]

Including relevant couplings, the fermion loop has the structure

$$A_{\text{photon}} \equiv 3 y_t e^2 \text{Tr}[S_F(l)\gamma^\mu S_F(l+k_2)\gamma^\nu S_F(l-p)] \tag{17}$$

in the case of photons, with S_F the top propagator and l the loop momentum (see also figure). The factor 3 accounts for the fact that there are three top colors to be taken into account. This is replaced by

$$A_{\text{gluon}} \equiv y_t g^2 \text{Tr}[S_F(l)\gamma^\mu t^b S_F(l+k_2)\gamma^\nu t^a S_F(l-p)] \tag{18}$$

in the case of gluon emission (red labels in the figure), where the t^a are the $SU(3)$ generators in the fundamental representation **[0.5 pt each for correct placement of t^a, for $e \to g$ and the factor of 3]**. Note that the matrix elements are identical, otherwise (apart from the fact that the gluon polarization vectors also carry a color index). Using that the quark propagators are diagonal in color space, we can simplify as follows:

$$\frac{3\alpha_{\text{em}}}{\alpha_s} A_{\text{gluon}} = \text{tr}[t^b t^a] A_{\text{photon}} = \delta^{ab} \times C(r) A_{\text{photon}} = \frac{1}{2}\delta^{ab} A_{\text{photon}} \;\textbf{[1 pt]}. \tag{19}$$

Here, the lower case 'tr' indicates a trace only in color space (which gave a factor of 3 in the photon case). Squaring, and summing over all final states thus gives

$$\sum |\mathcal{M}_{\text{gluon}}|^2 = \frac{\alpha_s^2}{9\alpha_{\text{em}}^2} A_{\text{gluon}} \sum |\mathcal{M}_{\text{photon}}|^2 \times \frac{1}{4} \underbrace{\delta^{ab}\delta^{ab}}_{=\delta^{aa}=d(G)=8}\,. \tag{20}$$

In total, we thus find $\Gamma_{h\to gg}/\Gamma_{h\to\gamma\gamma} = \frac{2}{9}\alpha_s^2/\alpha_{\text{em}}^2 \sim \frac{2}{9} \times 0.12^2/137^{-2} \sim 2 \times 10^{-7}$. **[0.5 pt for final result, bonus 0.5 for correct order-of-magnitude estimate]**