# Chapter 2

# The Poincaré algebra and its extensions

We now take a look at the symmetry groups behind Special Relativity (SR), the Lorentz and Poincaré groups. We will first see what sort of states transform properly under SR, which has surprising connections to already familiar physics. We will then look for ways to extend these external symmetries of the coordinates to internal symmetries of quantum fields, *i.e.* the symmetries of gauge groups.

## 2.1 The Lorentz Group

Einstein's requirement in Special Relativity was that the laws of physics should be invariant under rotations and/or boosts (changes of velocity) between different inertial reference frames. A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by a four-vector $x^\mu = (t, x, y, z)$. The resulting allowed transformations of the space-time coordinates are captured in the Lorentz group.

> **Definition**: The **Lorentz group** is the group of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 \equiv x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous** or **restricted Lorentz group** is a subgroup of the Lorentz group where $\det\Lambda = 1$ (**proper**) and $\Lambda^0{}_0 \geq 1$(**orthochronous**).

The physical interpretation of the orthochronous property is that it keeps the direction (sign) of time of the four vector, while a proper group preserves orientation in rotations.

Since the definition of the Lorentz group effectively gives a composition function we can easily conclude that it is a Lie group. In fact, if we allow for a slight extension of the orthogonal group $O(n)$ to the **indefinite orthogonal group** $O(m, n)$, where instead of the orthogonality property for group members $O$, $O^{-1} = O^T$, we demand $O^{-1} = g^{-1}O^T g$ where

$$g = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m}),$$

is the "metric",[1] then we can write the Lorentz group as $SO^+(1, 3)$, where the plus sign signifies the orthochronous property. The counting of the free parameters of $SO(n, m)$ works just

[1] Indeed, we can recognise this matrix relationship as one of the defining (necessary) properties of Lorentz transformations $\Lambda^T g\Lambda = g$.

as for $SO(n)$, giving a total of six free parameters for $SO^+(1,3)$. Physically, we can identify these with the three parameters needed to specify a general rotation in three dimensions, and the three parameters needed to specify a boost (the velocity components).

Since the rotation operations are closed, *i.e.* two rotations result in another rotation, one can prove that this forms a subgroup of $SO^+(1,3)$. We have earlier claimed that the generators of $SO(3)$ (rotations in three dimensions) are identical to the generators of $SU(2)$. This now allows us to identify three of the generators of $SO^+(1,3)$ as the $J_i$ that fulfil the $su(2)$ algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.1}$$

The boost operations are not closed, and one can show that their generators $K_i$ (exercises) have the following relationships with the rotation generators

$$\begin{aligned} [K_j, J_i] &= i\epsilon_{ijk}K_k, & (2.2)\\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. & (2.3) \end{aligned}$$

where (2.1) and (2.3) then defines the complete algebra of $SO^+(1,3)$.

To simplify notation these generators can further be structured into a matrix $M$ given by

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix}.$$

In terms of $M$ the commutation relations (2.1) and (2.3) can be written:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{2.4}$$

From the discussion in Sec. 1.7 and here, any $\Lambda \in SO^+(1,3)$ can now be written as

$$\Lambda^\mu{}_\nu = \left[\exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{2.5}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the six free parameters of the transformation and $M_{\rho\sigma}$ are the generators of the group $SO(1,3)$ and form the basis of the Lie algebra for $O(1,3)$. In fact, this is also the algebra of $O(1,3)$ since the orthochronous and proper requirements do not change the number of free parameters, but rather restricts us to a subset of the matrices. This also nicely illustrates the local property of the exponential map: using these generators we can in fact not get outside of the $SO^+(1,3)$ subgroup of $O(1,3)$. The larger group $O(1,3)$ can be seen as four connected components with $\det\Lambda \pm 1$ and $|\Lambda^0{}_0| \geq 1$ that are joined by the time $T$ and parity $P$ inversion operators.

## 2.2 The Poincaré group

We can now extend $O(1,3)$ by adding translation by a constant four-vector $a^\mu$ to the transformation of the Lorentz group: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$. This transformation leaves lengths $(x-y)^2$ invariant in $\mathbb{M}_4$.

**Definition**: The **Poincaré group** is the group of all transformations of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

We can also construct the **restricted Poincaré group** by restricting the matrices $\Lambda$ in the same way as in $SO^+(1,3)$.

Writing a group member in terms of its parameters $(\Lambda, a)$, we can see from the explicit form of the transformation that the composition of two elements in this group is:

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is *not* a direct product of the Lorentz group and the translation group, but rather a *semi-direct product* of $O(1,3)$ and the (indefinite) translation group $T(1,3)$, $O(1,3) \ltimes T(1,3)$. The translation group is a normal subgroup, and while the Lorentz group is a subgroup, it is not normal. The restricted Poincaré group is written in the same way as $SO^+(1,3) \ltimes T(1,3)$.

The translation part of the Poincaré group adds four parameters to the six parameters of the rotations and boosts. This means that there are four more generators compared to the Lorentz group. Given our earlier discussion of the translation group in Sec. 1.7.1 we can convince ourselves that we can use the momentum operators $P_\mu = -i\partial_\mu$. These generators have a trivial commutation relationship:

$$[P_\mu, P_\nu] = 0. \tag{2.6}$$

Finally, one can show the following commutators with the generators of the Lorentz group:[2]

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu). \tag{2.7}$$

Equations (2.4), (2.6) and (2.7) together form the **Poincaré algebra**, a Lie algebra. This allows us to write a general member $g$ of the restricted Poincaré group by using the exponential map

$$g = \exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma} + ia^\mu P_\mu\right), \tag{2.8}$$

where $a^\mu$ are the additional parameters of the translation.

## 2.3 Irreducible representations of the Poincaré group

We would now like to ask the question: what sort of particles, or, if you like, quantum fields, can exist if we require that they are representations of the Poincaré group?[3]

To answer that question we will need to classify all the irreducible representations of the Poincaré group. This seems like a dramatically difficult task, however, we can now use Schur's lemma that we saw in Sec. 1.4.1. To do this we need to find the **Casimir operators** of the algebra.

[2]For a rigorous derivation of this see Chapter 1.2 of [3]. The proof is constructed by looking at the infinitesimal action of the generators.

[3]In the sense of being described by a vector space that the group representations act on.

**Definition**: The **Casimir operators** of a Lie algebra are the elements that commute with all other elements of the algebra

From Schur's lemma the Casimir operators should then be proportional to the identity for irreducible representations, and, most importantly, the constants of proportionality classify the irrep.

Let us take an example to demonstrate how this works. We saw earlier that $SO(3)$ and $SU(2)$ had the same algebra, with three members that we can generically write as $J_i$. We can show that $J^2 = J_1^2 + J_2^2 + J_3^2$ is a Casimir invariant of this algebra, meaning that in a given representation, we can write $J^2 = \lambda I$, where $\lambda$ is this constant and $I$ is the identity matrix. It may not surprise you to find out that the constant is $\lambda = \ell(\ell+1)$, where $\ell$ can take half-integer values, meaning that we have the relationship $J^2 = \ell(\ell+1)I$ that is familiar from quantum mechanics. We can now go back and test this, checking the relationship for the Pauli matrices and the $J_i$ matrices in (1.2). We will find that for the Pauli matrices $J^2 = \frac{3}{4}$, corresponding to $\ell = \frac{1}{2}$, and for the $J_i$ matrices, $\ell = 1$. The point here is that $\ell$ labels the representation, here the spin-$\frac{1}{2}$ and spin-1 representations.

We can now use the constants of proportionality to classify the (irreducible) representations of our Lie algebra (and group). For the Poincaré algebra $P^2 = P_\mu P^\mu$ is a Casimir operator because the following holds:

$$\left[P_\mu, P^2\right] = 0, \tag{2.9}$$

$$\left[M_{\mu\nu}, P^2\right] = 0. \tag{2.10}$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number that we will (randomly, or maybe not) name $m^2 \in \mathbb{R}$, writing a corresponding state in the vector space as $|m\rangle$, such that:[4]

$$P^2|m\rangle = m^2|m\rangle.$$

If we go to the rest frame of a particle the state has eigenvalues $(m, \vec{0})$ for the operator $P_\mu$, where $m$ is the mass (rest energy) of the particle.[5] This demonstrates that the label $m^2$ can indeed be interpreted as the (square) of the mass,

The number of Casimir operators is equal to the **rank** of the algebra, *e.g.* the rank of $su(n)$ is $n-1$. It turns out that the Poincare algebra has rank 2, and thus two Casimir operators. To demonstrate this is rather involved, and we will not make an attempt here, but note that it can be shown that $SO^+(1,3)$ is homomorphic to $SU(2)\times SU(2)$, because of the structure of the boost and rotation generators, where the algebra of each $SU(2)$ has rank 1.

So, what is the second Casimir of the Poincaré algebra?

**Definition**: We define the **Pauli-Ljubanski polarisation vector** by:

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.11}$$

[4]Note that in general $m^2$ is not restricted to be larger than zero.

[5]This does not loose generality since physics should be independent of frame, however, this argument needs to be modified somewhat for massless particles.

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of the Poincaré algebra since we can show that

$$\left[M_{\mu\nu}, W^2\right] = 0, \quad (2.12)$$
$$\left[P_\mu, W^2\right] = 0. \quad (2.13)$$

Note that these relationships are not trivial to demonstrate. See [3] for a complete proof.

If we again look at the situation in the rest frame we can write

$$W_i = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = m S_i,$$

where $S_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ is the **spin operator**. [6] By showing that $WP = 0$ we also have $W_0 = 0$ in this reference frame. This gives $W^2 = -\mathbf{W}^2 = -m^2\mathbf{S}^2$. Since the spin operator acts on a state with spin $s$ as $\mathbf{S}^2|s\rangle = s(s+1)|s\rangle$, we have that

$$W^2|m,s\rangle = -m^2 s(s+1)|m,s\rangle$$

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by special relativity, can be classified by two quantum numbers: mass and spin.

## 2.4 The no-go theorem and graded Lie algebras

Since we now know the Poincaré group and its representations well, we can ask: Can the external space-time symmetries be extended, perhaps also to include the internal gauge symmetries? Unfortunately no. In 1967 Coleman and Mandula [4] showed that any extension of the Pointcaré group to include gauge symmetries is isomorphic to $G_{SM} \times P_+^\uparrow$, *i.e.* the generators $B_i$ of standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

Not to be defeated by a simple mathematical proof this was countered by Haag, Łopuszański and Sohnius (HLS) in 1975 in [5] where they introduced the concept of graded Lie algebras to get around the no-go theorem.

**Definition**: A ($\mathbb{Z}_2$) **graded Lie algebra** or **superalgebra** is a vector space $L$ that is a direct sum of two vector spaces $L_0$ and $L_1$, $L = L_0 \oplus L_1$ with a binary operation $\bullet : L \times L \to L$ such that for $\forall x_i \in L_i$

i) $x_i \bullet x_j \in L_{i+j \mod 2}$ (grading)[a]

ii) $x_i \bullet x_j = -(-1)^{ij} x_j \bullet x_i$ (supersymmetrization)

iii) $x_i \bullet (x_j \bullet x_k)(-1)^{ik} + x_j \bullet (x_k \bullet x_i)(-1)^{ji} + x_k \bullet (x_i \bullet x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)

This definition can be generalised to $\mathbb{Z}_n$ by a direct sum over $n$ vector spaces $L_i$, $L = \oplus_{i=0}^{n-1} L_i$, such that $x_i \bullet x_j \in L_{i+j \mod n}$ with the same requirements for supersymmetrization and Jacobi identity as for the $\mathbb{Z}_2$ graded algebra.

[a] This means that $x_0 \bullet x_0 \in L_0$, $x_1 \bullet x_1 \in L_0$ and $x_0 \bullet x_1 \in L_1$.

[6] Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.