Chapter 2

The Poincaré algebra and its extensions

We now take a look at the symmetry groups behind Special Relativity (SR), the Lorentz and Poincaré groups. We will first see what sort of states transform properly under SR, which has surprising connections to already familiar physics. We will then look for ways to extend these external symmetries of the coordinates to internal symmetries of quantum fields, *i.e.* the symmetries of gauge groups.

2.1 The Lorentz Group

Einstein's requirement in Special Relativity was that the laws of physics should be invariant under rotations and/or boosts (changes of velocity) between different inertial reference frames. A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by a four-vector $x^\mu = (t, x, y, z)$. The resulting allowed transformations of the space-time coordinates are captured in the Lorentz group.

> **Definition**: The **Lorentz group** is the group of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 \equiv x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous** or **restricted Lorentz group** is a subgroup of the Lorentz group where $\det \Lambda = 1$ (**proper**) and $\Lambda^0{}_0 \geq 1$(**orthochronous**).

The physical interpretation of the orthochronous property is that it keeps the direction (sign) of time of the four vector, while a proper group preserves orientation in rotations.

Since the definition of the Lorentz group effectively gives a composition function we can easily conclude that it is a Lie group. In fact, if we allow for a slight extension of the orthogonal group $O(n)$ to the **indefinite orthogonal group** $O(m,n)$, where instead of the orthogonality property for group members O, $O^{-1} = O^T$, we demand $O^{-1} = g^{-1}O^T g$ where

$$g = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m}),$$

is the "metric",[1] then we can write the Lorentz group as $SO^+(1,3)$, where the plus sign signifies the orthochronous property. The counting of the free parameters of $SO(n,m)$ works just

[1] Indeed, we can recognise this matrix relationship as one of the defining (necessary) properties of Lorentz transformations $\Lambda^T g \Lambda = g$.

as for $SO(n)$, giving a total of six free parameters for $SO^+(1,3)$. Physically, we can identify these with the three parameters needed to specify a general rotation in three dimensions, and the three parameters needed to specify a boost (the velocity components).

Since the rotation operations are closed, *i.e.* two rotations result in another rotation, one can prove that this forms a subgroup of $SO^+(1,3)$. We have earlier claimed that the generators of $SO(3)$ (rotations in three dimensions) are identical to the generators of $SU(2)$. This now allows us to identify three of the generators of $SO^+(1,3)$ as the J_i that fulfil the $su(2)$ algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.1}$$

The boost operations are not closed, and one can show that their generators K_i (exercises) have the following relationships with the rotation generators

$$\begin{aligned} [K_j, J_i] &= i\epsilon_{ijk}K_k, &(2.2)\\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. &(2.3) \end{aligned}$$

where (2.1) and (2.3) then defines the complete algebra of $SO^+(1,3)$.

To simplify notation these generators can further be structured into a matrix M given by

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix}.$$

In terms of M the commutation relations (2.1) and (2.3) can be written:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{2.4}$$

From the discussion in Sec. 1.7 and here, any $\Lambda \in SO^+(1,3)$ can now be written as

$$\Lambda^\mu{}_\nu = \left[\exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{2.5}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the six free parameters of the transformation and $M_{\rho\sigma}$ are the generators of the group $SO(1,3)$ and form the basis of the Lie algebra for $O(1,3)$. In fact, this is also the algebra of $O(1,3)$ since the orthochronous and proper requirements do not change the number of free parameters, but rather restricts us to a subset of the matrices. This also nicely illustrates the local property of the exponential map: using these generators we can in fact not get outside of the $SO^+(1,3)$ subgroup of $O(1,3)$. The larger group $O(1,3)$ can be seen as four connected components with $\det\Lambda \pm 1$ and $|\Lambda^0{}_0| \geq 1$ that are joined by the time T and parity P inversion operators.

2.2 The Poincaré group

We can now extend $O(1,3)$ by adding translation by a constant four-vector a^μ to the transformation of the Lorentz group: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$. This transformation leaves lengths $(x-y)^2$ invariant in $\mathbb{M}_4$.

Definition: The **Poincaré group** is the group of all transformations of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

We can also construct the **restricted Poincaré group** by restricting the matrices Λ in the same way as in $SO^+(1,3)$.

Writing a group member in terms of its parameters (Λ, a), we can see from the explicit form of the transformation that the composition of two elements in this group is:

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is *not* a direct product of the Lorentz group and the translation group, but rather a *semi-direct product* of $O(1,3)$ and the (indefinite) translation group $T(1,3)$, $O(1,3) \ltimes T(1,3)$. The translation group is a normal subgroup, and while the Lorentz group is a subgroup, it is not normal. The restricted Poincaré group is written in the same way as $SO^+(1,3) \ltimes T(1,3)$.

The translation part of the Poincaré group adds four parameters to the six parameters of the rotations and boosts. This means that there are four more generators compared to the Lorentz group. Given our earlier discussion of the translation group in Sec. 1.7.1 we can convince ourselves that we can use the momentum operators $P_\mu = -i\partial_\mu$. These generators have a trivial commutation relationship:

$$[P_\mu, P_\nu] = 0. \tag{2.6}$$

Finally, one can show the following commutators with the generators of the Lorentz group:[2]

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu). \tag{2.7}$$

Equations (2.4), (2.6) and (2.7) together form the **Poincaré algebra**, a Lie algebra. This allows us to write a general member g of the restricted Poincaré group by using the exponential map

$$g = \exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma} + ia^\mu P_\mu\right), \tag{2.8}$$

where a^μ are the additional parameters of the translation.

2.3 Irreducible representations of the Poincaré group

We would now like to ask the question: what sort of particles, or, if you like, quantum fields, can exist if we require that they are representations of the Poincaré group?[3]

To answer that question we will need to classify all the irreducible representations of the Poincaré group. This seems like a dramatically difficult task, however, we can now use Schur's lemma that we saw in Sec. 1.4.1. To do this we need to find the **Casimir operators** of the algebra.

[2]For a rigorous derivation of this see Chapter 1.2 of [3]. The proof is constructed by looking at the infinitesimal action of the generators.

[3]In the sense of being described by a vector space that the group representations act on.

Definition: The **Casimir operators** of a Lie algebra are the elements that commute with all other elements of the algebra

From Schur's lemma the Casimir operators should then be proportional to the identity for irreducible representations, and, most importantly, the constants of proportionality classify the irrep.

Let us take an example to demonstrate how this works. We saw earlier that $SO(3)$ and $SU(2)$ had the same algebra, with three members that we can generically write as J_i. We can show that $J^2 = J_1^2 + J_2^2 + J_3^2$ is a Casimir invariant of this algebra, meaning that in a given representation, we can write $J^2 = \lambda I$, where λ is this constant and I is the identity matrix. It may not surprise you to find out that the constant is $\lambda = \ell(\ell+1)$, where ℓ can take half-integer values, meaning that we have the relationship $J^2 = \ell(\ell+1)I$ that is familiar from quantum mechanics. We can now go back and test this, checking the relationship for the Pauli matrices and the J_i matrices in (1.2). We will find that if $J_i = \frac{1}{2}\sigma_i$, where σ_i are the Pauli matrices $J^2 = \frac{3}{4}$, corresponding to $\ell = \frac{1}{2}$, and for the J_i matrices, $\ell = 1$. The point here is that ℓ labels the representation, here the spin-$\frac{1}{2}$ and spin-1 representations.

We can now use the constants of proportionality to classify the (irreducible) representations of our Lie algebra (and group). For the Poincaré algebra $P^2 = P_\mu P^\mu$ is a Casimir operator because the following holds:

$$\left[P_\mu, P^2\right] = 0, \qquad (2.9)$$
$$\left[M_{\mu\nu}, P^2\right] = 0. \qquad (2.10)$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number that we will (randomly, or maybe not) name $m^2 \in \mathbb{R}$, writing a corresponding state in the vector space as $|m\rangle$, such that:[4]

$$P^2|m\rangle = m^2|m\rangle.$$

If we go to the rest frame of a particle the state has eigenvalues $(m, \vec{0})$ for the operator P_μ, where m is the mass (rest energy) of the particle.[5] This demonstrates that the label m^2 can indeed be interpreted as the (square) of the mass,

The number of Casimir operators is equal to the **rank** of the algebra, *e.g.* the rank of $su(n)$ is $n-1$. It turns out that the Poincare algebra has rank 2, and thus two Casimir operators. To demonstrate this is rather involved, and we will not make an attempt here, but note that it can be shown that $SO^+(1,3)$ is homomorphic to $SU(2) \times SU(2)$, because of the structure of the boost and rotation generators, where the algebra of each $SU(2)$ has rank 1.

So, what is the second Casimir of the Poincaré algebra?

Definition: We define the **Pauli-Ljubanski polarisation vector** by:

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}. \qquad (2.11)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor with $\epsilon_{0123} = 1$.

[4]Note that in general m^2 is not restricted to be larger than zero.

[5]This does not loose generality since physics should be independent of frame, however, this argument needs to be modified somewhat for massless particles.

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of the Poincaré algebra since we can show that

$$\left[M_{\mu\nu}, W^2\right] = 0, \tag{2.12}$$
$$\left[P_\mu, W^2\right] = 0. \tag{2.13}$$

Note that these relationships are not trivial to demonstrate. See [3] for a complete proof.

If we again look at the situation in the rest frame we can write

$$W_i = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = m S_i, \tag{2.14}$$

where $S_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ is the **spin operator**. [6] By showing that $WP = 0$ we also have $W_0 = 0$ in this reference frame. This gives $W^2 = -\mathbf{W}^2 = -m^2\mathbf{S}^2$. Since the spin operator acts on a state with spin s as $\mathbf{S}^2|s\rangle = s(s+1)|s\rangle$, we have that

$$W^2|m,s\rangle = -m^2 s(s+1)|m,s\rangle$$

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by special relativity, can be classified by two quantum numbers: mass m^2 and spin s.

What do these (irreducible) representations then look like? If we start with spin-0 representations, $s = 0$, we can write the corresponding states without any vector structure as $|m,0\rangle \sim e^{\pm ipx}$, where p_μ is the four-momentum of the particle, since $P^2|m,0\rangle = -\partial_\mu\partial^\mu|m,0\rangle \sim p^2|m,0\rangle = m^2|m,0\rangle$. This exponential part of states can then always be used to take care of the eigenvalues of the P^2-Casimir, and is often just implicitly implied in the states/fields.

We can also immediately write down the $s = 1$ vector representation of the Poincaré group, $|m,1\rangle \sim \epsilon_\mu e^{ipx}$. We simply use a four-vector ϵ_μ that transforms under the fundamental (four-dimensional) representation of the Lorentz group $SO^+(1,3)$. In order to fulfil the eigenvalue equation of the W^2-Casimir, this vector (called the polarisation vector) needs to fulfil certain requirements which we do not detail here (see a course on quantum field theory).

However, to find a spin-$\frac{1}{2}$ representation we need to take some more care. In fact, we will find representations both in four and two dimensions. For those familiar with quantum field theory, these will as expected be the Dirac and Weyl spinor representations.

2.4 Weyl spinors

Interestingly, there exists a homomorphism between the groups $SO^+(1,3)$ and $SL(2,\mathbb{C})$. This homomorphism, with $\Lambda^\mu{}_\nu \in SO^+(1,3)$ and $M \in SL(2,\mathbb{C})$, can be explicitly given by:[7]

$$\Lambda^\mu{}_\nu(M) = \frac{1}{2}\mathrm{Tr}[\bar{\sigma}^\mu M \sigma_\nu M^\dagger], \tag{2.15}$$
$$M(\Lambda^\mu{}_\nu) = \pm\frac{1}{\sqrt{\det(\Lambda^\mu{}_\nu\sigma_\mu\bar{\sigma}^\nu)}}\Lambda^\mu{}_\nu\sigma_\mu\bar{\sigma}^\nu, \tag{2.16}$$

[6] Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.

[7] The choice of sign in Eq. (2.16) is the reason that this is a homomorphism, instead of an isomorphism. Each element in $SO^+(1,3)$ can be assigned to two in $SL(2,\mathbb{C})$.

where $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and $\sigma^\mu = (1, \vec{\sigma})$.

This two-to-one correspondence means that $SO^+(1,3) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$. Thus we can look at the representations of $SL(2,\mathbb{C})$ instead of the Poincaré group, when we describe particles, but what are those representations? It turns out that there exist two inequivalent fundamental representations ρ of $SL(2,\mathbb{C})$:

i) The self-representation $\rho(M) = M$ acting on an element ψ of a representation vector space V:
$$\psi'_A = {M_A}^B \psi_B, \quad A, B = 1, 2.$$

ii) The complex conjugate self-representation $\rho(M) = M^*$ working on a vector $\bar{\psi}$ in a space $\dot{V}$:
$$\bar{\psi}'_{\dot{A}} = {(M^*)_{\dot{A}}}^{\dot{B}} \bar{\psi}_{\dot{B}}, \quad \dot{A}, \dot{B} = 1, 2.$$

The vectors ψ and $\bar{\psi}$ in these representation spaces are called, respectively, **left- and right-handed Weyl spinors**.

The indices here follow the same summation rules as four-vectors. Indices can be lowered and raised with:
$$\epsilon_{AB} = \epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\epsilon^{AB} = \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The dots on the indices for the complex conjugate representation are there to help us remember which representation we are using and does not carry any additional importance. For a consistent index notation, the relationship between the vectors ψ and $\bar{\psi}$ can be expressed with:
$$\bar{\sigma^0}^{\dot{A}A} (\psi_A)^* = \bar{\psi}^{\dot{A}}.$$

This may be seen as a bit of an overkill in indices as $\bar{\sigma^0}^{\dot{A}A} = \delta^{\dot{A}A}$, and we will in the following often omit the matrix and simply write $(\psi_A)^* = \bar{\psi}^{\dot{A}}$. Note that from the above the following relationships hold for the hermitian conjugate:
$$(\psi_A)^\dagger = \bar{\psi}_{\dot{A}}$$
$$(\bar{\psi}_{\dot{A}})^\dagger = \psi_A.$$

We further define contractions of Weyl spinors that are invariant under $SL(2,\mathbb{C})$ transformations – just as contractions of four-vectors are invariant under Lorentz transformations – as follows:

Definition: The contraction of two Weyl spinors is given by $\psi\chi \equiv \psi^A \chi_A$ and $\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{A}} \bar{\chi}^{\dot{A}}$.

With this in hand we see that
$$\psi^2 \equiv \psi\psi = \psi^A \psi_A = \epsilon^{AB} \psi_B \psi_A = \epsilon^{12} \psi_2 \psi_1 + \epsilon^{21} \psi_1 \psi_2 = \psi_2 \psi_1 - \psi_1 \psi_2.$$

This quantity is zero if the Weyl spinors commute. In order to avoid this we make the following assumption which is consistent with how we treat fermions as anti-commuting operators:

Postulate: All Weyl spinors anticommute:[a] $\{\psi_A, \psi_B\} = \{\bar{\psi}_{\dot{A}}, \bar{\psi}_{\dot{B}}\} = \{\psi_A, \bar{\psi}_{\dot{B}}\} = \{\bar{\psi}_{\dot{A}}, \psi_B\} = 0$.

[a] This means that Weyl spinors are so-called **Grassmann numbers**.

This means that the contraction evaluates as

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = -2\psi_1\psi_2.$$

Because of (2.16) we can now find how the Lorentz part of the Poincare group acts on the Weyl spinors, and can use this to show that they fulfil the requirements of the spin-$\frac{1}{2}$ representation of the Poincaré group. In the rest frame of a particle, this is relatively straight forward since the spin operators S_i in (2.14) in the fundamental representation of $SL(2,\mathbb{C})$ can be written in terms of the Pauli matrices σ_i as $S_i = \frac{1}{2}\sigma_i$, and we already know that $\sigma^2 = 3I$, so $S^2 = \frac{3}{4}I$, which corresponds to $s = \frac{1}{2}$.

The Weyl spinors can in turn be used in a four-dimensional representation of the Poincaré group, stacking two Weyl spinors into a four-component Dirac spinor ψ_a:

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix}.$$

Here, we have in general $(\psi_A)^* \neq \bar{\chi}^{\dot{A}}$. In order to describe a Dirac fermion, which has both particle and antiparticle states, with this Dirac spinor we need two distinct Weyl spinors with different handednesses. For Majorana fermions that are their own antiparticles we have:

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\psi}^{\dot{A}} \end{pmatrix}.$$

2.5 The no-go theorem and graded Lie algebras

The Poincaré group contains the complete set of transformations for the symmetries of special relativity (invariance under rotations, translations and boosts), and we have seen that this implies certain properties for the particles, or rather fields, that want to live in representations of the Poincaré group. At the same time we know that the quantum fields have (internal) gauge symmetries. It would then be tempting so ask if these are somehow related and can be described in a larger symmetry.

Unfortunately, the answer to that question is 'no', at least as long as we keep to describing our symmetries using Lie algebras. In 1967 Coleman and Mandula [4] showed that under reasonable assumptions any extension of the restricted Pointcaré group P to include gauge symmetries is isomorphic to $G_{\text{gauge}} \times P$. A direct product like this means that the generators of the two groups all commute, meaning that the generators B_i of the standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

The result is that there can be no real interaction between the external and internal symmetries.

Not to be defeated by a simple mathematical proof, in 1975 Haag, Łopuszański and Sohnius (HLS) [5] showed that there is a way around Coleman and Mandula's no-go theorem, if one introduces the concept of $\mathbb{Z}_2$ **graded Lie superalgebras**.[8]

Definition: A **graded Lie superalgebra** is a vector space L that is a direct sum of two vector spaces L_0 and L_1, $L = L_0 \oplus L_1$, with a binary operation $\circ : L \times L \to L$ such that for $\forall x_i \in L_i$

i) $x_i \circ x_j \in L_{i+j \mod 2}$ (grading)[a]

ii) $x_i \circ x_j = -(-1)^{ij} x_j \circ x_i$ (supersymmetrization)

iii) $x_i \circ (x_j \circ x_k)(-1)^{ik} + x_j \circ (x_k \circ x_i)(-1)^{ji} + x_k \circ (x_i \circ x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)

[a] For $x_0 \in L_0$ and $x_1 \in L_1$, this means that $x_0 \circ x_0 \in L_0$, $x_1 \circ x_1 \in L_0$ and $x_0 \circ x_1 \in L_1$.

The second requirement generalises the definition of a Lie algebra in Sec. 1.7 to allow for anti-commutators, $x \circ y = \{x, y\} \equiv xy + yx$, as the binary operation for elements in L_1.

We can now start, following HLS, with the Poincaré Lie algebra ($L_0 = P$) and add a new vector space L_1 spanned by some generators Q_a. It can be shown that the superalgebra requirements are fulfilled if there are four generators, $a = 1, 2, 3, 4$, that together form a four-component Majorana spinor. The algebra is then

$$[Q_a, P_\mu] = 0 \tag{2.17}$$
$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu} Q)_a \tag{2.18}$$
$$\{Q_a, \bar{Q}_b\} = 2\not{P}_{ab}, \tag{2.19}$$

where $\sigma_{\mu\nu}$ is given in terms of the γ-matrices, $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$, and as usual $\not{P} = P_\mu \gamma^\mu$ and $\bar{Q}_a = (Q^\dagger \gamma_0)_a$.[9] This is called the **super-Poincaré algebra**.

Because of (2.18) this new algebra is a non-trivial extension of the Poincaré algebra that avoids the no-go theorem. However, in the Q_a we have introduced new operators that (disappointingly) do not correspond to the generators of the gauge groups (which should be related by commutators, not anti-commutators). The gauge group generators *can* appear in the algebra if we extend the algebra with a set of $N > 1$ new spinors Q_a^α, where $\alpha = 1, \ldots, N$. This gives rise to so-called $N > 1$ supersymmetries. However, this seem impossible to realise in nature at energies accessible in experiments due to an extensive number of extra particles.[10] This extension, including the potential for $N > 1$, can be proven, under some reasonable assumptions, to be the **largest possible** extension of the symmetries of special relativity.

We can also write the super-Poincaré algebra in terms of the Weyl spinors introduced in

[8] The definition of graded Lie algebras can be extended to $\mathbb{Z}_n$ by a direct sum over n vector spaces L_i, $L = \oplus_{i=0}^{n-1} L_i$, such that $x_i \circ x_j \in L_{i+j \mod n}$, with the same requirements for supersymmetrization and Jacobi identity as for the $\mathbb{Z}_2$ graded algebra.

[9] Alternatively, (2.19) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

[10] Note that $N > 8$ would also include elementary particles with spin greater than 2, which seems to be in contradiction with quantum field theory.

Sec. 2.4. With

$$Q_a = \begin{pmatrix} Q_A \\ \bar{Q}^{\dot{A}} \end{pmatrix}, \tag{2.20}$$

for the Majorana spinor charges, we have instead

$$\begin{aligned} [Q_A, P_\mu] &= [\bar{Q}_{\dot{A}}, P_\mu] = 0, & (2.21) \\ [Q_A, M^{\mu\nu}] &= \sigma^{\mu\nu}{}_A{}^B Q_B, & (2.22) \\ \{Q_A, Q_B\} &= \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0, & (2.23) \\ \{Q_A, \bar{Q}_{\dot{B}}\} &= 2\sigma^\mu_{A\dot{B}} P_\mu, & (2.24) \end{aligned}$$

where now $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$.

2.6 The Casimir operators of the super-Poincaré algebra

It is easy to see that P^2 is still a Casimir operator of the superalgebra. From Eq. (2.21) P_μ commutes with the Qs, so in turn P^2 must commute.[11] However, W^2 is not a Casimir because of the following result:[12]

$$[W^2, Q_a] = W_\mu(\not{P}\gamma_\mu\gamma^5 Q)_a + \frac{3}{4}P^2 Q_a.$$

We want to find an extension of W that commutes with the Qs while retaining the commutators we alread have. The construction

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu,$$

where

$$B_\mu \equiv W_\mu + \frac{1}{4}X_\mu,$$

and with

$$X_\mu \equiv \frac{1}{2}\bar{Q}\gamma_\mu\gamma^5 Q,$$

has the required relation:

$$[C_{\mu\nu}, Q_a] = 0.$$

We can show that C^2 indeed commutes with all the generators in the algebra:

$$\begin{aligned} [C^2, Q_a] &= 0, && \text{(trivial)} \\ [C^2, P_\mu] &= 0, && \text{(excessive algebra)} \\ [C^2, M_{\mu\nu}] &= 0. && \text{(because } C^2 \text{ is a Lorentz scalar)} \end{aligned}$$

Thus C^2 is a Casimir operator for the superalgebra.

[11] Although the fact that Eq. (2.21) holds crucially depends on Q_a being four-dimensional. P_μ and Q_a would not commute if there had been five Qs.

[12] Which, by the way, is really hard work!