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Lecture spring 2024:

Quantum field theory II

Compulsory problem set – solutions

Problem I

a) Without taking into account the necessity to add additional terms, the theory exhibits 3 counter terms, corresponding to two field re-normalizations and one coupling re-normalization, respectively. They can most conveniently be introduced by writing the bare field and couplings as

$$\psi^{(0)} \equiv \sqrt{1+\delta_\psi}\,\psi, \qquad \phi^{(0)} \equiv \sqrt{1+\delta_\phi}\,\phi, \qquad y^{(0)} \equiv \frac{(1+\delta_y)}{(1+\delta_\psi)\sqrt{1+\delta_\phi}}\,y, \tag{1}$$

which results in the renormalized Lagrangian being

$$\mathcal{L}_\psi = i\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}\phi\Box\phi + y\phi\overline{\psi}\psi + i\delta_\psi\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}\delta_\phi\phi\Box\phi + \delta_y\phi\overline{\psi}\psi\,. \tag{2}$$

The infinite parts of these counterterms are uniquely determined by requiring the self-energies and 3-point coupling to be finite.

Starting with the fermion self-energy, we have at leading order[1]

$$i\mathcal{M} \equiv -i\Sigma(p\!\!\!/) = i\delta_\psi p\!\!\!/ - i\Sigma_2(p\!\!\!/) + \mathcal{O}(\alpha^2)\,, \tag{3}$$

where the first term is the counter term contribution and the second the loop contribution. Adapting (18.6) to the case of a Yukawa coupling (and scalar rather than vector propagator), we can write (with $d = 4-\epsilon$)

$$\begin{aligned} -i\Sigma_2(p\!\!\!/) &= y^2\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k\!\!\!/}{k^2+i\varepsilon}\frac{1}{(p-k)^2+i\varepsilon} && (4)\\ &= y^2\int_0^1 dx\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k\!\!\!/}{\left[k^2(1-x)+(p-k)^2x+i\varepsilon\right]^2} && (5)\\ &= y^2\int_0^1 dx\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{xp\!\!\!/}{\left[k^2-\Delta+i\varepsilon\right]^2}\,, && (6)\end{aligned}$$

where $\Delta = -(1-x)p^2x$ and we completed the square by shifting $k \to k + px$ in order to arrive at this expression. Hence,

[1] Note that the sign of Σ is simply a definition – different in P&S and Schwartz – but it is useful to write the total amplitude (at least once) in order to check *relative* signs for the counter terms.

$$-i\Sigma_2(\not p) = \not p\, y^2 \int_0^1 dx\, x \underbrace{\mu^{-\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-\Delta+i\varepsilon]^2}}_{\overset{(B.45)}{=} i\frac{1}{16\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)+\ln(\mu^2/\Delta)+\mathcal{O}(\epsilon)\right]} \tag{7}$$

$$= i\not p\frac{\alpha}{4\pi}\frac{1}{2}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)+\ln\frac{\mu^2}{p^2}-\underbrace{\int_0^1 dx\, x\ln\big(x(x-1)\big)}_{-1-i\pi}\right] \tag{8}$$

Note that in the last step we had to be careful with the expression $\ln(\Delta)$ since $\Delta < 0$. Remembering that Δ really enters as $\Delta - i\varepsilon$ in (B.45), we have used $\ln(\Delta) = \ln(\Delta - i\varepsilon) = \ln|\Delta| - i\pi$. We can now read off the $\overline{\text{MS}}$ counter term as

$$\delta_\psi = -\frac{\alpha}{8\pi}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right]. \tag{9}$$

For the scalar self-energy, correspondingly, we have

$$-i\Pi(p^2) = i\delta_\phi p^2 - i\Pi_2(p^2) + \mathcal{O}(\alpha^2), \tag{10}$$

with

$$-i\Pi_2(p^2) = y^2\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\frac{\not k}{k^2+i\varepsilon}\frac{\not p-\not k}{(p-k)^2+i\varepsilon}\right] \tag{11}$$

$$= y^2\int_0^1 dx\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\frac{x(1-x)\not p\not p-\not k\not k}{[k^2-\Delta+i\varepsilon]^2}\right] \tag{12}$$

$$\overset{(B.45)}{=} -i\frac{\alpha}{4\pi}\int_0^1 dx\,\Delta\,(4-\epsilon)\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)+\ln(\mu^2/\Delta)\right]$$
$$-y^2\int_0^1 dx\,\underbrace{\mu^{-\epsilon}(4-\epsilon)\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{[k^2-\Delta+i\varepsilon]^2}}_{\overset{(B.37)}{=} i\frac{\Delta}{2\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)+\ln(\mu^2/\Delta)\right]} \tag{13}$$

$$= -i\frac{\alpha}{4\pi}12\int_0^1 dx\,\Delta\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)-\frac{1}{3}+\ln(\mu^2/\Delta)\right] \tag{14}$$

$$= ip^2\frac{\alpha}{2\pi}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{p^2}+\frac{4}{3}-6i\pi\right] \tag{15}$$

where in the last step we used $\int dx(1-x)x\ln(x^{-1}(x-1)^{-1}) = \frac{5}{18}-i\pi$, similar to above. In the $\overline{\text{MS}}$ scheme, we thus get

$$\delta_\phi = -\frac{\alpha}{2\pi}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right]. \tag{16}$$

Finally, let's look at the 3-point coupling. In analogy to the vertex correction in QED, cf. Eq. (17.15), counter term and loop sum to the 3-point amplitude as $i\mathcal{M} \equiv i\overline{u}(y+\Gamma_y)u$, with

$$i\Gamma_y = iy\delta_y + (iy)^3\mu^{-\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i}{(k-q_1)^3+i\varepsilon}\frac{i(\not p+\not k)}{(k+p)^2+i\varepsilon}\frac{i\not k}{k^2+i\varepsilon} \tag{17}$$

$$= iy\delta_y - 2y^3\int_0^1 dxdydz\,\delta(-1+x+y+z)\,\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{\not k^2+\not p(z\not q_1-y\not p)}{(k^2-\Delta+i\varepsilon)^3} \tag{18}$$

where now $\Delta = -xyp^2$ as in Eq. (17.21). Being only interested in the infinite parts, things simplify considerably:

$$\delta_y = \text{finite} - 2iy^2\underbrace{\int_0^1 dxdydz\,\delta(-1+x+y+z)}_{1/2}\underbrace{\mu^{-\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{d\,k^2}{(k^2+i\varepsilon)^3}}_{\stackrel{(B.38)}{\to}\frac{i}{4\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)-1\right]} \tag{19}$$

$$\stackrel{\overline{\rm MS}}{\to} 4\frac{\alpha}{4\pi}\left[\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right]. \tag{20}$$

b) The kinetic term for the fermions exhibits a chiral symmetry

$$P_L\psi \to e^{i\beta_L}P_L\psi\,, \tag{21}$$
$$P_R\psi \to e^{i\beta_R}P_R\psi\,, \tag{22}$$

which would be broken in the presence of a (Dirac) mass term $m_\psi(\psi_R\psi_L+\psi_L\psi_R)$. The full theory still has a discrete remnant of this symmetry:

$$\begin{aligned} P_L\psi &\to e^{i\beta_L}P_L\psi\,,\\ P_R\psi &\to e^{i(\pi+\beta_L)}P_R\psi\,,\\ \phi &\to -\phi\,, \end{aligned} \tag{23}$$

which would also be broken by a mass term. Even in the presence of an explicit mass term, however, this *custodial symmetry* protects against parametrically large radiative corrections. In practice, this implies that radiative corrections to the mass will be proportional to m_ψ – which is the formal requirement to make it 'technically natural' to have a small parameter m_ψ (small w.r.t. other dimensionfull quantities, in particular – see below – the scalar mass).

c) For the scalar mass, there is *no* corresponding argument, i.e. no symmetry argument that one could use to argue for a very small or vanishing mass. Indeed, as we will see in problem II, the presence of a vanishing scalar mass is an artefact of mistakenly not stating the full theory: even if the $\overline{\rm MS}$ mass would vanish at *some* energy scale, it will not do so at other energies.

Problem II

a) At 1-loop level a ϕ^4 operator is generated (via fermion loops). These diagrams have superficial degrees of divergence of $D=0$, and there is no symmetry that would force a scalar 4-point amplitude to be zero. Those loops thus diverge, and are hence not calculable – the scalar 4-point coupling is genuine free parameter of the theory and must be determined by measurement. This means that we have to add a quartic self-interaction term to the Lagrangian,

$$\mathcal{L} \to \mathcal{L} - \frac{\lambda}{4!}\phi^4 \,, \tag{24}$$

with a corresponding counterterm to cancel those divergences (note that we here absorb the field strength renormalizations of ϕ into the definition of δ_λ):

$$\mathcal{L}^{\text{c.t.}} \to \mathcal{L}^{\text{c.t.}} - \frac{1}{4!}\delta_\lambda \phi^4 \,. \tag{25}$$

A cubic scalar self-interaction, ϕ^3, would be the only other Lorentz-invariant option (apart from mass-terms, see the next problem) – but is not consistent with the symmetry stated in Eq. (23). (At the technical level, the corresponding loop of three massless fermions would involve a trace of three gamma matrices, which vanishes identically.)

b) For the *fermion self-energies*, the newly introduced interactions do not contribute at leading order. The discussion of fermion masses is thus not affected.

For the *scalar self-energies*, on the other hand, there are now two types of 1-loop contributions:

$$i\Pi_2(p^2) = i\Pi_{2,\psi}(p^2) + i\Pi_{2,\lambda}(p^2) \,, \tag{26}$$

where the first term is the fermion loop contribution that we calculated in the previous problem. The other diagram will introduce a divergence that is *not* proportional to p^2 (or any other power of p^2); to cancel it, we thus have to follow the same reasoning as above and accept that the scalar mass is indeed a free parameter of the theory – that needs to be measured and properly be accounted for. In other words, we have to further enlarge the Lagrangian by

$$\mathcal{L} \to \mathcal{L} - \frac{1}{2}m_\phi^2\phi^2 \quad \text{and} \quad \mathcal{L}^{\text{c.t.}} \to \mathcal{L}^{\text{c.t.}} - \frac{1}{2}(\delta_m + \delta_\phi) m_\phi^2 \phi^2 \,. \tag{27}$$

Concretely, we have a scalar loop attached to a 4-point vertex (with only one propagator):

$$-i\Pi_{2,\lambda} = \frac{1}{2}\lambda\,\mu^{-\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2 - m_\phi^2 + i\varepsilon} \tag{28}$$

$$\overset{(B.35)}{=} -i\frac{\lambda}{32\pi^2}m_\phi^2\left[\frac{2}{\epsilon} - \gamma_E + 1 + \ln\frac{4\pi\mu^2}{m_\phi^2}\right] . \tag{29}$$

As mentioned in the problem, only $\Pi_{2,\psi}$ has a non-vanishing imaginary part – which, according to the *optical theorem* indicates that the loop particles can go on-shell. Indeed, the decay $\phi \to \psi\bar{\psi}$ is always kinematically possible for massless fermions. The tadpole diagram $\Pi_{2,\lambda}$, on the other hand, cannot be 'cut', explaining the absence of imaginary contributions.

c) The *actual* Lagrangian of the theory introduced above, containing all parameters that must be determined by measurements, is given by

$$\mathcal{L}_{\phi,\psi} = i\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}\phi(\Box + m_\phi^2)\phi + y\phi\overline{\psi}\psi - \frac{\lambda}{4!}\phi^4 \,. \tag{30}$$

The MS counterterms of the dim-2 operators are

$$\delta_\psi \stackrel{(9)}{=} \frac{\alpha}{4\pi}\frac{2}{\epsilon} \times \left(-\frac{1}{2}\right) \tag{31}$$

$$\delta_\phi \stackrel{(16)}{=} \frac{\alpha}{4\pi}\frac{2}{\epsilon} \times (-2) \tag{32}$$

$$\delta_m + \delta_\phi \stackrel{(29)}{=} \frac{1}{16\pi^2}\frac{2}{\epsilon} \times \left(-\frac{1}{2}\lambda\right)$$

$$\rightsquigarrow \delta_m = \frac{1}{16\pi^2}\frac{2}{\epsilon} \times \left(2\alpha - \frac{1}{2}\lambda\right) \tag{33}$$

The divergence structure of the 3-point amplitude is unaffected by the presence of a scalar mass, hence

$$\delta_y \stackrel{(20)}{=} 4\frac{\alpha}{4\pi}\frac{2}{\epsilon} \,. \tag{34}$$

We thus only need to calculate corrections to the scalar 4-point function, receiving contributions from fermion and scalar loops, respectively:

$$i\mathcal{M}_4 = -i\lambda - i\delta_\lambda + i\mathcal{M}_{4,\psi} + i\mathcal{M}_{4,\lambda} \,. \tag{35}$$

For a single fermion loop diagram, we have (where p_1 & p_2 are ingoing, p_3 & p_4 outgoing momenta)

$$\begin{aligned} i\mathcal{M}_{4,\psi}^{\text{single}} &= -(iy)^4 \mathrm{Tr}\left[\int \frac{d^4k}{(2\pi)^4}\frac{ik\!\!\!/}{k^2+i\epsilon}\frac{i(k\!\!\!/ + p\!\!\!/_1)}{(k+p_1)^2+i\epsilon}\frac{i(k\!\!\!/ + p\!\!\!/_1 - p\!\!\!/_3)}{(k+p_1-p_3)^2+i\epsilon}\frac{i(k\!\!\!/ - p\!\!\!/_2)}{(k-p_2)^2+i\epsilon}\right] \\ &\stackrel{k\to\infty}{\sim} -y^4\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[k\!\!\!/ k\!\!\!/ k\!\!\!/ k\!\!\!/\right]}{k^8+i\epsilon} \end{aligned} \tag{36}$$

$$= -y^4 d\,\mu^\epsilon \int \frac{d^dk}{(2\pi)^4}\frac{1}{k^4+i\epsilon} = -i\frac{y^4}{4\pi^2}\frac{2}{\epsilon} \tag{37}$$

For the entire amplitude, we have to multiply by a factor 2 from reversing the fermion arrow direction. Further independent(!) diagrams arise from exchanging $p_1 \leftrightarrow p_2$ as well as $p_1 \leftrightarrow p_3$. In total there are thus 6 different diagrams – which however all contribute the same amount to the divergent part of the amplitude.

For a single scalar loop diagram, we have (note the symmetry factor!)

$$i\mathcal{M}^{\text{single}} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_\phi^2+i\epsilon}\frac{i}{(k+p_1+p_2)^2-m_\phi^2+i\epsilon} \tag{38}$$

$$\stackrel{k\to\infty}{\sim} \frac{\lambda^2}{2}\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4+i\epsilon} = i\frac{1}{8}\frac{\lambda^2}{4\pi^2}\frac{2}{\epsilon} \tag{39}$$

This result has to be multiplied by 3, since there are three different diagrams that we can get by exchanging $p_2 \to p_3$ and $p_2 \to p_4$, respectively. In total, we thus find

$$\delta_\lambda = \frac{1}{16\pi^2}\frac{2}{\epsilon} \times \left(\frac{3}{2}\lambda^2 - 24y^4\right) \tag{40}$$

in the MS scheme.

The theory only has dim ≤ 4 operators and is hence renormalizable. If a dim > 4 operator existed explicitly in the Lagrangian, the theory would not be renormalizable. Hence, all dim > 4 operators must be finite and calculable (as functions of the parameters in the stated Lagrangian).

d) In this sub-problem, we denote with $\tilde{m}_\phi$ the pole mass and with $\overline{m}_\phi$ the $\overline{\text{MS}}$ mass of the scalar, respectively. The former is defined by the on-shell conditions

1.
$$0 = \frac{d}{dp^2}\Pi(p^2 = \tilde{m}_\phi^2) \tag{41}$$
$$= -\delta_\phi + \frac{d}{dp^2}\Pi_{2,\psi}(p^2 = \tilde{m}_\phi^2) \tag{42}$$
$$\stackrel{(15)}{=} -\delta_\phi + \frac{\Pi_{2,\psi}(p^2 = \tilde{m}_\phi^2)}{\tilde{m}_\phi^2} - \underbrace{p^2 \frac{\alpha}{2\pi}\partial_{p^2} \ln p^{-2}}_{-\alpha/(2\pi)} \tag{43}$$
$$\rightsquigarrow \delta_\phi \stackrel{(15)}{=} -\frac{\alpha}{2\pi}\left[\frac{2}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{\tilde{m}_\phi^2} + \frac{1}{3} - 6i\pi\right] \tag{44}$$

2.
$$0 = \Pi(p^2 = \tilde{m}_\phi^2) \tag{45}$$
$$= \delta_m \tilde{m}_\phi^2 + \Pi_{2,\psi}(p^2 = \tilde{m}_\phi^2) + \Pi_{2,\lambda} + \Pi_{2,\mu}(p^2 = \tilde{m}_\phi^2)\,, \tag{46}$$

resulting in

$$\delta_m \stackrel{(15,29)}{=} \frac{\alpha}{2\pi}\left[\frac{2}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{\tilde{m}_\phi^2} + \frac{1}{3} - 6i\pi\right] - \frac{\lambda}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma_E + 1 + \ln\frac{4\pi\mu^2}{\tilde{m}_\phi^2}\right] \tag{47}$$

The pole mass satisfies by definition

$$\tilde{m}_\phi^2 = m_\phi^2 + \Pi(p^2 = \tilde{m}_\phi^2) = m_\phi^2 + \Pi_2(p^2 = \tilde{m}_\phi^2) - \tilde{m}_\phi^2\delta_\phi + m_\phi^2(\delta_\phi + \delta_m) \tag{48}$$

in *any* scheme. In the on-shell scheme, we have

$$0 = \Pi^{\text{o.s.}}(p^2 = \tilde{m}_\phi^2) = \Pi_2^{\text{o.s.}}(p^2 = \tilde{m}^2) + \tilde{m}_\phi^2\delta_m^{\text{o.s.}}\,. \tag{49}$$

Hence, we can 'insert a zero' and write, to leading order,

$$\tilde{m}_\phi^2 - m_\phi^2 = \underbrace{(\Pi_2 - \Pi_2^{\text{o.s.}})_{p^2=\tilde{m}^2}}_{\approx 0} - \tilde{m}_\phi^2\delta_m^{\text{o.s.}} - \tilde{m}_\phi^2\delta_\phi + m_\phi^2(\delta_\phi + \delta_m) \approx m_\phi^2(\delta_m - \delta_m^{\text{o.s.}})\,, \tag{50}$$

where the difference between the Π_2 terms is only a higher-order correction (Π_2, δ_i and Δm between different schemes are all quantities at the same leading order in perturbation theory). Applied to the $\overline{\text{MS}}$ scheme, and using Eqs. (33,47), this gives directly

$$\frac{\overline{m}_\phi^2 - \tilde{m}_\phi^2}{\tilde{m}_\phi^2} = \frac{\alpha}{2\pi}\left[\ln\frac{\mu^2}{\tilde{m}_\phi^2} + \frac{1}{3} - 6i\pi\right] - \frac{\lambda}{32\pi^2}\left[\ln\frac{\mu^2}{\tilde{m}_\phi^2} + 1\right], \qquad (51)$$

where we have again used $\tilde{m}_\phi^2 \simeq \overline{m}_\phi^2$ to leading order on the r.h.s..

We thus find, again to leading order

$$\frac{\overline{m}_\phi^2(\mu) - \overline{m}_\phi^2(\mu_0)}{\overline{m}_\phi^2(\mu_0)} = \left(\frac{\alpha}{2\pi} - \frac{\lambda}{32\pi^2}\right)\ln\frac{\mu^2}{\mu_0^2} \qquad (52)$$

Notably, this is a *finite* expression that predicts the relevant mass parameter at energy scale μ as a function of an input ('measured') value $\overline{m}_\phi^2(\mu)$.

Problem III

a) The Lagrangian from problem I was symmetric under the parity transformation P, with

$$\phi(t,\mathbf{x}) \to P\phi(t,\mathbf{x})P = \phi(t,-\mathbf{x})\,. \qquad (53)$$

In particular – see p. 71 in P&S – we have $P\bar\psi\psi P = \bar\psi\psi$ and $P\bar\psi\partial\!\!\!/\psi P = \bar\psi\partial\!\!\!/\psi$.

The operator $i\bar\psi\gamma\psi$, on the other hand, has *negative* parity (and wouldn't be Hermitian without the i): $Pi\bar\psi\gamma\psi P = -i\bar\psi\gamma\psi$. This means that our new Lagrangian is still symmetric under P, iff ϕ is a *pseudoscalar* rather than a scalar:

$$P\phi(t,\mathbf{x})P = -\phi(t,-\mathbf{x})\,. \qquad (54)$$

All *other* symmetries from the previous problem(s) remain unaffected. In particular, the symmetry stated in Eq. (23) still holds because $\overline{\psi}\gamma^5\psi = \overline{\psi}_L\psi_R - \overline{\psi}_R\psi_L$.

This also implies that there are no qualitative changes, i.e. the interaction will generate the same new terms as before. Quantitatively, there will be an additional factor of $i\gamma^5$ for any vertex involving y. However, all results involve this coupling *twice*, entering in the Feynman rules in the form of $y\gamma^\mu y \to (iy\gamma^5)\gamma^\mu(iy\gamma^5) = y^2\gamma^\mu$. Hence, *none of the results changes*, not even by a sign.

b) Adding fermion masses affects all loops containing fermion propagators, by adding an explicit m_ψ-dependence in the respective Δ term in the Feynman parameterization. This is the only change to the loop calculations presented above.

The most important *qualitative* change is that a fermion mass term breaks the (residual) chiral symmetry stated in Eq. (23). For the theory studied in problems I&II, this means that a scalar 3-point coupling is no longer protected by any symmetry. At the technical level, the corresponding fermion loop will be proportional to $\mathrm{Tr}[(\not k - m_\psi)(\not k - \not p_1 - m_\psi)(\not k + \not p_2 - m_\psi)] \propto m_\psi \mathrm{Tr}[\gamma^\mu\gamma^\nu]$ and no longer vanish. In particular, the $-3m_\psi \mathrm{Tr}[\not k^2] = -12 m_\psi k^2$ part of that trace will results in a log-divergent contribution to the loop-integral – making it necessary to add a term $\frac{1}{3!}\mu_\phi\phi^3$ to the Lagrangian, including a counterterm, in full analogy to the case of the 4-point interaction that was discussed in detail above.

A scalar three-point coupling would however *not* be symmetric under the new symmetry P as stated in Eq. (54), and hence cannot be generated in this theory. At the technical level, this becomes manifest by noting that we would now have $\mathrm{Tr}[\gamma^5(\not k - m_\psi)\gamma^5(\not k - \not p_1 - m_\psi)\gamma^5(\not k + \not p_2 - m_\psi)] \propto m_\psi \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0$.

c) A fermion mass term would be generated as a contribution $\Sigma(\not p) \propto m_\psi$ to the fermion self-energy. We have calculated $\Sigma(\not p)$ at one-loop level and explicitly seen that no such contribution is generated. In fact, we also understood that the reason is the symmetry of Eq. (23). At *any* higher-loop level, there can thus also not be any contributions that *only* involve the couplings studied so far. The lowest-order contribution involving the new coupling is a tadpole diagram:

$$-i\Sigma \supset -i\Sigma_{\rm tad} = \frac{1}{2} i\mu_\phi (iy) \frac{i}{0^2 - m_\phi^2} \underbrace{\mu^{-\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m_\phi^2}}_{\frac{m_\phi^2}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + 1 + \ln\frac{4\pi\mu^2}{m_\phi^2}\right)} \tag{55}$$

$$= i\frac{y\mu_\phi}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + 1 + \ln\frac{4\pi\mu^2}{m_\phi^2}\right). \tag{56}$$

Note that there is no corresponding tadpole diagrams for massless fermions, since the fermion loop provides an additional factor of $\mathrm{Tr}[\not k + m_\psi] = 4m_\psi$.

We do however create a *one-point function* $\langle\phi\rangle$, with

$$\langle\phi(p)\rangle = \frac{i}{p^2 - m_\phi^2}(i\mathcal{M}_1) = \frac{i\mathcal{M}_1}{im_\phi^2}, \tag{57}$$

where $p = 0$ because of momentum conservation. In analogy to the expression above, the amplitude is calculated as

$$i\mathcal{M}_1 = \frac{\mu_\phi m_\phi^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + 1 + \ln\frac{4\pi\mu^2}{m_\phi^2}\right). \tag{58}$$

The fact that $v \equiv \langle\phi\rangle \neq 0$ indicates *spontaneous symmetry breaking* – implying that we should shift the scalar field as $\phi \equiv \Phi - v$ in order to describe a *physical* scalar field Φ. Notably, this generates a fermion mass term $m_\psi = yv$ directly from the Yukawa coupling (as discussed in the context of the Higgs mechanism). The fact that this fermion mass is identical to the one that follows from Eq. (56) is both re-assuring and required by consistency: the fermion mass in *any* scheme should be independent of whether we choose to describe our theory in terms of ϕ or Φ.