Chapter 1

The algebraic origin of SUSY

The goal of these lectures is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as a main example. Rather than starting with the problems of the SM, we will focus on the algebraic origin of SUSY in the sense of an extension of the symmetries of Einsten's Special Relativity (SR), which was the original motivation for SUSY.

1.1 What is a group?

Definition: The set $G = \{g_i\}$ and operation $\bullet$ form a **group** if and only if for $\forall\, g_i \in G$

i) $g_i \bullet g_j \in G$ (closure)

ii) $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$ (associativity)

iii) $\exists e \in G$ such that $g_i \bullet e = e \bullet g_i = g_i$ (identity element)

iv) $\exists g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = e$ (inverse)

A simple example of a group is $G = \mathbb{Z}$ with usual addition as the operation, $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition with modulo n. In this group, $g_i^{-1} = n - g_i$ and the unit element is $e = 0$. Note that $\mathbb{Z}$ is an *infinite* group, while $\mathbb{Z}_n$ is finite, with *order* n (meaning n members). Both are *abelian* groups, meaning that $g_i \bullet g_j = g_j \bullet g_i$.

All of this is "only" mathematics. Physicists are often more interested in groups where the elements of G *act* on some elements of a set $s \in S$, $g(s) = s' \in S$.[1] S here can for example be the state of a system, say a wave-function in quantum mechanics. We will return to this in a moment, let us just mention that the operation $g_i \bullet g_j$ acts as $(g_i \bullet g_j)(s) = g_i \bullet (g_j(s))$ and the identity acts as $e(s) = s$.[2]

[1] As a result mathematics courses in group theory are not always so relevant to a physicist.

[2] We can prove this from iii) in the definition. Note that we use e as the identity in an abstract group, while

A more sophisticated example of a group can be found in a use for the Taylor expansion[3]

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} f(x) \\ &= e^{a\frac{d}{dx}} f(x) \end{aligned}$$

The operator $T_a = e^{a\frac{d}{dx}}$ is called the **translation operator** (in this case in one dimension). Together with the operation $T_a \bullet T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where $T_a^{-1} = T_{-a}$. In N dimensions the group $T(N)$ has the elements $T_{\vec{a}} = e^{\vec{a}\cdot\vec{\nabla}}$.

Definition: A subset $H \subset G$, is a **subgroup** if and only if:[a]

i) $h_i \bullet h_j \in H$ for $\forall h_i, h_j \in H$

ii) $h_i^{-1} \in H$ for $\forall h_i \in H$

[a] An alternative, more compact, way of writing these two requirements is $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in G$. This is often utilised in proofs.

Definition: H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$. A subgroup H is a **normal** (invariant) subgroup, if and only if for $\forall g \in G$,

$$ghg^{-1} \in H \text{ for } \forall h \in H$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no abelian normal subgroup.

The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U, i.e. matrices such that $U^\dagger U = 1$ or $U^{-1} = U^\dagger$. This has the neat property that for $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged:

$$\begin{aligned} \vec{x}' \cdot \vec{y}' &\equiv \vec{x}'^\dagger \vec{y}' = (U\vec{x})^\dagger U\vec{y} \\ &= \vec{x}^\dagger U^\dagger U \vec{y} = \vec{x}^\dagger \vec{y} = \vec{x} \cdot \vec{y} \end{aligned}$$

If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$. Let $U_i, U_j \in SU(n)$, then

$$\det(U_i U_j^{-1}) = \det(U_i) \det(U_j^{-1}) = 1.$$

1 is used as the identity matrix in matrix representations.

[3] This is the first of many points where any real mathematician would start to cry loudly and leave the room.

This means that $U_i U_j^{-1} \in SU(N)$. In other words, $SU(n)$ is a **proper subgroup** of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V)\det(U)\det(V^{-1}) = \frac{\det(V)}{\det(V)}\det(U) = 1.$$

In other words, $SU(n)$ is also a **normal subgroup** of $U(n)$.

> **Definition**: A **(left) coset** of a subgroup $H \subset G$ is a set $\{gh : h \in H\}$ where $g \in G$ and a **(right) coset** of a subgroup $H \subset G$ is a set $\{hg : h \in H\}$ where $g \in G$. For normal subgroups H the left and right cosets coincide and form the **coset group** G/H which has the members $\{gh : h \in H\}$ for $\forall g \in G$ and the binary operation $*$ with $gh * g'h' \in \{(g \bullet g')h : h \in H\}$.

> **Definition**: The **direct product** of groups G and H, $G \times H$, is defined as the *ordered pairs* (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet h_j)$. $G \times H$ is then a group and G and H are normal subgroups of $G \times H$. For **semi-direct products** $G \rtimes H$, H is not a normal subgroup of $G \rtimes H$.

The SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is an example of a direct product. Direct products are "trivial" structures because there is no "interaction" between the subgroups. Can we imagine a group $G \supset SU(3)_c \times SU(2)_L \times U(1)_Y$ that can be broken down to the SM group but has a non-trivial unified gauge structure? There is, $SU(5)$ being one example.

1.2 Representations

> **Definition**: A **representation** of a group G on a vector space V is a map $\rho : G \to GL(V)$, where $GL(V)$ is the **general linear group** on V, i.e. matrices of the field of V, such that for $\forall g_i, g_i \in G$, $\rho(g_i g_j) = \rho(g_i)\rho(g_j)$ (homeomorphism).

For $U(1)$ $e^{i\chi\alpha(x)}$ is a representation on a wavefunction $\psi(x)$ (one dimensional vector space over complex numbers). For $SU(2)$ $e^{i\alpha_i\sigma_i}$, with σ being the Pauli matrices, is the **fundamental representation** on weak doublets $\psi = (\nu_l, l)$.[4]

> **Definition**: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \to V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

[4]This is a bit daft, since both $U(1)$ and $SU(2)$ are defined in terms of matrices. However, we will also have use for other representations, e.g. the **adjoint representation**, which is not the fundamental or defining representation.

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, i.e. there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.[a]

[a] In other words, we can not split the matrix representation of G in two parts that do not "mix".

Let $\rho(g)$ for $g \in G$ act on a vector space V as a matrix. If $\rho(g)$ can be decomposed into $\rho_1(g)$ and $\rho_2(g)$ such that

$$\rho(g)v = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix} v$$

for $\forall v \in V$, then ρ is **reducible**.

Definition: $T(R)$ is the **Dynkin index** of the representation R in terms of matrices T_a, given by $\mathrm{Tr}[T_a, T_b] = T(R)\delta_{ab}$. $C(R)$ is the **Casimir invariant** given by $C(R)\delta_{ij} = (T^a T^a)_i j$

1.3 Lie groups

Definition: A **Lie group** is a finite-dimensional (n) **smooth manifold** C^∞, i.e. for $\forall g \in G$, g can locally be mapped onto (parametrised by) $\mathbb{R}^n$ or $\mathbb{C}^n$, and group multiplication and inversion are smooth functions, meaning that given $g(\vec{a}), g'(\vec{a}) \in G$, $g(\vec{a}') \bullet g'(\vec{a}') = g''(\vec{b})$ where $\vec{b}(\vec{a}, \vec{a}')$ is analytic, and $g^{-1}(\vec{a}) = g'(\vec{a}')$ where $\vec{a}'(\vec{a})$ is analytic.
In terms of a Lie group G acting on a vector space V, $\dim(V) = m$ (or more generally an m-dimensional manifold), this means we can write the map $G \times V \to V$ for $\vec{x} \in V$ as $x_i \to x_i' = f_i(x_i, a_j)$ where f_i is analytic in x_i and a_j. Additionally f_i should have an inverse.

The translation group $T(1)$ with $g(a) = e^{a\frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is analytic. Here we can write $f(x, a) = x + a$. $SU(n)$ are Lie groups as they have a fundamental representation $e^{i\vec{\alpha}\vec{\lambda}}$ where λ is a set of $n \times n$-matrices, and $f_i(\vec{x}, \vec{\alpha}) = [e^{i\vec{\alpha}\vec{\lambda}}\vec{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i = f_i(x_i, 0)$. By an infinitesimal transformation da_i we then get

$$\begin{aligned} x_i' &= x_i + dx_i = f_i(x_i, da_i) \\ &= f_i(x_i, 0) + \frac{\partial f_i}{\partial a_j} da_j + \ldots \\ &= x_i + \frac{\partial f_i}{\partial a_j} da_j \end{aligned}$$

This is the transformation by the group member da_j from the identity. Now, let F be a function from V to $\mathbb{R}$ or $\mathbb{C}$. The group transformation da_i changes F by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x_i} dx_i \\ &= \frac{\partial F}{\partial x_i}\frac{\partial f_i}{\partial a_j} da_j \\ &= da_j X_j F \end{aligned}$$

where

$$X_j = \frac{\partial f_i}{\partial a_j}\frac{\partial}{\partial x_i}$$

are the n **generators** of the Lie group.

As an example we can now go in the opposite direction and look at the two-parameter transformation

$$x' = f(x) = a_1 x + a_2$$

that gives

$$X_1 = \frac{\partial f}{\partial a_1}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x},$$

which is the generator for **dilation** (scale change), and

$$X_2 = \frac{\partial}{\partial x},$$

which is the generator for $T(1)$. Note that $[X_1, X_2] = -X_2$.

Theorem: (Lie's theorems)

i) For a Lie group $\frac{\partial f_i}{\partial a_j}$ is analytic.

ii) The generators X_i satisfy $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are **structure constants**.

iii) $C_{ij}^k = -C_{ji}^k$ and $C_{ij}^k C_{kl}^m + C_{jl}^k C_{ki}^m + C_{li}^k C_{kj}^m = 0.$[a]

[a] This follows from the Jacobi identity $[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$

1.4 Lie algebras

Definition: An **algebra** A on a field (say $\mathbb{R}$ or $\mathbb{C}$) is a linear vector space with a binary operation $\circ : A \times A \to A$.

The vectorspace $\mathbb{R}^3$ together with the crossproduct constitutes an algebra.

Definition: A **Lie algebra** L is an algebra where the binary operator, called Lie bracket, has the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

i) (associativity)

$$[ax + by, z] = a[x, z] + b[y, z]$$

$$[z, ax + by] = a[z, x] + b[z, y]$$

ii) (anti-commutation)

$$[x, y] = -[y, x]$$

iii) (Jacobi identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

We usually restrict ourselves to algebras of linear operators with $[x, y] = xy - yx$, where iii) is automatic. From Lie's theorems the generators of an n-dimensional Lie group form an n-dimensional Lie algebra.

We mentioned the fundamental representation of a matrix based group earlier. These representations have the lowest possible dimension. Another important representation is the **adjoint**. This consists of the matrices:

$$(M_i)_j^k = -C_{ij}^k$$

where C_{ij}^k are the structure constants. From the Jacobi identity we have $[M_i, M_j] = C_{ij}^k M_k$, meaning that the adjoint representation fulfills the same algebra as the fundamental (generators). Note that the dimension of the fundamental representation n for $SO(n)$ and $SU(n)$ is always smaller than the adjoint, which is equal to the degrees of freedom, $\frac{1}{2}n(n-1)$ and $n^2 - 1$ respectively.