Supersymmetry

Lecture notes for FYS5190/FYS9190 Supersymmetry

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Preface

The goal of these lecture notes is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as our main example. The notes are based on lectures given at the University of Oslo in 2011, 2013, 2015, 2017, 2019, 2021, and 2023, and lectures at the NORDITA Winter School on Theoretical Particle Physics in 2012.

Several student have contributed significantly to the notes. We owe a particular gratitude to Paul Batzing, who took notes during the 2011 lectures, forming the start of this document, and to Carl Martin Favang who fixed many mistakes.

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Introduction

Rather than the traditional approach of starting with the current problems of the Standard Model of particle physics, and how supersymmetry can solve these, in these notes we will focus on the algebraic origin of supersymmetry in the sense of an extension of the symmetries of Einstein's special relativity (SR). This was the original motivation for work on what we today call supersymmetry.

We first need to introduce some basic mathematical concepts used in physics for exploring symmetries, mainly groups and Lie algebras, which we will take care of in Chapter 1. In Chapter 2 we will then study the symmetries of special relativity, through the Poincaré group, and look at how these symmetries can be extended into the super-Poincaré group by adding so-called supercharges to the Poincaré Lie algebra. We will then introduce superspace, a coordinate system where supersymmetry is manifest, and use this to derive differential representations for the supercharges in Chapter 3. Here, we also define superfields, which function as representations of the super-Poincaré group, and are the building blocks of the supersymmetric Largrangians we construct in Chapter 4. In Chapter 5 we discuss how to break supersymmetry using spontaneous symmetry breaking.

From Chapter 6 and onwards we turn more towards phenomenology, constructing the minimal realisation of a supersymmetric Standard Model, before we discuss its phenomenology inChapter 7, before ending with a discussion of supersymmetric Dark Matter candidates in Chapter 8.

Note that sections marked with an asterisk are somewhat tangential to the main argument of the text and can be read for light entertainment only. Solutions to some of the exercises can be found in Appendix A.

Chapter 1

Groups and algebras

The study of symmetries plays a central part in theoretical physics, and the mathematical language we use is that of groups. The action of the group elements on our (quantum) states effects the transformation that has the symmetry, while the invariance of the physical properties of the system under that transformation is the symmetry itself. For example, rotations in three-dimensions can be carried out by the application of a 3×3 rotation matrix on the coordinates of an object. As we will see, these matrices form the group called $SO(3)$. For a sphere, which is invariant under these rotations, $SO(3)$ is then the symmetry group.

Of special interest to us are the Lie groups, which are the groups that represent continuous transformations, such as the $SO(3)$ rotations. The properties of Lie groups can be further studied by finding their generators which form a (Lie) algebra. The generators *almost* – in a very specific sense of the word almost – describes the whole group, and allows us to reconstruct the group elements by what is called the exponential map.

Here, we will begin by defining groups and looking at some of their most important properties. What is crucial in physics are the *representations* of groups, meaning what the operators of the transformations on the states actually look like. Returning to the rotation example these are 3×3 matrices, but with some restrictions on their elements. After discussing representations we will move on to defining Lie groups, before we end on a discussion of their generators and corresponding algebras.

1.1 Group definition

A group is an abstract mathematical structure that consists of a set of objects (elements), and a multiplication rule acting between pairs of these objects. We define a group as follows.

Definition: The set of elements $G = \{g_i\}$ and operation $\circ$ (sometimes called multiplication) form a **group** if and only if for all $g_i \in G$:

i) $g_i \circ g_j \in G$, (closure)

ii) $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$, (associativity)

iii) $\exists e \in G$ such that $g_i \circ e = e \circ g_i = g_i$, (identity element)

iv) $\exists g_i^{-1} \in G$ such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = e$. (inverse)

Below, where no confusion can occur, we will often drop the multiplication symbol for the group multiplication (and other abstract multiplications), writing $g_i \circ g_j = g_i g_j$.

A straight forward example of a group is $G = \mathbb{Z}$ (the integers), with standard addition as the operation $\circ$. Then $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is now addition modulo n. In this group, $g_i^{-1} = n - g_i$ and the unit element is again $e = 0$.[1] Here, $\mathbb{Z}$ is an example of an **infinite** group, the set has an infinite number of members, while $\mathbb{Z}_n$ is **finite**, with **order** n, meaning n members. Both are **abelian** groups, meaning that the elements commute: $g_i \circ g_j = g_j \circ g_i$, because the standard addition commutes.

The simplest, non-trivial, of these $\mathbb{Z}_n$-groups is $\mathbb{Z}_2$ which only has the members $e = 0$ and 1. The "multiplication" operation is completely defined by the three possibilities $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 0$. Now, compare this to the set $G = \{-1, 1\}$ with the ordinary multiplication operation. Here, all the possible operations are $1 \cdot 1 = 1$, $1 \cdot (-1) = -1$ and $(-1) \cdot (-1) = 1$. This has exactly the same structure as $\mathbb{Z}_2$, only that the identity element is now 1. We say that these two groups are **isomorphic**, because there is a one-to-one correspondence between all the (two) elements, $0 \leftrightarrow 1$ and $1 \leftrightarrow -1$, and the results of the multiplication operation is the same, and in fact we consider them as the *same group* despite the considerable apparent visual differences.[2] This notion of isomorphic ("identical") groups is very important, and we will return to it in more detail in Sec. 1.3.

A somewhat more sophisticated example of a group can be found in the Taylor expansion of a function F, where

$$\begin{aligned} F(x+a) &= F(x) + aF'(x) + \frac{1}{2}a^2 F''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{\partial^n}{\partial x^n} F(x) \\ &= e^{a \frac{\partial}{\partial x}} F(x). \end{aligned}$$

The last equality uses the formal definition of the exponential series, but may drive some mathematicians crazy.[3] The resulting operator $T_a = e^{a\frac{\partial}{\partial x}}$ is called the **translation operator**,

[1] Note that we here use e for the identity in an abstract group, while we will later use I or 1 as the identity matrix in matrix representations of groups.

[2] This observation generalises to the set $G = \{e^{2\pi i k/n} | k = 1, \dots, n-1\}$, the **n-th roots of unity**, which, together with the standard multiplication operation, is isomorphic to $\mathbb{Z}_n$.

[3] We will not discuss this further, but there is a deep question here whether the operator formed by this exponentiation is well defined.

in this case in one dimension, since it shifts the coordinate x of the function F it is operating on by an amount a. Defining the (natural) multiplication operation $T_a \circ T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where we can show that T_0 is the identity element and the inverse is $T_a^{-1} = T_{-a}$.[4] In n dimensions the group $T(n)$ has the elements $T_{\mathbf{a}} = e^{\mathbf{a}\cdot\nabla}$. Whereas we say that the groups $\mathbb{Z}$ and $\mathbb{Z}_n$ are **discrete groups**, since we can count the number of elements, T_a is a **continuous group** since the parameter a can be any real number.

1.2 Matrix groups

We next define some groups that are very important in physics and to the discussion in these notes. They have in common that they are defined in terms of square matrices.

1.2.1 General and special linear groups

The largest matrix group for a given matrix dimension n is the general linear group.

> **Definition**: The **general linear group** $GL(n)$ is defined by the set of all invertible $n \times n$ matrices A under matrix multiplication. If we additionally require that $\det(A) = 1$, the matrices form the **special linear group** $SL(n)$.

Th closure property of groups is guaranteed by the fact that the product of two invertible matrices is also invertible, and matrix multiplication is always associative. The existence of the group identity is guaranteed by the identity matrix I being an invertible matrix (with I as the inverse). Since the existence of an inverse is also necessary in the group definition, we can not construct larger matrix groups. The general linear group also give us our first example of a **non-abelian group**, since matrix multiplication does not in general commute. For two matrices A and B, we may have $AB \neq BA$.

We usually take the matrices in matrix groups to be defined over the field of complex numbers $\mathbb{C}$. If we want to specify the field we may use the notation $GL(n, \mathbb{R})$, signifying that the group is defined over the real numbers. Defined over the complex numbers the $GL(n)$ groups have $2n^2$ free parameters since each of the n^2 elements of the matrices can be a complex number, needing two parameters. The $SL(n)$ group has $2n^2 - 2$ free parameters since the requirement on the determinant fixes both the real and imaginary part of the determinant.

1.2.2 Unitary and special unitary groups

We first remind you that the **Hermitian conjugate** or conjugate transpose of a matrix is given by transposing the matrix and taking the complex conjugate of its elements. Here, we will use the dagger symbol † for this operation, so that for a matrix A, $A^\dagger = (A^T)^*$.

We now define the unitary groups.

> **Definition**: The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U, *i.e.* matrices such that $U^\dagger U = I$ or $U^{-1} = U^\dagger$. If we additionally require that $\det U = 1$ the matrices form the **special unitary group** $SU(n)$.

[4] We could instead have defined the operation between two group elements to be ordinary multiplication and used that to show the relationship $T_a \circ T_b = T_{a+b}$. However, it is important to notice that showing this is not entirely trivial because ordinary arithmetic rules for exponentials fail for operators. In this particular case the proof is fairly simple, but this is in general not so. This will return to trouble us later in the notes.

Since for $U \in U(n)$,

$$\det(UU^\dagger) = \det(U)\det(U^\dagger) = \det(U)\det(U^T)^* = \det(U)\det(U)^* = \det(I) = 1,$$

we have that the determinant of these matrices must be complex numbers on the unit circle, *i.e.* $\det(U) = e^{i\theta}$. It can be shown, see Ex. 4, that the $U(n)$, groups have n^2 independent parameters, while the $SU(n)$ groups have $n^2 - 1$.

It is these unitary groups that form the gauge symmetry groups of the Standard Model: $SU(3)$, $SU(2)$ and $U(1)$. The group $U(1)$ makes perfect sense despite the odd matrix dimensions. This is simply the set of all complex numbers of unit length with ordinary multiplication, *i.e.* $U(1) = \{e^{i\alpha} | \alpha \in \mathbb{R}\}$, but notice that $SU(1)$ would be trivial since it contains only the element 1.

The members of the unitary group has has the important property that for all complex vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ (think finite-dimensional quantum states) multiplication by a unitary matrix leaves scalar products (inner products) unchanged. If $\mathbf{x}' = U\mathbf{x}$ and $\mathbf{y}' = U\mathbf{y}$, then

$$\begin{aligned} \mathbf{x}' \cdot \mathbf{y}' &\equiv \mathbf{x}'^\dagger \mathbf{y}' = (U\mathbf{x})^\dagger U\mathbf{y} \\ &= \mathbf{x}^\dagger U^\dagger U \mathbf{y} = \mathbf{x}^\dagger \mathbf{y} = \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

Thus, as this implies $|\mathbf{x}'| = |\mathbf{x}|$, its members do not change the length of the vectors they act on. Since we would like to let our group representations act on vectors that describe quantum mechanical states, the unitary groups then conserve probability for these states. For example, when acting on a complex number (a complex scalar), such as a wavefunction $\psi(x)$, the elements of $U(1)$ rotate the phase of ψ, however, the magnitude is conserved since $\psi' = e^{i\alpha}\psi$ gives $|\psi'|^2 = \psi^* e^{-i\alpha} e^{i\alpha} \psi = |\psi|^2$.

We can construct unitary matrices from either Hermitian or anti-Hermitian matrices by using the **matrix exponential**. This formally interprets the exponential series in terms of a real or complex valued $n \times n$ matrix M as

$$\exp(M) \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!} = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \ldots.$$

This series can be shown to always converge, so the series is well defined. Since this is a series of non-commuting objects we again have to be careful with using properties of the exponential from ordinary arithmetic. The following useful properties of the matrix exponential can be proven:

i) $\exp(A^T) = \exp(A)^T$

ii) $\exp(A^*) = \exp(A)^*$

iii) If B is an invertible matrix then $\exp(BAB^{-1}) = B\exp(A)B^{-1}$

iv) If $[A, B] = 0$ then $\exp(A)\exp(B) = \exp(A + B)$

v) $\det e^A = e^{\operatorname{Tr} A}$.

Given a Hermitian matrix M, $M^\dagger = M$, the matrix $U = e^{iM}$ is then automatically unitary since $U^\dagger = e^{-iM^\dagger} = e^{-iM}$ so that $U^\dagger U = e^{-iM}e^{iM} = I$, where the last equality is due to M commuting with itself. We shall later in this chapter show that in fact all unitary matrices can be written in terms of Hermitian matrices like this. This is the physics construction that we will mostly use in these notes because of our quantum fondness for Hermitian operators. For an anti-Hermitian matrix M, $M^\dagger = -M$, we could instead use that $U = e^M$ is automatically unitary since $U^\dagger = e^{-M}$ so that $U^\dagger U = e^{-M}e^M = I$. This is the mathematical construction found in a lot of mathematical literature.

$SU(2)$

A general member S of $SU(2)$ can be written in terms of two complex parameters $\alpha, \beta \in \mathbb{C}$ as

$$S(\alpha, \beta) = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}, \tag{1.1}$$

with the additional constraint $|\alpha|^2 + |\beta|^2 = 1$, leaving three free parameters as expected.

There is a deep connection between the $SU(2)$ group and spin because the Hermitian **Pauli matrices**

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1.2}$$

that make up the spin-operators $S^i = \frac{\hbar}{2}\sigma^i$, are members of $SU(2)$, and indeed, as we shall see, they can be used to generate (almost) any member of $SU(2)$ by the exponential construction $U = e^{i\alpha_i\sigma^i}$, where $\alpha_i \in \mathbb{R}$ are three real parameters.[5]

1.2.3 Orthogonal and special orthogonal groups

If we restrict the unitary matrices to be real, we get the orthogonal groups.

> **Definition**: The **orthogonal group** $O(n)$ is the group of real $n \times n$ orthogonal matrices O, *i.e.* matrices where $O^TO = I$. If we additionally require that $\det(O) = 1$ the matrices form the **special orthogonal group** $SO(n)$.

It follows from the definition of the orthogonal group that the determinant of the members is either 1 or -1, thus the special orthogonal group is simply one half of the members. For $\mathbf{x} \in \mathbb{R}^n$ the orthogonal group has the same property as the unitary group of leaving the length of vectors invariant.

Matrices in the $O(n)$ and $SO(n)$ groups have $n(n-1)/2$ independent parameters since an $n \times n$ matrix with real entries has n^2 elements, and there are $n(n+1)/2$ equations to be satisfied by the orthogonality condition.[6]

[5] Make a note of the notation here, the position of the indices is used to prepare the ground for using these in a four-vector $\sigma^\mu = (\sigma^0, \boldsymbol{\sigma})$ together with the identity matrix

$$\sigma^0 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

[6] Only the upper triangular part of O^TO has independent equations since O^TO is a symmetric matrix, $(O^TO)^T = O^T(O^T)^T = O^TO$.

The special orthogonal groups $SO(2)$ and $SO(3)$ are much used because their elements represent rotations in two and three dimensions, respectively, while $SO(n)$ extends this to higher dimensions and represents the symmetries of a sphere in n dimensions. To see this we can start from the fact that rotations, by definition, conserve angles and distances (and orientation). This means that the original set of orthogonal axis – or orthogonal basis vectors if you wish – must transform into another orthogonal set of axis under the rotation. The matrix performing the rotation must then be orthogonal, and thus the collection of rotations must be $O(n)$. If we additionally require that orientation is preserved, this removes the matrices with negative determinant, leaving the $SO(n)$ group.

SO(2)

Given that $SO(2)$ has only one parameter, we can write a general group member R as parameterised by θ

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{1.3}$$

As expected, we recognise this as the matrix of rotations of an angle θ around a point in the plane, and it represents the (orientation preserving) symmetries of a circle. Some tinkering with this representation will show that $SO(2)$ is in fact an abelian group, despite the matrix definition. From a physical viewpoint this should be expected: the order of rotations in the plane should not matter.

It is interesting to observe that the elements of $SO(2)$ rotate points in the plane, while the elements of $U(1)$ rotate complex numbers, which can be represented by points in the plane. Indeed, a one-to-one correspondence can be found between the members of the two groups so that the groups are indeed the same, or, as we say, isomorphic, $SO(2) \cong U(1)$.

SO(3)

This group has three free parameters. Already at this point writing down the explicit form of a general group member is not very enlightening. There are also a number of different conventions in use, so proper care is advised when using results from the literature. In terms of a general rotation in three dimensions this can either be viewed as rotation angles around three fixed axis, or as the fixing of a rotation axis by two angles, with a third rotation angle around that axis.

One particular explicit form, where the angles correspond to the three **Euler angles** of rotation in three dimensions, α, β and γ, is

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix},$$

where $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta \leq \pi$. These rotations do not commute, so the group is non-abelian.

We have seen above that $SU(2)$ has three free parameters, just the same as $SO(3)$. You may at this point guess that $SO(3)$ is isomorphic to $SU(2)$. That would be a very good guess, however, it would also be wrong. We will return to this later, but this is one of those things in mathematics that turn out to be disappointingly only almost true.

1.2.4 Symplectic and compact symplectic groups*

Definition: The **symplectic group** $Sp(2n)$ is the group of $2n \times 2n$ symplectic matrices M, *i.e.* matrices where $M^T \Omega M = \Omega$, with

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

The **compact symplectic group** $Sp(n)$ is the intersection of the symplectic group $Sp(2n)$ and the unitary group $U(2n)$, *i.e.*

$$Sp(n) \equiv Sp(2n) \cap U(2n),$$

so that its matrices are members of both groups.[a]

[a] The n as apposed to $2n$ here is intentional, the reason is that the compact symplectic group can be described in terms of the unitary group $U(n, \mathbb{H})$ of $n \times n$ matrices over the quaternions (see Sec. 1.6.1).

The choice here of Ω can be generalised to any nonsingular skew-symmetric matrix, *i.e.* an invertible matrix Ω where $\Omega^T = -\Omega$, however, this does not change the group structure. Note that we can easily show $\det \Omega = 1$ and $\Omega^{-1} = \Omega^T = -\Omega$ for our choice of Ω. This in turn implies that the symplectic matrices have $\det M = \pm 1$, but this can be strengthened to show that indeed $\det M = 1$.

The compact symplectic group has interesting applications in classical mechanics. If a system of n particles has generalised coordinates q_i and momenta p_i we can stack the coordinates in the vector $\boldsymbol{\eta} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$. It can then be shown that any symplectic matrix M acting on $\boldsymbol{\eta}$ is a **canonical transformation**, meaning it is a transformation that leaves Hamiltons equations unchanged, so the symplectic groups are the symmetry groups of Hamilton's equations.

In quantum mechanics we can write the canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$ as

$$[\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\eta}}^T] = i\hbar\Omega.$$

1.3 Group properties

1.3.1 Subgroups

We now extend our vocabulary for groups by defining the **subgroup** of a group G.

Definition: A subset $H \subset G$ is a **subgroup** if and only if:[a]

i) $h_i \circ h_j \in H$ for all $h_i, h_j \in H$, (closure)

ii) $h_i^{-1} \in H$ for all $h_i \in H$. (inverse)

H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$.

[a] An alternative, equivalent, and more compact way of writing these two requirements is the single requirement $h_i \circ h_j^{-1} \in H$ for all $h_i, h_j \in H$. This is often utilised in proofs.

We have already seen some examples of subgroups: the $SU(n)$ groups are subgroups of $U(n)$, and the $SO(n)$ groups are subgroups of the $O(n)$ groups. This can easily be shown using the properties of determinants.

There is a very important class of subgroup called the **normal** subgroup. The importance will become clear in a moment.

Definition: A subgroup H is a **normal** (invariant) subgroup, if and only if the **conjugation** of any element $h \in H$ by any $g \in G$ is in H,[a] meaning

$$ghg^{-1} \in H \text{ for all } h \in H.$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no proper abelian normal subgroup.

[a] Another, pretty, but slightly abusive, way of writing the definition of a normal group is to say that $gHg^{-1} = H$. This implies (correctly), that the image if H under the conjugation operation is guaranteed to be the whole of H.

We can for example show that for $n > 1$, $SU(n)$ is a normal subgroup of $U(n)$, see Ex. 8.

1.3.2 Quotient groups

The normal subgroup can be seen as a factor in the original group that can be divided out to form a simpler group that only retains the structure that was not in the normal group. To be more precise we need the concept of **cosets**.

Definition: A **left coset** of a subgroup $H \subset G$ with respect to $g \in G$ is the set of members $\{gh|h \in H\}$, and a **right coset** of the subgroup is the set $\{hg|h \in H\}$. These are sometimes written gH and Hg, respectively.

Despite the appearance of containing many of the same members we can show any two cosets are either disjoint or identical sets.

For normal subgroups H it can be shown that the sets of left and right cosets gH and Hg coincide and form a group. This is called the **quotient** or **coset group** and denoted G/H.[7] This has as its members all the *distinct sets* $\{gh|h \in H\}$, that can be generated by a $g \in G$, and has the binary operation $*$ with $\{gh|h \in H\} * \{g'h|h \in H\} = \{(g \circ g')h|h \in H\}$. To simplify notation this can be written $gH * g'H = (g \circ g')H$.

To make some sense of this, let us briefly discuss an example of a quotient group. We

[7] Sometimes also called the **factor group**. The notation G/H is pronounced “G mod H”, where “mod” is short for modulo.

already know that $SU(n)$ is a normal subgroup to $U(n)$. This means that $U(n)/SU(n)$ is a group. What sort of group is this? Notice that two matrices U_i and U_j live in the same coset of $SU(n)$ if and only if $\det(U_i) = \det(U_j)$. In other words, each coset constructed from $SU(n)$ is simply the set of all matrices with a given determinant, which we saw for the $U(n)$ group can be any unit complex number. Observe that this means that many of the cosets are the same, all cosets generated by members in $U(n)$ with the same determinant is the same coset. When these cosets act on each other with the group operation of $U(n)/SU(n)$ they form new cosets of matrices with a determinant that is the product of their individual determinants, *i.e.* with $U, U' \in U(n)$ we have the product of quotient group member $\{US|S \in SU(n)\} * \{U'S|S \in SU(n)\} = \{UU'S|S \in SU(n)\}$. Thus, the group behaves exactly as $U(1)$, and is in fact isomorphic to it.

1.3.3 Product groups

Now that we have introduced group division, we also need to introduce products of groups.

> **Definition**: The **direct product** of groups G and H, $G \times H$, is defined as the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \circ (g_j, h_j) = (g_i \circ g_j, h_i \circ h_j)$. $G \times H$ is then a group and G and H can be shown to be normal subgroups of $G \times H$.

We should note here that the subgroups are strictly $G \times \{e_H\}$ and $\{e_G\} \times H$, but these are isomorphic to G and H.

Because it has an important appearance in this text we also need the definition of the **semi-direct product**.

> **Definition**: The **semi-direct product** of groups G and H, $G \rtimes H$, where G is also a mapping $G : H \to H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \circ (g_j, h_j) = (g_i \circ g_j, h_i \circ g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$, but G is.

Note how the semi-direct product is not symmetric between the factors.

The famous Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is an example of a direct product. Direct products are "trivial" structures because there is no "interaction" between the subgroups, the elements of each group act only on elements of the same group. For the Standard Model this means that the each of the gauge transformations can act independently on the states unaffected by any other gauge transformation. This is not true for semi-direct products.

1.3.4 Isomorphic groups

We have already talked about how two groups are the same if they have a one-to-one correspondence between their members and the same results for the multiplication operation, and we have called this an isometry. Let us now try to put this notion of when two groups are the same into a more formal language.

> **Definition**: Two groups G and H are **homomorphic** if there exists a map between the elements of the groups $\rho : G \to H$, such that for all $g, g' \in G$, $\rho(g \circ g') = \rho(g) \circ \rho(g')$.

For homomorphic groups we say that the mapping conserves the structure of the group, or in other words, all the rules for the group operation/multiplication. This leads to our notion of group equality, namely **isomorphic** groups:

> **Definition**: Two groups G and H are **isomomorphic**, written $G \cong H$, if they are homomorphic and the relevant mapping is one-to-one (injective) and onto (surjective).

The one-to-one and onto (hitting all elements of H) mapping ensures that there is a one-to-one correspondence between the elements of the two groups, so that isomorphic groups effectively contain both the same members and have the same multiplication operation.

For matrix groups, a good way of checking the plausibility of isomorphism is to count the number of free parameters. The difference of the parameters for the two factors should be equal to the number of parameters for the quotient group. In our example $U(n)/SU(n)$, $U(n)$ has n^2, while $SU(n)$ has $n^2 - 1$, and this gives $n^2 - (n^2 - 1) = 1$ parameters for the quotient group, which matches the one parameter of $U(1)$.

At the end of this subsection, let us make a warning: despite the enticing notation it is *not* in general the case that if H is a normal subgroup to G, then $G \cong G/H \times H$.

1.4 Representations

In some contrast to the treatment in most introductory group theory texts in mathematics, physicists are mostly interested in the properties of groups G where the elements of G *act* to transform some elements of a vector space $v \in V$, $g(v) = v' \in V$. This part of group theory is called **representation theory**. Here, the members of V can for example be the state of a system, say a wave-function in quantum mechanics or a field in quantum field theory. To be useful in physics, we would like that the result of the group operation $g_i \circ g_j$ acts as $(g_i \circ g_j)(v) = g_i(g_j(v))$ and the group identity acts as $e(v) = v$.

1.4.1 Definition

We begin with the abstract definition of a representation that ensures these properties.

> **Definition**: A **representation** of a group G on a vector space V over the field K is a map $\rho : G \to GL(V, K)$, where $GL(V, K)$ is the **general linear group of the vector space** V,[a] such that for all $g_i, g_j \in G$,
>
> $$\rho(g_i \circ g_j) = \rho(g_i)\rho(g_j). \text{ (homomorphism)}$$
>
> If this map is also isomorphic, we say that the representation is **faithful**.
>
> [a]The general linear group of the abstract vector space V is the set of all one-to-one and onto linear transformations $V \to V$, together with functional composition as group operation. For finite dimensional spaces this group is isomorphic to the general linear group $GL(n, K)$ that we introduced earlier.

Here V is called the **representation space** and the dimension n of V is called the dimension of the representation. While somewhat confusing, it is common to talk about V as the representation itself, even though it is really the space on which the representation acts.

To see that this fulfils our wanted properties, notice that $\rho(g) = \rho(e \circ g) = \rho(e)\rho(g)$, which means that $\rho(e) = \mathbb{1}$ must be the identity transformation on V. The concatenation property follows directly from the homomorphism.

Writing this definition in terms of a generic vector space V and field K may be a bit obfuscating, but we need to keep in mind that our group elements can be acting on members of abstract vector space, *e.g.* spaces where the members of the space are functions, such as in the earlier example of the translation group acting on functions. In the case where V is finite dimensional it is common to choose a concrete basis for V and identify $GL(V, K)$ with $GL(n, K)$, the group of $n \times n$ invertible matrices with elements from the field K. So, in many cases, the representations we are interested in are matrices acting on coordinate space vectors over the fields $\mathbb{R}$ or $\mathbb{C}$, *i.e.* vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$.

From a physics point of view, the underlying point here is that (the members of) our groups will be used on quantum mechanical states, or fields in field theory, which can be just complex numbers or functions, or multi-component vectors of such. They are thus members of a vector space, and the definition of representations allows the transformation properties of the group to be written in terms of matrices for finite dimensional representation spaces. Furthermore, the mapping from the group, or, if you like, the concrete way of writing the abstract group elements, must be homomorphic (structure preserving), meaning that if we can write a group element as the product of two others, the matrix for that element must be the product of the two matrices for the individual group elements it can be written in terms of.

In physics an important category of representations are the **unitary representations**, which are representations ρ of a group G on a complex Hilbert space V where $\rho(g)$ is a unitary operator for every $g \in G$, *i.e.* an operator that preserves the length of vectors in V. This generalises the unitary (matrix) group, which are examples of unitary representations, beyond finite dimensional vector spaces.

1.4.2 Representation examples

You may by now have realised that the matrix groups defined in Sec. 1.2 have the property that they are defined in terms of one of their representations. These are called the **fundamental** or **defining representations**. However, we will also have use for other representations, *e.g.* the **adjoint representation** which we will introduce later.

Let us now take a few examples that connect to our definition and to relevant physics. We saw earlier that for $U(1)$ the group members can be written in the fundamental representation as the complex numbers on the unit circle $e^{i\theta}$, which can be used as phase transformations on wavefunctions $\psi(x)$. This can be viewed as the action on a one-dimensional space over $\mathbb{C}$ where the basis vector is the wavefunction.

For $SU(2)$ we saw that we needed three real numbers to parametrise the group elements. It then seems reasonable that we should be able to write a general matrix in $SU(2)$ in terms of the linear combination of three unitary "basis" matrices, for example the **Pauli matrices** in (1.2). Since the three Pauli matrices are linearly independent, the sum $\alpha_i \sigma^i$, $\alpha_i \in \mathbb{R}$, should (hopefully) in some sense span all of $SU(2)$.[8] As it turns out, the group elements of $SU(2)$ in the fundamental representation can indeed written as the exponentials $e^{i\alpha_i\sigma^i}$. We will return to why we use an exponential in Sec. 1.7. In physics the fundamental representation

[8] We will later see to what extent this is true, but we must emphasise here that $SU(2)$ is not a vector space, so it is not spanning in the vector space sense.

of $SU(2)$ is often denoted **2**, following the pattern that the representations are written up with the dimension of their matrices in boldface. In the Standard Model the fundamental representation is used on the vector space of doublets of fermion fields, *e.g.* the electron–neutrino doublet $\psi = (\nu_e, e)^T$ that form a two-dimensional vector space, as the $SU(2)_L$ gauge transformation.

However, we can construct many more representations than the fundamental from a single group such as $SU(2)$. Using the three free parameters in $SU(2)$ it turns out that we can also represent the elements in the group in terms of three 3×3-matrices that act on vectors in a three-dimensional space. In place of the Pauli matrices we can use the three (anti-Hermitian) matrices

$$L^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.4}$$

You may recognise these as related to the angular momentum operators in quantum mechanics in the Heisenberg picture. As we will see later, these matrices will form the basis for creating the three-dimensional adjoint representation of $SU(2)$, named **3**. In the Standard Model this is the representation used for operations on the gauge fields that transform under $SU(2)_L$, which are represented as three-component vectors. The central point here is that the group structure is the same (isometric), even if the objects in the representation are different.

At this point you may be worried about the translation group $T(1)$ that we saw earlier and its extensions to higher dimensions. There we wrote down the elements using differential operators $T_a = e^{a\frac{\partial}{\partial x}}$. This does not look a lot like matrices. However, here the abstract vector space that is acted on by the group elements is made up of smooth (infinitely differentiable) functions, and in our generic definition of a representation above there is no need for the general linear group of such a space to be written in terms of matrices. Unless we are very strict with what functions we allow, this is typically an infinite dimensional space. It is a bit tricky to prove that our differential operators T_a form a representation, however, they do, and these kinds of group representations are called **differential representations**.

The existence of multiple representations for the same group necessitates a definition of when representations are actually equivalent, or isomorphic. This should not be confused with whether groups are isomorphic, but removes differences in representations that are simply due to a change in basis for the vector space the group is acting on, or trivial changes in the dimension of the vector space.[9]

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if there exists a map $A : V \to V'$, that is one-to-one and onto, such that for all $g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

We again give the definition for an abstract representation space. For a finite-dimensional space with a given basis, A and $\rho(g)$ are simply matrices.

[9] Imagine, for example, that you create a representation for $U(1)$ that consists of diagonal 2×2-matrices with the unit complex numbers repeated twice on the diagonal. This is not essentially different from the one-dimensional representation, and should not be considered as such.

1.4.3 Irreducible representations

The building blocks of representations are so-called **irreducible representations**, also called **irreps**. These are the essential ingredients in representation theory, and are defined as follows:

> **Definition**: An **irreducible representation** ρ of a group G is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, *i.e.* there is no $W \subset V$ such that for all $w \in W$, and all $g \in G$ we have $\rho(g)w \in W$.[a]
>
> [a] In other words, we can not split the matrix representation of G in two parts that do not "mix".

Let us take an example to try to clear up what a *reducible* representation means in contrast to an irreducible. Assume the representation $\rho(g)$ for $g \in G$ acts on a vector space V as matrices. If these matrices $\rho(g)$ can all be decomposed into $\rho_1(g)$, $\rho_2(g)$ and $\rho_{12}(g)$ such that for $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V$

$$\rho(g)\mathbf{v} = \begin{bmatrix} \rho_1(g) & \rho_{12}(g) \\ 0 & \rho_2(g) \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix},$$

then ρ is **reducible**. The subspace W of V spanned by $\mathbf{v}_1$ violates the irreducibility condition above.

If we also have $\rho_{12}(g) = 0$ in our example we say that the representation is **completely reducible**. It can be shown that in most cases a reducible representation is also completely reducible. In fact, representations for which this is not true tend to be mathematical curiosities. For example, if the representation is unitary and the vector space is a Hilbert space (states in quantum mechanics), we can prove that the representation is always completely reducible. As a result, there is a tendency in physics to use the term "reducible" where we should maybe use the term "completely reducible".

In the case of a completely reducible representation we can split the vector space V into a direct sum of two vector spaces $V = V_1 \oplus V_2$, where $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$, and define a representation of G on each of them using ρ_1 and ρ_2, which in turn could either be reduced more, or would themselves be irreducible. This process can then be continued until the representation has been broken down into only irreducible components.

We end this section with an important theorem that helps us decide whether a representation is irreducible, and ultimately gives a property identifying the representation. As with many important theorems, it started its life as a lemma.

> **Theorem**: **(Schur's Lemma [1])**
> If we have an irreducible representation ρ of a group G on a vector space V and a linear operator $A : V \to V$ that commutes with $\rho(g)$ for all $g \in G$, then A is proportional to the identity map, $A = \lambda\mathbb{1}$. The constant of proportionality λ can be used to label the representation.

Schur's lemma as stated here has again been formulated in the language of general vector spaces. In terms of matrix representations it says that any matrix A that commutes with all the matrices in an irreducible representation must be equal to a constant times the identity matrix, *i.e.* $A = \lambda I$.

The constant of proportionality λ is unique to the representation up to a common normalisation constant that can be incorporated into the linear operator/matrix (a constant multiple of A would still have the same commutation properties). For us, these linear operators A will generally turn out to have a very specific physical interpretation, so their natural normalisation will be clear. As a result the constants can be used to index the different irreducible representations. The proof of this property is somewhat beyond the scope of these notes.

1.5 Lie groups

1.5.1 Definitions

In physics we are particularly interested in a special type of continuous groups that we can parametrise, the **Lie groups**, which are the basic tools we use to describe continuous symmetries. The abstract mathematical notion of continuity comes from topology, along with a notion of open sets and proximity, so continuous groups are also called **topological groups**. For topological groups the group multiplication and inverse need to be continuous maps. We further say that a topological group is a **compact group** if its topology is compact, meaning that it has no punctures or missing endpoints, *i.e.* it includes all limiting values of the group members. In practise the topology of our groups will be subspaces of Euclidean space E^n, in which case the space is compact if and only if it is closed and bounded. We will return to some examples of this later.

A Lie group extends the continuity of topological groups by requiring the multiplication and inverse maps to be *smooth* (infinitely differentiable C^∞). In order to define Lie groups we will need to use the technical term **(smooth) manifold**, meaning a mathematical object (formally a topological space) that locally[10] can be parametrised as a function of $\mathbb{R}^n$ or $\mathbb{C}^n$. We will thus describe a Lie group G as a manifold in terms of a parameterisation of the members $g(\mathbf{a}) \in G$, where $\mathbf{a} \in \mathbb{R}^n$ (or $\mathbb{C}^n$). Additionally, in order to describe continuous symmetries these parameterisations need to be smooth.

Definition: A **Lie group** G is a finite-dimensional **smooth manifold** where group multiplication and inversion are smooth functions, meaning that given elements $g(\mathbf{a}), g(\mathbf{a}') \in G$, $g(\mathbf{a}) \circ g(\mathbf{a}') = g(\mathbf{b})$ where $\mathbf{b}(\mathbf{a}, \mathbf{a}')$ is a smooth function of $\mathbf{a}$ and $\mathbf{a}'$, and $g^{-1}(\mathbf{a}) = g(\mathbf{b})$ where $\mathbf{b}(\mathbf{a})$ is a smooth function of $\mathbf{a}$.

The dimension of a Lie group is the dimension n (or $2n$) of the manifold.

From our earlier example we immediately see that the translation group $T(1)$, given the parameterisation of the elements $g(a) = e^{a\frac{\partial}{\partial x}}$, is a Lie group since $g(a)g(a') = g(b) = g(a+a')$ and $b = a + a'$ is an analytic function of a and a', and for the inverse $g^{-1}(a) = g(b) = g(-a)$ where $b = -a$ is a smooth function of a. Since $a \in \mathbb{R}$ is unbounded the group members are also unbounded, and thus the group is not compact.

All the matrix groups are also Lie groups. While their dimension and parameterisation is clear, for example a member of $GL(n, \mathbb{R})$ has n^2 parameters by its elements, to prove that they are in fact Lie groups is tricky because of the requirement on the non-zero determinant. The proof is done by showing it for the general linear group, and then using the **closed subgroup theorem**, saying that any closed (in the topological sense) subgroup of a Lie group is a Lie group, which applies to the other matrix groups.

[10] This insistence on local means that the parameterisation is not necessarily the same for the whole group.

We saw several examples of parameterisations of matrix groups earlier, for example $U(1)$ could be parameterised by a single parameter $\theta \in [0, 2\pi]$ by writing an element $U \in U(1)$ as the complex number $U = e^{i\theta}$. We can then identify the group topologically with a subspace of E^2, and since $|U| = 1$ this group is bounded and closed, and thus a compact group. Similar arguments can be made for the compactness of all the matrix groups $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Sp(2n)$, and $Sp(n)$, however, $GL(n)$ and $SL(n)$ are not compact because their elements are not bounded.

1.5.2 Generators

The situation we are usually in in physics is that of a d-dimensional Lie group G acting on a vector space V through a representation. This representation vector space may be finite dimensional, in which case its members $\mathbf{x} \in V$ are written as vectors in some specific basis, or it may be infinite dimensional and consist of functions that in turn have a domain $\mathbb{R}^n$ or $\mathbb{C}^n$.

For finite-dimensional representations we can locally write the map of the representation $G \times V \to V$ for $\mathbf{x} \in V$ in terms of an explicit function $\mathbf{f}$ called the **composition function**. We have $x_i \to x_i' = f_i(\mathbf{x}, \mathbf{a})$, $i = 1, \ldots, n$, where the composition function f_i is analytic[11] in x_i and a_j, $j = 1, \ldots, d$. Additionally f_i should have an inverse. Note here that the dimensions of $\mathbf{x}$ and $\mathbf{a}$ are different, the first is given by the dimensions of the space V, n, while the second is given by the dimensions d of the Lie group.

By the analyticity of the explicit function f we can always construct the parametrisation so that the zero parameter corresponds to the identity element of the group, $g(0) = e$, which means that $f_i(\mathbf{x}, 0) = x_i$.[12] By an infinitesimal change $d\mathbf{a}$ of the parameters we then get the following Taylor expansion[13]

$$\begin{aligned} x_i' &= x_i + dx_i = f_i(\mathbf{x}, d\mathbf{a}) \\ &= f_i(\mathbf{x}, 0) + \frac{\partial f_i}{\partial a_j} da_j \bigg|_{\mathbf{a}=0} + \ldots \\ &= x_i + da_j \frac{\partial f_i}{\partial a_j} \bigg|_{\mathbf{a}=0}. \end{aligned}$$

This is the result of the transformation by the member of the group that in the parameterisation sits $d\mathbf{a}$ from the identity. For a group where the matrix A of the representation is parametrised as $A(\mathbf{a})$ the resulting change in the vector space is

$$dx_i = da_j \frac{\partial f_i}{\partial a_j}\bigg|_{\mathbf{a}=0} = da_j \left[\frac{\partial A(\mathbf{a})}{\partial a_j}\mathbf{x}\right]_{i,\mathbf{a}=0} = [i da_j X^j \mathbf{x}]_i,$$

where we have defined

$$iX^j \equiv \frac{\partial A(\mathbf{a})}{\partial a_j}\bigg|_{\mathbf{a}=0},$$

as the d matrix **generators** X^j of the Lie group.

[11] Analytic means infinitely differentiable and in possession of a convergent Taylor expansion. As a result analytic functions (on $\mathbb{R}$) are smooth, but the reverse does not hold.

[12] If this was not true for our parameterisation we could Taylor expand $\mathbf{f}$ around the parameter giving the identity element and then redefine the parameterisation by a linear shift.

[13] The fact that f_i is analytic means that this Taylor expansion must converge in some radius around $f_i(\mathbf{x}, 0)$.

If we now instead let $F(\mathbf{x})$ be a function taken from the (infinite dimensional) vector space V that we are interested in, mapping to either the real $\mathbb{R}$ or complex numbers $\mathbb{C}$, where $\mathbf{x}$ is now in the n-dimensional domain of the function, and letting $\mathbf{f}$ be the transformation of the domain effected by the group elements, then the group transformation defined by $d\mathbf{a}$ near the identity changes F by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x_i} dx_i = \frac{\partial F}{\partial x_i} \left.\frac{\partial f_i}{\partial a_j}\right|_{\mathbf{a}=0} da_j \\ &\equiv i da_j X^j F, \end{aligned}$$

where the operators defined by

$$iX^j \equiv \left.\frac{\partial f_i}{\partial a_j}\right|_{\mathbf{a}=0} \frac{\partial}{\partial x_i}, \tag{1.5}$$

are called the d differential **generators** of the Lie group. We see here that in both cases the number of generators is the same as the dimension of the manifold, which is also the number of free parameters in the parameterisation of the group.

It is these generators X that then define the effect of the Lie group members in a given representation (near the zero parameter), while the d numbers a_j are mere parameters. We say that the generators determine the local structure of the group. It turns out that this linearisation, often called the **tangent space**, of a group using an infinitesimal change near the origin is sufficient to recover the whole group locally. Note that the generators are not unique, they do depend on the parameterisation of the group, but as we shall see later some of their properties are independent of this.

For the translation group $T(1)$ the action of the group on the domain $\mathbb{R}^1$ of the functions in the representation space, is $x' = f(x, a) = x + a$. The resulting (single) generator is

$$iX^1 = \frac{\partial f}{\partial a} \frac{\partial}{\partial x} = \frac{\partial}{\partial x}.$$

Notice here how we represented a generic element of $T(1)$ by the parameter and the generator as

$$T_a = e^{ia_j X^j} = e^{a \frac{\partial}{\partial x}}.$$

This is of course no coincidence, and shows the way to a general recipe that we will use.[14]

As another example of the above we can now go in the opposite direction and look at the two-parameter coordinate transformation *defined* by

$$x' = f(x) = a_1 x + a_2,$$

which gives the generators

$$iX^1 = \frac{\partial f}{\partial a_1} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x},$$

which is the generator for **dilation** (scale change) in one dimension $D(1)$, and

$$iX^2 = \frac{\partial f}{\partial a_2} \frac{\partial}{\partial x} = \frac{\partial}{\partial x},$$

[14]While we are in the business of *deja vu*, notice also how the generator for translations is $X = -i\frac{\partial}{\partial x}$, which can be compared with the quantum mechanical momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, keeping in mind that the conservation of momentum through Noether's theorem is intimately linked to the invariance of our models under translation. This is a point we will return to later.

which is again the generator for the translation $T(1)$. Notice that we can show the following relationship for these two generators: $[X^1, X^2] \equiv X^1X^2 - X^2X^1 = iX^2$. This can be seen to hold independently of the parameterisation of the group, something we will generalise later.

As we have seen, the group $SU(2)$ has three free parameters a_i, so it must have three generators X^i. We can now show that the generators for $SU(2)$ in the two-dimensional representation are indeed proportional to the Pauli matrices in (1.2) as intimated earlier, namely $X^i = \frac{1}{2}\sigma^i$, see Ex. 18. By multiplying out we can also show the following commutation relationships between the generators (Pauli matrices):

$$[X_i, X_j] = i\epsilon_{ijk}X^k. \tag{1.6}$$

These commutators should look familiar to us as they have the same structure as the commutators for the spin S_i operators in quantum mechanics.[15]

For matrix groups there is an alternative method to the generating functions for finding the generators that sees a lot of use. Since we are looking for infinitesimal deviations from the identity (matrix) I, we add an infinitesimal matrix dM (the generator) and look at the properties of dM given the definition of the group. Let us take $SO(2)$ as an example. Here a matrix in the group O must fulfil $O^TO = I$ and $\det O = 1$. Writing near the identity $O = I + dM$ we get the requirement $O^TO = I + dM + dM^T = I$ to first order in the infinitesimal. Thus the generator must fulfil $dM = -dM^T$ (it is a real, anti-symmetric matrix). This means that the diagonal of dM must be zero, and in two dimensions, up to a constant, the only matrix that fits the role of the generator is

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so that the group element can be written $O = I + d\theta X$, where $d\theta$ is some infinitesimal parameter of the group. To check the determinant we need to consider how we use the generator to recreate the group since the generator is defined through an infinitesimal change. For reasons that will become clearer later we use the exponential $O = \exp(\theta X)$, which to first order in an infinitesimal parameter $d\theta$ recreates the behaviour near the identity we used. By the properties of matrix exponentiation we can now check that $\det O = \det\exp(\theta X) = \exp(\theta \operatorname{Tr} X) = \exp(0) = 1$. Since X is an anti-Hermitian matrix, and we as physicists are more comfortable as Hermitian, we tend to use instead

$$X = -i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

as the generator, writing the group element as $O = \exp(i\theta X)$.

A similar calculation for $SO(3)$, which has three free parameters, gives the generators $X^i = \frac{1}{2}L^i$, where L^i are the three matrices in Eq (1.4). These generators have the exact same commutator as the $SU(2)$ generators, $[X_i, X_j] = \epsilon_{ijk}X^k$, up to a constant factor, indicating a structural similarity between the groups $SU(2)$ and $SO(3)$. For $SO(3)$ we can further find a corresponding set of differential generators

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{1.7}$$

[15] This would be $[S_i, S_j] = i\hbar\epsilon_{ijk}S^k$.

see Ex. 17. Up to a constant factor of $\hbar$ you may recognise these operators as the angular momentum operators in quantum mechanics. This is not a coincidence, the angular momentum operators in quantum mechanics are a representation of the generators of the group of rotations in three dimensions. In fact, the possibility of going back and forth between differential representations and matrix representations of the Lie group is the essence of the duality between the Schrödinger and Heisenberg pictures of quantum mechanics.

1.5.3 Structure constants

In general, the commutator of the generators of a Lie group satisfy

$$[X_i, X_j] = iC_{ij}{}^k X_k,$$

where $C_{ij}{}^k$ are called the **structure constants** of the group.[16] This implies that the generators close under the commutation operation.

We can easily see that these commutators are antisymmetric in the indices i and j,

$$C_{ij}{}^k = -C_{ji}{}^k.$$

We can also show that there is a **Jacobi identity** among the generators that just directly follows from the properties of the commutator,

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0, \tag{1.8}$$

which in turn leads to a corresponding identity for the structure constants:

$$C_{il}{}^m C_{jk}{}^l + C_{jl}{}^m C_{ki}{}^l + C_{kl}{}^m C_{ij}{}^l = 0. \tag{1.9}$$

Note that just like the generators are basis dependent, so are the structure constants. However, the closure in the commutation relationship is not. For example, we could have chosen the Pauli matrices to be the generators of $SU(2)$. They would then fulfil the commutator relationship

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma^k,$$

where the structure constant is $2\epsilon_{ijk}$ instead of ϵ_{ijk}. For the N-dimensional unitary groups in physics it is common to use the following normalisation of the structure constants,

$$C_{kl}{}^i C^{klj} = N\delta^{ij}, \tag{1.10}$$

which coincides with the choices made above for $SU(2)$.

1.6 Algebras

To further study the structure of groups we begin by defining an **algebra**. An algebra extends the familiar structure of vector spaces by adding a multiplication operation for the vectors which gives a new vector.

[16] There is an annoying difference in notation here between physics and mathematics, where the i is commonly dropped. This has the same origin as the Hermitian versus anti-Hermitian operator used to create unitary operators discussed earlier.

> **Definition**: An **algebra** A over a field (say $\mathbb{R}$ or $\mathbb{C}$) is a linear vector space with a binary (multiplication) operation $\circ : A \times A \to A$.

It is important to remember here that the vector space part of the definition implies that there is field, so for example from $\mathbf{x} \in A$ and $a \in \mathbb{R}$ (as the field) we can always form new members $a\mathbf{x} \in A$.

As a very simple example, the vector space $\mathbb{R}^3$ together with the standard cross-product constitutes an algebra since the cross product results in a new vector in $\mathbb{R}^3$. Even more trivially perhaps is that $\mathbb{R}$ with ordinary multiplication as the binary operation fulfils the algebra requirements (more on this case below).

Connecting to the discussion of Lie groups in Section 1.5, we see that the generators X_i of a Lie group also span an algebra as basis vectors of a vector space, if we use the commutator as the multiplication operation between two members. We leave it as an exercise to ponder out how all the criteria for a vector space is fulfilled by the generators, but this is not very difficult.

1.6.1 Normed division algebras*

As a slight aside to the main argument of the text, we want to give an interesting example of algebras. We start with a **division algebra**, which informally is an algebra where the binary operation of the algebra also has a meaningful (implicit) concept of division for all members (except division by zero).

> **Definition**: An algebra D is a **division algebra** if for any element x in D and any non-zero element y in D there exists precisely one element z in D with $x = y \circ z$ and precisely one element z' in D such that $x = z' \circ y$. In this sense y is a divisor of x.

We see that division algebras have the addition and subtraction properties of "ordinary numbers" (reals) since they are vector spaces, and they have the multiplication (algebra) and division (division algebra) properties of the reals as well. So, in a sense, division algebras are structures close to the reals in terms of properties – and, of course, the reals are again an example of a division algebra.

We can now add to the division algebra the notion of a **norm**, or length, of the members $\|x\|$. This is a map from the algebra to the field, $\| \, \| : D \to \mathbb{R}$, so that we can discuss for example convergence of the members as we do for the reals. We have to require here that the norm is homomorphic, meaning that it preserves the structure of the algebra, so that for example the "product" of two objects with large norm has a large norm. We then get a **normed division algebra**.

> **Definition**: If there exists a homomorphic norm for the division algebra, *i.e.* one where $\|x \circ y\| = \|x\|\|y\|$ for all $x, y \in D$, then the division algebra is a **normed division algebra**.

There is an important theorem by Hurwitz (1923) that demonstrates that only four of these real number "lookalikes" exist.

Theorem: Hurwitz's theorem. There are only four normed division algebras over the reals (up to isomorphism), the reals themselves $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.

In addition it is (relatively speaking) easy to show that $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$. One perspective on the relationship between these algebras is that the reals is the only ordered normed division algebra, *i.e.*, where we can compare uniquely the elements $a > b$. This is not possible for the complex numbers, but they keep the commutative and associate properties of the reals. The quaternions in turn break the commutativity of the complex numbers, while being associative, while the most unruly of the bunch, the octonions, are not even associative.

1.7 Lie algebras

We will now turn to the most crucial type of algebras for physics, namely **Lie algebras**. To distinguish these from the more general algebras that we have introduced above, we will use the notation $[\,,\,]$ for the binary operation in Lie algebras.

Definition: A **Lie algebra** $\mathfrak{l}$ is an algebra where the binary operator $[\,,\,]$, called the **Lie bracket**, has the properties that for $x, y, z \in \mathfrak{l}$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

i) (bilinearity)
$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

ii) (anti-commutation)
$$[x, y] = -[y, x]$$

iii) (Jacobi identity)
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

If the algebra is over $\mathbb{R}$ it is a real Lie algebra, and over $\mathbb{C}$ a complex Lie algebra. We write the name of Lie algebras $\mathfrak{l}$ in lower case **fraktur** style.

Since Lie algebras are vector spaces, a **subalgebra** $\mathfrak{m}$ of a Lie algebra $\mathfrak{l}$ is simply a subspace $\mathfrak{m} \subseteq \mathfrak{l}$. An **ideal** is a subalgebra that satisfies $[\mathfrak{l}, \mathfrak{m}] \subseteq \mathfrak{m}$. A **simple** Lie algebra is a non-abelian Lie algebra (non-zero structure constants) without any proper ideals. A Lie algebra is **semi-simple** if it is a direct sum of simple Lie algebras.

Again the vectors of $\mathbb{R}^3$ with the Lie bracket defined in terms of the cross product, $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}$, is a simple example of a Lie algebra. However, we usually restrict ourselves to algebras of linear operators where the Lie bracket is the standard commutator $[x, y] \equiv xy - yx$, where the defining properties follow automatically. Thus also explaining the notation that we have used for the binary operator.

From what we learnt in Section 1.5 the generators X_i of an d-dimensional Lie group then span a d-dimensional Lie algebra with the commutator as the Lie bracket.[17] Since this is also a vector space, any element X of the algebra can be written in terms of the generators X_i, as $X = a_i X^i$. This is called the **tangent space** of the group at the identity. The differences between choices in the definition of the generators and the normalisation of the structure constants disappear as they can be absorbed into a rescaling and linear combinations of the basis $\{X_i\}_{i=1}^d$, this makes the Lie algebra of the group *unique*.

However, the reverse is not true. There can be multiple Lie groups that have the same algebra. The often quoted example is the groups $SO(3)$ and $SU(2)$ where we have seen that the commutator of the generators is the same up to constants, so the algebra is the same, we say $\mathfrak{so}(3) \cong \mathfrak{su}(2)$. However, while closely related, the groups are not isomorphic.

1.7.1 Representations of a Lie algebra

We saw in the previous subsection that – just as for groups – we have many different way of writing down a single Lie algebra. We then need a formal definition of representations of Lie algebras.

Definition: A **representation** π **of a Lie algebra** $\mathfrak{l}$ on a vector space V is a map

$$\pi : \mathfrak{l} \to \mathfrak{gl}(V),$$

where $\mathfrak{gl}$ is the Lie algebra of all linear maps $V \to V$ (endomorphisms) with its bracket defined as the composition $[r, s] \equiv r \circ s - s \circ r$, for all $r, s \in \mathfrak{gl}$, and where the map π is homomorphic (structure preserving) for the Lie algebra bracket so that for all $x, y \in \mathfrak{l}$,

$$\pi([x, y]) = \pi(x) \circ \pi(y) - \pi(y) \circ \pi(x).$$

Again this is an abstract definition that is actually simple than it looks. For finite dimensional vector spaces $\mathfrak{gl}$ is the Lie algebra of the corresponding general linear matrix group, just as the definition of representations of groups was a map to the general linear group $GL(n)$. This algebra of $GL(n)$, called $\mathfrak{gl}(n)$, can be written up in terms of $n \times n$ matrices and their commutators, and in this case the multiplication operation $\circ$ is just matrix multiplication in hiding. One word of warning here though, while the group $GL(n)$ contained invertible matrices, there is no guarantee that its generators are invertible matrices, *e.g.* we saw earlier that the generators of $SO(3)$ were the matrices L_i in Eq (1.4) that are not invertible since $\det L_i = 0$.

What makes this definition of representations even simpler for Lie algebras is that **Ado's theorem** [2] proves that every finite-dimensional Lie algebra has a faithful (isomorphic to the Lie algebra) representation on a finite-dimensional vector space.[18] In other words, the Lie algebras for the groups we most care about, Lie groups with a finite number of parameters and thus finite number of generators, is guaranteed to have representations using matrices.

As for groups, we can talk about irreducible representations, irreps, of the Lie algebras,

[17] If you are really paranoid here, you may question wether these generators are guaranteed to be linearly independent. However, since the Lie group is parameterised by d parameters, if the generators were not linearly independent we should be able to remove one of the parameters, leading to a contradiction. Stop it!

[18] For a modern version of a proof of Ado's theorem, see Terence Tao's blog `https://terrytao.wordpress.com/2011/05/10/ados-theorem/`.

which are representations such that V has no proper invariant subspace under the representation, meaning that we cannot construct proper sub-representations. An important result for representations of Lie groups is **Weyl's complete reducibility theorem** which says that the any representation of a finite-dimensional semi-simple Lie algebra on a finite-dimensional space is isomorphic to a direct sum of irreducible representations. This means that the classification of the representations of semi-simple Lie algebras can be done in a very systematic way. In particular, every semi-simple Lie algebra is a subalgebra of $\mathfrak{sl}$, the Lie algebra of the special linear group, and all our matrix groups have simple or semi-simple Lie algebras.

In Sec. 1.5.2 we saw the two sets of generators of the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$, derived from the defining matrix representations of the groups that have different dimensions. Since this is the same algebra this shows two distinct representations of the same algebra.

1.7.2 Exponential map

As we discussed in Section 1.5, the generators describe the local structure of the group near the identity element. We can now finally look at how the group (and matrix representation) is reconstructed from the algebra. For this we use what is called the **exponential map**.

Definition: The **exponential map** from a finite-dimensional Lie algebra $\mathfrak{l}$ of a group G to G is defined by $\exp : \mathfrak{l} \to G$, where for $X \in \mathfrak{l}$ we get the element $g \in G$ given by

$$g = \exp(iX) \equiv \sum_{n=0}^{\infty} \frac{(ia_i X^i)^n}{n!}. \tag{1.11}$$

Here the X_i are the generators of G and thus members of the Lie algebra, and $X = a_i X^i$.

By Ado's theorem any (finite-dimensional) Lie algebra can be represented by matrices, in which case the infinite sum in this definition is nothing more than the formal series definition of the matrix exponential introduced in Sec. 1.2.2. However, we can also use the exponential map for differential representations where the X_i are differential operators, and for infinite-dimensional Lie groups, but the technical aspects why this extension works is a little beyond the scope of these notes.

We have already seen multiple example of the exponential map construction, but we may ask, why use the exponential? One reason is that it gives us a natural expansion of the group around the identity element. With the parameters $\mathbf{a} = 0$ the resulting matrix is the identity matrix, which means that the group element is the identity element $g = e$. Similarly, for an infinitesimal $d\mathbf{a}$ step away from the parameters of the identity we have the group elements near the identity, $g = I + ida_i X^i$ that we used to derive the generators. And now, if we apply a small finite step a_i away from the identity $g = I + i\frac{a_i}{n} X^i$ a total of n times we will get

$$g = \left(I + i\frac{a_i}{n} X^i\right)^n.$$

By the limit definition of the exponential, with smaller and smaller steps letting $n \to \infty$ this is the exponential map

$$g = \lim_{n\to\infty} \left(I + i\frac{a_i}{n} X^i\right)^n = \exp(ia_i X^i).$$

We should now ask two important questions, is the structure of these elements g actually a group, and is it (isomorphic to) the group that gave the Lie algebra? For the first question we may check the group definition. The exponential map does give an identity element which is just the identity matrix. Associativity similarly holds because we are using matrices and matrix multiplication. There is indeed an inverse of an element $g = \exp(iX)$, which is $g^{-1} = \exp(-iX)$, since by the properties of matrix exponentiation we have $gg^{-1} = \exp(iX)\exp(-iX) = I$. The tricky point is whether group multiplication is closed.

If we have two elements from the map $g = \exp(iX)$ and $g' = \exp(iY)$, then $gg' = \exp(iX)\exp(iY)$, and we would like to show that $\exp(iX)\exp(iY) = \exp(iZ)$, where $Z \in \mathfrak{l}$. However, unless X and Y commute we cannot easily join them in the same exponential. We here use the **Baker–Campbell–Haussdorff formula**,

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[[X,Y],Y] - \frac{1}{12}[[X,Y],X] + \ldots, \tag{1.12}$$

where the omitted terms involve Lie brackets of four or more elements, and where every element in the series contains the commutator $[X,Y]$. If X and Y are close enough to the zero in $\mathfrak{l}$ (corresponding to the identity element in G) then this series can be shown to converge to an element in $\mathfrak{l}$.[19] Thus the exponential constructs a group at least locally around the identity element.

The question of whether the exponential map reaches all of the members of the group, *i.e.* that it is surjective on G, depends on the properties of the group. We certainly have counter examples, since the groups $SO(3)$ and $SU(2)$ have the same Lie algebra and thus the same exponential map, but are different groups, the map cannot reach all of the elements of both groups. What we do know is that locally, meaning sufficiently close to the identity group element, the exponential map generates the group.

Let us end this subsection by returning to some of our examples in view of what we now know about the exponential map. For $U(1)$ we saw that we could write a generic group member as $e^{i\theta}$. Comparing to the exponential map we see that the single generator must here simply be 1, while θ is the parameter. The Lie algebra is then a very simple one-dimensional vector space $\mathbb{R}$ where the multiplication is ordinary arithmetic multiplication. Similarly, for the translation group $T(1)$ we saw that a group member could be written as $e^{a\frac{\partial}{\partial x}} = e^{ia(-i\frac{\partial}{\partial x})}$. Thus the generator is the differential operator $P = -i\frac{\partial}{\partial x}$ and again the vector space of the Lie algebra is one dimensional.

For $SU(2)$ the generators taken from the fundamental representation $\mathbf{2}$ were proportional to the Pauli matrices $X_i = \frac{1}{2}\sigma_i$, and the exponential map generating a group member U is thus, as we alluded to earlier, $U = e^{i\alpha_i X^i}$, where α_i are the parameters. For $SO(3)$ the (Hermitian) generators taken from the fundamental representation $\mathbf{3}$ are the matrices $Y_i = -iL_i$, where the L_i were given in (1.4), and the exponential map is $O = e^{i\alpha_i Y^i}$. Since $SU(2)$ and $SO(3)$ have the same algebra for X_i and Y_i, but the groups are *not* isomorphic, in what sense, if any, does these exponential maps generate different elements? The answer to that is that $SU(2)$ is a **double cover** of $SO(3)$, it has double the number of elements. There exists a two-to-one group homomorphism $f : SU(2) \to SO(3)$. This can be represented by $U(\alpha_1, \alpha_2, \alpha_3) \to O(\alpha_1, \alpha_2, \alpha_3)$. To see that this is two-to-one, assume that $\alpha_1 = \alpha_2 = 0$.

[19] This formula holds even if the elements in the Lie algebra $\mathfrak{l}$ are not represented as finite-dimensional matrices.

Then $O(0,0,\alpha_3) = O(0,0,\alpha_3+2\pi)$ since a rotation by an extra 2π does nothing. For $SU(2)$ we instead have

$$U(0,0,\alpha_3) = \begin{bmatrix} e^{i\frac{\alpha_3}{2}} & 0 \\ 0 & -e^{i\frac{\alpha_3}{2}} \end{bmatrix} \quad \text{and} \quad U(0,0,\alpha_3+2\pi) = \begin{bmatrix} e^{i\frac{\alpha_3}{2}+i\pi} & 0 \\ 0 & -e^{i\frac{\alpha_3}{2}+i\pi} \end{bmatrix} = -U(0,0,\alpha_3)$$

Thus the $SU(2)$ matrices has a period 4π versus the period 2π for $SO(3)$.[20] Notice that this means that the three-dimensional representation of the Lie algebra coming from the generators of $SO(3)$ does not generate all the elements of $SU(2)$, while the two-dimensional one does.

Beyond the fundamental representation $SU(2)$ has irreducible representations in all dimensions **3**, **4**, **5**, ... This is not true in general, for example $SU(3)$ has irreducible representations only in some dimensions, starting with **3**, **6**, **8**, and **10**, ...

From the exponential map and the properties of matrix exponentiation it immediately follows that the generators T^a for all groups $SU(n)$ are matrices with zero trace. To see this write an element in $SU(n)$ as $U = \exp(i\alpha_a T^a)$, then by the defining group property $\det U = 1$, we have $\det U = \exp(\mathrm{Tr}[i\alpha_a T^a]) = \exp(i\alpha_a \mathrm{Tr}\, T^a)$ which is only one if $\mathrm{Tr}\, T^a = 0$ for all a.

We saw in Sec. 1.2.2 that we could construct unitary matrices from Hermitian ones by matrix exponentiation, but in fact for unitary groups the generators need to be Hermitian. To see this let a group member be $U = \exp(ia_i M^i)$. Then $U^\dagger = \exp(ia_i M^i)^\dagger = \exp(-ia_i M^{i\dagger})$, but for a unitary matrix $U^\dagger = U^{-1} = \exp(-ia_i M^i)$, thus $M^{i\dagger} = M^i$ and the generators M_i are Hermitian.

1.7.3 Adjoint representations

The definition of the structure constants in Sec. 1.5.3 allows us to introduce another representation, the **adjoint representation** of the algebra – and by the exponential map the adjoint representation of the group – where the representation of the algebra consists of the matrices M_i with elements:

$$(M_i)_j{}^k = -iC_{ij}{}^k,$$

where $C_{ij}{}^k$, are the structure constants. If the dimension of the Lie algebra is n, then this is the number of generators, and it also means that the dimension of the matrices M_i is $n \times n$. To prove that this is actually a representation, we need to check that the mapping from the elements of the algebra X_i to the matrices M_i satisfies the homomorphism criteria of the Lie algebra representations. From the Jacobi identity of the algebra it follows that

$$[M_i, M_j] = iC_{ij}{}^k M_k, \tag{1.13}$$

meaning that the matrices of the adjoint representation fulfils the same commutator relationship as the fundamental (generators). As a direct consequence the M_i form a representation.

Note that the dimension n of the matrices in the adjoint representation for $\mathfrak{so}(m)$ and $\mathfrak{su}(m)$, which is equal to the number of independent parameters, is $n = m(m-1)/2$ and $n = m^2 - 1$, respectively. For the matrices in the representation derived from the defining representation of the group the dimensions is the same as the dimension of the defining

[20] The argument can be made more formal and independent of the parameterisation, but is to extensive to repeat here.

matrices, m. This means that in most cases the matrices in the adjoint representation is larger, with the exception of $\mathfrak{so}(2)$ and $\mathfrak{so}(3)$.

We can now briefly return to the $SU(2)$ and $SO(3)$ examples. These groups have the same structure constants because they have the same commutators between generators. Thus the adjoint representations of their algebras are the same. With structure constants ${C_{ij}}^k = {\epsilon_{ij}}^k$ we get the elements of the adjoint matrices $(M_i)_{jk} = -i\epsilon_{ijk}$, which gives $M_i = -iL_i$, with the L_i being the matrices in (1.4).

From the adjoint representation of the Lie algebra we can construct the adjoint representations of the Lie group. For example the adjoint representation of $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$ has the exponential map $\exp(ia_iM^i) = \exp(a_iL^i)$. This happens to generate the $SO(3)$ fundamental representations, but this is not generally the case, it is "accidental" that the dimensions of the fundamental representation of $SO(3)$, $n = 3$, and the dimension of its adjoint representation $n = 3(3-1)/2 = 3$ is the same.

1.7.4 Dual representations

Every representation of a Lie algebra on a vector space V (and every representation of a group) has a **dual representation** (also known as a **contragredient representation**) that is a representation on the **dual vector space** V^*. The definition of a dual vector space is somewhat abstract, it is formally the space consisting of all linear maps $\phi : V \to K$, where K is the field of V. Since the map takes members of V and maps them linearly to the field K, for a finite-dimensional vector space we must be able to write any member of V^* in terms of a row-vector over the same field, so that if $\mathbf{v} \in V$ and $\mathbf{w}^T \in V^*$, the map is given by the matrix multiplication $\mathbf{w}^T\mathbf{v}$ which gives a scalar. This means that in practice V^* consists of the corresponding row-vectors to the column vectors in V.[21]

The definition of the dual representation for a Lie algebra is as follows:[22]

Definition: If $\mathfrak{l}$ is a Lie algebra with a representation ρ on a vector space V, then the **dual representation** on the dual space V^* is ρ^* given by

$$\rho^*(x) = -\rho(x)^T,$$

for all $x \in \mathfrak{l}$.

To see why this makes sense we need to look at the exponential map of the dual representation. If X_i is a generator in the representation ρ, then the dual representation ρ^* tells us to use instead $-X_i^T$. In the exponentiation the group element $g = \exp(iX_i)$ becomes the group element $\exp(-iX_i^T) = \exp(-iX_i)^T = (g^{-1})^T$ with the dual representation. This is the only sensible representation that fulfils the group multiplication properties as matrices acting on row-vectors.

For unitary groups like $U(n)$ or $SU(n)$ we have matrix representations of the Lie algebra with Hermitian generators $M_i = M_i^\dagger$, generating the members in the group as $U = \exp(ia_iM^i)$. From the definition of the dual representation the group members generated in

[21] It is slightly unfortunate and confusing that the symbol for the dual space uses a star since this seems to imply complex conjugation, where instead we are actually talking about transpose, however, as we will see below, the complex conjugation is central to dual representations of unitary groups.

[22] This definition also applies beyond finite-dimensional vector space with a basis since the transpose operation can be generalised beyond matrices.

the dual representation are then

$$U = \exp(-ia_i M^{iT}) = \exp(-ia_i (M^{i\dagger})^*) = \exp(-ia_i M^{i*}),$$

meaning that the Lie algebra in the dual representation can be written in terms of Hermitian matrices $M'_i = -M^*_i$. As a check we can see that M'_i indeed fulfils the same algebra:

$$[M'_i, M'_j] = [M^*_i, M^*_j] = -iC_{ij}{}^k M^*_k = iC_{ij}{}^k M'_k,$$

which is true only if the structure constants $C_{ij}{}^k$ are real.[23] For the matrix groups we denote the dual representations using a bar, so that for example the dual of a representation $\mathbf{2}$ is $\bar{\mathbf{2}}$. The dual representations play a special role in quantum field theories, as the known fermions and anti-fermions transform under a representation of the gauge group and its dual, respectively.

For some groups, such as $SU(2)$, the original and dual representations can be shown to be isomorphic. However, this is not in general the case. For $SU(3)$ the dual representations $\bar{\mathbf{3}}$, $\bar{\mathbf{6}}$, and $\overline{\mathbf{10}}$, are not isomorphic to $\mathbf{3}$, $\mathbf{6}$, and $\mathbf{10}$. On the other hand, $\mathbf{8}$ and $\bar{\mathbf{8}}$ are isomorphic.[24] This becomes important in the Standard Model as the quarks transform under the $\mathbf{3}$ representation of the $SU(3)_c$ gauge group, and the anti-quarks under $\bar{\mathbf{3}}$.

1.8 Exercises

Exercise 1.1

Show that for the translation group $T(1)$ the identity element is T_0 and that $T_a^{-1} = T_{-a}$, and use this to show that $T(1)$ is group.

Exercise 1.2

Show that the set of complex unitary $n \times n$ matrices U with matrix multiplication is a group.

Exercise 1.3

What are the elements of $O(1)$? Can you find a group that it is isomorphic to?

Exercise 1.4

Show that the $U(n)$ and $SU(n)$ groups have n^2 and $n^2 - 1$ independent (real) parameters, respectively. *Hint:* Consider the fact that $M = U^\dagger U$ is a hermitian matrix, *i.e.* $M^\dagger = M$.

Exercise 1.5

Show that Eq. (1.1) is a parametrisation of $SU(2)$.

Exercise 1.6

Show that for a subset $H \subset G$, if $h_i \circ h_j^{-1} \in H$ for all $h_i, h_j \in H$, then H is a subgroup of G.

[23] You can take this as a proof that the structure constants are real if you prove that the dual representation is actually a representation, or you can go the other way and prove that this is indeed a representation, if you can first show that the structure constants of unitary groups are real.

[24] However, the dual of the dual of any representation is always isomorphic to the original representation.

Exercise 1.7

Show that if H is a subgroup of G, then $h_i \circ h_j^{-1} \in H$ for all $h_i, h_j \in H$.

Exercise 1.8

Show that $SU(n)$ is a proper subgroup of $U(n)$ and that $U(n)$ is not simple.

Exercise 1.9

Show any two (left or right) cosets are either disjoint or identical sets.

Exercise 1.10

If H is a normal subgroup of G, show that its left and right cosets are the same, and show that the set formed of the cosets is a group under an appropriate group operation.

Exercise 1.11

Show that the factors in a direct product of groups are normal groups to the product.

Exercise 1.12

Show that the group G and the group $(G \times H)/H$ are isomorphic.

Exercise 1.13

Show that $U(1) \cong \mathbb{R}/\mathbb{Z} \cong SO(2)$.

Exercise 1.14

Prove that all unitary representations on Hilbert spaces are completely reducible. *Hint:* For Hilbert spaces we always have an inner product defined.

Exercise 1.15

Use the parameterisation of $SO(2)$ in (1.3) to find the generator of the group using generating functions and show that you get the same answer as when using the infinitesimal matrix demonstrated in the notes.

Exercise 1.16

Find the fundamental representation for $SO(3)$.

Exercise 1.17

Find the differential generators for $SO(3)$ and their commutation relationships by studying how a function on $\mathbb{R}^3$ changes under $SO(3)$ transformations. Show that they satisfy the same commutators as the matrix generators.

Exercise 1.18

Find an expression for the generators of $SU(2)$ and their commutation relationships. *Hint:* One answer uses a composition function but this approach has some dangers, try also to derive the properties of the generators from a member of the group an infinitesimal distance from the identity.

Exercise 1.19

What are the structure constants of SU(2)?

Exercise 1.20

Find the adjoint representation for $SU(2)$. Compare this to the fundamental representation of $SO(3)$.

Exercise 1.21

Let A be an algebra based on a finite-dimensional vector space over a field F ,with a basis $B = \{b_i | i = 1, \ldots, n\}$. Show that the multiplication of elements in A is completely determined by the n^2 products $b_i b_j$ for each pair of basis vectors in B.

Exercise 1.22

Let V be a finite-dimensional vector space over a field F with a basis $B = \{b_i | i = 1, \ldots, n\}$. Let $\{c_{rst} | r, s, t = 1, \ldots, n\}$ be a collection of n^3 elements in F. Show that there exists one, and only one, multiplication operation on V so that V is an algebra over F under this multiplication and

$$b_r b_s = c_{rst} b_t,$$

for every pair of basis vectors in B. The elements c_{rst} are the structure constants of the algebra.

Exercise 1.23

Show that $\mathbb{R}^3$ with the binary operator $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}$ is a Lie algebra.

Exercise 1.24

Show Eq. (1.13).

Exercise 1.25

Let A_i be the generators of the group G and B_j be the generators of group H. Explain in what sense the A_i and B_j are generators of the direct product group $G \times H$ and show that $[A_i, B_j] = 0$.

Exercise 1.26 Properties of matrix exponentiation

Prove the following useful properties of matrix exponentiation. A and B are matrices.

1. If A and B commute, $e^{A+B} = e^A e^B$.
2. If B has an inverse, $e^{BAB^{-1}} = Be^A B^{-1}$.
3. $e^{A^*} = (e^A)^*$
4. $e^{A^T} = (e^A)^T$
5. $e^{A^\dagger} = (e^A)^\dagger$
6. $e^{-A} = (e^A)^{-1}$
7. $\det e^A = e^{\mathrm{Tr}[A]}$.

Chapter 2

The Poincaré group and its extensions

We now take a look at the symmetry groups behind Special Relativity (SR), the Lorentz and Poincaré groups. We will first see what sort of states transform properly under SR, which has surprising connections to already familiar physics. We will then look for ways to extend these external symmetries of the coordinates to internal symmetries of quantum fields, *i.e.* the symmetries of gauge groups.

2.1 Representations of $SU(2)$

As a warm up, and to introduce some of the methods we will be using, we will start with the $SU(2)$ group and constructing its representations. As alluded to in the previous chapter the $SU(2)$ group is the symmetry group for spin. Since their Lie algebras are identical, this will also provide us information about the rotation group in three dimensions $SO(3)$. We have seen that the (abstract) generators X_i of both groups fulfil the Lie algebra $\mathfrak{su}(2)$,

$$[X_i, X_j] = i\epsilon_{ijk}X^k, \quad i, j, k = 1, 2, 3,$$

which should be well known to us as the commutator for spin operators S_i from our first quantum mechanics courses.[1] So apparently the spin operators are part of something rather fundamental.

To find a representation of $SU(2)$ through the exponential map we need to find representations π of the algebra in terms of finite dimensional matrices (which exist given Ado's theorem). Let us very suggestively call the matrices in such a representation $J_i = \pi(X_i)$. These will then act on some vector space which we will also need to find. We can remind ourselves here that the spin operators we first learned about acted on a two-dimensional Hilbert space for the spin states of spin-$\frac{1}{2}$ particles. The quantum mechanical notation for the vectors in this space was the Dirac-ket $|\psi\rangle$, which we will stick to in the following.

Our first move in the construction is to consider that since the generators in any representation of the Lie algebra are not unique, we can change their basis by a **similarity transform** $J_i' = SJ_iS^{-1}$, where S is an invertible matrix. We can easily show that the J_i' fulfil the same algebra and that this is thus an isomorphism of the representation. We can then uses this

[1] To be more exact, this was $[S_i, S_j] = i\hbar\epsilon_{ijk}S^k$.

freedom to now diagonalise one of the three matrices, J_3, by a choice of basis. We can not diagonalise more than one because we know from linear algebra – and operators in quantum mechanics – that two matrices that are diagonalised by the same similarity transform must commute (since diagonal matrices commute), and this would contradict our commutator for the J_i.

In the following sections of this chapter we will be particularly interested in the eigenvectors of the generators that we can simultaneously diagonalise, in the $SU(2)$ case only J_3, which we will call **states**. This follows what we did in quantum mechanics where we found the set of all commuting operators for a given system, and found their common eigenstates/eigenfunctions, which we then used as basis states/vectors under the quantum mechanical axiom that the eigenstates form a complete set of basis states. The eigenvalues in these cases are the *only* possible measurements for the corresponding observables. Returning to our spin example, for the spin-$\frac{1}{2}$ spin operator we have two states, the spin-up and spin-down states, and can measure the spin in the z-direction (S_3) to be in the up or down direction, $S_3 = \pm\frac{\hbar}{2}$. Any spin-$\frac{1}{2}$ fermion must then be in a linear combination of the two states. We should like to emphasise here that this says something dramatic about what exists according to quantum mechanics, there are only two states labeled by the quantum number s_3, $|s_3 = \frac{1}{2}\rangle$ and $|s_3 = -\frac{1}{2}\rangle$.

2.1.1 Ladder operators

We now return to our $\mathfrak{su}(2)$ algebra and its representations. What follows might again cause some *deja vu*, as you might feel you have done this before in an earlier life. Let us call the eigenstates of J_3 for $|m\rangle$ and its eigenvalues m, *i.e.* $J_3|m\rangle = m|m\rangle$, and assume that the states are normalised, $\langle m|m\rangle = 1$. We know that $m \in \mathbb{R}$ because the eigenvalues of a Hermitian matrix, as we saw J_3 was in the previous chapter, are real.

We are constructing finite-dimensional representations, so we also know that J_3 has a finite number of eigenvalues and eigenstates. We pick the state with the largest eigenvalue, called the **highest weight state**, and denote it by $|j\rangle$.[2] We now define **lowering and raising operators** $J_\pm$,

$$J_\pm \equiv \frac{1}{\sqrt{2}}(J_1 \pm iJ_2).$$

We can show that these have commutators

$$[J_+, J_-] = J_3, \quad \text{and} \quad [J_3, J_\pm] = \pm J_\pm.$$

Notice also that $J_\pm^\dagger = J_\mp$. The lowering and raising operators have the property that operating on a state $|m\rangle$ we get

$$J_3 J_\pm|m\rangle = (J_\pm J_3 \pm J_\pm)|m\rangle = (m \pm 1)J_\pm|m\rangle,$$

meaning that $J_\pm|m\rangle$ are also eigenstates of J_3, but with eigenvalue $m \pm 1$. We normalise the new eigenstates using the constants $N_m^\pm$:

$$J_\pm|m\rangle = N_m^\pm|m \pm 1\rangle.$$

[2] Although j suggests an integer, at this point all we know is that j is some real number.

Starting from the highest weight state $|j\rangle$ we can use J_- to walk down a ladder of states $|j\rangle, |j-1\rangle, \dots, |\Omega\rangle$, but we know that the ladder must end at some point, since we are constructing a finite-dimensional representation. Thus there must be an eigenstate $|\Omega\rangle$ such that $J_-|\Omega\rangle = 0$. The next step is to figure out what Ω is for a given j and what values of j that are admissible. This will also give us the dimension of the representation we are constructing. For this we shall need a very powerful structure in Lie group theory, the Casimir invariant.

2.1.2 The Casimir invariant

A **Casimir element** or **Casimir invariant** is a combination of the elements of the Lie algebra that commutes with all the elements of the algebra. However, in constructing a Casimir element we may take products of matrices in a given representation, and hence we leave the algebra (which is a vector space, thus allowing only linear combinations).[3] A semi-simple Lie algebra, such as the algebras of our matrix groups, has a number of Casimir invariants equal to its **rank**.[4]

In the context of $\mathfrak{su}(2)$, this algebra has the single Casimir invariant

$$\mathbf{J}^2 = J^2 \equiv J_1^2 + J_2^2 + J_3^2 = J_+J_- + J_-J_+ + J_3^2,$$

and therefore has rank-1. It is relatively easy to show that $[J^2, J_1] = [J^2, J_2] = [J^2, J_3] = 0$, so that we can indeed confirm that this is a Casimir element, and we can convince ourselves that there are no other combinations of the J_i that will work. Physically we should recognise this as the total spin or angular momentum (squared). The significance of the Casimir invariant comes from Schur's lemma in Sec. 1.4.3. There is a corresponding version of Schur's lemma for Lie algebras that shows that a linear map on the representation space that commutes with all the elements of the algebra in an irreducible representation (such as the Casimir) is proportional to the identity map. Thus J^2 must be a multiple of the identity matrix, $J^2 = \lambda I$.

If we let the Casimir invariant act on the highest weight state $|j\rangle$ we get

$$J^2|j\rangle = (J_+J_- + J_-J_+ + J_3^2)|j\rangle = (J_+J_- + J_3^2)|j\rangle = (N_{j-1}^+ N_j^- + j^2)|j\rangle = j(j+1)|j\rangle.$$

This identifies the constant of proportionality as $\lambda = j(j+1)$. However, this must be independent of which state is acted on, so $J^2|m\rangle = j(j+1)I|m\rangle = j(j+1)|m\rangle$ for all states $|m\rangle$ in the representation. In the above we have used that $J_+|j\rangle = 0$, otherwise the highest weight assumption is violated, and that $N_{j-1}^+ N_j^- = j$, see Ex. 1.

The property of the Casimir invariant also allows us to find explicit expressions for the normalisation constants using

$$|N_m^\pm|^2 = \langle m|J_\mp J_\pm|m\rangle = \frac{1}{2}\langle m|J^2 - J_3^2 \mp J_3|m\rangle = \frac{1}{2}(j(j+1) - m^2 \mp m),$$

which gives[5]

$$N_m^\pm = \frac{1}{\sqrt{2}}\sqrt{j(j+1) - m^2 \mp m}.$$

[3] For those interested in further reading, the technical statement is that the Casimir elements live in the **universal enveloping algebra** of a Lie algebra. A detailed discussion of this is significantly beyond the scope of these notes.

[4] Rank itself is a rather technical term and comes from the dimension of the Cartan subalgebra of the algebra.

[5] We choose an arbitrary phase factor to be real and positive. There is an alternative way of arriving at the same result that uses recursion relationships for the normalisations constants, but this is very heavy and tedious algebraically.

In turn, this finally allows us to determine where the ladder of states ends. The ladder ends with $J_-|m\rangle = 0$ if and only if $m = -j$ since $N^-_{-j} = 0$. However, this also implies that $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, *i.e.* it is half-integer, since otherwise we would not reach $m = -j$ from $m = j$ in a whole number of steps and then the ladder would never end, which contradicts the finite-dimensional nature of the representation.

What we have now is proof that there exists an infinite tower of finite-dimensional representations of $\mathfrak{su}(2)$ that are indexed, or labeled, by the non-negative half-integer j, with one representation for every dimension. Although it is beyond the scope of these notes, these representations are irreducible, and they are indeed *all* the irreducible representations of $\mathfrak{su}(2)$. This fulfils the promise of Schur's lemma, namely that the irreducible representations are labeled by the eigenvalues of the Casimir, and show that these eigenvalues have a physical interpretation, in this case the total spin or angular momentum. Each of these representations has dimension $2j+1$, with that many states denoted by $|j,m\rangle$, $m = -j, j+1, \ldots, j-1, j$, that are eigenstates of the J_3 generator. We call these representations the spin-j representations because they are indeed the representations of $SU(2)$ used for the different spin possibilities. These representations can also be used for $SO(3)$ in odd dimensions, where j is integer

2.1.3 The lowest-dimensional representations

The spin-0 representation is the trivial representation where $j = 0$ and the representation space is one dimensional and everything acts as the identity element.

The first non-trivial representation is for spin-$\frac{1}{2}$ with $j = \frac{1}{2}$. Here the two states are $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$. To write down a representation of the algebra in terms of 2×2 matrices we think of the states as two basis vectors

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix},$$

and to fulfil the properties for the $J_\pm$ and J_3 generators

$$\begin{aligned} J_\pm|j,m\rangle &= \frac{1}{\sqrt{2}}\sqrt{j(j+1) - m^2 \mp m}|j, m \pm 1\rangle \\ J_3|j,m\rangle &= m|j,m\rangle, \end{aligned}$$

we must have

$$J_+ = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad J_- = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \quad J_3 = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$$

Going back to the J_i, $i = 1, 2, 3$, basis we then get

$$J_1 = \frac{1}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \quad J_2 = \frac{1}{2}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \quad J_3 = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix},$$

which, unsurprisingly, leaves us back at the Pauli matrices $J_i = \frac{1}{2}\sigma_i$ that were the generators we found from the defining representation of $SU(2)$.

For the spin-1 representation we have $j = 1$ and the three states are $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$. The basis vectors are then

$$|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

and the $J_\pm$ and J_3 generators

$$J_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Going back to the J_i, $i = 1, 2, 3$, basis we get

$$J_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_2 = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We can check that these matrices indeed fulfil the $\mathfrak{su}(2)$ algebra, and we can also find a similarity transform that transforms them into the generators derived from the defining representation of the $SO(3)$ group in (1.4). Our arguments here can be repeated for higher and higher dimensions at will.

One final note on differential representations of the generators. There is in principle nothing in the arguments for the properties of the representations of the algebra in the previous subsection that can not be carried over into differential representations. If for the generators J_i we take the differential generators we found for $SO(3)$ in Eq. (1.7) we have here shown that there exists states, now functions $Y_m^l(\theta, \phi)$ in a vector space, that fulfil the relationships

$$\begin{aligned} L_z Y_l^m(\theta, \phi) &= m Y_l^m(\theta, \phi), \\ L^2 Y_l^m(\theta, \phi) &= l(l+1) Y_l^m(\theta, \phi), \end{aligned}$$

for $l = 0, 1, 2, \ldots$, and $m = -l, -l+1, \ldots, l-1, l$. Here we write the functions in spherical coordinates since rotations under $SO(3)$ can be easily expressed in the two spherical coordinate angles. The functions $Y_l^m(\theta, \phi)$ are well known as the **spherical harmonics**.

We have now shown what the representations of $SU(2)$ look like, but we have not seen why this group appears in physics in the first place. This will become clearer in the next sections.

2.2 The Lorentz Group

Einstein's requirement in Special Relativity was that the laws of physics should be invariant under rotations and/or boosts (changes of velocity) between different inertial reference frames. A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by a four-vector $x^\mu = (t, x, y, z)$. The resulting transformations of the space-time coordinates are captured in the Lorentz group.

Definition: The **Lorentz group** L is the group of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 \equiv x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous** or **restricted Lorentz group** is a subgroup of the Lorentz group where $\det \Lambda = 1$ (**proper**) and $\Lambda^0{}_0 \geq 1$(**orthochronous**).

The physical interpretation of the orthochronous property is that it keeps the direction (sign) of time of the four vector, while a proper group preserves orientation in rotations.

Since the definition of the Lorentz group is in terms of a continuous transformation of coordinates we have strong reason to suspect that it is a Lie group. In fact, if we allow for a slight extension of the orthogonal group $O(n)$ to the **indefinite orthogonal group** $O(m,n)$, where instead of the orthogonality property for group members O, meaning $O^{-1} = O^T$, we demand $O^{-1} = g^{-1}O^T g$ where

$$g = \mathrm{diag}(\underbrace{1,\ldots,1}_{n},\underbrace{-1,\ldots,-1}_{m}),$$

is the **metric**,[6] then we can write the Lorentz group as $SO^+(1,3)$, where the plus sign signifies the orthochronous property.[7] This is also the part of the group that contains the identity element. As a subgroup of the general linear group $GL(4)$ this is indeed a Lie group.

Compared to the $SO(4)$ group which it resembles the Lorentz group is more complicated. Unlike $SO(4)$, because of the metric, it is not compact which means that it does not have any non-trivial finite-dimensional unitary representations. Fortunately, its algebra is semi-simple, so its finite-dimensional representations are equivalent to direct sums of irreducible representations by Weyl's complete reducibility theorem. So we need only to find the irreducible representations to construct any representation.

The counting of the free parameters of $SO(n,m)$ works just as for $SO(n)$, giving a total of six free parameters for $SO^+(1,3)$, and then six generators. Physically, we can identify these with the three parameters needed to specify a general rotation in three dimensions, with generators named J_i, $i = 1,2,3$, and the three parameters needed to specify a boost (the velocity components), with generators K_i, $i = 1,2,3$.

Since the rotation operations are known to be closed, *i.e.* two rotations result in another rotation, this forms a subgroup of $SO^+(1,3)$. We know that the generators fulfil the $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.1}$$

The boost operations are not closed, and one can show that their anti-Hermitian generators K_i, see Ex. 3, have the following relationships with the rotation generators

$$[K_i, J_j] = i\epsilon_{ijk}K_k, \tag{2.2}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.3}$$

where (2.1)–(2.3) then defines the complete algebra of $SO^+(1,3)$. This is consistent with what we know about boost: two boosts are in general equivalent to a boost and a rotation, so the generators of boosts commute into generators of rotation.

To simplify notation these generators can further be structured into an anti-symmetric tensor $M_{\mu\nu}$ given by

$$M_{\mu\nu} = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix}. \tag{2.4}$$

In terms of M the commutation relations of the algebra (2.1)–(2.3) can be written:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{2.5}$$

[6]Indeed, we can recognise this matrix relationship as one of the defining (necessary) properties of Lorentz transformations $\Lambda^T g\Lambda = g$.

[7]Because of the metric, $\det O = 1$ alone no longer insures that we do not have time or parity reversal.

The differential representation of these generators is

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{2.6}$$

where we can more or less see how the differential generators for $SO(3)$ in (1.7) appear as the M_{ij} components. For a complete demonstration, see Ex. 4.

With the generators from the defining representation we can now write a general element $\Lambda \in SO^+(1,3)$ as[8]

$$\Lambda^\mu{}_\nu = \left[\exp\left(\frac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{2.7}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the six free parameters of the transformation (the parameters that were used to derive the generators). The anti-symmetry of ω is a consequence of the anti-symmetry of M. The factor of a half takes into account that all generators appear twice in M.

The generators $M_{\rho\sigma}$ form the Lie algebra $\mathfrak{so}(1,3)$. In fact, this is also the algebra of $O(1,3)$ since the orthochronous and proper requirements do not change the number of free parameters, but rather restricts us to a subset of the matrices that do not change the sign on the time and position components of the four-vector. As we have seen, in general the exponential map from the algebra to the group is not guaranteed to be one-to-one, but describes the group locally around the identity. Using the $\mathfrak{so}(1,3)$ generators we can in fact not get outside of the $SO^+(1,3)$ subgroup of $O(1,3)$. The larger group $O(1,3)$ can be seen as four disconnected parts with $\det\Lambda = \pm 1$ and $|\Lambda^0{}_0| \geq 1$ that are joined by the time T and parity P inversion operators. However, the exponential map (2.7) is surjective (onto) $SO^+(1,3)$. Thus, any group element in the connected component around the identity can be expressed as an exponential of an element of the Lie algebra.

To learn more about the representations of the Lorentz group we want to look at the structure of the Lie algebra $\mathfrak{so}(1,3)$. If study the algebra as given in (2.1)–(2.3) carefully we may notice that a small change in basis would allow us to rewrite the algebra in a more symmetric fashion. We define a new basis of six generators from a linear combination (not to be confused with the earlier ladder operators):

$$J_{\pm,i} = \frac{1}{2}(J_i \pm iK_i).$$

This gives the algebra

$$\begin{aligned}
[J_{+,i}, J_{-,j}] &= 0,\\
[J_{+,i}, J_{+,j}] &= i\epsilon_{ijk} J_{+,k},\\
[J_{-,i}, J_{-,j}] &= i\epsilon_{ijk} J_{-,k}.
\end{aligned}$$

What has happened is that we have separated the algebra into two instances of the $\mathfrak{su}(2)$ algebra that do not interact (the generators commute). We write this as a direct sum of the (vector spaces of the) algebras, $\mathfrak{so}(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$. However, what we have done is to do a linear transformation with complex coefficients, thus the algebra is now a complex Lie algebra instead of a real one (this is called the complexification of the algebra).

[8] So this is definitely a finite-dimensional representation, but is it not also unitary? (Meaning conserves the length of vectors.) It does conserve Lorentz invariants, yes, however, those are not defined as the length of vectors.

We can can find further relations by using the isomorphism $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2)$, where $\mathfrak{sl}(2)$ is the Lie algebra of the special linear group $SL(2, \mathbb{C})$. This can be demonstrated by looking at their commutation relations. Then we write $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \cong \mathfrak{sl}(2)_{\mathbb{C}}$ where we complexify the real Lie algebra $\mathfrak{sl}(2)$, using the real and imaginary parts to represent the two $\mathfrak{sl}(2)$ algebras in the direct sum.

The result is an isomorphism between the complexifications of two real Lie algebras. However, if we restrict these two algebras to real Lie algebras we have just demonstrated the isomorphism $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2)$. Thus the Lie groups $SO(1,3)$ and $SL(2, \mathbb{C})$ have the same Lie algebra, even though the dimension of the matrices are very different. It turns out that the map $SL(2, \mathbb{C}) \to SO(1,3)$ is a double covering, just as $SU(2)$ was a double cover of $SO(3)$. We will find an explicit expression for this map later in Sec. 2.5.

We will not immediately use this information to study the representations of the Lorentz group, but delay this to the next sections where we extend the symmetry of the Lorentz group to the Poincaré group.

2.3 The Poincaré group

We can now extend $O(1,3)$ by adding translation by a constant four-vector a^μ to the transformation of the Lorentz group: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$. This transformation leaves lengths $(x-y)^2$ invariant in $\mathbb{M}_4$, and invariance under this group add symmetry of time and space translation to the symmetries of the Lorentz group.

Definition: The **Poincaré group** P is the group of all transformations of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

We can also construct the **restricted Poincaré group** by restricting the matrices Λ in the same way as in $SO^+(1,3)$.

Writing a group member in terms of its parameters (Λ, a), we can see from the explicit form of the transformation that the composition of two elements in this group is:

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is *not* a direct product of the Lorentz group and the translation group, but rather a *semi-direct product* of $O(1,3)$ and the (indefinite) translation group $T(1,3)$, $O(1,3) \ltimes T(1,3)$. The translation group is a normal subgroup, and while the Lorentz group is a subgroup, it is not normal. The restricted Poincaré group is written in the same way as the restricted Lorentz group, $SO^+(1,3) \ltimes T(1,3)$.

The translation part of the Poincaré group adds four parameters to the six parameters of the rotations and boosts. This means that there are four more generators compared to the Lorentz group. Given our earlier discussion of the translation group in Sec. 1.7.2 we can convince ourselves that we can use the momentum operators $P_\mu = -i\partial_\mu$ as a differential representation. These generators have a trivial commutation relationship:

$$[P_\mu, P_\nu] = 0. \tag{2.8}$$

Using the differential representation of $M_{\mu\nu}$ one can also show the following commutators with the generators of the Lorentz group:

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu). \tag{2.9}$$

Equations (2.5), (2.8) and (2.9) together form the **Poincaré algebra**, a Lie algebra for the Poincaré group. This allows us to write a general member g of the restricted Poincaré group by using the exponential map

$$g = \exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma} + ia^\mu P_\mu\right), \tag{2.10}$$

where a^μ are the additional parameters of the translation.

The Poincaré group, like the Lorentz group, is not compact, and thus does not have any non-trivial finite-dimensional unitary representations, and we must instead search for infinite-dimensional ones. The original classification of unitary representations of the Poincaré group is due to Wigner [4].

2.4 Irreducible representations of the Poincaré group

We would now like to ask the following fundamental question: what sort of physical objects – in particles physics what particles or maybe what quantum fields – can exist if we require that they are representations of the Poincaré group, and what properties do they as a result have?[9] We have already learnt that this is an infinite-dimensional representation because of the non-compact nature of the Poincaré algebra.

To answer that question we will classify the irreducible representations of the Poincaré group. This seems like a dramatically difficult task, however, we will follow the arguments used for $SU(2)$ in Sec. 2.1 to find and classify the representations by the eigenvalues of the Casimir invariants (via Schur's lemma), and the states in each representation by the eigenvalues of the set of commuting generators.

For the Poincaré algebra $P^2 = P_\mu P^\mu$ is a Casimir operator because the following holds:

$$[P_\mu, P^2] = 0, \tag{2.11}$$

$$[M_{\mu\nu}, P^2] = 0. \tag{2.12}$$

Let λ be the eigenvalue of P^2 for a given irreducible representation $|\lambda\rangle$, what can we say about λ? Since all four-momentum operators P_μ commute, and commute with P^2, they have simultaneous eigenvalues and eigenstates. We know from quantum mechanics that these are Hermitian operators with real eigenvalues equal to the momenta p_μ. Thus the eigenvalue of P^2 is

$$P^2|\lambda\rangle = (P_0^2 - P_1^2 - P_2^2 - P_3^3)|\lambda\rangle = (E^2 - p_x^2 - p_y^2 - p_z^2)|\lambda\rangle \equiv m^2|\lambda\rangle.$$

So, in fact the eigenvalue of P^2 is the number $m^2 \in \mathbb{R}$ that we have called mass, and we use this to label our representations,

$$P^2|m\rangle = m^2|m\rangle,$$

of which there is a continuum. Note that nothing restricts m^2 to be larger or equal to zero, a negative m^2 is acceptable since a four vector can be time-like, light-like or space-like. In fact,

[9] Exist here in the sense of being described by a vector space that the group representations act on.

we can happily insert $m^2 < 0$ into Special Relativity, but such objects would correspond to objects moving faster than light, so-called **tachyons**, which do not seem to exist in nature. Since there are no restrictions on the eigenvalues of the momentum operator p_μ for a given m^2 we have infinite-dimensional irreducible representations of the translation subgroup of the Poincaré group, where the states are labeled by the momentum value $|m, p_\mu\rangle$.

What we have done up to now does not reconstruct the full representation of the group (algebra). To proceed we look for further Casimir invariants. Any Casimir invariant needs to commute with the Lorentz group generators that are part of the algebra, thus it needs to be a Lorentz invariant (the four-indices are all contracted). Since we only have the operators $M_{\mu\nu}$ and P_μ in the algebra, and we have already used up P^2, we need to consider $M_{\mu\nu}M^{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma}$. However, these both fail to commute with P_μ due to (2.9). The remaining possibilities are combinations of $M_{\mu\nu}$ and P_μ, and we need a combination that commutes with P_μ.

Definition: We define the **Pauli-Ljubanski polarisation vector** by:

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.13}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor with $\epsilon_{0123} = 1$.

We can show that this vector is the one combination that is translation invariant, *i.e.* that it commutes with the translation operator,

$$[P_\mu, W_\nu] = 0. \tag{2.14}$$

This gives us two possibilities for Casimir invariants with the two Lorentz invariants $W_\mu P^\mu$ and $W_\mu W^\mu$, unfortunately from the definition of W_μ it is easy to see that $W_\mu P^\mu = 0$. However, the remaining option $W^2 = W_\mu W^\mu$ is a Casimir operator of the Poincaré algebra. While we argued from the Lorentz and translation invariance of the operators we can also show explicitly that

$$\begin{aligned} [P_\mu, W^2] &= 0, && (2.15)\\ [M_{\mu\nu}, W^2] &= 0. && (2.16) \end{aligned}$$

The second of these relationships is not trivial to demonstrate. See [3] for a complete proof. This exhausts the list of possibilities, and the Poincaré algebra is a rank-2 algebra. The W_μ do not in general commute among themselves, in fact

$$[W_\mu, W_\nu] = \epsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma. \tag{2.17}$$

Thus, to find the full representation we can now look for states that are simultaneous eigenstates of W^2, P^2, the P_μ, as well as one component of W_μ.

We do this by starting from the representations of the translation subgroup with fixed eigenvalues p_μ. What we are using here is know as the method of **little groups** in physics. The idea is to consider the subgroup of the Poincare group that indeed leaves p_μ invariant: this is what is called the **little group**.[10] This will allow us to find unitary irreducible

[10] In mathematics, it is often called the **stabiliser subgroup**.

representations of the whole group from the unitary irreducible representations of the little group. Because the behaviour of the little group is different depending on the value of p_μ, we need to separate the problem into several discrete cases :

i) $m = 0$ and $p_\mu = (0, \mathbf{0})$

ii) $m = 0$ and $p_\mu \neq (0, \mathbf{0})$

iii) $m^2 > 0$

iv) $m^2 < 0$

Case i) Here, $P^2 = 0$ and $W^2 = 0$,[11] and there is only one representation $|0, 0\rangle$. The little group that leaves $p_\mu = (0, \mathbf{0})$ invariant is the whole Lorentz group since boosts and rotations on a zero momentum point-like object just gives the same thing back. However, since the Lorentz group is not compact, we know that the only finite-dimensional unitary representation of the little group is the trivial representation. Thus this representation corresponds to the trivial representation for the Poincare group as well. We consider a state that transforms this way as the vacuum.

Case iii) If we go to the rest frame of the particle, the states have momentum eigenvalues $p_\mu = (m, \mathbf{0})$.[12] In this case, the little group turns out to be the group of rotations in three-dimensional space, $SO(3)$. To see this consider the Pauli-Ljubanski vector acting on such a state

$$W_i |m, p_\mu\rangle = \frac{1}{2}\epsilon_{i0jk} m M^{jk} |m, p_\mu\rangle = -m J_i |m, p_\mu\rangle, \tag{2.18}$$

where $J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$ is the **spin operator** that forms the $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ algebra. Since $WP = 0$ we also have $W_0 = 0$ in this reference frame.[13] This gives $W^2 = -\mathbf{W}^2 = -m^2 \mathbf{J}^2$. So in this case we want to find the eigenstates of $W^2 = -m^2 J^2$ and one component of W_μ which we choose to be $W_3 = -m J_3$ with eigenvalue named j_3.

We already found the representations of $\mathfrak{su}(2)$ in Sec. 2.1, so for this case we immediately know that in total the irreducible representations can be labeled by two numbers, $m^3 \in \mathbb{R}$ and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, as $|m, j\rangle$, and while each representation is infinite dimensional because it has continuous number for the momentum eigenvalues p_μ, for each momentum eigenvalue it has has $2j + 1$ spin states with J_3 eigenvalue $j_3 = -j, -j+1, \ldots, j-1, j$, which we write as $|m, j, p_\mu, j_3\rangle$. Since the total spin operator acts on a state with spin j as $\mathbf{J}^2 |j\rangle = j(j+1)|j\rangle$, we also have that

$$W^2 |m, j\rangle = -m^2 j(j+1) |m, j\rangle.$$

We have now arrived at the conclusion that massive particles transforming under the Poincaré group, meaning the objects that obey Special Relativity, can be classified by two numbers: mass m^2 and spin j. The appearance of spin in physics is thus intimately connected to the symmetries of Special Relativity.

[11] In the sense that their eigenvalues are always zero.

[12] This does not loose generality since the physics of the representation should be independent of frame.

[13] In the sense that W_0 has eigenvalue 0 for all states.

Case ii) We now want to look at massless particles, but with non-zero p_μ. Here we can not go to a rest frame and instead choose a frame such that $p_\mu = (p, 0, 0, p)$. We again have $P^2 = 0$ in the usual sense, and one can prove that W_μ and P_μ are proportional, $W_\mu = HP_\mu$. Thus $W^2 = 0$ and the proportionality factor can be found from

$$\begin{aligned} W_0|0, p_\mu\rangle &= \frac{1}{2}\epsilon_{0ijk}P^i M^{jk}|0, p_\mu\rangle = P^i J_i|0, p_\mu\rangle = \mathbf{P}\cdot\mathbf{J}|0, p_\mu\rangle, \\ P_0|0, p_\mu\rangle &= p|0, p_\mu\rangle, \end{aligned}$$

as

$$H = \frac{\mathbf{P}\cdot\mathbf{J}}{p},$$

which is the definition of the **helicity** of a massless particle.

The little group that leaves p_μ unchanged here is not so obvious to see. But it turns out to be given by the so-called special Euclidean group $SE(2)$, which consists of rotations and translations in two dimensional Euclidean space. The eigenvalues h of H, $H|0, p_\mu\rangle = h|0, p_\mu\rangle$ can then be shown to be $h = \pm j_3$, where j_3 is another half-integer, $j_3 = 0, \frac{1}{2}, 1, \ldots$.

So, in summary, there is one representation $|0, 0\rangle$, which is infinite-dimensional since the values of p_μ form a continuum, and for each of these momentum eigenstates there are two helicity eigenstates which we write in similarity with the massive case as $|0, 0, p_\mu, \pm j_3\rangle$.

Case vi) Because of time constraints and since tachyons do not seem to appear in nature we will not treat this case further.

Let us finally try to ask the question, what do these (irreducible) infinite-dimensional unitary representations actually look like? If we start with spin-0 representations, $j = 0$, we can write the corresponding infinite-dimensional representation of massive states without any vector structure as $|m, 0\rangle \sim e^{\pm ipx}$, where p_μ is the four-momentum of the particle, since then

$$P^2|m, 0\rangle = -\partial_\mu\partial^\mu|m, 0\rangle = p^2|m, 0\rangle = m^2|m, 0\rangle.$$

This exponential part of states can then always be used to take care of the eigenvalues of the P^2-Casimir, and is often just implicitly implied in the states/fields.

We can also immediately write down the $j = 1$ vector representation of the Poincaré group for massive particles, $|m, 1\rangle \sim \epsilon_\mu e^{ipx}$. We simply use a four-vector ϵ_μ that transforms under the fundamental (four-dimensional) representation of the Lorentz group $SO^+(1, 3)$. In order to fulfil the eigenvalue equation of the W^2-Casimir, and describe the three spin states $j_3 = -1, 0, 1$, this vector (called the **polarisation vector**) needs to fulfil certain requirements which we do not detail here (see a course on quantum field theory).

However, in order to find a spin-$\frac{1}{2}$ representation for fermions we need to take some more care. In fact, we will find representations both in four and two dimensions. For those familiar with quantum field theory, these will as expected be the Dirac and Weyl spinor representations.

2.5 Weyl spinors

As we discussed at the end of Sec. 2.2 there exists a two-to-one homomorphism between the $SL(2, \mathbb{C})$ and the Lorentz group $SO^+(1, 3)$. This homomorphism, with $\Lambda^\mu{}_\nu \in SO^+(1, 3)$ and

$M \in SL(2,\mathbb{C})$, can be explicitly given by:[14]

$$\begin{aligned}\Lambda^{\mu}{}_{\nu}(M) &= \frac{1}{2}\mathrm{Tr}[\bar{\sigma}^{\mu} M \sigma_{\nu} M^{\dagger}], &(2.19)\\ M(\Lambda^{\mu}{}_{\nu}) &= \pm\frac{1}{\sqrt{\det(\Lambda^{\mu}{}_{\nu}\sigma_{\mu}\bar{\sigma}^{\nu})}}\Lambda^{\mu}{}_{\nu}\sigma_{\mu}\bar{\sigma}^{\nu}, &(2.20)\end{aligned}$$

where $\bar{\sigma}^{\mu} = (1, -\vec{\sigma})$ and $\sigma^{\mu} = (1, \vec{\sigma})$. The generators of $SO^{+}(1,3)$ can be shown to transform to (be proportional to) the Pauli matrices in $SL(2,\mathbb{C})$:

$$J_i = \frac{1}{2}\sigma_i, \quad K_i = \frac{i}{2}\sigma_i.$$

This two-to-one correspondence means that $SO^{+}(1,3) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ and the groups have the same algebra (as discussed in Sec. 2.2). Thus we can look at the representations of $SL(2,\mathbb{C})$ instead of the Lorentz group, when we describe spin-$\frac{1}{2}$ particles,[15] and working in $SL(2,\mathbb{C})$ if often much easier, but what are those representations? It turns out that there are two inequivalent fundamental representations ρ of $SL(2,\mathbb{C})$ in terms of 2×2 matrices $M \in SL(2,\mathbb{C})$:

i) The self-representation $\rho(M) = M$ acting on a member ψ of a representation vector space V:
$$\psi'_A = M_A{}^B\psi_B, \quad A, B = 1, 2.$$

ii) The complex conjugate self-representation $\rho(M) = M^*$ working on a vector $\bar{\psi}$ in a space $\dot{V}$:
$$\bar{\psi}'_{\dot{A}} = (M^*)_{\dot{A}}{}^{\dot{B}}\bar{\psi}_{\dot{B}}, \quad \dot{A}, \dot{B} = 1, 2.$$

The vectors ψ and $\bar{\psi}$ in these representation spaces are called, respectively, **left- and right-handed Weyl spinors**, and the induced representation of the Lorentz group is called the **spinor representation**. In addition to these two representations there are two dual representations, see Sec. 1.7.4, with $\rho(M) = M^{-1T}$ acting on vectors ψ^A in V^*, and $\rho(M) = M^{*-1T}$ on vectors $\bar{\psi}^{\dot{A}}$ in $\dot{V}^*$, that are equivalent to i) and ii), respectively.

The indices here follow the same summation rules as four-vectors. Indices can be lowered and raised with:

$$\begin{aligned}\epsilon_{AB} &= \epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, &(2.21)\\ \epsilon^{AB} &= \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, &(2.22)\end{aligned}$$

which work as maps between the dual spaces. The dots on the indices for the complex conjugate representation are there to help us remember which representation we are using and does not carry any additional importance, other than being a different index.

[14] The choice of sign in Eq. (2.20) is the reason that this is a homomorphism, instead of an isomorphism. Each element in $SO^{+}(1,3)$ can be assigned to two in $SL(2,\mathbb{C})$.

[15] We can of course also use the $SL(2,\mathbb{C})$ representations to construct representations for higher spin.

Since (2.20) gives M in terms of the Pauli matrices, their index structure must be $\bar{\sigma}^{\mu\dot{A}A}$ and $\sigma^{\mu}_{A\dot{A}}$. For a consistent index notation, the relationship between the vectors ψ and $\bar{\psi}$ can be expressed with:

$$\psi^A = \bar{\psi}^*_{\dot{A}}\bar{\sigma}^{0\dot{A}A}, \quad \psi_A = \sigma^0_{A\dot{A}}\bar{\psi}^{\dot{A}*}, \quad \bar{\psi}^{\dot{A}} = \bar{\sigma}^{0\dot{A}A}\psi^*_A, \quad \text{and} \quad \bar{\psi}_{\dot{A}} = \psi^{A*}\sigma^0_{A\dot{A}}.$$

This may be seen as a bit of an overkill in indices as $\bar{\sigma^0}^{\dot{A}A} = \delta^{\dot{A}A}$, and we will in the following often omit the matrix and simply write $(\psi_A)^* = \bar{\psi}^{\dot{A}}$. Note that from the above the following relationships hold for the hermitian conjugate:

$$(\psi_A)^\dagger = \bar{\psi}_{\dot{A}} \tag{2.23}$$

$$(\bar{\psi}_{\dot{A}})^\dagger = \psi_A. \tag{2.24}$$

We further define contractions of Weyl spinors that are invariant under $SL(2,\mathbb{C})$ transformations – just as contractions of four-vectors are invariant under Lorentz transformations – as follows:

Definition: The contraction of two Weyl spinors ψ and χ is given by $\psi\chi \equiv \psi^A\chi_A$ and $\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{A}}\bar{\chi}^{\dot{A}}$.

With this in hand we see that

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = \epsilon^{AB}\psi_B\psi_A = \epsilon^{12}\psi_2\psi_1 + \epsilon^{21}\psi_1\psi_2 = \psi_2\psi_1 - \psi_1\psi_2.$$

This quantity is zero if the Weyl spinors commute. In order to avoid this we make the following assumption which is consistent with how we treat fermions as anti-commuting operators:

Postulate: All Weyl spinors anticommute:[a] $\{\psi_A, \psi_B\} = \{\bar{\psi}_{\dot{A}}, \bar{\psi}_{\dot{B}}\} = \{\psi_A, \bar{\psi}_{\dot{B}}\} = \{\bar{\psi}_{\dot{A}}, \psi_B\} = 0$.

[a] This means that Weyl spinors are so-called **Grassmann numbers**.

This means that the contraction evaluates as

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = -2\psi_1\psi_2.$$

2.5.1 Useful relationships for Weyl spinors

For Weyl spinors ψ, η, and ϕ we can prove the following relationships[16]

$$\begin{aligned}
\eta\psi &= \psi\eta, && (2.25)\\
\bar{\eta}\bar{\psi} &= \bar{\psi}\bar{\eta}, && (2.26)\\
(\eta\psi)^\dagger &= \bar{\psi}\bar{\eta}, && (2.27)\\
(\eta\psi)(\eta\phi) &= -\frac{1}{2}(\eta\eta)(\psi\phi), && (2.28)\\
\eta\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\eta, && (2.29)\\
(\sigma^\mu\bar{\eta})_A(\eta\sigma^\nu\bar{\eta}) &= \frac{1}{2}g^{\mu\nu}\eta_A(\bar{\eta}\bar{\eta}), && (2.30)\\
(\eta\sigma^\mu\bar{\eta})(\eta\sigma^\nu\bar{\eta}) &= \frac{1}{2}g^{\mu\nu}(\eta\eta)(\bar{\eta}\bar{\eta}), && (2.31)\\
(\eta\sigma^\mu\partial_\mu\bar{\psi})(\eta\psi) &= -\frac{1}{2}(\psi\sigma^\mu\partial_\mu\bar{\psi})(\eta\eta), && (2.32)\\
(\partial_\mu\psi\sigma^\mu\bar{\eta})(\bar{\eta}\bar{\psi}) &= -\frac{1}{2}(\partial_\mu\psi\sigma^\mu\bar{\psi})(\bar{\eta}\bar{\eta}), && (2.33)\\
\eta\sigma^{\mu\nu}\psi &= -\psi\sigma^{\mu\nu}\eta. && (2.34)
\end{aligned}$$

Here $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$.

2.5.2 Dirac spinors

The Weyl spinors can in turn be used in a four-dimensional representation of the Poincaré group for spin-$\frac{1}{2}$ fermions, stacking two Weyl different spinors, one from the self-representation ϕ_A, and one from the complex conjugate, $\bar{\chi}^{\dot{A}}$, into a four-component **Dirac spinor** ψ_a,

$$\psi_a = \begin{pmatrix}\phi_A\\ \bar{\chi}^{\dot{A}}\end{pmatrix},$$

making a new vector space that is a direct sum of the two vector spaces $W = V \oplus \dot{V}^*$. Here, we have in general $(\phi_A)^* \neq \bar{\chi}^{\dot{A}}$. In order to describe a Dirac fermion, which has both particle and antiparticle states, using this Dirac spinor we need two distinct Weyl spinors with different handedness. For Majorana fermions that are their own antiparticles we can instead use the simpler:

$$\psi_a = \begin{pmatrix}\psi_A\\ \bar{\psi}^{\dot{A}}\end{pmatrix}.$$

The representation of $SL(2,\mathbb{C})$ on W is

$$\rho(M) = \begin{bmatrix} M & 0\\ 0 & M^{*-1T}\end{bmatrix}.$$

When we deal with four-component spinors we have a use for γ-matrices. These are defined as objects that fulfil a type of **Clifford algebra** given by[17]

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{2.35}$$

[16] For clarity we have inserted parenthesis to show the different contractions.

[17] Be aware that the expression on the right hand side should be read as consisting of a rank-2 tensor *with each element* being a 4×4 identity matrix.

There exists different representations of this algebra, just as for the Lie algebras. In these notes we will use what is called the **Weyl-representation** where the γ-matrices are 4×4 matrices given in terms of the Pauli matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.36}$$

The γ-matrices can also be used to form a 'fifth' γ-matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}.$$

This can be used to project out the Weyl spinors in the Dirac spinors through the projection operators $P_L = \frac{1}{2}(1-\gamma^5)$ and $P_R = \frac{1}{2}(1+\gamma^5)$, which projects out the left-handed and right-handed Weyl spinor, respectively,

$$P_L\psi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\psi = \begin{pmatrix} \psi_A \\ 0 \end{pmatrix}, \quad P_R\psi = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\psi = \begin{pmatrix} 0 \\ \bar{\psi}^{\dot{A}} \end{pmatrix}.$$

Note the projection properties $P_L + P_R = I$, $P_L^2 = P_L$, $P_R^2 = P_R$, and $P_L P_R = 0$.

The equation of motion for spin-$\frac{1}{2}$ particles with mass m in relativistic quantum mechanics is the **Dirac equation**

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{2.37}$$

where the 'slash' notation signifies contraction with a γ-matrix, $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$, and where ψ is a four component spinor constructed as above. Using Weyl spinors and the Weyl-representation of the γ-matrices this can be written as the coupled differential equations

$$\begin{aligned} i\sigma^\mu\partial_\mu\bar{\chi} - m\phi &= 0, \\ i\bar{\sigma}^\mu\partial_\mu\phi - m\bar{\psi} &= 0. \end{aligned}$$

In the massless limit or the extreme relativistic limit, $m \to 0$, these equations decouple into one separate equation per Weyl-spinor and become the **Weyl-equations**

$$\begin{aligned} (i\partial_t - \boldsymbol{\sigma}\cdot\mathbf{P})\bar{\chi} &= 0, \\ (i\partial_t + \boldsymbol{\sigma}\cdot\mathbf{P})\phi &= 0. \end{aligned}$$

These equations have plane wave solutions $\phi \sim e^{-ipx}$ and $\bar{\chi} \sim e^{-ipx}$, which are the helicity eigenstates for the massless particles discussed in Sec. 2.4, case ii), with eigenvalues $\pm\frac{1}{2}$. To see this, notice that $i\partial_t\phi = E\phi = |\mathbf{p}|\phi$, since $m = 0$, giving

$$|\mathbf{p}|\phi + \boldsymbol{\sigma}\cdot\mathbf{P}\phi = 0 \quad \text{or} \quad \frac{1}{2}\frac{\boldsymbol{\sigma}\cdot\mathbf{P}}{|\mathbf{p}|}\phi = -\frac{1}{2}\phi,$$

and similarly for $\bar{\chi}$

$$\frac{1}{2}\frac{\boldsymbol{\sigma}\cdot\mathbf{P}}{|\mathbf{p}|}\bar{\chi} = \frac{1}{2}\bar{\chi}.$$

2.6 The no-go theorem and graded Lie superalgebras

The Poincaré group contains the complete set of transformations for the symmetries of special relativity (invariance under rotations, translations and boosts), and we have seen that this implies certain properties for the particles, or rather fields, that want to live in representations of the Poincaré group. At the same time we know that the quantum fields have (internal) gauge symmetries. It would then be tempting so ask if these are somehow related and can be described in a larger symmetry.

Unfortunately, the answer to that question is 'no', at least as long as we keep to describing our symmetries using Lie algebras. In 1967 Coleman and Mandula [5] showed that under reasonable assumptions any extension of the restricted Pointcaré group P to include gauge symmetries is isomorphic to $G_{\text{gauge}} \times P$, where G_{gauge} is whatever gauge group the Standard Model has. A direct product like this means that the generators of the two groups all commute, meaning that the generators B_i of the standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

The result is that there can be no real interaction between the external and internal symmetries.

Not to be defeated by a simple mathematical proof, in 1975 Haag, Łopuszański and Sohnius (HLS) [6] showed that there is a way around Coleman and Mandula's no-go theorem, if one introduces the concept of $\mathbb{Z}_2$ **graded Lie superalgebras**.[18]

> **Definition**: A **graded Lie superalgebra** is a vector space $\mathfrak{l}$ that is a direct sum of two vector spaces $\mathfrak{l}_0$ and $\mathfrak{l}_1$, $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$, with a binary operation $\circ : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ such that for all $x_i \in \mathfrak{l}_i$
>
> i) $x_i \circ x_j \in \mathfrak{l}_{i+j \mod 2}$ (grading)[a]
>
> ii) $x_i \circ x_j = -(-1)^{ij} x_j \circ x_i$ (supersymmetrisation)
>
> iii) $x_i \circ (x_j \circ x_k)(-1)^{ik} + x_j \circ (x_k \circ x_i)(-1)^{ji} + x_k \circ (x_i \circ x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)
>
> [a]For $x_0 \in \mathfrak{l}_0$ and $x_1 \in \mathfrak{l}_1$, this means that $x_0 \circ x_0 \in \mathfrak{l}_0$, $x_1 \circ x_1 \in \mathfrak{l}_0$ and $x_0 \circ x_1 \in \mathfrak{l}_1$.

The second requirement generalises the definition of a Lie algebra in Sec. 1.7 to allow for anti-commutators, $x \circ y = \{x, y\} \equiv xy + yx$, as the binary operation for elements in $\mathfrak{l}_1$.

We can now start, following HLS, with the Poincaré Lie algebra ($\mathfrak{l}_0 = \mathfrak{p}$) and add a new vector space $\mathfrak{l}_1$ spanned by some generators Q_a. It can be shown that the superalgebra requirements are fulfilled if there are four generators, $a = 1, 2, 3, 4$, that together form a

[18]The definition of graded Lie algebras can be extended to $\mathbb{Z}_n$ by a direct sum over n vector spaces $\mathfrak{l}_i$, $\mathfrak{l} = \oplus_{i=0}^{n-1} \mathfrak{l}_i$, such that $x_i \circ x_j \in \mathfrak{l}_{i+j \mod n}$, with the same requirements for supersymmetrization and Jacobi identity as for the $\mathbb{Z}_2$ graded algebra.

four-component Majorana spinor,[19] also called the **supercharges**. The algebra is then

$$\begin{aligned} [Q_a, P_\mu] &= 0, && (2.38)\\ [Q_a, M_{\mu\nu}] &= (\sigma_{\mu\nu} Q)_a, && (2.39)\\ \{Q_a, \bar{Q}_b\} &= 2\not{P}_{ab}, && (2.40) \end{aligned}$$

where $\sigma_{\mu\nu}$ is given in terms of the γ-matrices, $\sigma_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu]$, and as usual $\not{P} \equiv P_\mu \gamma^\mu$ and $\bar{Q}_a \equiv (Q^\dagger \gamma_0)_a$.[20] Together with the commutators in (2.5), (2.8) and (2.9) this is called the **super-Poincaré algebra $\mathfrak{sp}$**.

Because of (2.40) this new algebra is a non-trivial extension of the Poincaré algebra that avoids the no-go theorem. This extension can be proven, under some reasonable assumptions, to be the *largest possible* extension of the symmetries of Special Relativity. However, in the Q_a we have introduced new operators that (disappointingly) do not correspond to the generators of the gauge groups, which should in any case be related by commutators, not anti-commutators. The gauge group generators *can* appear in the algebra if we instead extend the algebra with $N > 1$ sets of new spinors Q_a^α, where $\alpha = 1, \ldots, N$. This gives rise to so-called $N > 1$ supersymmetries, while a single set of Q_a is called $N = 1$ supersymmetry. Given a gauge group algebra $[B_i, B_j] = iC_{ij}{}^k B_k$, we can then extended the superalgebra by the non-trivial commutator $[Q_a^\alpha, B_l] = iS_l^{\alpha\beta} Q_a^\beta$, where S_l are matrix representations of the gauge symmetry group, which does not work for $N = 1$.

However, the $N > 1$ supersymmetries seem impossible to realise in nature due to an extensive number of extra particles that do not conform to the particles and gauge symmetries of the Standard Model. Note that $N > 8$ would include elementary particles with spin greater than 2, which seems to be in contradiction with quantum field theory. The largest consistent supersymmetry, $N = 8$, has a minimum of one spin-2 state (identified with the graviton), 8 spin-$\frac{3}{2}$ states, 28 vector bosons (spin-1), 56 spin-$\frac{1}{2}$ fermions and 70 scalar fields. One fundamental problem with this, besides the plethora of particles, is that the vector bosons here form an $O(8)$ group which is too small to contain the Standard Model $SU(3) \times SU(2) \times U(1)$ symmetry. However, $N = 8$, supersymmetry has some very interesting theoretical properties. It is currently unknown whether the theory is finite or not (has infinities that need renormalisation). This has been checked up to four loops, surprisingly without any divergences appearing [7].

We can also write the super-Poincaré algebra in terms of the Weyl spinors introduced in Sec. 2.5. With

$$Q_a = \begin{pmatrix} Q_A \\ \bar{Q}^{\dot{A}} \end{pmatrix}, \tag{2.41}$$

for the Majorana spinor charges, we have instead

$$\begin{aligned} [Q_A, P_\mu] &= [\bar{Q}_{\dot{A}}, P_\mu] = 0, && (2.42)\\ [Q_A, M^{\mu\nu}] &= \sigma^{\mu\nu}_A{}^B Q_B, && (2.43)\\ \{Q_A, Q_B\} &= \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0, && (2.44)\\ \{Q_A, \bar{Q}_{\dot{B}}\} &= 2\sigma^\mu_{A\dot{B}} P_\mu, && (2.45) \end{aligned}$$

where now the $\sigma^{\mu\nu}$ are given in terms of the Pauli matrices $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$.

[19] Thus the four generators are not independent.

[20] Alternatively, (2.40) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

2.7 Conformal symmetry*

Despite the Coleman and Mandula no-go theorem, there exists a larger potential space-time symmetry, namely **conformal symmetry**. This extends the boosts, rotations and translations of the Poincaré symmetry with the **special conformal transformations** and **dilation**, with the generators K_μ and D, respectively.

We saw the one-dimensional differential representation of the dilation operator that changes scale in Sec. 1.5. Generalised to four space-time dimensions this is $D = -x_\mu \partial^\mu$. The composition function for the special conformal transformation is

$$x'_\mu = f_\mu(x_\mu, a_\mu) = \frac{x_\mu - a_\mu x^2}{1 - 2ax + a^2x^2}, \tag{2.46}$$

which gives the representation $K_\mu = i(x^2\partial_\mu - 2x_\mu x_\nu \partial^\nu)$.

The extra products in the algebra are then

$$\begin{aligned}
[K_\mu, K_\nu] &= 0, &(2.47)\\
[K_\mu, D] &= iK_\mu, &(2.48)\\
[K_\mu, P_\nu] &= 2i(g_{\mu\nu}D - M_{\mu\nu}), &(2.49)\\
[K_\mu, M_{\nu\rho}] &= i(g_{\mu\nu}K_\rho - g_{\mu\rho}K_\nu), &(2.50)\\
[D, P_\mu] &= iP_\mu, &(2.51)\\
[D, M_{\mu\nu}] &= 0. &(2.52)
\end{aligned}$$

Unfortunately, the scale invariance in conformal symmetry means that all the particles in the theory must be massless.[21] The usual reason given is that a theory with a particular particle mass scale has a corresponding length scale, and since the dilation symmetry would require the action to be invariant under length scale transformation, this breaks the correspondence.[22] As a result, conformal symmetry can not be a symmetry of the Standard Model, however, conformal symmetries are an interesting area of study, because they appear in other important theories such as Maxwell's equations for electromagnetism, general relativity in two dimensions and the so-called $N = 4$ supersymmetric Yang-Mills theory.

2.8 Irreducible representations of the super-Poincaré group

We now want to find the (irreducible) representations, irreps, of the super-Poincaré algebra and compare it to the known representations of the Poincaré algebra so see what sort of particles/states this leads to.

2.8.1 The Casimir operators of the superalgebra

It is easy to see that P^2 is also a Casimir operator of the superalgebra. From Eq. (2.38) P_μ commutes with the Qs, so in turn P^2 must commute.[23] The algebra, just as the Poincaré

[21] It is technically possible to have a quantum field theory that is scale invariant, but not conformally invariant, but examples are rare.

[22] However, be aware that this is not the complete story. It possible to get away with both theories with a continuous mass spectrum and theories with infinite mass, that have conformal symmetry, however, neither of these fit with the observed microcosmos.

[23] Although the fact that Eq. (2.38) holds crucially depends on Q_a being four-dimensional. P_μ and Q_a would not commute if there had been five Qs.

algebra, then also has irreducible representations labeled by the eigenvalue $m^2 \in \mathbb{R}$ and an infinite number of states $|m, p_\mu\rangle$ that are eigenstates of the momentum operator P_μ. However, W^2 is not a Casimir because of the following result:[24]

$$[W^2, Q_a] = W_\mu(\not{P}\gamma^\mu\gamma^5 Q)_a + \frac{3}{4}P^2 Q_a.$$

We want to find an extension of W that commutes with the Qs while retaining the commutators we already have with P_μ and $M_{\mu\nu}$. The construction

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu,$$

where

$$B_\mu \equiv W_\mu + \frac{1}{4}X_\mu, \quad X_\mu \equiv \frac{1}{2}\bar{Q}\gamma_\mu\gamma^5 Q,$$

can be shown to have the required relation:

$$[C_{\mu\nu}, Q_a] = 0.$$

Note that by (2.38) we also have

$$[X_\mu, P_\nu] = 0. \tag{2.53}$$

We can show that C^2 then indeed commutes with all the generators in the algebra:

$$\begin{array}{rcll} [C^2, Q_a] &=& 0, & \text{(trivial by the above)} \\ [C^2, P_\mu] &=& 0, & \text{(proof by excessive algebra)} \\ [C^2, M_{\mu\nu}] &=& 0. & \text{(because } C^2 \text{ is a Lorentz scalar)} \end{array}$$

Thus C^2 is a Casimir operator for the superalgebra.

To find the possible eigenvalues of C^2, let us again assume that we are in the rest frame (RF) of the particle.[25] For C^2 we have to do a bit of calculation:

$$\begin{array}{rcl} C^2 &=& 2B_\mu P_\nu B^\mu P^\nu - 2B_\mu P_\nu B^\nu P^\mu \\ &\overset{RF}{=}& 2m^2 B_\mu B^\mu - 2m^2 B_0^2 \\ &=& 2m^2 B_k B^k, \end{array}$$

where we used that $[B_\mu, P_\nu] = 0$, which we get from (2.14) and (2.53). From the definition of B_μ:

$$B_k = W_k + \frac{1}{4}X_k = mS_k + \frac{1}{8}\bar{Q}\gamma_k\gamma^5 Q \equiv mJ_k. \tag{2.54}$$

The operator we just defined, $J_k \equiv \frac{1}{m}B_k$, is an extension of the ordinary spin operator which we have here renamed to S_k due to a shortage of letters. This gives us, still in the rest frame,

$$C^2 = 2m^4 J_k J^k = -2m^4 \mathbf{J}^2,$$

[24] Which, by the way, is really hard work!

[25] We can again carry out a similar argument in a different frame for massless particles as we saw for the Poincaré algebra.

so J^2 also commutes with all the elements in the algebra since C^2 is a Casimir. We can also show that the J_k commute with the Qs[26]

$$[J_k, Q_a] = 0, \tag{2.55}$$

and just like the spin operator J_k can be shown to fulfil the $\mathfrak{su}(2)$ algebra:

$$[J_i, J_j] = i\epsilon_{ijk} J^k.$$

We then know that the eigenvalue equation for the second Casimir is:

$$C^2 |m, j\rangle = -m^4 j(j+1)|m, j\rangle,$$

for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. In addition, for each irrep with a value of j there are $2j+1$ distinct *states* with labels $j_3 = -j, -j+1, \ldots, j-1, j$, so that we may further write $|m, j, p_\mu, j_3\rangle$, labelling also the states of the irrep. The above follows from the identical argument we made for the Poincaré algebra, which in turn relies just on the properties of the $\mathfrak{su}(2)$ algebra.

2.8.2 The states of the irreps of the super-Poincaré group

What we have learned above is that the irreducible representations of the superalgebra can be labeled by (m, j), and any given set of m and j will give us $2j+1$ eigenstates of J_3 with different eigenvalues j_3, as well as an infinite number of momentum eigenstates.[27] However, unlike for spin, because we have introduced another generator Q this does not exhaust the number of states for the representation. We can have simultaneous eigenstates of P^2, C^2, P_μ, J^2, and J_3, but also of the original spin operator S_3 which we can see commutes with all the other operators in this list.[28]

To find all the states it is useful to write the generators Q in terms of two-component Weyl spinors instead of four-component Dirac spinors, making explicit use of their Majorana nature, as we did in Section 2.5. We note that from Eq. (2.55) above

$$[J_k, Q_A] = [J_k, \bar{Q}_{\dot{B}}] = 0.$$

We begin by claiming that for any eigenstate of J_3 with eigenvalue j_3 there must then exist a state $|\Omega\rangle$ – possibly the same state – that has the same eigenvalue j_3 and for which

$$Q_A |\Omega\rangle = 0. \tag{2.56}$$

This state is called the **Clifford vacuum**.[29]

To show this, start with $|\beta\rangle$, an eigenstate of J_3 with eigenvalue j_3. Then the construction

$$|\Omega\rangle = Q_1 Q_2 |\beta\rangle,$$

has these properties. Using (2.44) we first we show that (2.56) holds:

$$Q_1 Q_1 Q_2 |\beta\rangle = -Q_1 Q_1 Q_2 |\beta\rangle = 0,$$

[26] Again the proof is algebraically extensive, and the interested reader is suggested to pursue [3].

[27] Make sure you remember that j here is *not* the spin of the particles, but a generalisation of spin.

[28] We know this for P_μ and $M_{\mu\nu}$ already, and commutation with J_k follows from (2.55).

[29] It is called the Clifford vacuum because the operators satisfy a Clifford algebra $\{Q_A, \bar{Q}_{\dot{B}}\} = 2m\sigma^0_{A\dot{B}}$. Do not confuse this with a vacuum state, it is only a name.

and

$$Q_2Q_1Q_2|\beta\rangle = -Q_1Q_2Q_2|\beta\rangle = Q_1Q_2Q_2|\beta\rangle = 0.$$

For this state we also have:

$$J_3|\Omega\rangle = J_3Q_1Q_2|\beta\rangle = Q_1Q_2J_3|\beta\rangle = j_3|\Omega\rangle,$$

in other words, $|\Omega\rangle$ has the same value for j_3 as the $|\beta\rangle$ it was constructed from and the Clifford vacuum exists. This proof demonstrates a general feature of the supercharges, if one supercharge with a particular index repeats in a term, then the term is zero by the anticommutation property of the supercharges.

We can now use the explicit expression for J_k in terms of the two-component supercharges

$$J_k = S_k - \frac{1}{4m}\bar{Q}_{\dot{B}}\bar{\sigma}_k^{\dot{B}A}Q_A, \tag{2.57}$$

in order to find the spin for this state. First we can see that

$$S_3|\Omega\rangle = J_3|\Omega\rangle = j_3|\Omega\rangle,$$

meaning that $s_3 = j_3$ is the eigenvalue of S_3 for the Clifford vacuum $|\Omega\rangle$. Further, since

$$S^2|\Omega\rangle = J^2|\Omega\rangle = j(j+1)|\Omega\rangle,$$

the eigenvalue of S^2 is $s(s+1) = j(j+1)$ for the Clifford vacuum.

We can construct three more states from the Clifford vacuum using the Qs:[30]

$$\bar{Q}^{\dot{1}}|\Omega\rangle, \quad \bar{Q}^{\dot{2}}|\Omega\rangle, \quad \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle.$$

This means that there are four possible states that can be constructed out of any state with the labels m, j, j_3. Taking a look at:

$$J_3\bar{Q}^{\dot{A}}|\Omega\rangle = \bar{Q}^{\dot{A}}J_3|\Omega\rangle = j_3\bar{Q}^{\dot{A}}|\Omega\rangle,$$

this means that all these states have the same j_3 (and j) quantum numbers.[31] We can now find their eigenvalues for S_3. From the superalgebra (2.43) we have:

$$[M^{ij}, \bar{Q}^{\dot{A}}] = -(\sigma^{ij})^{\dot{A}}{}_{\dot{B}}\bar{Q}^{\dot{B}}, \quad \text{or} \quad [S_k, \bar{Q}^{\dot{A}}] = -\frac{1}{2}\epsilon_{kij}(\sigma^{ij})^{\dot{A}}{}_{\dot{B}}\bar{Q}^{\dot{B}},$$

so that:

$$\begin{aligned} S_3\bar{Q}^{\dot{A}}|\Omega\rangle &= \bar{Q}^{\dot{A}}S_3|\Omega\rangle + \frac{i}{8}(\epsilon_{3ij}[\sigma^i,\sigma^j])^{\dot{A}}{}_{\dot{B}}\bar{Q}^{\dot{B}}|\Omega\rangle \\ &= \bar{Q}^{\dot{A}}S_3|\Omega\rangle - \frac{1}{2}(\bar{\sigma}_3\sigma^0)^{\dot{A}}{}_{\dot{B}}\bar{Q}^{\dot{B}}|\Omega\rangle \\ &= \left(j_3 \mp \frac{1}{2}\right)\bar{Q}^{\dot{A}}|\Omega\rangle, \end{aligned}$$

[30] All other possible combinations of Qs and $|\Omega\rangle$ give either one of the other four states, or zero.

[31] The same can easily be shown for $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$.

where $-$ is for $\dot{A} = \dot{1}$ and $+$ is for $\dot{A} = \dot{2}$. We can similarly show that

$$S_3\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle = j_3\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle.$$

This means that for en irrep with labels m and j, there are $2j+1$ different values of j_3, each giving two states with $s_3 = j_3$, and two with $s_3 = j_3 \pm \frac{1}{2}$, meaning two bosonic and two fermionic states with the same mass m, and in total $4(2j+1)$ states per irrep.

We should be careful to note here that we have only found the spin-components s_3 of these states, not their spins s. For the state $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$, s is the same as for the Clifford vacuum, *i.e.* $s = j$. This is because in the application of S_k from (2.57) to the state the terms with supercharges will all be zero since at least one of the $\bar{Q}^{\dot{A}}$ will repeat in each term, thus the eigenvalues of S^2 are the same as the eigenvalues of J^2. For the other states we may need to combine states into definite spin states using Clebsch-Gordan coefficients.

The above explains the much repeated statement that any supersymmetry theory has an equal number of bosons and fermions, which, incidentally, is not true. What is true, is that there must be an equal number of bosonic and fermionic states in all representations.

Theorem: For any representation of the superalgebra where P_μ is a one-to-one operator there is an equal number of boson and fermion states.

To show this, divide the representation into two sets of states, one with bosons and one with fermions. Let $\{Q_A, \bar{Q}_{\dot{B}}\}$ act on the members of the set of bosons. $\bar{Q}_{\dot{B}}$ transforms bosons to fermions and Q_A does the reverse mapping. If P_μ is one-to-one, then so is $\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma^\mu_{A\dot{B}}P_\mu$. Thus there must be an equal number in both sets.

2.8.3 Examples of irreducible representations

Finally, let us briefly look at two examples of irreducible representations for a fixed positive value of m.

$j = 0$

For $j = 0$, we must have $j_3 = 0$ and as a result the Clifford vacuum $|\Omega\rangle$ has $s = 0$, $s_3 = 0$, and is a bosonic state. We can then create two states $\bar{Q}^{\dot{A}}|\Omega\rangle$ with $s_3 = \pm\frac{1}{2}$ and $s = \frac{1}{2}$, and one state $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$ with $s_3 = 0$ and $s = 0$. Note that we should really check the total spin s of each of the fermion states, which would involve some algebra. In total there are two scalar states and two spin-$\frac{1}{2}$ fermion states. We will later represent this set of states by the so-called **scalar superfield**.

We should use be a little careful about using the term particle about these states since what we have found for the fermions are in fact Weyl spinor states. From what we saw in Sec. 2.5 a Dirac fermion can then only be described by a $j = 0$ representation together with a different $j = 0$ complex conjugate representation, thus consisting of four states. The complex conjugate representation of the first representation together with the self-representation of the second then form the anti-particle of the fermion, and provide an additional two scalars. So the total particle count from the two irreducible representations is a fermion–anti-fermion pair, and four scalars. Note that all of the resulting particles have the same mass m.

For a Majorana fermion the situation is simpler, since we only need one self-representation and its complex conjugate representation.

$j = \frac{1}{2}$

For $j = \frac{1}{2}$ we have two Clifford vacua $|\Omega\rangle$ with $j_3 = \pm\frac{1}{2}$, and with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$, thus they are fermionic states. For the moment we label them as $|\Omega; \frac{1}{2}\rangle$ and $|\Omega; -\frac{1}{2}\rangle$. From each of these we can construct two further fermion states $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega; \pm\frac{1}{2}\rangle$ where we know $s = \frac{1}{2}$ and $s_3 = \mp\frac{1}{2}$. Together these four states can form two fermions with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$.

In addition to this we have the two states $\bar{Q}^{\dot{1}}|\Omega; \frac{1}{2}\rangle$ and $\bar{Q}^{\dot{2}}|\Omega; -\frac{1}{2}\rangle$ with $s_3 = 0$, the state $\bar{Q}^{\dot{2}}|\Omega; \frac{1}{2}\rangle$ with $s_3 = 1$, and the state $\bar{Q}^{\dot{1}}|\Omega; -\frac{1}{2}\rangle$which has $s_3 = -1$. By linear combinations of these we can create three states with $s = 1$, and $s_3 = 1, 0, -1$, and one state with $s = 0$ and $s_3 = 0$, representing one massive vector particle and one scalar. Carefull consideration of the transformation properties of these particles will show that the scalar is a pseudo-scalar (a particle that changes sign under a parity transformation).

In total this representation then has one (massive) spin-1 vector with three spin-states, two spin-$\frac{1}{2}$ fermions and one spin-0 scalar. We will later refer to this set of states as the **vector superfield**.

2.9 Exercises

Exercise 2.1

Show the following relationship for the normalisation constants $N_m^{\pm}$ of the ladder operators for $\mathfrak{su}(2)$,

$$N_{m+1}^{-}N_m^{+} + m = N_{m-1}^{+}N_m^{-},$$

and use this to conclude that $N_{j-1}^{+}N_j^{-} = j$ where j is the largest eigenvalue of the J_3 operator.

Exercise 2.2

Find the spin-$\frac{3}{2}$ representation of $\mathfrak{su}(2)$.

Exercise 2.3

Find an explicit expression for the boost generators K_i and show the commutation properties of the Lorentz group generators J_i and K_i. *Hint:* We advise that you use rapidity to parametrise the boosts to avoid excessive algebra.

Exercise 2.4

Show that (2.6) are the differential generators of the Lorentz group.

Exercise 2.5

Use Eq. (2.7) to write out an explicit expression for a Lorentz boost in the x-direction with rapidity η.

Exercise 2.6

Show that a general boost in the direction of the unit vector $\mathbf{n}$ with rapidity η can be written as

$$B(\eta, \mathbf{n}) = I + (\cosh\eta - 1)(\mathbf{n}\cdot\mathbf{K})^2 - i\sinh\eta(\mathbf{n}\cdot\mathbf{K}).$$

Exercise 2.7

Show the commutation properties of the Poincaré group generators P_μ and $M_{\mu\nu}$.

Exercise 2.8

Show that

$$\begin{aligned}\left[P_\mu, P^2\right] &= 0,\\ \left[M_{\mu\nu}, P^2\right] &= 0.\end{aligned}$$

Exercise 2.9

Show that $[P_\mu, W_\nu] = 0$.

Exercise 2.10

Show that

$$\left[P_\mu, W^2\right] = 0 \tag{2.58}$$

$$\left[M_{\mu\nu}, W^2\right] = 0. \tag{2.59}$$

Hint: You can use that[32]

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M^{\rho\sigma}M_{\nu\sigma}P_\rho P^\nu,$$

Exercise 2.11

Starting from the four-component form of the super-Poincaré algebra, derive the two-component (Weyl spinor) form.

Exercise 2.12

Show that $[X_\mu, P_\nu] = 0$.

Exercise 2.13

Show that $SO^+(1,3)$ and $SL(2,\mathbb{C})$ are indeed homomorphic, *i.e.* that the mapping defined by (2.19) or (2.20) has the property that $\Lambda(M_1M_2) = \Lambda(M_1)\Lambda(M_2)$ or $M(\Lambda_1\Lambda_2) = M(\Lambda_1)M(\Lambda_2)$.

Exercise 2.14

Show that the generalisation of the spin operator, $J_k \equiv S_k + \frac{1}{8m}\bar{Q}\gamma_\mu\gamma^5 Q$, fulfils the algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

Exercise 2.15

What are the states for $j = 1$?

[32] This is non-trivial to demonstrate, see Chapter 1.2 of [3].

Chapter 3

Superspace

In this chapter we will introduce a very handy notation system for considering supersymmetry transformations effected by the Q elements of the superalgebra, or, more correctly, the elements of the super-Poincaré group and their representations. This notation uses a coordinate system called superspace, and allows us to define so-called superfields as a replacement of ordinary field theory fields. This mirrors the Lorentz invariance built into relativistic field theory by using four-vectors. In order to do this we need to know a little more about the properties of Grassman (anti-commuting) numbers. However, we begin by taking another look at the familiar four-vectors in light of what we have learnt about continuous groups and Lie algebras.

3.1 An initial skirmish: four-vectors as a coset space

Traditionally, we are introduced to four-vectors as a record keeping device for time and space-position in Special Relativity. In this notation we introduce (greek) four-vector indices, and some (odd) rules for manipulating these.

Let us now go back to the Poincaré group and its generators in Sec. 2.3, and in particular the exponential map in Eq. (2.10). We here followed the conventions of four-vectors, but we could have equally well written it up using ordinary vector component notation, starting with the generators we derived for the different transformations of the group, *i.e.* rotations J_i, boosts K_i, time-translation P_0, and space-translation P_i.

We can now form the (right) coset of the Poincaré group P with its Lorentz subgroup L, P/L. This is not a group since L is not normal to P, however, the coset space is a vector space formed by the elements $\{\Lambda g | \Lambda \in L\}$ where $g \in P$. As before, we parameterise a general element g in the Poincaré group P as

$$g = \exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma} + ia^{\mu}P_{\mu}\right),$$

but keeping in mind the discussion above. Since any two cosets are either disjoint or identical, we can now represent each element in the coset space by one of the members in the coset picking the member where $\omega^{\rho\sigma} = 0$ giving $\Lambda = I$. What remains in the coset is the four-dimensional translation operator $T(a) = e^{ia^{\mu}P_{\mu}}$. Physically, this space is the set of all translations independent of reference frames (boosts and rotations), and it is equivalent (isomorphic) to the vector space formed by the four components of a^{μ} (parameters of the translation) since there is a one-to-one map between a^μ and the elements of the coset space. Thus we can say that the four-vectors form the coset space between the Poincaré group and the Lorentz group.

The transformation properties of four-vectors that we have learnt about can be demonstrated by the properties of the Poincaré group. It is possible to show, using only the properties of the generators and their commutators, that with a member of the Lorentz group $\Lambda = \exp(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma})$ we have

$$\Lambda T(a) = \exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)T(a) = T(\Lambda a)\exp\left(\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right) = T(\Lambda a)\Lambda,$$

where $\Lambda a \equiv \Lambda^\mu{}_\nu a^\nu$ is the same Lorentz transformation of the four-vector translation parameter as the one we are accustomed to.

If we now want to see how a Lorentz transformation Λ acts for example on a scalar function $F(x)$ we can write $F(x) = \exp(ix^\mu P_\mu)F(0)$. This means that the transformation is essentially captured as x' in $\Lambda\exp(ix^\mu P_\mu) = \exp(ix'^\mu P_\mu)$. Now, from the above we have

$$\Lambda\exp(ix^\mu P_\mu) = \Lambda T(x) = T(\Lambda x)\Lambda,$$

and since the Lorentz transformation does nothing to a scalar coordinate independent quantity such as $F(0)$, we have $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, and the scalar function transforms the way we are used to under Lorentz transformations, namely $\Lambda F(x) = F(x')$. This argument can easily be extended to vector fields $A_\mu(x)$ and higher rank tensors, for example

$$\Lambda A^\nu(x) = \Lambda^\mu{}_\nu A^\nu(x').$$

The way we have defined four-vectors have built the Lorentz transformations of Special Relativity into our equations.

3.2 Superspace definition

Superspace[1] is a coordinate system where supersymmetry transformations are *manifest*, in other words, the action of elements in the super-Poincaré group (SP) are treated like Lorentz-transformations are in Minkowski space.

> **Definition**: **Superspace** is an eight-dimension manifold that can be constructed from the **coset space** of the super-Poincaré group, SP, and the Lorentz group, L, SP/L, by giving coordinates $z^\pi = (x^\mu, \theta^A, \bar{\theta}_{\dot{A}})$, $\pi = \mu, A, \dot{A}$, where x^μ are the ordinary Minkowski coordinates, and where θ^A and $\bar{\theta}_{\dot{A}}$ are four Grassman (anti-commuting) numbers in the form of Weyl spinors, being the parameters of the supercharges Q_A and $\bar{Q}^{\dot{A}}$, respectively, in the exponential map of the superalgebra.

To understand what all of this means we start from the same perspective as for the four-vectors in the previous section, and begin by writing a general element of SP, $g \in SP$, using the exponential map defined in Section 1.7:

$$g = \exp\left(ix^\mu P_\mu + i\theta^A Q_A + i\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}} + \frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right),$$

[1] First introduced by Salam & Strathdee [8].

where x^μ, θ^A, $\bar{\theta}_{\dot{A}}$ and $\omega_{\rho\nu}$ constitute the parametrisation of the group, and P_μ, Q_A, $\bar{Q}^{\dot{A}}$ and $M_{\rho\nu}$ are the generators. Following the same argument as above we can now parametrise the coset space SP/L simply by setting $\omega_{\mu\nu} = 0$.[2] The remaining parameters of SP/L are then the coordinates that span superspace.

In order for the exponential map to make sense the parameters θ need to anti-commute just like the Qs so that the contractions $\theta^A Q_A$ and $\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}$ follow ordinary commutation rules. Otherwise the commutation properties of the exponential map would be different order by order.[3] As physicists we also want to know the dimensions of our new parameters. To do this we first look at Eq. (2.45):

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma^\mu_{A\dot{B}} P_\mu,$$

where we know that P_μ has mass dimension $[P_\mu] = M$. This means that $[Q^2] = M$ and $[Q] = M^{1/2}$. In the exponential, all terms must have mass dimension zero to make sense. This means that $[\theta Q] = 0$, and therefore $[\theta] = M^{-1/2}$.

We now want to find the effect of supersymmetry transformations (transformations by the super-Poincaré group) on the superspace coordinates, and we begin by noting that any SP transformation can effectively be written in the following way without the boosts and rotations of the Lorentz group:

$$g_0(a, \alpha) = \exp[ia^\mu P_\mu + i\alpha^A Q_A + i\bar{\alpha}_{\dot{A}}\bar{Q}^{\dot{A}}],$$

effectively setting $\omega_{\rho\nu} = 0$, because we can again show that[4]

$$\exp\left[\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right] g_0(a, \alpha) = g_0(\Lambda a, S(\Lambda)\alpha) \exp\left[\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right], \tag{3.1}$$

i.e. that all that a Lorentz boost does is to transform spacetime coordinates by Λ and Weyl spinors by $S(\Lambda)$, which is the spinor representation of Λ ($SL(2, \mathbb{C})$). Thus, in more colloquial terms, for the supersymmetry transformation it does not matter which reference frame we are working in, we know how the transformation changes between the frames, given by Λa and $S(\Lambda)\alpha$.

We can now find the transformation of superspace coordinates under a supersymmetry transformation, just as we have seen the Minkowski coordinates transform under Lorentz transformations. It might be tempting to look directly at the effects of an element $g_0(a, \alpha)$ on a function on superspace coordinates, $F(z^\pi) = F(x^\mu, \theta^A, \bar{\theta}_{\dot{A}})$, just as we did for the translation group. However, the powers of the sum of generators in the infinite series will become very unwieldy and messy. Instead, we pull out the coordinate dependence from the function $F(z^\pi) = e^{iz^\pi K_\pi}F(0)$, where $K_\pi = (P_\mu, Q_A, \bar{Q}^{\dot{A}})$, and we look at the transformation $z^\pi \to z'^\pi$ given by

$$g_0 e^{iz^\pi K_\pi} = e^{iz'^\pi K_\pi}.$$

[2] SP/L is again *not* a coset group, because L is not a normal subgroup of SP, but it still forms a vector space (the coset space) which we call superspace.

[3] We might already see how this can be useful: if we consistently use $\theta^A Q_A$ and $\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}$ instead of only Q_A and $\bar{Q}^{\dot{A}}$ in Eqs. (2.42)–(2.45) we can actually rewrite the superalgebra as an ordinary Lie algebra, but with Grassman elements, because of these commutation properties.

[4] Fortunately we are not going to do this because it is messy, but it can be done using the algebra of the group and the series expansion of the exponential function. Note, however, that the proof rests on the Ps and Qs forming a closed set, which we saw in the algebra Eqs. (2.42)–(2.45).

In moving the problem to the exponential, we can use the power of some results on the exponentials of non-commuting quantities, in particular the Campbell-Baker-Hausdorff expansion from (1.12). We have[5]

$$\begin{aligned} g_0 e^{iz^\pi K_\pi} &= \exp(ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}})\exp(iz^\pi K_\pi) \\ &= \exp(ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}} + iz^\pi K_\pi \\ &\quad + \frac{1}{2}[ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}}, iz^\pi K_\pi] + \,) \end{aligned}$$

Now we take a closer look at the commutator:[6]

$$\begin{aligned} [\;,\;] &= -[\alpha^B Q_B, \bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}] - [\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}}, \theta^A Q_A] \\ &= \alpha^B\bar{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\{Q_B, \bar{Q}_{\dot{C}}\} + \bar{\alpha}_{\dot{B}}\theta^A\epsilon^{\dot{B}\dot{C}}\{\bar{Q}_{\dot{C}}, Q_A\} \\ &= 2\alpha^B\bar{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\sigma^\mu_{B\dot{C}}P_\mu + 2\bar{\alpha}_{\dot{B}}\theta^A\epsilon^{\dot{B}\dot{C}}\sigma^\mu_{A\dot{C}}P_\mu \\ &= 2(\alpha^B\bar{\theta}^{\dot{C}}\sigma^\mu_{B\dot{C}} + \bar{\alpha}^{\dot{C}}\theta^A\sigma^\mu_{A\dot{C}})P_\mu. \end{aligned}$$

We can relabel $B = A$ and $\dot{C} = \dot{A}$ which leads to

$$\frac{1}{2}[\;,\;] = (\alpha^A\sigma^\mu{}_{A\dot{A}}\bar{\theta}^{\dot{A}} - \theta^A\sigma^\mu{}_{A\dot{A}}\bar{\alpha}^{\dot{A}})P_\mu.$$

The commutator is proportional with P_μ, and will therefore commute with all the operators in the problem, in particular the higher terms in the Campbell-Baker-Hausdorff expansion, meaning that the series reduces to

$$g_0 e^{iz^\pi K_\pi} = \exp[i(x^\mu + a^\mu - i\alpha^A\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}} + i\theta^A\sigma^\mu_{A\dot{A}}\bar{\alpha}^{\dot{A}})P_\mu + i(\theta^A + \alpha^A)Q_A + i(\bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}})\bar{Q}^{\dot{A}}].$$

So superspace coordinates transform under supersymmetry transformations as:

$$(x^\mu, \theta^A, \bar{\theta}_{\dot{A}}) \to (x^\mu + a^\mu - i\alpha^A\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}} + i\theta^A\sigma^\mu_{A\dot{A}}\bar{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}), \tag{3.2}$$

or given more explicitly as a composition function

$$f_\pi(x^\mu, \theta^A, \bar{\theta}_{\dot{A}}, a^\mu, \alpha^A, \bar{\alpha}_{\dot{A}}) = (x^\mu + a^\mu - i\alpha^A\sigma^\mu{}_{A\dot{A}}\bar{\theta}^{\dot{A}} + i\theta^A\sigma^\mu{}_{A\dot{A}}\bar{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}). \tag{3.3}$$

As a crucial by-product we can now write down a differential representation for the supersymmetry generators by applying the standard expression for the generators X_i of a Lie algebra, given the composition functions f_π:

$$iX_j = \frac{\partial f_\pi}{\partial a_j}\frac{\partial}{\partial z_\pi},$$

which gives us:

$$\begin{aligned} iP_\mu &= \partial_\mu, &(3.4)\\ iQ_A &= -i(\sigma^\mu\bar{\theta})_A\partial_\mu + \partial_A, &(3.5)\\ i\bar{Q}^{\dot{A}} &= -i(\bar{\sigma}^\mu\theta)^{\dot{A}}\partial_\mu + \partial^{\dot{A}}. &(3.6) \end{aligned}$$

The interested reader can now use these expressions to check the (anti-)commutation relations for the supercharges in Eqs. (2.44) and (2.45).

[5] Here we use Campbell-Baker-Hausdorff expansion $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$ where the terms that follow all contain commutators of the first commutator $[A, B]$ and the operators A and B.

[6] Using that P_μ commutes with all the other generators present, as well as $[\theta^A Q_A, \xi^B Q_B] = -\theta^A\xi^B\{Q_A, Q_B\} = 0$, and similarly for $\bar{Q}^{\dot{B}}$.

3.3 Superspace calculus

It should be clear from the following section that we need to know something about the calculus of anti-commuting Grassmann numbers in order to make sense of differentiation (and integration) with respect to them. In this section we will briefly discuss their most important properties, focusing on the coordinates of superspace.

As Grassmann numbers the superspace coordinates obey the following commutation rules:

$$\{\theta_A, \theta_B\} = \{\theta_A, \bar{\theta}_{\dot{B}}\} = \{\bar{\theta}_{\dot{A}}, \theta_B\} = \{\bar{\theta}_{\dot{A}}, \bar{\theta}_{\dot{B}}\} = 0.$$

From this we get the relationships:[7]

$$\begin{aligned} \theta_A^2 &= \theta_A\theta_A = -\theta_A\theta_A = 0, &(3.7)\\ \bar{\theta}_{\dot{A}}^2 &= \bar{\theta}_{\dot{A}}\bar{\theta}_{\dot{A}} = -\bar{\theta}_{\dot{A}}\bar{\theta}_{\dot{A}} = 0, &(3.8)\\ \theta^2 &\equiv \theta\theta \equiv \theta^A\theta_A = -2\theta_1\theta_2, &(3.9)\\ \bar{\theta}^2 &\equiv \bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{A}}\bar{\theta}^{\dot{A}} = 2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}. &(3.10) \end{aligned}$$

Notice that if we have a function f of a Grassman number, say θ_A, then the all-order expansion of that function in terms of θ_A, is

$$f(\theta_A) = a_0 + a_1\theta_A, \tag{3.11}$$

as there are simply no more terms because of (3.7).

We now need to define differentiation and integration on these numbers in order to create a calculus for them.[8]

Definition: We define differentiation in superspace by:

$$\partial_A\theta^B \equiv \frac{\partial}{\partial\theta^A}\theta^B \equiv \delta_A{}^B,$$

with a product rule

$$\begin{aligned} \partial_A(\theta^{B_1}\theta^{B_2}\theta^{B_3}\dots\theta^{B_n}) \equiv\ & (\partial_A\theta^{B_1})\theta^{B_2}\theta^{B_3}\dots\theta^{B_n}\\ & -\theta^{B_1}(\partial_A\theta^{B_2})\theta^{B_3}\dots\theta^{B_n}\\ & +\dots+(-1)^{n-1}\theta^{B_1}\theta^{B_2}\dots(\partial_A\theta^{B_n}). \end{aligned} \tag{3.12}$$

This implies that the differential operator ∂_A is itself a Grassmann number and anti-commutes. We can show that the indices of these operators can be raised and lowered with the ϵ in Eqs. (2.21) and (2.22).

[7]There is no summation implied in the first two lines, only a repetition of the same superspace coordinate. These are of course the same relations we already used for the Weyl spinors.

[8]These definitions have no infinitesimal interpretations, nor is there any separate notion of definite integrals.

Definition: The integral of a function f of a superspace coordinate θ_A is defined as a functional $I[f]$

$$I[f] = \int d\theta_A \, f(\theta_A),$$

which evaluates to a **c-number** (number in $\mathbb{C}$). The evaluation is defined by the relations

$$\int d\theta_A \equiv 0, \quad \int d\theta_A \, \theta_A \equiv 1,$$

and similarly for $\bar{\theta}_{\dot{A}}$, and we demand linearity:

$$\int d\theta_A [af(\theta_A) + bg(\theta_A)] \equiv a \int d\theta_A f(\theta_A) + b \int d\theta_A g(\theta_A).$$

In these definitions there is no implied summation over the index A.

This integral definition has a surprising property. If we take the integral of (3.11) we get:

$$\int d\theta_A f(\theta_A) = a_1 = \partial^A f(\theta_A),$$

meaning that differentiation and integration has the same effect on functions of Grassmann numbers.

To integrate over multiple Grassmann numbers we define volume elements as

$$\begin{aligned}
d^2\theta &\equiv -\frac{1}{4} d\theta^A d\theta_A, \\
d^2\bar{\theta} &\equiv -\frac{1}{4} d\bar{\theta}_{\dot{A}} d\bar{\theta}^{\dot{A}}, \\
d^4\theta &\equiv d^2\theta d^2\bar{\theta},
\end{aligned}$$

and we demand that the integral operators anti-commute, just as the differential operators

$$\{\int d\theta_A, \int d\theta_B\} = \{\int d\theta_A, \theta_B\} = 0.$$

This specific volume element definition is made to normalise the following integrals

$$\begin{aligned}
\int d^2\theta \; \theta\theta &= 1, \\
\int d^2\bar{\theta} \; \bar{\theta}\bar{\theta} &= 1, \\
\int d^4\theta \; (\theta\theta)(\bar{\theta}\bar{\theta}) &= 1.
\end{aligned}$$

Delta functions of Grassmann variables are given by:

$$\begin{gathered}
\delta(\theta_A) = \theta_A, \\
\delta^2(\theta_A) = \theta\theta, \\
\delta^2(\bar{\theta}^{\dot{A}}) = \bar{\theta}\bar{\theta},
\end{gathered}$$

and we can easily show that these functions satisfy, just as the usual definition of delta functions,

$$\int d\theta_A f(\theta_A)\delta(\theta_A) = f(0).$$

3.4 Covariant derivatives

Similar to the properties of covariant derivatives for gauge transformations in gauge theories, it would be nice to have a derivative that is invariant under supersymmetry transformations, *i.e.* commutes with the supersymmetry generators. Obviously $P_\mu = -i\partial_\mu$ does this, but more general covariant derivatives can be made.

Definition: The following **covariant derivatives** commute with supersymmetry transformations:

$$\begin{aligned} D_A &\equiv \partial_A + i(\sigma^\mu\bar{\theta})_A\partial_\mu, && (3.13)\\ \bar{D}_{\dot{A}} &\equiv -\partial_{\dot{A}} - i(\theta\sigma^\mu)_{\dot{A}}\partial_\mu. && (3.14)\end{aligned}$$

These can be shown to satisfy the following relations that are useful in calculations:

$$\begin{aligned} \{D_A, D_B\} &= \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0, && (3.15)\\ \{D_A, \bar{D}_{\dot{B}}\} &= -2\sigma^\mu_{A\dot{B}}P_\mu, && (3.16)\\ D^3 = \bar{D}^3 &= 0, && (3.17)\\ D^A\bar{D}^2D_A &= \bar{D}_{\dot{A}}D^2\bar{D}^{\dot{A}}. && (3.18)\end{aligned}$$

Here, D^3 and $\bar{D}^3$ means the application of at least three of these covariant derivatives.

From the covariant derivatives we can also construct a set of three projection operators.

Definition: The operators

$$\begin{aligned} \pi_+ &\equiv -\frac{1}{16\Box}\bar{D}^2D^2, && (3.19)\\ \pi_- &\equiv -\frac{1}{16\Box}D^2\bar{D}^2, && (3.20)\\ \pi_T &\equiv \frac{1}{8\Box}\bar{D}_{\dot{A}}D^2\bar{D}^{\dot{A}}, && (3.21)\end{aligned}$$

with $\Box \equiv \partial_\mu\partial^\mu$, are orthogonal projection operators, *i.e.* they fulfil:

$$\begin{aligned} \pi^2_{\pm,T} &= \pi_{\pm,T}, && (3.22)\\ \pi_+\pi_- &= \pi_+\pi_T = \pi_-\pi_T = 0, && (3.23)\\ \pi_+ + \pi_- + \pi_T &= 1. && (3.24)\end{aligned}$$

3.5 Superfields

Using the superspace coordinates we can now define functions of these to use in a field theory. Naturally we should call these objects superfields.

> **Definition**: A **superfield** Φ is an operator valued function on superspace $\Phi(x, \theta, \bar{\theta})$.

Notice how we write, just as for ordinary fields depending on four-vector coordinates, the coordinates sans indices.

We can expand any such superfield $\Phi(x, \theta, \bar{\theta})$ as a power series in θ and $\bar{\theta}$. For a superfield without explicit spinor indices this gives in general,

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) &= f(x) + \theta^A \phi_A(x) + \bar{\theta}_{\dot{A}} \bar{\chi}^{\dot{A}}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) \\ &\quad + \theta\sigma^\mu \bar{\theta} V_\mu(x) + \theta\theta \bar{\theta}_{\dot{A}} \bar{\lambda}^{\dot{A}}(x) + \bar{\theta}\bar{\theta}\theta^A \psi_A(x) + \theta\theta\bar{\theta}\bar{\theta} d(x),\end{aligned} \tag{3.25}$$

where the functions of space-time coordinates x are called the **component fields**. Any superfield without explicit spinor indices, such as the one above, commutes with any other superfield, because all the Grassmann numbers appear in contracted pairs.

The properties of the component fields can be deduced from the requirement that Φ must be an (operator valued) Lorentz scalar or pseudoscalar. These are shown in Table 3.1 along with the corresponding degrees of freedom each field has.

Component field	**Type**	**d.o.f.**
$f(x)$, $m(x)$, $n(x)$	Complex (pseudo) scalar	2
$\psi_A(x)$, $\phi_A(x)$	Left-handed Weyl spinor	4
$\bar{\chi}^{\dot{A}}(x)$, $\bar{\lambda}^{\dot{A}}(x)$	Right-handed Weyl spinor	4
$V_\mu(x)$	Lorentz 4-vector (complex)	8
$d(x)$	Complex scalar	2

Table 3.1: Component field content of a general superfield.

One can show that under supersymmetry transformations these component fields transform linearly into each other, thus we say that superfields (with the differential form of the supersymmetry generators) are *representations* of the super-Poincaré group – in the sense of being states in a representation space – just as ordinary quantum fields are representations of the Poincaré group, albeit *highly reducible* representations since there are subsets of the component fields that are closed on the supersymmetry transformation![9] For example, the scalar fields f and m, and the Weyl-spinor ϕ transform as[10]

$$\delta_S f = \alpha\phi + \bar{\alpha}\bar{\chi}, \tag{3.26}$$

$$\delta_S \phi_A = 2\alpha_A m + (\sigma^\mu \bar{\alpha})_A (i\partial_\mu f + V_\mu), \tag{3.27}$$

$$\delta_S m = \bar{\alpha}\bar{\lambda} - \frac{i}{2}\partial_\mu \phi \sigma^\mu \bar{\alpha}. \tag{3.28}$$

[9] Indeed, they are linear representations since a sum of superfields is a superfield, and the differential supersymmetry operators act linearly.

[10] A word of warning is in order here. Considering Φ as a *quantum* field with operator value it transforms under the supersymmetry transformations in the *passive sense*

$$\Phi(x) \to \exp(-i\alpha Q - i\bar{\alpha}\bar{Q})\Phi(x)\exp(i\alpha Q + i\bar{\alpha}\bar{Q}).$$

We can recover the known irreducible representations, see Section 2.8.2, by imposing some restrictions on the fields. To do this we define the following three types of superfields that we will discuss below:

$$\bar{D}_{\dot{A}}\Phi(x,\theta,\bar{\theta}) = 0 \qquad \textbf{(left-handed scalar superfield)} \tag{3.29}$$
$$D_A\Phi^\dagger(x,\theta,\bar{\theta}) = 0 \qquad \textbf{(right-handed scalar superfield)} \tag{3.30}$$
$$\Phi^\dagger(x,\theta,\bar{\theta}) = \Phi(x,\theta,\bar{\theta}) \qquad \textbf{(vector superfield)} \tag{3.31}$$

Note that it is $\Phi^\dagger$ which is the right handed superfield in Eq. (3.30), not Φ. However, we can show that the hermitian conjugate of a left-handed scalar superfield fulfils the condition for a right-handed scalar superfield. We always keep the "dagger" operator on the right-handed fields to remember what they are since the difference between left- and right-handed superfields will become crucial later. Supersymmetry transformations can also be shown to transform left-handed superfields into left-handed superfields, right-handed superfields into right-handed superfields and vector superfields into vector superfields. This means that they are separate representations.

Products of the same type of superfield is a superfield of the same type since for left-handed scalar superfields Φ_i and Φ_j,

$$\bar{D}_{\dot{A}}(\Phi_i\Phi_j) = (\bar{D}_{\dot{A}}\Phi_i)\Phi_j + \Phi_i(\bar{D}_{\dot{A}}\Phi_j) = 0,$$

and similarly for a right-handed scalar superfields, and for vector superfields Φ_i and Φ_j,

$$(\Phi_i\Phi_j)^\dagger = \Phi_j^\dagger\Phi_i^\dagger = \Phi_j\Phi_i = \Phi_i\Phi_j.$$

The product of a left-handed scalar superfield Φ and its hermitian conjugate $\Phi^\dagger$, $V = \Phi\Phi^\dagger$, is a vector superfield since

$$V^\dagger = (\Phi\Phi^\dagger)^\dagger = \Phi\Phi^\dagger = V.$$

The same is true for sums of superfields of the same type. These properties will be important when creating a superfield version of a Lagrangian.

Note that the projection operators that we defined in Section 3.4, $\pi_\pm$, project out left-/right-handed superfields, respectively, from the general superfield, because:

$$\bar{D}_{\dot{A}}\pi_+\Phi = D_A\pi_-\Phi^\dagger = 0,$$

which follows from using Eq. (3.17). This is analogous to the properties of $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ that we saw in Sec. 2.5.2.

3.5.1 Scalar superfields

What is the connection between the scalar superfields and the $j = 0$ irreducible representation? To see this we use a cute trick:[11] Change to the variable $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then the covariant derivatives simplify to

$$D_A = \partial_A + 2i(\sigma^\mu\bar{\theta})_A\frac{\partial}{\partial y^\mu}, \tag{3.32}$$
$$\bar{D}_{\dot{A}} = -\partial_{\dot{A}}. \tag{3.33}$$

[11] Here cute is used in the widest possible sense.

This means that a field fulfilling $\bar{D}_{\dot{A}}\Phi = 0$ in the new set of coordinates must be independent of the $\bar{\theta}$ coordinates. Thus we can write this field as:

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{3.34}$$

and looking at the much restricted component field content we get the result in Table 3.2.

Component field	**Type**	**d.o.f.**
$A(x)$, $F(x)$	Complex scalar	2
$\psi_A(x)$	Left-handed Weyl spinor	4

Table 3.2: Component fields contained in a left-handed scalar superfield.

Since we wish to interpret the Weyl spinor here as a fermion quantum field with dimension $M^{3/2}$, and given that $[\theta] = M^{-1/2}$, the scalar superfield itself must have mass dimension $[\Phi] = 1$. This means that the scalar field A has the expected mass dimension M^1 of an ordinary scalar quantum field, however, the scalar F has the odd mass dimension $[F] = 2$.[12] The Weyl spinor is used to represent (half of) one of the Dirac fermions of the Standard Model, while the scalar A is typically given the same name as the fermion with an 's'-prefix, and the scalar F is called an **auxiliary** field, for reasons which will become clear later.

We can undo the coordinate change in (3.34) by inserting for y and expanding in powers of θ and $\bar{\theta}$, giving

$$\Phi(x,\theta,\bar{\theta}) = A(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x). \tag{3.35}$$

By using the transformation $y^\mu \equiv x^\mu - i\theta\sigma^\mu\bar{\theta}$ we can show a similar field content for the right-handed scalar superfield. The general form of a right handed scalar superfield is then as could be expected:

$$\Phi^\dagger(x,\theta,\bar{\theta}) = A^*(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x). \tag{3.36}$$

We can now compare the above to the $j = 0$ representation of the super-Poincaré group that had two scalar states and two fermionic states (d.o.f.). After applying the equations of motions (e.o.m.) the auxiliary field $F(x)$, with the strange mass dimension, can be completely eliminated as it does not have any derivatives.[13] The e.o.m. also eliminate two of the fermion d.o.f. using the Direc/Weyl equations from Sec. 2.5.2.[14] This does not happen for the scalar $A(x)$ since their e.o.m are not linear in the time-derivative. Thus, after the equations of motion, we are left with the same states as in the $j = 0$ representation.

[12] Odd as in strange, 2 is known to be an even number.

[13] Remember that the classical equation of motion for a field/particle, the Lagrange equation, has a term with the derivative of the Lagrangian w.r.t. to the field derivatives/velocities and one w.r.t. the fields/coordinates. For F only the latter is non-zero and can be used to solve for F. We will show this in more detail in Sec. 4.6 when we construct supersymmetric Lagrangians.

[14] Since the Lagrangian is linear in time derivatives of ψ – it will have to be when we construct it from these superfields that are linear – the generalised momenta $\pi = \partial_0\psi$ can be re-expressed in terms of the generalised coordinates without time derivatives and are not independent coordinates.

However, the scalar superfields will not correspond directly to particle states for the known Standard Model particles since, as we discussed in Sec. 2.8.3, a Weyl spinor on its own cannot describe a Dirac fermion. When we construct particle representations we will take one left-handed scalar superfield and one *different* right-handed scalar superfield. These will form a Dirac fermion and two scalars (and their anti-particles) after application of the e.o.m.

3.5.2 Vector superfields

If we take the general superfield in (3.25) and compare the Φ and $\Phi^\dagger$ expressions we can see that the following is the restricted component field structure of a vector superfield:

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + \theta\phi(x) + \bar{\theta}\bar{\phi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}M^*(x) \\ &\quad + \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}D(x). \end{aligned}$$

The properties of the component fields are summarised in Table 3.3. Repeating the arguments of mass dimension, if the V_μ is to represent a vector quantum field with $[V_\mu] = M^1$, then the dimension of the vector superfield must be $[V] = M^0$, the Wey-spinor λ is a normal looking $[\lambda] = M^{3/2}$, while the scalar D again has the odd $[D] = M^2$. For the C and ϕ fields the mass dimension is even stranger.

Component field	**Type**	**d.o.f.**
$C(x)$, $D(x)$	Real scalar field	1
$\phi_A(x)$, $\lambda_A(x)$	Weyl spinor	4
$M(x)$	Complex scalar field	2
$V_\mu(x)$	Real Lorentz 4-vector	4

Table 3.3: Field content of a general vector superfield.

With the large number of component fields, and their strange mass dimensions, you may now be a little suspicious that this vector superfield does not correspond to the promised degrees of freedom in the $j = \frac{1}{2}$ representation of the superalgebra, even after the application of the equations of motion. However, gauge-freedom now comes to our rescue.

3.6 Supergauge

We first define what we will mean by an abelian (super)gauge transformation of a superfield.[15] Later we will see how it relates to the ordinary gauge transformations of quantum fields. We begin with the scalar superfields.

Definition: The abelian **supergauge transformation** (local or global) on a left handed scalar superfield Φ_i is defined as:

$$\Phi_i \to \Phi_i' = e^{iq_i\Lambda}\Phi_i \tag{3.37}$$

where q_i is the charge of Φ_i under that gauge group and Λ (global), or $\Lambda(x)$ (local), is the parameter of the gauge transformation.

[15] We promise that we will get back to the corresponding definition for non-abelian transformations.

The definition is of course completely equivalent for right-handed scalar fields and generalises the standard definition of (abelian) gauge transformation in quantum field theory. For the definition to make sense the transformed field Φ_i' must be a left-handed scalar superfield itself, thus

$$\bar{D}_{\dot{A}}\Phi_i' = 0,$$

and this requires:

$$\bar{D}_{\dot{A}}\Phi_i' = \bar{D}_{\dot{A}}e^{iq_i\Lambda}\Phi_i = e^{iq_i\Lambda}\bar{D}_{\dot{A}}\Phi_i + iq_i(\bar{D}_{\dot{A}}\Lambda)e^{iq_i\Lambda}\Phi_i = iq_i(\bar{D}_{\dot{A}}\Lambda)\Phi_i' = 0.$$

Thus we must have $\bar{D}_{\dot{A}}\Lambda = 0$, which means that the parameter Λ is also a left-handed superfield. Note, however, that Λ has must have a mass dimension $[\Lambda] = M^0$ in order for the exponentiation to make sense.

Next, we move to the vector superfields.

Definition: Given a vector superfield $V(x,\theta,\bar{\theta})$, we define the **abelian supergauge transformation** as

$$V(x,\theta,\bar{\theta}) \to V'(x,\theta,\bar{\theta}) = V(x,\theta,\bar{\theta}) - i[\Lambda(x,\theta,\bar{\theta}) - \Lambda^\dagger(x,\theta,\bar{\theta})], \quad (3.38)$$

where the parameter of the transformation Λ is a scalar superfield.

With the expressions for scalar superfields in (3.35) and (3.36), using $\Phi = i\Lambda$, we can show that under supergauge transformations the vector superfield components transform as:

$$\begin{aligned}
C(x) &\to C'(x) = C(x) + A(x) + A^*(x) && (3.39)\\
\phi(x) &\to \phi'(x) = \phi(x) + \sqrt{2}\psi(x) && (3.40)\\
M(x) &\to M'(x) = M(x) + F(x) && (3.41)\\
V_\mu(x) &\to V_\mu'(x) = V_\mu(x) + i\partial_\mu(A(x) - A^*(x)) && (3.42)\\
\lambda(x) &\to \lambda'(x) = \lambda(x) && (3.43)\\
D(x) &\to D'(x) = D(x). && (3.44)
\end{aligned}$$

If we look at the transformation this implies for the vector field, this is equivalent to the ordinary abelian gauge transformation for a vector field $V_\mu(x) \to V_\mu'(x) = V_\mu(x) + \partial_\mu g(x)$, with the gauge parameter given by the scalar component field of Λ, $g(x) = i[A(x) - A^*(x)] = -2\,\mathrm{Im}\,(A(x))$. It then immediately follows that the standard field strength for a vector field, $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, is supergauge invariant.

If we demand that our theory is invariant under these gauge transformations we can choose the component fields of Λ in order to eliminate some the remaining reducibility in the representation.

Definition: The **Wess-Zumiono (WZ) gauge** is a supergauge transformation of a vector superfield by a scalar superfield with

$$\begin{aligned}
\psi(x) &= -\frac{1}{\sqrt{2}}\phi(x), && (3.45)\\
F(x) &= -M(x), && (3.46)\\
A(x) + A^*(x) &= 2\,\mathrm{Re}\,(A(x)) = -C(x). && (3.47)
\end{aligned}$$

A vector superfield in the WZ-gauge can then be written:

$$V_{WZ}(x,\theta,\bar{\theta}) = (\theta\sigma^\mu\bar{\theta})[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))] + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}D(x). \quad (3.48)$$

Again, the equations of motion will eliminate the auxiliary D-field, as well as one d.o.f. from the gauge field leaving the two d.o.f. of a massless on-shell gauge boson as found in the Standard Model,[16] and two d.o.f. from the (Majorana) fermion formed by λ, usually called the **gaugino** partner of the gauge boson. This does contain the correct number of degrees of freedom that corresponds to the representation $j = \frac{1}{2}$, however, for the massless $m = 0$ representation.[17]

Notice that the WZ gauge is particularly convenient for calculations because:

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))][V^\mu(x) + i\partial^\mu(A(x) - A^*(x))], \quad (3.49)$$

and, since multiplying in any θ or $\bar{\theta}$ into V_{WZ}^2 will then yield zero, we have

$$V_{WZ}^3 = 0,$$

so that

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2.$$

Unfortunately, supersymmetry transformations break the Wess-Zumiono gauge, meaning that a vector superfield in the WZ-gauge will no longer be in the WZ-gauge after a supersymmetry transformation.

3.7 Exercises

Exercise 3.1

Show that

$$\int d^2\theta\, \theta\theta = 1.$$

Exercise 3.2

Check that the explicit differential forms of the generators in Eqs. (3.4)–(3.6) fulfil the superalgebra in Eqs. (2.44)–(2.42).

Exercise 3.3

Demonstrate the correctness of the general expression for the left-handed scalar superfield in Eq. (3.35). *Hint:* You may have use for the spinor identities

$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta},$$

$$\theta\partial_\mu\psi\theta\sigma^\mu\bar{\theta} = \frac{1}{2}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta}.$$

[16]Hang on, where did that last d.o.f. go from $V(x)$? We have a remaining gauge freedom in the choice of the imaginary component of $A(x)$, which is the ordinary gauge freedom of a $U(1)$ field theory. This can be used to eliminate one d.o.f. from the vector field.

[17]The vector bosons will get mass through electroweak symmetry breaking just as in the Standard Model.

Exercise 3.4

Show the vector superfield supergauge transformation properties for the component fields. *Hint:* Use the field redefinitions:

$$\lambda(x) \to \lambda(x) - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\phi}(x),$$

$$D(x) \to D(x) - \frac{1}{4}\Box C(x).$$

Exercise 3.5

Derive the expression for V_{WZ}^2 in Eq. (3.49).

Chapter 4

Construction of a low-energy supersymmetric Lagrangian

We would now like to construct models, in the form of a field theory Lagrangians, that are invariant under supersymmetry transformations, much in the same way that the Standard Model Lagrangian is invariant under Poincaré transformations. However, just as for ordinary quantum field theory Lagrangians we will need to be able to impose gauge invariance for a choice of gauge group, and we want to limit the models to models that are *renormalisable*, *i.e.* models where any infinities can be cancelled by a finite number of counter-terms. Along the way we will also find a way to deal with the problem of equal masses for the particles in an irreducible representation – in other words, where are all the supersymmetric partners of the Standard Model particles? – taking inspiration from the Standard Model Higgs mechanism using spontaneous symmetry breaking.

4.1 Supersymmetry invariant Lagrangians and actions

As should be well known the relativistic field theory **action**

$$S \equiv \int_R d^4x\, \mathcal{L}, \tag{4.1}$$

is invariant under supersymmetry transformations if this transforms the Lagrangian by a total derivative term $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial^\mu f(x)$, where $f(x) \to 0$ on the surface $S(R)$ of the integration region R. The question then becomes: how can we construct a Lagrangian from superfields with this property?

An infinitesimal **supersymmetry transformation** of a function on superspace $F(x, \theta, \bar\theta)$ can be written as

$$F'(x,\theta,\bar\theta) = \exp(i\alpha Q + i\bar\alpha\bar Q)F(x,\theta,\bar\theta) \simeq F(x,\theta,\bar\theta) + (i\alpha Q + i\bar\alpha\bar Q)F(x,\theta,\bar\theta), \tag{4.2}$$

where α^A and $\bar\alpha_{\dot A}$ are the infinitesimal parameters of the transformation. If these parameters are constant then we say that this is a **global** supersymmetry transformation which is what we are (mostly) going to concern ourselves with in these notes. Replacing $\alpha \to \alpha(x)$ gives a **local** supersymmetry transformation.[1] Here we have ignored the action of the P_μ operators

[1] Allowing for local supersymmetry transformations leads to a gauge theory of supersymmetry called supergravity, which can incorporate gravity in a natural way.

since they commute with all the Qs, so that this part of the group acts independently, and only results in a translation of the Minkowski coordinate. The total change in the function under the supersymmetry transformation is $\delta_s F = F' - F$, where

$$\delta_s = i(\alpha Q + \bar{\alpha}\bar{Q}). \tag{4.3}$$

We can show that the highest order component fields in θ and $\bar{\theta}$ of a superfield always transform as a total derivative under δ_s, *e.g.* for the general superfield the highest order component field $d(x)$ transforms as[2]

$$\delta_s d(x) = d'(x) - d(x) = \frac{i}{2}(\partial_\mu \psi(x)\sigma^\mu \bar{\alpha} - \partial_\mu \bar{\lambda}(x)\sigma^\mu \alpha), \tag{4.4}$$

and the same is naturally true for the D component-field of a vector superfield. For a scalar superfield it is the F-field which has this property:

$$\delta_s F(x) = -i\sqrt{2}\partial_\mu \psi(x)\sigma^\mu \bar{\alpha}. \tag{4.5}$$

These highest power θ-components can be isolated by using the projection property of integration in Grassmann calculus writing

$$S = \int_R d^4x \int d^4\theta \, \mathcal{L},$$

where the Lagrangian density $\mathcal{L}$, which is a function of superfields, is guaranteed to give a supersymmetry invariant action. Note that this constitutes a redefinition of what we mean by the Lagrangian $\mathcal{L}$ when working with superfields. We should in particular be careful when counting the dimension of terms. We have, see Section 3.2, $[\theta] = M^{-1/2}$, which, since $\int d\theta\theta = 1$, leads to $[\int d\theta] = M^{1/2}$. We then have $[\int d^4\theta] = M^2$. Since we must have $[\int d^4\theta \, \mathcal{L}] = M^4$ for the action to be dimensionless, we need $[\mathcal{L}] = M^2$, which is different from the standard field theory Lagrangian with dimension M^4.

We can now write down the generic form for a supersymmetric Lagrangian consisting of scalar superfields Φ_i, where the indices indicate the highest power of θ in the term:

$$\mathcal{L} = \mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}}(\Phi_i, \Phi_i^\dagger) + \bar{\theta}\bar{\theta}\mathcal{L}_{\theta\theta}(\Phi_i) + \theta\theta\mathcal{L}_{\bar{\theta}\bar{\theta}}(\Phi_i^\dagger).$$

Here $\mathcal{L}_{\theta\theta}$ ($\mathcal{L}_{\bar{\theta}\bar{\theta}}$) is a function of *only* left-handed (right-handed) scalar superfields where we project out the F-field by multiplying by $\bar{\theta}\bar{\theta}$ ($\theta\theta$) and integrating over all the θ. The function form is limited to **holonomic functions** which means that the term is itself a scalar superfield. These two terms form what we call the **superpotential**. Meanwhile, the $\mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}}$ term is a real valued function of the scalar superfields where we project out the d-field, called the **Kähler potential**. Possible terms include $\Phi_i^\dagger\Phi_i$, but not $\Phi_i + \Phi_i^\dagger$, since this would belong in the superpotential.

The requirement of renormalisability of the resulting quantum field theory puts further restrictions on the fields in $\mathcal{L}$. We can at most have three factors of scalar superfields in each term of the superpotential, and two factors of scalar superfields in the Kähler potentia, for

[2] A word of warning: we really have to be slightly more careful here with the supersymmetry transformation since the superfields are operator valued functions acting on a state-space. A unitary transformation $U|\psi\rangle$ of the state implies a unitary transformation $U^\dagger A U$ of an operator A on that state.

details see *e.g.* Wess & Bagger [9].[3] Since the total action must be real, the (almost) most general supersymmetry Lagrangian that can be written in terms of scalar superfields is:

$$\mathcal{L} = \Phi_i^\dagger \Phi_i + \bar\theta\bar\theta W[\Phi_i] + \theta\theta W[\Phi_i^\dagger].$$

Here the first term is called the **kinetic term**,[4] and W is the symbol for the **superpotential**, which by renormalisability is restricted to

$$W[\Phi] = g_i\Phi_i + m_{ij}\Phi_i\Phi_j + \lambda_{ijk}\Phi_i\Phi_j\Phi_k. \tag{4.6}$$

This means that to specify a supersymmetric Lagrangian we only need to specify the superfield content of the model and the form of the superpotential.

Dimension counting, starting from $[\mathcal{L}] = M^2$ and $[\Phi] = 1$, means that the superpotential terms must each be M^3, for the couplings this gives $[g_i] = M^2$, $[m_{ij}] = M$ and $[\lambda_{ijk}] = 1$. Notice also that m_{ij} and λ_{ijk} must be symmetric since the superfields commute.

4.2 Abelian gauge theories

We will of course require not only a supersymmetry invariant Lagrangian, but also a gauge invariant Lagrangian, and we start with the requirements for an abelian gauge group.

Let us first look at the transformation of the superpotential W under the gauge transformation of the scalar superfields in (3.37):

$$W[\Phi] \to W[\Phi'] = t_i e^{iq_i\Lambda}\Phi_i + m_{ij}e^{i(q_i+q_j)\Lambda}\Phi_i\Phi_j + \lambda_{ijk}e^{i(q_i+q_j+q_k)\Lambda}\Phi_i\Phi_j\Phi_k.$$

Requiring gauge invariance, $W[\Phi] = W[\Phi']$, we must have:

$$\begin{aligned} q_i &= 0 \;\text{ or }\; t_i = 0, && (4.7)\\ q_i + q_j &= 0 \;\text{ or }\; m_{ij} = 0, && (4.8)\\ q_i + q_j + q_k &= 0 \;\text{ or }\; \lambda_{ijk} = 0. && (4.9) \end{aligned}$$

This puts great restrictions on the form of the superpotential and the charge assignments of the superfields, as in ordinary gauge theories.

What then about the kinetic term? This transforms as

$$\Phi_i^\dagger\Phi_i \to \Phi_i^\dagger e^{-iq_i\Lambda^\dagger}e^{iq_i\Lambda}\Phi_i = e^{iq_i(\Lambda-\Lambda^\dagger)}\Phi_i^\dagger\Phi_i.$$

As in ordinary gauge theories we can introduce a **gauge compensating vector (super)field** V with the appropriate gauge transformation to make the kinetic term invariant under supersymmetry transformations. We write the kinetic term instead as $\Phi_i^\dagger e^{q_iV}\Phi_i$, which gives us an invariant term:[5]

$$\Phi_i^\dagger e^{q_iV}\Phi_i \to \Phi_i^\dagger e^{-iq_i\Lambda^\dagger}e^{q_i(V-i\Lambda+i\Lambda^\dagger)}e^{iq_i\Lambda}\Phi_i = \Phi_i^\dagger e^{q_iV}\Phi_i.$$

[3] The argument is similar to ordinary quantum field theory, terms with couplings with negative mass dimension is forbidden.

[4] The constant in front of this term can always be chosen to be one because we can rescale the whole Lagrangian. It is possible to start from more general Kähler potentials, but this is mostly beyond the scope of these notes, although we will return to modify this term to account for gauge invariance.

[5] In case you were worried: we can use the WZ gauge to show that the new kinetic term $\Phi_i^\dagger e^{q_iV}\Phi_i$ has no term with dimension higher then four, and is thus renormalisable.

This definition of gauge transformation can be shown to recover the Standard Model **minimal coupling** for the component fields through the covariant derivative

$$D^i_\mu = \partial_\mu - \frac{i}{2} q_i V_\mu,$$

where V_μ is the vector component field of the vector superfield. The factor of a half here is admittedly a little odd, but is in fact related to how we (in these notes) defined the supergauge transformations. We can now simply redefine the charge, or go back and fiddle with the numerical factor in front of the vector superfields in the kinetic terms. Your choice!

4.3 Non-Abelian gauge theories

How do we extend the above to deal with the much more complicated non-abelian gauge theories? Let us take a gauge group G with the Lie algebra of group generators t_a

$$[t_a, t_b] = i f_{ab}{}^c t_c, \tag{4.10}$$

where $f_{ab}{}^c$ are the structure constants. We recap that for an element g in the group G we want to write down a unitary representation $U(g)$ that transforms a vector Ψ in the representation by $\Psi \to \Psi' = U(g)\Psi$.[6] With an exponential map we can write this representation as $U(g) = e^{i\lambda^a t_a}$, where λ^a are the parameters and t_a are hermitian operators, as you may perhaps have expected.

As in ordinary gauge theories, we simply copy the structure of the abelian (super)gauge transformation, and transform (a vector of) scalar superfields Φ under a non-abelian group as

$$\Phi \to \Phi' = e^{iq\Lambda^a T_a}\Phi,$$

in a non-abelian supergauge transformation. Here q is the charge of Φ under G. At this point we need to choose a particular representation, reflected in a particular choice for the generators which we write as T_a. Since for gauge groups we are almost exclusively interested in groups defined by a matrix representation, $U(g)$ will be a matrix with dimension fixed by the dimension chosen for the representation. Again we can easily show that we must require that the Λ^a has the defining property of a left-handed scalar superfield for Φ to transform to a left-handed scalar superfield.

For the superpotential to be gauge invariant we must now have:

$$t_i U_{ir} = t_r \quad \text{or} \quad t_i = 0, \tag{4.11}$$

$$m_{ij} U_{ir} U_{js} = m_{rs} \quad \text{or} \quad m_{ij} = 0, \tag{4.12}$$

$$\lambda_{ijk} U_{ir} U_{js} U_{kt} = \lambda_{rst} \quad \text{or} \quad \lambda_{ijk} = 0, \tag{4.13}$$

where the indices on U are its matrix indices.

We also want a similar construction for the kinetic terms as for abelian gauge theories, $\Phi^\dagger e^{V^a T_a}\Phi$, to be invariant under non-abelian gauge transformations. Now, using that the generators T_a are hermitian,

$$\Phi^\dagger e^{qV^a T_a}\Phi \to \Phi'^\dagger e^{qV'^a T_a}\Phi' = \Phi^\dagger e^{-iq\Lambda^{a\dagger} T_a} e^{qV'^a T_a} e^{iq\Lambda^a T_a}\Phi,$$

[6] Of, course, you may ask, how do we even know that we can find a unitary representation for a particular Lie group? For the unitary group this is by definition, but is it possible in general? It turns out that this is always true for **compact Lie groups** which includes many of the matrix groups $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Sp(2n)$, and $Sp(n)$.

so, in order to have gauge invariance, we have to require that the vector superfield V transforms as:[7]

$$e^{qV'^aT_a} = e^{iq\Lambda^{a\dagger}T_a}e^{qV^aT_a}e^{-iq\Lambda^aT_a}. \tag{4.14}$$

Using the Baker-Campbell-Hausdorff formula in (1.12) we can write this as

$$V'^aT_a = V^aT_a - i(\Lambda^a - \Lambda^{a\dagger})T_a - \frac{i}{2}q[V^aT_a, (\Lambda^b + \Lambda^{b\dagger})T_b] + \mathcal{O}(\Lambda^2),$$

where the higher order terms all contain the commutator

$$[V^aT_a, (\Lambda^b + \Lambda^{b\dagger})T_b] = V^a(\Lambda^b + \Lambda^{b\dagger})[T_a, T_b] = V^a(\Lambda^b + \Lambda^{b\dagger})if_{ab}{}^cT_c,$$

so that the vector superfield transforms as

$$V'^a = V^a - i(\Lambda^a - \Lambda^{a\dagger}) - \frac{1}{2}qf_{bc}{}^aV^b(\Lambda^{c\dagger} + \Lambda^c) + \mathcal{O}(\Lambda^2). \tag{4.15}$$

This sensibly reduces to the definition for abelian groups in (3.38) since all the higher order terms contain the structure constant $f_{bc}{}^a$, which is zero for abelian groups. If we look at the component vector fields of V^a, V^a_μ, these transform just like in a standard non-abelian gauge theory:

$$V^a_\mu \to V'^a_\mu = V^a_\mu + i\partial_\mu(A^a - A^{a*}) - qf_{bc}{}^aV^b_\mu(A^c - A^{c*}),$$

in the adjoint representation of the gauge group.

The supergauge transformations of vector superfields can be written more efficiently in a representation independent way as

$$e^{V'} = e^{i\Lambda^\dagger}e^Ve^{-i\Lambda},$$

where we have defined matrix superfields $\Lambda \equiv q\Lambda^aT_a$ and $V \equiv qV^aT_a$, and the inverse transformation is then given by

$$e^{-V'} = e^{i\Lambda}e^{-V}e^{-i\Lambda^\dagger},$$

such that $e^Ve^{-V} = e^{V'}e^{-V'} = 1$.[8]

4.4 Supersymmetric field strength

There is one type of term missing from the supersymmetric Lagrangian we are constructing, namely field strength terms that make the gauge fields dynamical, *e.g.* terms to describe the electromagnetic field strength term $\mathcal{L} \sim -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ in QED.

Definition: The **supersymmetric field strength** for a vector superfield V is defined by the spinor matrix scalar superfields W_A and $\bar{W}_{\dot{A}}$ given by

$$W_A \equiv -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_Ae^V, \tag{4.16}$$

$$\bar{W}_{\dot{A}} \equiv -\frac{1}{4}DDe^{-V}\bar{D}_{\dot{A}}e^V, \tag{4.17}$$

where again $V \equiv qV^aT_a$.

[7] This is independent of our choice of representation for the gauge group for the supergauge transformation.

[8] Notice that despite the non-commutative nature of the matrices involved, the identity $e^Ae^{-A} = 1$ holds. See Exercise 1.26.

For an abelian gauge field this definition reduces to

$$\begin{aligned} W_A &= -\frac{1}{4}\bar{D}\bar{D}D_A V, \\ \bar{W}_{\dot{A}} &= -\frac{1}{4}DD\bar{D}_{\dot{A}} V, \end{aligned}$$

where V is simply the vector superfield of the gauge group.

We can show that the components of W_A ($\bar{W}_{\dot{A}}$) are left-handed (right-handed) scalar superfields, and that both $\mathrm{Tr}[W^A W_A]$ and $\mathrm{Tr}[\bar{W}_{\dot{A}}\bar{W}^{\dot{A}}]$ are supergauge invariant constructions, and thus potential terms in the supersymmetry Lagrangian. Firstly,

$$\bar{D}_{\dot{A}} W_A = -\frac{1}{4}\bar{D}_{\dot{A}}\bar{D}\bar{D}e^{-V}D_A e^V = 0,$$

because from Eq. (3.17), $\bar{D}^3 = 0$, and similarly for $\bar{W}_{\dot{A}}$, showing that they are both scalar superfields of their particular type, and as such can be used to form a supersymmetric Lagrangian.

Under a supergauge transformation we have:

$$\begin{aligned} W_A \to W'_A &= -\frac{1}{4}\bar{D}\bar{D}e^{i\Lambda}e^{-V}e^{-i\Lambda^\dagger}D_A e^{i\Lambda^\dagger}e^V e^{-i\Lambda} \\ (\bar{D}_{\dot{A}}\Lambda = 0) \qquad &= -\frac{1}{4}e^{i\Lambda}\bar{D}\bar{D}e^{-V}e^{-i\Lambda^\dagger}D_A e^{i\Lambda^\dagger}e^V e^{-i\Lambda} \\ (D_A\Lambda^\dagger = 0) \qquad &= -\frac{1}{4}e^{i\Lambda}\bar{D}\bar{D}e^{-V}D_A e^V e^{-i\Lambda} \\ &= -\frac{1}{4}e^{i\Lambda}\bar{D}\bar{D}e^{-V}[(D_A e^V)e^{-i\Lambda} + e^V(D_A e^{-i\Lambda})] \\ &= e^{i\Lambda}W_A e^{-i\Lambda} - \frac{1}{4}e^{i\Lambda}\bar{D}\bar{D}D_A e^{-i\Lambda}. \end{aligned} \tag{4.18}$$

We are free to add zero to (4.18) in the form of $-\frac{1}{4}e^{i\Lambda}\bar{D}D_A\bar{D}e^{-i\Lambda} = 0$,[9] giving

$$\begin{aligned} W'_A &= e^{i\Lambda}W_A e^{-i\Lambda} - \frac{1}{4}e^{i\Lambda}\bar{D}\{\bar{D}, D_A\}e^{-i\Lambda} \\ &= e^{i\Lambda}W_A e^{-i\Lambda} + \frac{1}{2}e^{i\Lambda}\bar{D}_{\dot{A}}\sigma^\mu{}_{A\dot{B}}\epsilon^{\dot{A}\dot{B}}P_\mu e^{-i\Lambda} \\ &= e^{i\Lambda}W_A e^{-i\Lambda}, \end{aligned}$$

where we have used Eq. (3.16) to replace the anti-commutator. This means that the following trace is gauge invariant:

$$\begin{aligned} \mathrm{Tr}[W'^A W'_A] &= \mathrm{Tr}[e^{i\Lambda}W^A e^{-i\Lambda}e^{i\Lambda}W_A e^{-i\Lambda}] \\ &= \mathrm{Tr}[e^{-i\Lambda}e^{i\Lambda}W^A W_A] = \mathrm{Tr}[W^A W_A], \end{aligned}$$

and can be used in the Lagrangian.

If we expand W_A in the component fields we find, as we might have hoped, that it contains the ordinary field strength tensor:

$$F^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + q f_{bc}{}^a V^b_\mu V^c_\mu$$

and that the trace indeed contains terms with $F^a_{\mu\nu}F^{\mu\nu a}$.

[9] Which is zero because again Λ is a left-handed scalar superfield, so $\bar{D}_{\dot{A}}\Lambda = 0$.

4.5 The (almost) complete supersymmetric Lagrangian

We can compile all our results up to now to write down the complete Lagrangian for a supersymmetric theory with (possibly) non-abelian gauge symmetries:[10]

$$\mathcal{L} = \Phi^\dagger e^V \Phi + \delta^2(\bar{\theta}) W[\Phi] + \delta^2(\theta) W[\Phi^\dagger] + \frac{1}{2T(R)q^2}\delta^2(\bar{\theta})\,\mathrm{Tr}[W^A W_A]. \tag{4.19}$$

Here, $T(R)$ given by $T(R)\delta_{ab} = \mathrm{Tr}[T_a T_b]$ is called the **Dynkin index** of the representation R of the gauge group using the generators T_a. This number is representation dependent, and deeply connected to the corresponding eigenvalues of the Casimir operators belonging to the irrep, but independent of which generator we choose to calculate it from.

The Dynkin index appears in the field strength term to correctly normalise the energy density for the chosen representation R of the gauge group. To see that this factor cancels in a natural way, note that since the matrix structure of W_A is spanned by T_a for a given representation, we can write $W_A = qW_A^a T_a$, where W_A^a are spinor superfields, and where we have taken out a common factor of the charge q that appears in all terms of W_A. Then

$$\mathrm{Tr}[W^A W_A] = q^2 W^{aA} W_A^b \mathrm{Tr}[T_a T_b] = q^2 W^{Aa} W_A^b \delta_{ab} T(R) = q^2 T(R) W^{aA} W_A^a. \tag{4.20}$$

In addition to the above, it is also possible to add a pure vector superfield term as part of the Kähler potential, which is not constructed from scalar superfields, of the form $\mathcal{L}_{FI} \sim -kV$ where V is the vector superfield and k some constant. This kind of term is called a **Fayet–Iliopoulos term**. However, this is not possible for non-abelian gauge groups since a term $-kV^a$ could not be supergauge invariant.[11]

4.6 Finding the equations of motion

To find the equations of motion from our supersymmetric Lagrangian construction we can now perform the integration over superspace coordinates θ and $\bar{\theta}$ in the action and then apply the standard Euler–Lagrange field equations that minimise the action for the generic component fields ϕ_i of the Lagrangian $\mathcal{L}$ after the superspace integration:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0. \tag{4.21}$$

We will now discuss some of the general properties of this solution without specifying field content or gauge groups for the Lagrangian.

As already intimated the auxiliary component fields with unusual mass dimension, F_i and D^a, from scalar Φ_i and vector V^a superfields, respectively, will be eliminated.[12] Starting with

[10]There is no hermitian conjugate of the field strength term, and a slightly odd normalisation using 1/2 instead of 1/4. This is because the term can be proven to be real, although this is sometimes overlooked in the literature when authors instead use

$$\mathcal{L} \sim \frac{1}{4T(R)q^2}\delta^2(\bar{\theta})\mathrm{Tr}[W^A W_A] + \frac{1}{4T(R)q^2}\delta^2(\theta)\mathrm{Tr}[\bar{W}_{\dot{A}} \bar{W}^{\dot{A}}],$$

which admittedly looks more symmetric with respect to the superpotential part.

[11]Why is the abelian case supergauge invariant then? Well, the superspace integration in the action will isolate the D-term from V, and this term is itself invariant under abelian supergauge transformations, see Eq. (3.44). For non-abelian supergauge transformations the various V^a fields mix as seen in (4.15).

[12]In the following we use the notation in Eqs. (3.35) and (3.48).

the F_i-fields these occur only in the kinetic terms and the superpotential. From the kinetic term $\Phi_i^\dagger e^V \Phi_i$ the contribution is $|F_i|^2$ as this is the only term with F_i that can survive the θ-integration if we write the vector superfields in the Wess-Zumino gauge. In the superpotential $W[\Phi_i]$ only the terms where a number of scalar fields A_j with no θ-coordinate multiply F_i can survive the integration. The derivatives of these terms with respect to F_i can be expressed in terms of the superpotential functional as

$$W_i \equiv \frac{\partial W[A_1, ..., A_n]}{\partial A_i}, \tag{4.22}$$

where the scalar superfields have been replaced by their scalar component fields. This means that the Euler–Lagrange equation for F_i^* is

$$\frac{\partial \mathcal{L}}{\partial F_i^*} = F_i + W_i^* = 0,$$

which is used to eliminate F_i.

For a concrete example of how this can be worked out to arrive at a complete action integral for a single scalar superfield, see Exercise 4.1. Generalising this to an expression for the action for any number of scalar superfields Φ_i in terms of their component fields, ignoring gauge interactions, gives:

$$S = \int d^4x \{ i\partial_\mu \bar{\psi}_i \sigma^\mu \psi_i - A_i^* \Box A_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} W_{ij}^* \bar{\psi}_i \bar{\psi}_j - |W_i|^2 \}$$

where W_{ij}, given by

$$W_{ij} \equiv \frac{\partial^2 W[A_1, ..., A_n]}{\partial A_i \partial A_j}, \tag{4.23}$$

is called the **fermionic mass matrix**.

The D^a-fields occur in the kinetic terms and the field strength terms. In the kinetic term, in the Wess–Zumino gauge, it is relatively easy to see that D^a appears only once in the expansion of the exponential as $qA_i^* T_{ij}^a A_j D^a$ in the action. In the field strength term one can show with some more effort that the contribution is $D^a D^a$ in the Wess–Zumino gauge. For abelian gauge fields there is also a possible kD contribution from the Fayet–Iliopoulos term. This leads to the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial D^a} = qA_i^* T_{ij}^a A_j + 2D^a = 0,$$

which can be used to eliminate D^a.

Note that the terms discussed here with F- and D-fields exhaust all the non-derivative terms that only have scalar fields since no terms with only A-fields can survive the θ-integration. Thus the complete non-derivative scalar field contribution to the Lagrangian, called the **scalar potential**, can be written as

$$V(A_i, A_i^*) = \sum_i |F_i|^2 + \sum_a D^a D^a = \sum_i \left| \frac{\partial W[A_1, ..., A_n]}{\partial A_i} \right|^2 + \frac{1}{4} \sum_a q^2 (A_i^* T_{ij}^a A_j)^2. \tag{4.24}$$

4.7 R-symmetry

The Lagrangian we have constructed is invariant under both internal gauge transformations and external (coordinate) super-Poincaré transformations, where the latter has been built in through the superspace coordinates. The Minkowski part of superspace, the four-vector coordinates, transform under the Lorentz group, and since all our Lagrangian ingredients are Lorentz scalars they are invariant. We can now ask the question if there is any transformation of the superspace coordinates θ and $\bar{\theta}$ that the Lagrangian is also invariant under.

In Sec. 2.6 we claimed there there is no nontrivial interaction between a gauge group and the supersymmetry generators for $N = 1$ supersymmetry. This is not entirely true as we may have non-trivial commutators for the generator R of an abelian group, *i.e.*

$$[Q_A, R] = Q_A, \quad [\bar{Q}_{\dot{A}}, R] = -\bar{Q}_{\dot{A}}. \tag{4.25}$$

Using this we define a $U(1)_R$ transformation called **R-symmetry** as

$$\theta_A \to \theta'_A = e^{i\alpha}\theta_A, \quad \bar{\theta}_{\dot{A}} \to \bar{\theta}'_{\dot{A}} = e^{-i\alpha}\bar{\theta}_{\dot{A}}, \tag{4.26}$$

where α is the parameter of the transformation and the charge of θ and $\bar{\theta}$ under the transformation is 1 and -1, respectively. If a Lagrangian is to be invariant under such a transformation then a superfield Φ must transform as

$$\Phi(x, \theta, \bar{\theta}) \to \Phi'(x, \theta, \bar{\theta}) = e^{ir_\Phi\alpha}\Phi(x, e^{-i\alpha}\theta, e^{i\alpha}\bar{\theta}), \tag{4.27}$$

where r_Φ is the charge of that superfield under the transformation. The charge of a product of two superfields is then just the sum of the charges of the fields. This is required so that the kinetic term in the Lagrangian is invariant under the transformation.

This means that for a scalar (left- or right-handed) superfield the component fields A, ψ, and F must transform as

$$A \to e^{ir_\Phi\alpha}A, \quad \psi \to e^{i(r_\Phi - 1)\alpha}\psi, \quad F \to e^{i(r_\Phi - 2)\alpha}F, \tag{4.28}$$

while vector superfields must have zero R-charge since they are real, and their component fields, in the Wess-Zumino gauge must thus transform as

$$V_\mu \to V_\mu, \quad \lambda \to e^{i\alpha}\lambda, \quad D \to D. \tag{4.29}$$

We can check that all the terms in the Lagrangian (4.19) are invariant under this transformation, with the exception of the superpotential W, which must have total R-charge 2. This will exclude most allowed charge assignments for the the scalar superfields, and in the minimal supersymmetruc models we shall look at there will be no viable continuous R-symmetry. However, R-symmetry has importance in building models of supersymmetry breaking, and could remain as a broken symmetry in the model.

4.8 The hierarchy problem

There is a fundamental problem with having scalar particles in a quantum field theory. Let us take the Standard Model Higgs boson h as an example, however, the following would be

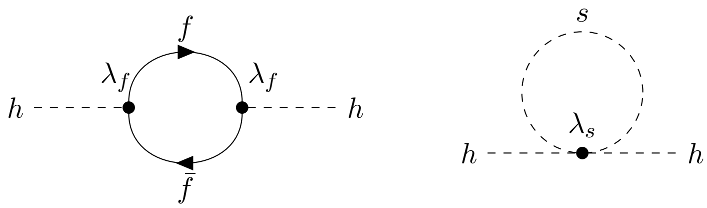

Figure 4.1: Higgs self-energy diagrams with fermions f (left) and scalars s (right). The dots indicate vertices with the given coupling strength.

true for any scalar particle. At tree level its behaviour is controlled by the Standard Model scalar potential

$$V(h) = -\mu^2|h|^2 + \lambda|h|^4, \tag{4.30}$$

where μ is the tree-level mass. and λ is its self-coupling. If we naively calculate loop-corrections to its mass in self-energy diagrams like the ones shown in Fig. 4.1, where f is a fermion and s some other scalar, they both diverge due to their loop momenta integrals, meaning they are infinite. This then needs what is called **regularisation** in order to yield a finite answer.

There are different ways of achieving regularisation. Since we know that the Standard Model is an incomplete theory, at least when we go up to Planck scale energies where we need an unknown quantum theory of gravity, we can introduce a **cut-off regularisation** where we limit the integral in the loop-correction to momenta below a scale Λ_{UV}. Then the loop-correction to the Higgs mass is, at leading order in Λ_{UV},

$$\Delta m_h^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + \frac{\lambda_s}{16\pi^2}\Lambda_{UV}^2 + \ldots \tag{4.31}$$

where λ_f and λ_s are the couplings of f and s to the Higgs, respectively, from the Lagrangian interaction terms $\mathcal{L} \sim \lambda_f\bar{\psi}\psi h + \text{h.c.} + \lambda_s|s|^2|h|^2$. The dots represents terms that are less than quadratically divergent in terms of the cut-off scale, the first missing terms being logarithmically divergent. Picking Λ_{UV} suggestively as the Planck scale, $\Lambda_{UV} = M_P = 2.44 \times 10^{18}$ GeV, meaning that the quantum corrections to the Higgs mass are some 15 orders of magnitude greater than its measured value.

We observe that the difference in sign between the fermions and bosons in (4.31) means that it could *in principle* be possible that these huge contributions cancel to keep $m_h \sim 125\,\text{GeV}$ as measured,[13] however, as physicists we should worry why the Universe seems to be organised in such a strange way without any specific cause. This is known as the **Higgs fine tuning problem**, or the scale **hierarchy problem** because it is fundamentally a problem of why the electroweak scale at around 100 GeV where the Higgs lives, is so much lighter than the Planck scale of 10^{18} GeV, when the former should in principle be pulled up to the latter by the sensitivity of scalars to loop corrections.[14]

[13]Could we not fix this by introducing a very large Lagrangian Higgs mass μ? Not really, in that the tree-level Higgs mass is constrained by the properties of the other electroweak masses, *i.e.* the Z and W-bosons.

[14]With some background in renormalisation you may ask: What about choosing dimensional regularisation instead where there is no cut-off scale? That would in principle work, however, as soon as you introduce *any* new particle (significantly) heavier than the Higgs there is still a quadratic correction of the form of (4.31) with the new particle masses as the scale, meaning that we cannot complete the Standard Model at a higher scale without reintroducing the problem.

Enter supersymmetry to the rescue: with unbroken supersymmetry we find that we automatically have $|\lambda_f|^2 = \lambda_s$ and exactly twice as many scalar as fermion degrees of freedom running around in loops. This provides a magical exact cancellation of the quadratic divergence in Eq. (4.31). To see that this relation between the couplings holds, notice that it is the superpotential term $W \sim \lambda_{ijk}\Phi_i\Phi_j\Phi_k$ which is responsible for generating Lagrangian Yukawa terms of the form $\lambda_{ijk}\psi_i\psi_j A_k$ that couple two fermions to a scalar A_k. If the superfield Φ_k is the one that contains the Higgs boson this means that A_k is the Higgs boson field h, and it couples to the right-handed Weyl-spinors ψ_i and ψ_j in the superfields Φ_i and Φ_j as $\lambda_{ijk}\psi_i\psi_j h$, with the coupling λ_{ijk}. From the superpotential with right-handed scalar superfields the same term appears with right-handed Weyl-spinors $\lambda_{ijk}\bar{\psi}_i\bar{\psi}_j h^*$. These two terms can be combined to the same interaction term with a single Dirac fermion ψ, $\lambda_{ijk}\bar{\psi}\psi h$, as in the Standard Model, giving $\lambda_f = \lambda_{ijk}$. There are no other such terms coupling two fermions to a scalar in the whole Lagrangian.

At the same time, in the scalar potential (4.24), that same superpotential term is responsible for the terms

$$V(A, A^*) \sim \left|\frac{\partial W}{\partial A_i}\right|^2 = |\lambda_{ijk}|^2 A_j^* A_k^* A_j A_k. \tag{4.32}$$

If A_j is the second scalar s in Fig. 4.1, then this term becomes $|\lambda_{ijk}|^2|s|^2|h|^2$ and we have $\lambda_s = |\lambda_{ijk}|^2$. As a result $\lambda_f^2 = \lambda_s$.

The fact that there are loop contributions from two scalars for each fermion due to the same number of states (two scalar states per fermion state), means that unbroken supersymmetry predicts an exact cancellation in Eq. (4.31).[15] Notice that there is nothing here that is special about the Higgs boson, this mechanism will in fact protect all scalar particles from quadratic corrections to their mass in supersymmetric models.

4.9 Vacuum energy

To explain the measured accelerated expansion of the universe one can introduce a constant term Λ in Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - g_{\mu\nu}\Lambda = \frac{1}{M_P^2}T_{\mu\nu} \tag{4.33}$$

and this contribution has been coined **dark energy**. The measured value of dark energy in terms of an energy scale is $\Lambda_{DE} \simeq 2.4 \times 10^{-3}$ eV. To get some sense of this energy scale the rest mass of the lightest charged particle, the electron, is around 0.511 MeV, while the energy of a photon in visible light is of the order of 1 eV.

This energy can be interpreted as vacuum energy, *i.e.* the energy predicted from the contribution of Feynman diagrams with *no* external particles. However, the predicted value of the vacuum energy in the Standard Model is $\Lambda_{VE} \sim M_P$.[16] If we want to compare energy densities, we should compare Λ_{DE}^4 to Λ_{VE}^4,[17] which means that the prediction is off from the measurement by some 120 orders of magnitude, which is said to be a record for the greatest

[15] To keep the argument simple we have avoided the contributions from vector bosons, however, we can show that these also cancel exactly against the contributions from their fermionic partners.

[16] The origin of this is just the same as the quadratic divergence for the Higgs mass. It is the same type of diagrams contributing, only without external legs.

[17] In natural units the unit of energy density has units $[\rho] = [E/V] = M/L^3 = M^4$.

ever discrepancy between theory and experiment. This problem is the **hierarchy problem for vacuum energy**.

Can supersymmetry save us here as well? Sort of. For an unbroken global supersymmetry we can use the supersymmetry **non-renormalisation theorem**[18] of Grisaru, Roach and Siegel (1979) [10] to show that the prediction in supersymmetry is exactly $\Lambda_{VE} = 0$.

Theorem: **Non-renormalisation theorem**
All higher order contributions to the effective supersymmetric action S_{eff} for a process can be written:

$$S_{\text{eff}} = \sum_n \int d^4x_i ... d^4x_n d^4\theta \, F_1(x_1, \theta, \bar{\theta}) \times ... \times F_n(x_1, \theta, \bar{\theta}) \times G(x_1, ..., x_n), \quad (4.34)$$

where F_i are products of the n external superfields in the process and their covariant derivatives, and G is a supersymmetry invariant function.

While this is a rather technical statement, that fact that vacuum diagrams do not have external superfields means that the higher order contributions are all zero, and since there are no tree-level contributions for vacuum diagrams – you must have loops to not have external legs – there can be no contribution at all.

This was a victory for supersymmetry before the discovery of dark energy, when the problem was instead to prove that $\Lambda = 0$, modulo the fact that the breaking of supersymmetry, as we will be doing in Chapter 5, changes this prediction. As we shall see there, the scale of the contribution to the vacuum energy in broken supersymmetry has to be the mass scale of the supersymmetric particles, so with for example $m_{SUSY} \simeq 2\,\text{TeV}$ as this scale, we have $m_{SUSY}/\Lambda_{DE} \simeq 10^{15}$, some 15 orders of magnitude too large, which is better than the Standard Model prediction of $M_P/\Lambda_{DE} = 10^{30}$, but still a bit off the measured value.[19]

So now we are left with showing that the contribution is very small but non-zero, which is in general thought to be a much harder problem than finding models where it is exactly zero. However, in supergravity something interesting happens. Introducing a local supersymmetry the scalar potential is not simply given by the superpotential derivatives in (4.24), but instead is (ignoring the effects of gauge fields)

$$V(A, A^*) = e^{K/M_P^2} \left[K_{ij}^{-1}(D_iW)(D_jW^*) - \frac{3}{M_P^2}|W|^2 \right], \quad (4.35)$$

where $K_{ij} = \partial_i \partial_j K[A, A^*]$ is the **Kähler metric**, D_i the **Kähler derivative** $D_i = \partial_i + \frac{1}{M_P^2}(\partial_i K)$, and all the derivatives are with respect to the scalar fields in the Kähler potential K and superpotential W. In the $M_P \to \infty$ limit, the low energy limit, we see that we recover

[18]The name of this theorem is a bit funny. Why? Well, mainly because the result is not about not being able to renormalise the theory, but about about not *needing* to renormalise certain parts of it. Another consequence of the theorem, which gives it its name, is that the couplings of the superpotential do not need separate renormalisation. The renormalisation of the superfields suffices.

[19]So, one would be tempted to say that supersymmetry has solved half the problem. On a more serious note, there is a significant difference in that the Standard Model prediction requires a cut-off to be finite, while supersymmetry predicts a finite but too large value.

the flat space result of Eq. (4.24). What is important to notice is that there is now a second *negative* term in the potential that can in principle cancel the positive supersymetry breaking contribution, however, given the large size of the breaking contribution this will come at the price of fantastic fine-tuning unless some mechanism can be found where this is natural.

4.10 Excercises

Exercise 4.1

Write down the Lagrangian and find the action of the simplest possible supersymmetric field theory with a single scalar superfield, without gauge transformations, in terms of component fields, and show that it contains no kinetic terms for the F-field. Then show how the F-field can be eliminated by the equations of motion. *Hint:* The kinetic part of the action will turn out to be

$$S_{\text{kin}} = \int d^4x \left\{ -A^*(x)\Box A(x) + |F(x)|^2 + i\partial_\mu\psi(x)\bar{\sigma}^\mu\psi(x) \right\}. \tag{4.36}$$

To show this you may have use of the identities in Sec. 2.5.1.

Exercise 4.2

Show that the supersymmetric field strength term for an abelian gauge field can be written in terms of component fields as

$$\mathcal{L} = -i\lambda(x)\sigma^\mu\partial_\mu\bar{\lambda}(x) + 2D^2(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{i}{4}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x). \tag{4.49}$$

Try to find an argument why the last term with the dual field tensor $\tilde{F}^{\mu\nu}(x)$ can be ignored when finding the equations of motion. *Hints:* To get started it may be productive to consider the coordinate change $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. You will also likely need the following algebraic relationship

$$(\sigma^{\mu\nu}\theta)^A F_{\mu\nu}(\sigma^{\rho\sigma}\theta)_A F_{\rho\sigma} = -\frac{1}{2}\theta\theta[F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}\tilde{F}^{\mu\nu}]. \tag{4.50}$$

Exercise 4.3

Extend Exercise 1 to include a single abelian gauge group under which the scalar superfield has charge q. Simplify your answer using the covariant derivative $D_\mu \equiv \partial_\mu - i\frac{q}{2}V_\mu$. *Hint:* The spinor relationship

$$\theta\psi\bar{\theta}\bar{\psi}(\theta\sigma^\mu\bar{\theta}) = -\frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\bar{\psi}\bar{\sigma}^\mu\psi, \tag{4.58}$$

may also come in handy.

Exercise 4.4

Find the Lagrangian for supersymmetric QED (SQED) in terms of component fields, eliminating any non-dynamical auxiliary fields. Supersymmetric QED is the smallest (in terms of field content) supersymmetric theory that has the particles and interactions in QED as a subset.

Chapter 5

Breaking supersymmetry

In Chapter 2 we saw that supersymmetry predicts scalar partner particles with the same mass as the known fermions, and new fermions with the same mass as the known vectors, living in the same irreducible representation. These, somewhat unfortunately, contradict experiment by not existing. In this chapter we will look at how we can break supersymmetry so that the new states become more massive, and avoid current experimental bounds.

5.1 Spontaneous supersymmetry breaking

In the Standard Model we have already faced this problem: the vector bosons should remain massless under the gauge symmetry of the model since explicit gauge boson mass terms break the symmetry. Yet, some of them are observed to be very massive indeed. This is solved with the introduction of the Higgs mechanism and **spontaneous symmetry breaking** in the **scalar potential**.[1] The idea is that while there is an internal symmetry of the Lagrangian – in the Standard Model the gauge symmetry $SU(3)_c \times SU(2)_L \times U(1)_Y$ – this may not be a symmetry for the particular vacuum state of the theory (the lowest energy state), thereby allowing the properties of the vacuum to supply the masses. In the Standard Model this is achieved by the shape of the scalar potential having a minimum for a non-zero value of the Higgs field which means that the value of the field there – its **vacuum expectation value** (vev) – can contribute to the masses. Would it not be great if we could have a similar *spontaneous symmetry breaking* in order to break supersymmetry this way and boost the masses of supersymmetric particles beyond current limits?

To find the properties of the vacuum state for our supersymmetric models we start by pointing out that we can write the supersymmetric Hamiltonian as

$$H = \frac{1}{4}(Q_1\bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}}Q_1 + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2).$$

To see this, consider first

$$\begin{aligned} \{Q_A, \bar{Q}_{\dot{B}}\}\bar{\sigma}^{\nu\dot{B}A} &= 2\sigma^{\mu}{}_{A\dot{B}}\bar{\sigma}^{\nu\dot{B}A}P_\mu \\ &= 2\,\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu]P_\mu \\ &= 4g^{\mu\nu}P_\mu = 4P^\nu. \end{aligned}$$

[1] The potential of the Lagrangian are those terms not containing derivatives of the fields (kinetic terms). The scalar potential are such terms that contain only scalar fields.

Now,

$$\begin{aligned} H &= P^0 = \frac{1}{4}\{Q_A, \bar{Q}_{\dot{B}}\}\bar{\sigma}^{0\dot{B}A} \\ &= \frac{1}{4}(Q_1\bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}}Q_1 + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2). \end{aligned}$$

As discussed in Section 2.5 we have $Q_A^\dagger = \bar{Q}_{\dot{A}}$. Thus the Hamiltonian is **semipositive definite**, *i.e.* $\langle H\rangle = \langle\Psi|H|\Psi\rangle \geq 0$ for any state $|\Psi\rangle$, so the energy of any state in supersymmetry is always non-negative.

Imagine now that there exists some lowest lying state (or possibly a set of degenerate states), the ground state(s) $|0\rangle$, that has vanishing energy $\langle 0|H|0\rangle = 0$. These states are supersymmetric – meaning invariant under the supersymmetry transformation – since, to fulfil the energy assumption, we must have

$$Q_A|0\rangle = \bar{Q}_{\dot{A}}|0\rangle = 0, \quad \text{for all } A, \dot{A}, \tag{5.1}$$

and are thus invariant under the supersymmetry transformations given by (4.3)

$$\delta_S|0\rangle = (\alpha^A Q_A + \bar{\alpha}_{\dot{A}}\bar{Q}^{\dot{A}})|0\rangle = 0. \tag{5.2}$$

The vanishing energy means that at this supersymmetric minimum of the potential the scalar potential must contribute zero, $\langle V(A, A^*)\rangle = 0$, and thus, from Eq. (4.24),

$$\frac{\partial W}{\partial A_i}|0\rangle = 0 \quad \text{and} \quad A_i^* T^a_{ij} A_j|0\rangle = 0.$$

Conversely, if the scalar potential does contribute to the energy for the vacuum (ground state) $|0\rangle$, so that it does *not* have vanishing energy, meaning that either

$$\frac{\partial W}{\partial A_i}|0\rangle \neq 0 \quad \text{or} \quad A_i^* T^a_{ij} A_j|0\rangle \neq 0,$$

in the minimum of the potential for some A_i, then *supersymmetry must be broken.* As in the Standard Model, the Lagrangian is still (super)symmetric, but $|0\rangle$ is not because (5.1) can no longer hold for all the Qs.

The **O'Raifeartaigh model** (1975) [11] is an example of a model that spontaneously breaks supersymmetry with three scalar superfields X, Y, Z, and the superpotential

$$W = MYZ + gX(Z^2 - m^2), \tag{5.3}$$

where M, g and m are real non-zero parameters. The scalar potential is

$$\begin{aligned} V(A, A^*) &= \left|\frac{\partial W}{\partial A_X}\right|^2 + \left|\frac{\partial W}{\partial A_Y}\right|^2 + \left|\frac{\partial W}{\partial A_Z}\right|^2 \\ &= |g(A_Z^2 - m^2)|^2 + |MA_Z|^2 + |MA_Y + 2gA_XA_Z|^2, \end{aligned} \tag{5.4}$$

which can never be zero because setting $A_Z = 0$, which is needed for the second term, gives a non-zero contribution g^2m^4 from the first term. Since the expectation value at the minimum that breaks supersymmetry is $\langle 0|\frac{\partial W}{\partial A_i}|0\rangle$, and $F_i = -\frac{\partial W^*}{\partial A_i}$, the condition for spontaneous supersymmetry breaking with the O'Raifeartaigh mechanism can be written

$$\langle F_i\rangle \equiv \langle 0|F_i(x)|0\rangle > 0, \tag{5.5}$$

hence it is often given the more generic name **F-term breaking** outside of the specific O'Raifeartaigh model. In general F-term breaking it is the vacuum expectation value of the auxiliary field of a scalar superfield that supplies the breaking.

In a gauge theory, a similar mechanism is found by adding the Fayet-Iliopolous term $\mathcal{L}_{FI} \sim 2kV$ where V is an abelian vector superfield. The vev of the $D(x)$ auxiliary field, $\langle D\rangle = \langle 0|D(x)|0\rangle$, will then create a non-zero scalar potential.[2] This is called the **Fayet-Iliopolous model**, or **D-term breaking**.

5.2 Supertrace

Unfortunately, the above does not work in practice if all particles are at a low energy scale. The problem is that at *tree level* the **supertrace**, STr, of the mass matrix $\mathcal{M}$, meaning the weighted sum of eigenvalues of the mass matrix of the particles in the model, can be shown to vanish, $\mathrm{STr}\,\mathcal{M}^2 = 0$ even after spontaneous supersymmetry breaking.[3]

Definition: The **supertrace** is given by

$$\mathrm{STr}\left(\mathcal{M}^2\right) \equiv \sum_s (-1)^{2s}(2s+1)\ \mathrm{Tr}[M_s^2], \tag{5.6}$$

where $\mathcal{M}$ is the complete mass matrix of the Lagrangian, s is the spin of particles and M_s is the mass matrix of all spin-s particles.

For a theory with only scalar superfields, with two fermionic and two bosonic degrees of freedom each, and with mass matrices $M_{1/2}$ and M_0 containing the masses of each type of particle on the diagonal, this means that $\mathrm{STr}\left(\mathcal{M}^2\right) = \mathrm{Tr}[M_0^2 - 2M_{1/2}^2] = 0$. While before spontaneous supersymmetry breaking each fermion and boson in the same superfield had equal masses, we now have that the sum of (the square of) the fermion and boson masses over all the superfields is the same.[4] Since in the Standard Model we have a lot more fermions than scalars, the crucial consequence is that not all the scalar partners can be heavier than our known fermions in order to balance out the supertrace relationship.

Adding vector superfields does not help because the fermions there have spin-1 vector bosons partners, and in the supertrace these contributions get opposite sign, and cancel each other out. This relationship is a tree level relationship, meaning it does not take into account quantum corrections. However, accounting for those and using strong couplings to move the supertrace value away from zero, the effect does not seems to be enough to build a consistent model.

The solution to the supertrace dilemma is to put the spontaneous supersymmetry breaking up at some higher energy scale $\sqrt{\langle F\rangle}$ – significantly higher than the electroweak scale – that

[2]It is *always* the auxiliary fields' fault!

[3]See Ferrara, Girardello and Palumbo (1979) [12].

[4]The factor of two takes care of the fact that there is only half as many fermions as bosons.

we do not currently have experimental access to, so that there are also superfields with fermion and boson partners with much higher masses that fulfil the supertrace relationship, and the electroweak scale fermions and bosons become only a small perturbation on it.

5.3 Soft breaking

There are many alternatives in the literature of how we can break supersymmetry spontaneously at a high energy scale. The names of some popular examples that we will return to in the next section are:

- Planck-scale Mediated Symmetry Breaking (PMSB)
- Gauge Mediated Symmetry Breaking (GMSB)
- Anomaly Mediated Symmetry Breaking (AMSB)

Common to all of them is that the mass scale of the particles in the supersymmetry breaking fields is so high, that we can not observe their effects directly. What we can do instead is to parameterise our ignorance by adding effective explicit supersymmetry breaking terms to our low-energy Lagrangian.

However, we cannot simply add arbitrary terms to the Lagrangian. The terms we can add are so-called **soft terms** with couplings of mass dimension one or higher. The disallowed terms with smaller mass dimension are terms that can lead to divergences in loop contributions to scalar masses (such as the Higgs) that are quadratic or worse (because of the high dimensionality of the fields in the loops). This will re-introduce the hierarchy problem.

The allowed terms are in superfield notation as follows:

$$\begin{aligned}\mathcal{L}_{\text{soft}} &= -\frac{1}{4T(R)q^2}M_i\theta\theta\bar{\theta}\bar{\theta}\text{Tr}\{W_i^A W_{iA}\} \\ &\quad -\frac{1}{6}a_{ijk}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j\Phi_k - \frac{1}{2}b_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j - s_i\theta\theta\bar{\theta}\bar{\theta}\Phi_i + \text{h.c.} \\ &\quad -m_{ij}^2\theta\theta\bar{\theta}\bar{\theta}\Phi_i^\dagger\Phi_j, \end{aligned} \tag{5.7}$$

where $M_i, a_{ijk}, b_{ij}, s_i \in \mathbb{C}$, and $m_{ij}^2 \in \mathbb{R}$ are the couplings. Note that these terms are explicitly *not* supersymmetric. From the $\theta\theta\bar{\theta}\bar{\theta}$-factors we see that only the lowest order in θ component fields of the superfields contribute.

There are also some terms that are called "maybe-soft" terms:

$$\mathcal{L}_{\text{maybe}} = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}c_{ijk}\Phi_i^\dagger\Phi_j\Phi_k + \text{h.c.}, \tag{5.8}$$

with $c_{ijk} \in \mathbb{C}$. This last – oft ignored – type of term is soft as long as none of the scalar superfields is a singlet under all gauge symmetries. It is, however, quite difficult to get large values for c_{ijk} with spontaneous supersymmetry breaking.

In the above terms we have not specified any gauge symmetry, which will, in the same way as it did for the superpotential, severely restrict the allowed terms. However, it still turns out that soft-terms are responsible for most of the parameters in supersymmetric theories!

We can instead write the soft terms in terms of their component fields as

$$\begin{aligned}\mathcal{L}_{\text{soft}} &= -\frac{1}{2}M_i\lambda_i^a\lambda_i^a - (\frac{1}{6}a_{ijk}A_iA_jA_k + \frac{1}{2}b_{ij}A_iA_j + s_iA_i + \frac{1}{2}c_{ijk}A_i^*A_jA_k + \text{c.c.}) \\ &\quad -m_{ij}^2A_i^*A_j.\end{aligned} \tag{5.9}$$

Here we clearly see that the m_{ij} couplings provide extra mass terms to the scalar partners A_i of the fermions in the scalar superfields, and M_i to the fermion partners λ_{iA}^a of the vectors in the vector superfields, [5] however, they provide no mass to the fermions ψ_A in the scalar superfield, nor to the vectors V_μ in the vector superfield. Thus, they split the masses of the partners as we wanted to achieve with breaking supersymmetry.

Now, what happens to the hierarchy problem? Restricting ourselves to soft supersymmetry breaking terms guarantees that we end up with contributions to the Higgs mass of at most

$$\Delta m_h^2 = -\frac{\lambda_s}{16\pi^2}m_s^2\ln\frac{\Lambda_{UV}^2}{m_s^2} + \ldots, \tag{5.10}$$

at the leading order in Λ_{UV}, where m_s is the mass scale of the soft term – or, in other words, the mass of the supersymmetric partners. The correction to the Higgs mass is thus proportional to the masses of the supersymmetric particles.

This is the most important argument in favour of supersymmetry existing at low energy scales where we can detect it, because m_s can not be too large if we want the above corrections to be reasonably small. This is called the **little hierarchy problem**, and numerically, given that $16\pi^2 \sim 100$, and the couplings λ_s are not expected to be above unity, this means that we want $m_s \sim \mathcal{O}(1 \text{ TeV})$ in order to keep the cancellations reasonable and of the order of the measured Higgs mass, with little fine-tuning.

5.4 Models for supersymmetry breaking

Let us take a closer look at the models we use to motivate supersymmetry breaking, and what their phenomenological consequences are. To be concrete, and to simplify the language used, we will relate this discussion to the Minimal Supersymmetric Standard Model (MSSM) that we will study in Chapter 6, however, the mechanisms for supersymmetry breaking will be the same for most relevant models, and you could read the MSSM below as a generic low-energy supersymmetric model.

Generically, breaking models can be illustrated as shown in Fig. 5.1. There is one or more **hidden sector** (HS) scalar superfield X – by hidden we mean that it has no or very small direct couplings to the MSSM fields – that has an *effective (non-renormalisable) supersymmetric coupling* to the MSSM scalar fields Φ_i of the form

$$\mathcal{L}_{\text{HS}} = -\frac{1}{M}(\bar{\theta}\bar{\theta})X\Phi_i\Phi_j\Phi_k, \tag{5.11}$$

where M is some large scale, *e.g.* the Planck scale, that suppresses the interaction. Figure 5.2 shows interactions that can lead to such terms, where M is the mass scale of some **mediator** particle Y.

[5] We use the index i in case there are multiple gauge groups. In addition the fermions from the vector superfield may carry an index a, λ_{iA}^a, if there is more than one generator in the gauge group.

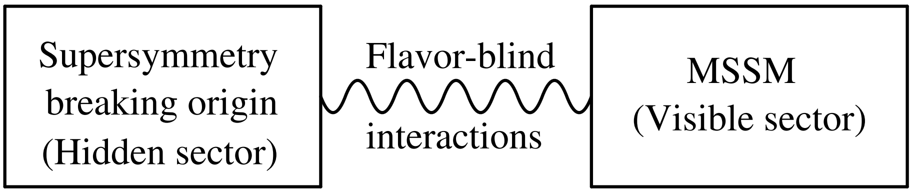

Figure 5.1: A generic illustration of how to generate soft breaking terms. Taken from Ref. [15].

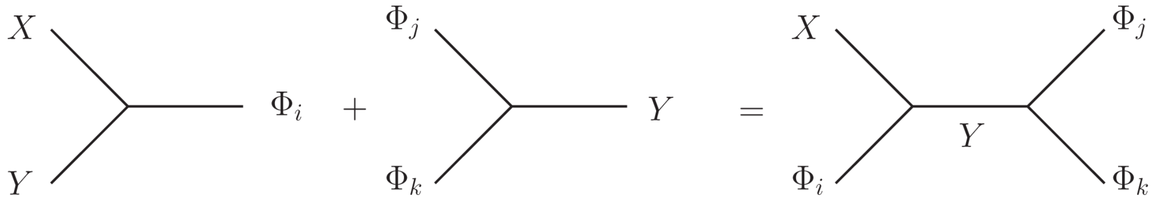

Figure 5.2: Interactions leading to effective 4-superfield couplings in our example.

If the hidden sector is constructed so that X develops a non-zero vev for its auxiliary F-component field, F_X,

$$\langle X \rangle = \theta\theta \langle F_X \rangle, \tag{5.12}$$

it breaks supersymmetry spontaneously, see the discussion leading up to Eq. (5.5). As a result, the interaction in (5.11) will in the low-energy limit produce a soft-term of the form of the second term in Eq. (5.9),

$$\mathcal{L}_{\text{soft}} = -\frac{\langle F_X \rangle}{M} A_i A_j A_k, \tag{5.13}$$

with the soft-mass parameter

$$m_{\text{soft}} = \frac{\langle F_X \rangle}{M}.$$

This has reasonable limits in that $m_{\text{soft}} \to 0$ as $\langle F_X \rangle \to 0$, which is the limit of no supersymmetry breaking, and $m_{\text{soft}} \to 0$ as $M \to \infty$, where the interaction with the hidden sector is decoupled because the mediating particle Y becomes too heavy to have any influence.

We will now look at two possible ways to construct such a hidden sector called Planck-scale Mediated Supersymmetry Breaking (PMSB) and Gauge Mediated Supersymmetry Breaking (GMSB).

5.4.1 Planck-scale Mediated Supersymmetry Breaking (PMSB)

In Planck-scale mediated supersymmetry breaking (PMSB) we blame some gravity mechanism for mediating the supersymmetry breaking from the hidden sector to the MSSM so that the scale of the breaking is $M = M_P = 2.4 \cdot 10^{18}$ GeV. Then we need to have $\sqrt{\langle F_X \rangle} \sim 10^{10} - 10^{11}$ GeV in order to get $m_{\text{soft}} \simeq 50 - 5000$ GeV, which is roughly of the right magnitude not to re-introduce the hierarchy problem. The use of $\sqrt{\langle F_X \rangle}$ is just a conventional shorthand notation for the magnitude of the vev of whichever F-term that breaks supersymmetry. This is called the **supersymmetry breaking scale**.

The complete soft terms for such a mechanism can then be shown to be

$$\begin{aligned}\mathcal{L}_{\text{soft}} &= -\frac{\langle F_X\rangle}{M_P}\left(\frac{1}{2}f_i\lambda_i^a\lambda_i^a + \frac{1}{6}y'_{ijk}A_iA_jA_k + \frac{1}{2}\mu'_{ij}A_iA_j + \frac{\langle F_X\rangle^*}{M_P^2}x_{ijk}A_i^*A_jA_k + \text{c.c.}\right)\\ &\quad -\frac{|\langle F_X\rangle|^2}{M_P^2}k_{ij}A_iA_j^*. \end{aligned} \tag{5.14}$$

Here f_i, y'_{ijk}, μ'_{ij}, x_{ijk}, and k_{ij} are breaking model dependent parameters. Incidentally, we can now see why we assumed the maybe-soft breaking terms to be unimportant, as in this model they are suppressed by $|\langle F_X\rangle|/M_P^2$ compared to the other couplings.

If one assumes a minimal form for the parameters at some high-scale, perhaps at the GUT scale motivated by the wish for unification, *e.g.* $f_i = f$, $y'_{ijk} = \alpha y_{ijk}$ where y_{ijk} are the Standard Model Yukawa couplings, $\mu'_{ij} = \beta\mu$, and $k_{ij} = k\delta_{ij}$, then all the soft terms are fixed by just four parameters

$$m_{1/2} = f\frac{\langle F_X\rangle}{M_P}, \qquad m_0^2 = k\frac{|\langle F_X\rangle|^2}{M_P^2}, \qquad A_0 = \alpha\frac{\langle F_X\rangle}{M_P}, \qquad B_0 = \beta\frac{\langle F_X\rangle}{M_P}.$$

The resulting phenomenology is called **minimal supergravity**, or **mSUGRA/CMSSM**. This is minimal in the sense of the form of the parameters, and is the most studied, but perhaps not best motivated, version of the MSSM. Usually, B_0 and $|\mu|$ are exchanged for the parameter $\tan\beta$ at low scales using the conditions for electroweak symmetry breaking that we will see in Eq. (6.31), so it is common to say that there are four and a half parameters in the CMSSM: $m_{1/2}$, m_0, A_0, $\tan\beta$ and $\operatorname{sgn}\mu$.

5.4.2 Gauge Mediated Supersymmetry Breaking (GMSB)

An alternative to PMSB is gauge-mediated supersymmetry breaking where soft terms come from *loop diagrams* with **messenger** superfields that get their own mass M by coupling to the hidden sector supersymmetry breaking vev $\langle F\rangle$, and that have Standard Model gauge interactions. By dimensional analysis we must have

$$m_{\text{soft}} = \frac{\alpha_i}{4\pi}\frac{\langle F\rangle}{M}.$$

If now the supersymmetry breaking scale $\sqrt{\langle F\rangle}$ and the messenger mass M are roughly comparable in size, which is reasonable given where the messenger mass comes from, then $\sqrt{\langle F\rangle} \simeq 100\,\text{TeV}$ can give a viable sparticle spectrum. Notice that there is now a lot less difference in scale between the breaking scale and the mass scale of the supersymmetric partners, m_{soft}, in the GMSB compared to the PMSB.

One way of thinking about how these mass terms appear is that the messenger field(s) get masses from hidden sector vevs and contribute to mass terms from the fermions in the vector superfields – called gauginos – through diagrams such as the one in Fig. 5.3, where messenger scalars and fermions run in the loop, and their masses from the hidden sector vevs are symbolised by the mass insertions. Note that scalars can only get mass contributions like this at two-loop order since the messenger interaction is a gauge interaction, involving gauge bosons or gauginos in the MSSM. In order not to spoil the unification of gauge couplings, see

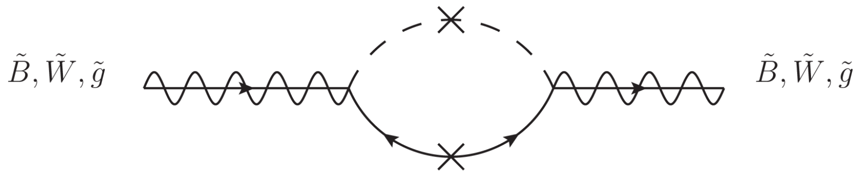

Figure 5.3: Diagram for GMSB giving masses to the gauginos $\tilde{g}$, $\tilde{B}$, and $\tilde{W}$, in the MSSM. The messenger scalars and fermions run in the loop.

Sec. 6.8, messengers are often assumed to have small mass splittings and come in N_5 complete $\mathbf{5}+\overline{\mathbf{5}}$ (fundamental) representations of $SU(5)$.

The minimal parametrisation of GMSB models is in terms of the scale $\Lambda = \frac{\langle F \rangle}{M}$, the messenger mass M, the number of representations N_5, and $\tan\beta$ for the electroweak symmetry breaking criterion. This gives the soft masses

$$M_i = \frac{\alpha_i}{4\pi}\Lambda N_5, \quad \text{(gaugino masses)} \tag{5.15}$$

$$m_j^2 = 2\Lambda^2 N_5 \sum C(R)_i \left(\frac{\alpha_i}{4\pi}\right)^2, \quad \text{(scalar masses)} \tag{5.16}$$

where the sum is over the gauge groups, with $C(R)$ being quadratic Casimir invariant for the scalar superfield Φ_j that the scalar field belongs to. We clearly see that the scalar soft-masses are a two-loop effect as discussed above.

While this parameterisation looks independent of the messenger mass M, the messenger scale sets the starting point of the renormalisation running of the sparticle masses, see Sec. 6.7, and thus influences their magnitude. For example, the tri-linear soft-term couplings a_{ijk} are expected to be very small at the messenger scale, and are effectively set to zero, however, due to the running they are small, but non-zero at the electroweak scale. Since the scalar masses m_j scale as $\sqrt{N_5}$ compared to N_5 for the gauginos, we in general expect the scalars to be lighter in GMSB models. One should also notice that this parameterisation gives a hierarchy of gaugino masses, $M_3 > M_2 > M_1$, since (5.15) is ordered in terms of the strength of the gauge couplings α_i.

5.5 Excercises

Exercise 5.1

Show that adding a Fayet-Iliopolous term to the SQED constructed in Exercise 4.4 will break supersymmetry spontaneously.

Exercise 5.2

Write down the supertrace relationship for a spontaneously broken SQED.

Chapter 6

The Minimal Supersymmetric Standard Model

The Minimal Supersymmetric Standard Model (MSSM) is a minimal supersymmetric model in the sense that it has the smallest field (and gauge) content consistent with the known Standard Model fields. We will now construct this model on the basis of what we have learnt the previous chapters, and look at some of its consequences.

6.1 MSSM field content

To specify a supersymmetric model we need to specify the superfield content and the gauge symmetries. The gauge symmetry is already given as the Standard Model $SU(3)_c \times SU(2)_L \times U(1)_Y$. We start now with writing down the fields we need for the Standard Model fermions. Previously we learnt that each (left-handed) scalar superfield S has a (left-handed) Weyl spinor ϕ_A, a complex scalar s and an auxiliary complex scalar field F, since they are a $j = 0$ representation of the superalgebra. After an application of the equations of motion ϕ_A and s have two fermionic and two bosonic degree of freedom remaining respectively, while the auxiliary field has been eliminated along with two fermionic degrees of freedom.

In order to construct a Dirac fermion, which are plentiful in the Standard Model, we need a right-handed Weyl spinor as well. We can acquire the needed right-handed Weyl spinor from the hermitian conjugate $\bar{T}^\dagger$ of a different scalar superfield $\bar{T}$ with the right-handed Weyl spinor $\bar{\chi}_{\dot{A}}$ and the complex scalar t^*.[1] With these four fermionic d.o.f. we can construct *two* Dirac fermions, a particle–anti-particle pair,

$$\psi_a = \begin{bmatrix} \phi_A \\ \bar{\chi}^{\dot{A}} \end{bmatrix}, \quad \bar{\psi}_a = \left[\chi^A,\, \bar{\phi}_{\dot{A}}\right],$$

and four scalars, two particle–anti-particle pairs, s, s^*, t and t^*.

We use these two superfield ingredients to construct all the known fermions:

[1]The bar here is used to (not) confuse us, it is part of the name of the superfields and does not denote any hermitian or complex conjugate. The bar signifies that $\bar{T}$ is the field where, when hermitian conjugated into $\bar{T}^\dagger$, we will pick the right-handed Weyl-spinor to use in the Dirac fermion, while the left-handed Weyl spinor in the bared field $\bar{T}$ itself belongs to the corresponding anti-particle. Since $SU(2)_L$ acts only on the left-handed Weyl spinors of particles (as opposed to anti-particles), another way to think about this is that the left-handed Weyl-spinor in the bared field $\bar{T}$ is the one that does not transform under $SU(2)_L$.

- To represent the Standard Model leptons we introduce the superfields l_i and $\bar{E}_i$ for the charged leptons (i is the generation index) and ν_i for the neutrinos, and form $SU(2)_L$ doublet superfield vectors $L_i = (\nu_i, l_i)$, which contain the left-handed Weyl spinors for the particles, while $\bar{E}_i$ contain the left-handed Weyl spinors that are singlets (do not transform) under $SU(2)_L$.

- There is an imbalance in the neutrinos in that we do not introduce the neutrino superfield $\bar{N}_i$ that through $\bar{N}_i^\dagger$ would contain right-handed neutrino spinors needed for massive Dirac neutrinos, instead leaving the neutrinos as massless Majorana particles in the MSSM.[2] This is a convention – the MSSM is older than the discovery of neutrino mass – and including $\bar{N}_i$ fields would have some interesting consequences. The $\bar{N}_i$ superfields and their component fields would not couple to any of the gauge fields since they are SM singlets,[3] however, the scalar field in the superfield could be a potential dark matter candidate.

- For quarks the situation is similar. Up-type and down-type quarks get the superfields u_i and d_i, forming the $SU(2)_L$ doublets $Q_i = (u_i, d_i)$, and the $SU(2)_L$ singlets $\bar{U}_i$ and $\bar{D}_i$.[4]

With all possible apologies, we now change notation for the component fields to what is closer to conventions in phenomenology, as opposed to pure theory. The left-handed Weyl spinors in these superfields will now be named for example l_i for the spinor in the superfield l_i, ν_i for the one in the ν_i superfield, and $\bar{e}_i$ for the one in $\bar{E}_i$, u_i for the one in the u_i superfield, d_i for the one in the d_i superfield, $\bar{u}_i$ for the one in the $\bar{U}_i$ superfield, and finally $\bar{d}_i$ for the one in the $\bar{D}_i$ superfield. This means that we leave our former notation where the bar signifies right-handed Weyl spinor, but now instead signifies the left-handed Weyl spinor for the anti-particle, or, in other words, a $SU(2)_L$ singlet. If we now want to indicate a right-handed Weyl spinor we use the hermitian conjugation in (2.24) as an alternative notation, so that for example the $l_i^\dagger$ superfield contains $\bar{l}_i^\dagger$ and $\bar{E}_i^\dagger$ contains $\bar{e}_i^\dagger$, and so on. The reason for this change is in part that we need a simple way to write down component field Lagrangians with fermions without greek letters and very many indices indicating particle type, as well as running out of letters since we need two superfields for one fermion.

The scalar component fields are named after the fermions using the tilde notation, for example in the superfield l_i we have the scalar $\tilde{l}_{iL}$ as the supersymmetric partner particle, often just called **sparticle**, of the fermion l_i. Similarly, $\bar{E}_i$ contains $\tilde{l}_{iR}^*$, $l_i^\dagger$ contains $\tilde{l}_{iL}^*$, and $\bar{E}_i^\dagger$ contains $\tilde{l}_{iR}$. Since scalars do not have any notion of handedness the L or R here is just part of the conventional name; we still call these particles for left-handed and right-handed scalar leptons though. The complex conjugates might be surprising, but remember that for example the superfield $\bar{E}_i$ contains the left-handed Weyl-spinor of the *anti*-leptons and thus has positive electric charge. The collective term for these scalars are **sfermions**.

Additionally, we need vector superfields, which, after the equations of motion have eliminated the auxiliary field, contain a *massless* vector boson V_μ component field with two scalar

[2]The anti-neutrino contained in the superfield $\nu_i^\dagger$ is a right-handed Weyl-spinor consistent with experiment.

[3]They can not be colour-charged, they are singlets under $SU(2)_L$ by construction thus they have zero weak isospin I_3, but since they should also have zero electric charge Q, the hypercharge Y must also be zero through the relationship $Q = \frac{1}{2}Y + I_3$.

[4]Here we should really also include a colour index a so that u_i^a is a component in an $SU(3)_c$ triplet superfield vector. We omit these for simplicity.

degrees of freedom and a Weyl-spinor λ_A, with two fermionic degrees of freedom. Together these form a $m = 0$, $j = \frac{1}{2}$ representation of the superalgebra. If the vector superfield is neutral, the Weyl-spinor can form a Majorana fermion, if not it can be combined with the Weyl-spinor from another vector superfield to form a Dirac fermion.

Looking at the construction $V \equiv qT_aV^a$ for the vector superfields in the supersymmetric Lagrangian we see that, as expected, we need one superfield V^a per generator T_a of the algebra, with the normal $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ vector bosons as vector component fields. We call these superfields C^a, W^i and B^0, where $a = 1, \ldots, 8$ and $i = 1, 2, 3$.[5] In order to be really confusing, we use the following symbols for the fermion Weyl-spinors: $\tilde{g}^a$, $\tilde{W}^+$, $\tilde{W}^0$, $\tilde{W}^-$ and $\tilde{B}^0$. The tilde here is supposed to tells us – just as for the scalar component fields of the scalar superfields – that they are supersymmetric partners of the known Standard Model particles. These particles are collectively known as the **gauginos**.

We also need superfields for the scalar Higgs boson. Now life gets interesting. The Higgs $SU(2)_L$ doublet scalar field H used in the Standard Model cannot give mass to all fermions because it relies on the construction $H^C \equiv -i(H^\dagger\sigma_2)^T$ to give masses to up-type quarks, and possibly neutrinos. The superfield version of this cannot appear in the superpotential because it would mix left- and right-handed superfields due to the hermitian conjugation in H^C. The minimal Higgs content we can get away with are two Higgs superfield $SU(2)_L$ doublets, which we will call H_u and H_d, indexing the quarks they give mass to.[6] These must have (more on the reason for that in Sec. 6.2) weak hypercharge $Y = \pm 1$ for H_u and H_d, respectively, so that we have the superfield doublets:

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}, \tag{6.1}$$

where we have given the electric charges of the scalar superfield components of the superfield doublets based on the standard $Q = \frac{1}{2}Y + T_3$ relationship after electroweak symmetry breaking, where T_3 is the weak isospin.[7] The scalar component fields of these fields, before their mixing following electroweak symmetry breaking, will have the same symbols as the superfields. (Yes, really!) The fermion component fields will be denoted $\tilde{H}_u^+$, $\tilde{H}_u^0$, $\tilde{H}_d^0$, and $\tilde{H}_d^-$, and are known as the **higgsinos**.

6.2 The kinetic terms

It is now straight forward to write down the kinetic terms of the MSSM Lagrangian giving the matter-gauge interaction terms

$$\begin{aligned} \mathcal{L}_{\text{kin}} \;=\; & L_i^\dagger e^{\frac{1}{2}g\sigma W - \frac{1}{2}g'B} L_i + Q_i^\dagger e^{\frac{1}{2}g_s\lambda C + \frac{1}{2}g\sigma W + \frac{1}{3}\cdot\frac{1}{2}g'B} Q_i \\ & + \bar{U}_i^\dagger e^{\frac{1}{2}g_s\lambda C - \frac{4}{3}\cdot\frac{1}{2}g'B} \bar{U}_i + \bar{D}_i^\dagger e^{\frac{1}{2}g_s\lambda C + \frac{2}{3}\cdot\frac{1}{2}g'B} \bar{D}_i \\ & + \bar{E}_i^\dagger e^{2\frac{1}{2}g'B} \bar{E}_i + H_u^\dagger e^{\frac{1}{2}g\sigma W + \frac{1}{2}g'B} H_u + H_d^\dagger e^{\frac{1}{2}g\sigma W - \frac{1}{2}g'B} H_d, \end{aligned} \tag{6.2}$$

where g', g and g_s are the couplings constants (strengths) of $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$, respectively, and $\frac{1}{2}\sigma_i$ and $\frac{1}{2}\lambda_a$ are the generators of $SU(2)_L$ and $SU(3)_c$. As a convention

[5] And there we have another W.

[6] In some further insanity some authors prefer H_1 and H_2 so that you have no idea which is which.

[7] The upper component of a doublet has $T_3 = \frac{1}{2}$ while the lower has $T_3 = -\frac{1}{2}$. This is again just the eigenvalues of the J_3 generator in two-dimensional representation of the $SU(2)$ group, see Sec. 2.1.3.

we assign the charges under $U(1)$, hypercharge, in units of $\frac{1}{2}g'$. For a list of the hypercharge assignments see Table 6.1. All non-singlets of $SU(2)_L$ and $SU(3)_C$ have the same charge, the factor $\frac{1}{2}$ here is used to get by without accumulation of numerical factors since the algebras for the Pauli (σ_i) and Gell-Mann matrices (λ_a) are:

$$\left[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j\right] = i\epsilon_{ijk}\frac{1}{2}\sigma_k,$$

and

$$\left[\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b\right] = if_{ab}{}^c\frac{1}{2}\lambda_c.$$

These conventions lead to the wanted Standard Model gauge transformations for the component fields and the familiar relations after electroweak symmetry breaking, $e = g\sin\theta_W = g'\cos\theta_W$, where e is the elementary electric charge (in natural units).

We mentioned earlier that the two Higgs superfields have opposite hypercharge. This is needed for so-called **anomaly cancellation** in the MSSM. Gauge anomaly is the possibility that at loop level contributions to processes such as in Fig. 6.1 break the gauge invariance that we have established at the classical level in the Lagrangian, and ruins the predictability of the theory. This rather miraculously does not happen in the SM because it has the exactly field content it has, so that all such possible gauge anomalies exactly cancel – we do not know of a deeper reason for why it has exactly this field content. If we have only *one* Higgs doublet this cancellation does not happen for the MSSM. With two Higgs doublets, and with opposite hypercharge, it does.

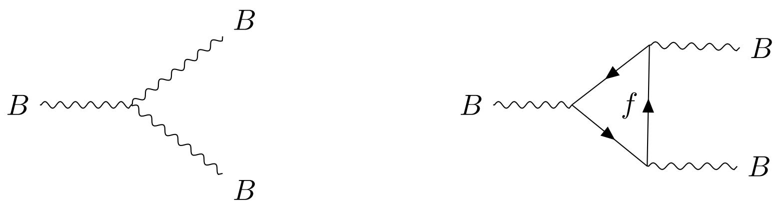

Figure 6.1: The tree level coupling between three gauge bosons B (left), and the one-loop fermion contribution to the same process (right).

6.3 Gauge terms

The pure gauge terms with supersymmetric field strengths are also fairly easy to write down:

$$\mathcal{L}_V = \frac{1}{2g_s^2}\bar{\theta}\bar{\theta}\,\mathrm{Tr}[C^A C_A] + \frac{1}{2g^2}\bar{\theta}\bar{\theta}\,\mathrm{Tr}[W^A W_A] + \frac{1}{2g'^2}\bar{\theta}\bar{\theta}B^A B_A + \text{h.c.}, \tag{6.3}$$

where we have used the Dynkin indices of the gauge group representations

$$T(R)_L = \mathrm{Tr}\left[\frac{1}{2}\sigma_1 \cdot \frac{1}{2}\sigma_1\right] = \frac{1}{2},$$

and

$$T(R)_c = \mathrm{Tr}\left[\frac{1}{2}\lambda_1 \cdot \frac{1}{2}\lambda_1\right] = \frac{1}{2},$$

in the normalization of the terms, and where the field strengths are given as:

$$C_A = -\frac{1}{4}\bar{D}\bar{D}e^{-C}D_A e^C , \qquad C = \frac{1}{2}g_s\lambda_a C^a, \tag{6.4}$$

$$W_A = -\frac{1}{4}\bar{D}\bar{D}e^{-W}D_A e^W , \qquad W = \frac{1}{2}g\sigma_i W^i, \tag{6.5}$$

$$B_A = -\frac{1}{4}\bar{D}\bar{D}D_A B , \qquad B = \frac{1}{2}g' B^0. \tag{6.6}$$

6.4 The MSSM superpotential

With the gauge structure of the Standard Model in place we are ready to write down all possible terms in the superpotential. First, we notice that there can be no **tadpole terms** $t_i\Phi_i$ (terms with only one superfield), since there are no superfields that are singlets (zero charge) under all Standard Model gauge groups. The only alternative would be if we introduced right-handed neutrino superfields $\bar{N}_i$.

For the possible **mass terms** $m_{ij}\Phi_i\Phi_j$ we first check the abelian gauge group $U(1)_Y$, where the requirement reduces to the relatively simple sum of hypercharges $Y_i + Y_j = 0$. In Table 6.1 we see that the only contributions with sum zero hypercharge using superfields that contain the Standard Model fermions are particle–anti-particle combinations such as $l_i l_i^\dagger$, but these come from superfields with different handedness (L_i and $L_i^\dagger$) and cannot be used together. Thus the superpotential *can not* be used to give masses to the Standard Model fermions, and we will need a Higgs mechanism in the MSSM as well.

Superfield	L_i	$\bar{E}_i$	Q_i	$\bar{U}_i$	$\bar{D}_i$	H_u	H_d
Fermion	ν_i, l_i	$\bar{e}_i$	u_i, d_i	$\bar{u}_i$	$\bar{d}_i$	$\tilde{H}_u^+$, $\tilde{H}_u^0$	$\tilde{H}_d^0$, $\tilde{H}_d^-$
Hypercharge Y	-1	2	$\frac{1}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$	1	-1
Electric charge Q	0, -1	1	$\frac{2}{3}$, $-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	1, 0	0, -1

Table 6.1: The MSSM superfields with their Standard Model fermion content, hypercharge Y, and electric charge Q.

Going beyond the superfields with Standard Model fermions we see that we can make a mass term with the two Higgs superfields that have opposite hypercharge $Y = \pm 1$. These fields are not charged under $SU(3)_c$, but in order to also be invariant under $SU(2)_L$ we have to write this superpotential term as

$$W_{\text{mass}} = \mu H_u^T i\sigma_2 H_d, \tag{6.7}$$

where $\mu \in \mathbb{C}$ is the superpotential mass parameter for this term. This construction is invariant under $SU(2)_L$ because, with the supergauge transformations $H_d \to e^{ig\frac{1}{2}\sigma_k W^k}H_d$ and $H_u^T \to H_u^T e^{ig\frac{1}{2}\sigma_k^T W^k}$, we get

$$\begin{aligned} H_u^T i\sigma_2 H_d &\to H_u^T e^{ig\frac{1}{2}\sigma_k^T W^k} i\sigma_2 e^{ig\frac{1}{2}\sigma_k W^k} H_d \\ &= H_u^T i\sigma_2 e^{-i\frac{1}{2}g\sigma_k W^k} e^{i\frac{1}{2}g\sigma_k W^k} H_d = H_u^T i\sigma_2 H_d, \end{aligned}$$

since $\sigma_k^T\sigma_2 = -\sigma_2\sigma_k$. Usually we ignore the $SU(2)_L$ specific structure and write terms like this as $\mu H_u H_d$, confusing the hell out of anyone that is not used to this convention since we really do mean Eq. (6.7). Notice that if we write (6.7) in terms of the component superfields in the two $SU(2)_L$ doublets we get

$$W_{\text{mass}} = \mu H_u^T i\sigma_2 H_d = \mu(H_u^+ H_d^- - H_u^0 H_d^0),$$

which we should have been able to guess because the Lagrangian must also conserve electric charge.

If you have paid very close attention to the argument above you may have noticed that there is one more possibility, namely

$$W_{\text{mass}} = \mu_i' L_i H_u \equiv \mu_i' L_i^T i\sigma_2 H_u = \mu_i'(\nu_i H_u^0 - l_i H_u^+),$$

where $\mu' \in \mathbb{C}$ is some other mass parameter in the superpotential. This is clearly an allowable term (and we will return to it below), however, it also raises a very interesting question: Could we have $L_i \equiv H_d$? Could the lepton superfields L_i play the rôle of Higgs superfields, thus reducing the field content needed to describe the SM particles in a supersymmetric theory? While not immediately forbidden as a superpotential term, this suggestions unfortunately leads to problems with anomaly cancelation, processes with large lepton flavour violation (LFV) and much too massive neutrinos, and has been abandoned.

We have now found all possible mass terms in the superpotential. What about the **Yukawa terms** $\lambda_{ijk}\Phi_i\Phi_j\Phi_k$? The hypercharge requirement here is $Y_i + Y_j + Y_k = 0$. From our table of hypercharges only the following terms are found to be viable:

$$W_{\text{Yukawa}} = y_{ij}^e L_i H_d E_j + y_{ij}^u Q_i H_u \bar{U}_j + y_{ij}^d Q_i H_d \bar{D}_j + \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k,$$

where we have named and indexed the Yukawa couplings in a hopefully natural way.[8] For all these terms we can simultaneously keep $SU(2)_L$ invariance with the $i\sigma_2$ construction implicitly inserted between any two superfield doublets. Note that because of the $SU(2)_L$ invariance, we must have $i \neq j$ for λ_{ijk}, since $i = j$ gives $L_i L_i \bar{E}_k = (\nu_i l_i - l_i \nu_i)\bar{E}_k = 0$.

For $SU(3)_c$ to be conserved for the Yukawa terms, we need to have colour singlets. Some of these terms are colour singlets by construction since they do not contain any coloured fields – the LHE and LLE terms. The terms with only two quark superfields contain left-handed Weyl spinors for quarks and anti-quarks, which form $SU(3)_c$ singlets if the superfields come in colour–anti-colour pairs. In representation language the superfields (and as a consequence their component fields) are in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations of $SU(3)_c$. Written with explicit colour indices we have for example $L_i Q_j \bar{D}_k = L_i i\sigma_2 Q_j^a \bar{D}_k^a$, where a is the colour index. The final term $\bar{U}_i \bar{D}_j \bar{D}_k$ is a colour singlet once we demand that it is totally anti-symmetric in the colour indices: $\bar{U}_i \bar{D}_j \bar{D}_k \equiv \epsilon_{abc} \bar{U}_i^a \bar{D}_j^b \bar{D}_k^c$. The anti-symmetry property of the Levi-Civita tensor ϵ_{abc} means that we must have $j \neq k$ in λ''_{ijk}.

Our complete superpotential is then:

$$\begin{aligned} W \;\; = \;\; & \mu H_u H_d + \mu_i' L_i H_u + y_{ij}^e L_i H_d E_j + y_{ij}^u Q_i H_u \bar{U}_j + y_{ij}^d Q_i H_d \bar{D}_j \\ & + \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k. \end{aligned} \tag{6.8}$$

The parameter μ, potentially complex, is a brand new supersymmetric parameter appearing in the superpotential, with no corresponding parameter existing in the Standard Model Lagrangian. However, the Yukawa couplings y_{ij} are identical to the Standard Model Yukawa

[8] For some particular opinion of what is natural.

couplings since they will be required to give mass to the Standard Model fermions after electroweak symmetry breaking when the Higgs fields get a vev, see Section 6.9. The fate of the other parameters will be discussed in the next section.

6.5 R-parity

The superpotential terms LH_u, LLE and $LQ\bar{D}$ that we have written down in Eq. (6.8) all violate lepton number conservation, and $\bar{U}\bar{D}\bar{D}$ violates baryon number conservation. Such terms do not exist in the Standard Model, although there is no symmetry forbidding their existence there. Instead a seemingly accidental combination of what fields exist and the gauge symmetries that limit their interactions means that there are no such tree-level interactions. We call this an **accidental symmetry** of the Standard Model.

Allowing such terms in the MSSM would lead to, among other phenomenological problems, processes like rapid proton decay, for example through $p \to e^+\pi^0$ as shown in Fig. 6.2, which breaks both baryon number and lepton number. These are not observed in nature. We can estimate the resulting proton life-time by noting that the exchange of a scalar particle (in this example a strange squark $\tilde{s}$) creates an effective dimension-6 four-fermion interaction Lagrangian term $\lambda\bar{u}\bar{d}eu$ with coupling

$$\lambda = \frac{\lambda'_{112}\lambda''_{112}}{m_{\tilde{s}}^2}, \tag{6.9}$$

where the sparticle mass $m_{\tilde{s}}$ comes from the scalar propagator in the diagram. The resulting matrix element for the total proton decay process must then be proportional to $|\lambda|^2$, which has mass dimension M^{-4}. Since decay width has mass dimension M, the phase space part of the calculation must provide something of mass dimension M^5. The only mass scale involved in the problem is the proton mass m_p, thus we approximate the phase space integration part of the proton decay width by m_p^5. We then have

$$\Gamma_{p\to e^+\pi^0} \sim |\lambda|^2 m_p^5 = |\lambda'_{112}\lambda''_{112}|^2 \frac{m_p^5}{m_{\tilde{s}}^4}. \tag{6.10}$$

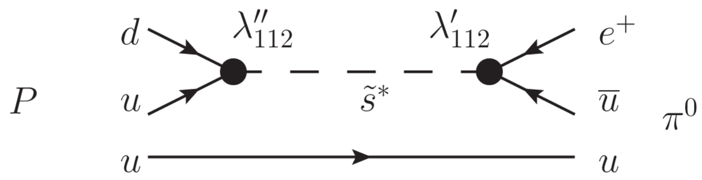

Figure 6.2: Possible Feynman diagram for proton decay with R-parity violating couplings λ''_{112} and λ'_{112}.

The current measured lower limit on the lifetime from watching a lot of protons not decay is $\tau_{p\to e^+\pi^0} > 1.6 \cdot 10^{34}$ y [13], or, converting to seconds, $\tau_{p\to e^+\pi^0} > \pi \cdot 10^7$ s/y $\times\, 1.6 \cdot 10^{34}$ y $= 5.0 \cdot 10^{41}$ s, which gives a limit on the width

$$\Gamma_{p\to e^+\pi^0} = \frac{\hbar}{\tau} < \frac{6.582 \cdot 10^{-25}\,\text{GeV s}}{5.0 \cdot 10^{41}\,\text{s}} \simeq 1.3 \cdot 10^{-66}\,\text{GeV},$$

so that we have the following very strict limit on the combination of the two couplings

$$|\lambda'_{112}\lambda''_{112}| < 1.3 \cdot 10^{-27} \left(\frac{m_{\tilde{s}}}{1\ \text{TeV}}\right)^2. \tag{6.11}$$

Such strict limits can be found on most of the lepton and baryon number couplings using the measured properties of the Standard Model particles, with some exceptions for coupling involving the second and third generation fermions.

To avoid all such lepton and baryon number couplings Fayet (1975) [14] introduced the conservation of R-partity.

> **Definition**: **R-parity** is a multiplicatively conserved quantum number given by
>
> $$R = (-1)^{2s+3B+L}$$
>
> where s is a particle's spin, B its baryon number and L its lepton number.

For all Standard Model particles, including all the scalar Higgs bosons, this gives $R = 1$, while the superpartners all have $R = -1$. One usually *defines* the MSSM as conserving R-parity. For the MSSM this excludes the terms LH_u, $LL\bar{E}$, $LQ\bar{D}$ and $\bar{U}\bar{D}\bar{D}$ from the superpotential,[9] leaving us with the R-partiy conserving superpotential

$$W = \mu H_u H_d + y^e_{ij} L_i H_d E_j + y^u_{ij} Q_i H_u \bar{U}_j + y^d_{ij} Q_i H_d \bar{D}_j. \tag{6.12}$$

The consequence of this somewhat *ad hoc* definition is that in all interactions the total number of incoming and outgoing supersymmetric particles must be an even number $2n$, so that the total R-number is $(-1)^{2n} = 1$. This leads to the following very important phenomenological consequences:

1. All sparticles must be produced in pairs (ignoring the very low probability of producing four or more).

2. Sparticles must annihilate in pairs.

3. The **lightest supersymmetric particle** (LSP) is absolutely stable, and every other sparticle must decay down to the LSP (possibly in multiple steps).

6.6 Supersymmetry breaking terms in the MSSM

We can directly apply our previous arguments on gauge invariance, that we used when discussing the superpotential, on the general soft-breaking terms in Eq. (5.7) in order to determine which supersymmetry breaking terms are allowed in the MSSM, keeping also in mind the requirement of R-party conservation.

---

[9] All the superpotential Yukawa terms lead to component field terms of the form $A_i\psi_j\psi_k$. If the scalar A_i here is *not* a Higgs boson, then it is a superpartner and if none of the fermions come from a Higgs superfield so that they are also a superpartner the term breaks R-parity conservation. This means that every Yukawa term needs one, and only one, Higgs superfield to conserve R-parity. The superpotential mass terms have component field terms of the form $\psi_i\psi_j$. If one of the fermions here comes from a Higgs superfield, then it is a superpartner, and if the other does not, the term breaks R-parity.

Mass terms of the form

$$-\frac{1}{4T(R)q^2}M\theta\theta\bar{\theta}\bar{\theta}\,\mathrm{Tr}[W^AW_A]+\text{c.c.},$$

are allowed because they have the same gauge structure as the supersymmetric field strength terms. In component fields only the fermions in the vector superfields survive, and are for the MSSM:

$$\mathcal{L}_{\text{soft}}=-\frac{1}{2}M_1\tilde{B}^0\tilde{B}^0-\frac{1}{2}M_2\tilde{W}^i\tilde{W}^i-\frac{1}{2}M_3\tilde{g}^a\tilde{g}^a+\text{c.c.},$$

where $M_i\in\mathbb{C}$. This gives six new parameters.

Yukawa terms

$$-\frac{1}{6}a_{ijk}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j\Phi_k+\text{h.c.},$$

are allowed when a corresponding term exist in the superpotential – meaning they are gauge invariant. In component fields only the scalar parts of the superfields survive, and the allowed terms are

$$\mathcal{L}_{\text{soft}}=-a^e_{ij}\tilde{L}_iH_d\tilde{l}^*_{jR}-a^u_{ij}\tilde{Q}_iH_u\tilde{u}^*_{jR}-a^d_{ij}\tilde{Q}_iH_d\tilde{d}^*_{jR}+\text{c.c.},$$

where we remind you that the H here refers to scalar parts of the Higgs superfield doublets,

$$H_d=\begin{pmatrix}H_u^+\\H_u^0\end{pmatrix}\quad\text{and}\quad H_d=\begin{pmatrix}H_d^0\\H_d^-\end{pmatrix},$$

and

$$\tilde{L}_i=\begin{pmatrix}\tilde{\nu}_{iL}\\\tilde{l}_{iL}\end{pmatrix}\quad\text{and}\quad\tilde{Q}_i=\begin{pmatrix}\tilde{u}_{iL}\\\tilde{d}_{iL}\end{pmatrix},$$

in the normal $SU(2)_L$ invariant construction. We can see that all of these terms are clearly R-parity conserving, since they consist of two sparticles and one (Higgs) particle. The couplings a_{ij} are all potentially complex valued, so this gives us 54 new parameters.

The mass terms

$$-\frac{1}{2}b_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j+\text{h.c.},$$

are again only allowed for corresponding terms in the superpotential, *i.e.*

$$\mathcal{L}_{\text{soft}}=-bH_uH_d+\text{c.c.},$$

where b is potentially complex valued, which gives us 2 new parameters.[10] Tadpole terms are not allowed, as there are no tadpoles in the superpotential.

Mass terms

$$-m^2_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i^\dagger\Phi_j,$$

are allowed because they have the same gauge structure as the supersymmetric kinetic terms. In component fields again only the scalar fields survive, and in the MSSM they are:

$$\begin{aligned}\mathcal{L}_{\text{soft}} &= -(m^L_{ij})^2\tilde{L}_i^\dagger\tilde{L}_j-(m^e_{ij})^2\tilde{l}^*_{iR}\tilde{l}_{jR}-(m^Q_{ij})^2\tilde{Q}_i^\dagger\tilde{Q}_j-(m^u_{ij})^2\tilde{u}^*_{iR}\tilde{u}_{jR}-(m^d_{ij})^2\tilde{d}^*_{iR}\tilde{d}_{jR}\\ &\quad -m^2_{H_u}H_u^\dagger H_u-m^2_{H_d}H_d^\dagger H_d,\end{aligned}\tag{6.13}$$

[10]The coupling b is sometimes written $B\mu$ where B is a factor that indicates how different the coupling is from the corresponding coupling in the superpotential.

where the m_{ij}^2 are potentially complex valued, however, also hermitian. This gives rise to 47 new parameters. Despite technically being allowed the MSSM ignores the "maybe-soft" terms in Eq. (5.8).

In total, after using our freedom to choose our basis for the fields wisely in order to remove what freedom we can, the MSSM has 105 new parameters compared to the Standard Model, 104 of these are soft-breaking terms and μ is the only new parameter in the superpotential.

6.7 Renormalisation group equations

Renormalisation, the removal of infinities from field theory predictions, introduces a fixed scale μ at which the fields and the parameters of the Lagrangian, the couplings, are defined. For example, the charge of the electron is not simply the **bare charge** e_0 given in the original Lagrangian, but a charge at a given energy scale μ, $e(\mu)$, which is the scale at which the theory wants to describe the electron, and which we can measure in an experiment at that scale. Describing the scattering an electron at very high energy will require a different value of $e(\mu)$ than at a low energy. This scale dependence in the coupling is an experimentally well verified fact.[11] The relationship between a bare field or coupling, and the (dressed) renormalised field or coupling can be found from the so-called **renormalisation constant** Z that renormalises the parameter. For example for a field ϕ, $\phi = Z_\phi \phi_0$, and for coupling g, $g = Z_g g_0$.

However, since μ is not an observable *per se* but in principle a choice of how to write down the theory to compare to an experiment (at which energy scale to write down the Lagrangian), the renormalised effective action S for a physical process should be invariant under a change of μ, which is expressed as:[12]

$$\mu \frac{d}{d\mu} S(\Phi, \lambda, \mu) = 0, \tag{6.14}$$

where λ is a generic name for the couplings of the theory and Φ represents the (super)fields that have been renormalised.[13] This equation can be re-written in terms of partial derivatives

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + n_\Phi \gamma_\Phi \right) S(\Phi, \lambda, \mu) = 0, \tag{6.15}$$

which is the **renormalisation group equation** (RGE). Here β_λ is the **$\boldsymbol{\beta}$-function**:

$$\beta_\lambda \equiv \mu \frac{\partial \lambda}{\partial \mu}. \tag{6.16}$$

which describes the behaviour of a Lagrangian parameter λ as a function of the energy scale μ away from the value where it was defined, often denoted μ_0. The **anomalous dimension**

[11] It is also impossible to avoid if we accept that the electron is a point particle. Since the potential has the form $V(r) \propto e/r$ an infinite energy would appear unless we were somehow to modify the charge at high energies, or equivalently, short distances.

[12] It is more common in the literature to find this expressed in terms of the Green's function $G^{(n)}$ for a given n-point correlation, *i.e.* a process with n-field insertions. In this form the equation is known as the **Callan–Symanzik equation**.

[13] In Sec. 4.9 we mentioned how the non-renormalisation theorem implies that we do not need to renormalise the coupling constants of the superpotential separately. In the MSSM this is the μ coupling (not to be confused with the energy scale here also called μ) and the Yukawa couplings. This will now have the consequence that their renormalisation can be expressed in terms of the renormalisation of the fields.

γ describes the scaling of the fields and the factor n gives the number of fields (of a given kind) in the effective action. The RGE balances the different contributions to sum to zero.

The reason for keeping around the factor of μ in the definition of the β-function is that it typically changes very slowly over large differences in energy scale, so it is practical to change variable to $t = \ln \frac{\mu}{\mu_0}$, so that $\mu \frac{\partial}{\partial \mu} = \frac{\partial}{\partial t}$ and $\beta_\lambda = \frac{\partial \lambda}{\partial t}$. If the β-functions of a quantum field theory are zero at some value of the couplings, then the value of the theory is said to be **scale-invariant**.

As an example of finding a β-function, take the relationship between a bare **gauge coupling constant** g_0 and the renormalised coupling g. This is given by (in $d = 4 - \epsilon$ dimensions):[14]

$$g_0 = Z_g g \mu^{-\epsilon/2}.$$

Then, differentiating both sides with respect to μ,

$$\begin{aligned}
0 &= \frac{\partial Z_g}{\partial \mu} g \mu^{-\epsilon/2} + Z_g \frac{\partial g}{\partial \mu} \mu^{-\epsilon/2} - \frac{\epsilon}{2} Z_g g \mu^{-\epsilon/2-1} \\
\mu \frac{\partial g}{\partial \mu} &= \frac{\epsilon}{2} g - \frac{g\mu}{Z_g} \frac{\partial Z_g}{\partial \mu} \\
\mu \frac{\partial g}{\partial \mu} &= \frac{\epsilon}{2} g - g\mu \frac{\partial}{\partial \mu} \ln Z_g,
\end{aligned}$$

and taking the limit $\epsilon \to 0$:

$$\beta_g \equiv \mu \frac{\partial g}{\partial \mu} = -g\gamma_g,$$

where we have defined the **anomalous dimension** of g

$$\gamma_g = \mu \frac{\partial}{\partial \mu} \ln Z_g. \tag{6.17}$$

The renormalisation constant Z_g can now be calculated to the required loop-order to find the β-function to that order, and in turn the running of the coupling constant with μ. By evaluating to one-loop order we can find that for our particular example of a gauge coupling constant for a generic supersymmetric model

$$\gamma_g \,|_{1-\mathrm{loop}} = -\frac{1}{16\pi^2} g^2 \left(\sum_R T(R) - 3C(A) \right), \tag{6.18}$$

where the sum of Dynkin indices $T(R)$ is over all superfields that transform under a representation R of the gauge group in question, and $C(A)$ is the **quadratic Casimir invariant** of the adjoint representation A of the vector field under the gauge group

$$C(A)\delta_{ij} = (T^a T^a)_{ij}.$$

For the adjoint representation of $U(1)$ this is 0, and for $SU(N)$ this is N. The running of the coupling constants is particularly important since it will later lead us to the concept of gauge coupling unification.

As a second relevant example, for the soft-breaking parameters M_i in (5.7) we have the one-loop β-functions

$$\beta_{M_i} \equiv \frac{d}{dt} M_i = \frac{1}{16\pi^2} g_i^2 M_i \left(2 \sum_R T(R) - 6C(A) \right). \tag{6.19}$$

[14]The factor $\mu^{-\epsilon/2}$ is there to ensure that the scale of g_0 is correct.

6.8 Gauge coupling unification

The one-loop β-functions for gauge couplings in a generic supersymmetric model were given in Eq. (6.18). With the MSSM field content and the gauge couplings discussed in this chapter:[15]

$$g_1 = \sqrt{\frac{5}{3}} g', \qquad g_2 = g, \qquad g_3 = g_s,$$

we arrive at

$$\beta_{g_i}|_{1-\text{loop}} = \frac{1}{16\pi^2}\, b_i g_i^3, \tag{6.20}$$

with in the MSSM

$$b_i^{\text{MSSM}} = \left(\frac{33}{5}, 1, -3\right).$$

For comparison, the same result in the Standard Model is

$$b_i^{\text{SM}} = \left(\frac{41}{10}, -\frac{19}{6}, -7\right).$$

The values of b_i for the MSSM are found from the Casimir invariant and the Dynkin index of the gauge group representations

$$C(A)_{SU(3)} = 3, \quad C(A)_{SU(2)} = 2, \quad C(A)_{U(1)} = 0,$$

and

$$T(R)_{SU(3)} = \frac{1}{2}, \qquad T(R)_{SU(2)} = \frac{1}{2}, \qquad T(R)_{U(1)} = \frac{3}{5} Y^2,$$

where for example $b_3 = \frac{1}{2} \cdot 12 - 3 \cdot 3 = -3$ in the MSSM, because, after careful counting, we have twelve quark/squark scalar superfields transforming under $SU(3)_C$.

At one-loop order we can do a neat rewrite using $\alpha_i \equiv \frac{g_i^2}{4\pi}$. Since

$$\frac{d}{dt}\alpha_i^{-1} = -2\frac{4\pi}{g_i^3}\frac{d}{dt}g_i,$$

we have:

$$\beta_{\alpha_i^{-1}} \equiv \frac{d}{dt}\alpha_i^{-1} = -\frac{8\pi}{g_i^3}\frac{1}{16\pi^2}g_i^3 b_i = -\frac{b_i}{2\pi}.$$

Thus α^{-1} runs linearly with t at one loop.

By running the couplings α_i^{-1} from their values measured at the electroweak scale to high energies it is observed that in the MSSM the coupling constants intersect at a single point, which they do not naturally do in the Standard Model. See Fig. 6.3, taken from Martin [15]. The common assumption made is then that a unified gauge group, *e.g.* $SU(5)$ or $SO(10)$, is broken at that scale, called the **grand unification theory** scale or GUT-scale, down to the Standard Model gauge group. This scale is $\mu_{\text{GUT}} \approx 2 \cdot 10^{16}$ GeV, about two orders of magnitude below the Planck scale.

[15] The normalisation choice for g_1 may seem a bit strange, however, this is the correct numerical factor when for example breaking a unified group such as SU(5) or SO(10) down to the Standard Model gauge group. This factor might be different with a different unified group.

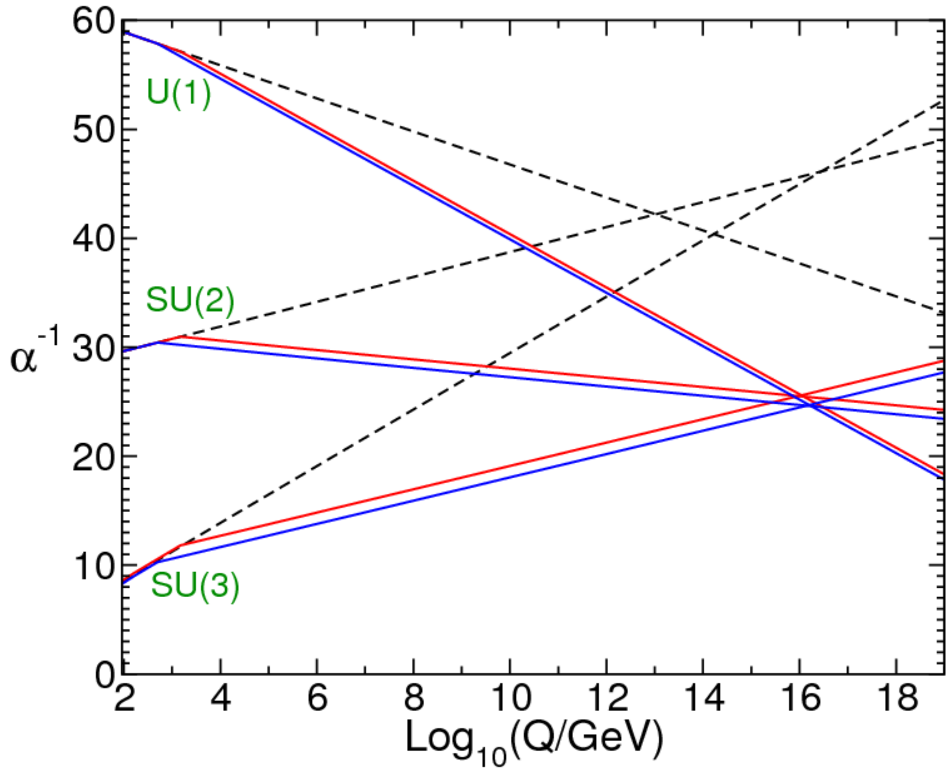

Figure 6.3: The RGE evolution of the inverse gauge couplings $\alpha_i^{-1}(Q)$ in the Standard Model (dashed lines) and the MSSM (solid lines). In the MSSM case, the sparticle mass thresholds are varied between 250 GeV and 1 TeV and $\alpha_3(m_Z)$ between 0.113 and 0.123 to create the bands shown by the red and blue lines. Two-loop effects are included.

Something funny happens to the gaugino soft-mass parameters M_i if we look at their running. From (6.19) the one-loop β-functions for the M_i in the MSSM are

$$\beta_{M_i}|_{1-\text{loop}} \equiv \frac{d}{dt}M_i = \frac{1}{8\pi^2}g_i^2 M_i b_i. \tag{6.21}$$

As a consequence all three ratios M_i/g_i^2 are scale independent at one loop. To see this let $R = M_i/g_i^2$, then

$$\beta_R \equiv \frac{dR}{dt} = \frac{\frac{dM_i}{dt}g_i^2 - M_i\frac{d}{dt}g_i^2}{g_i^4} = \frac{\frac{1}{8\pi^2}g_i^2 M_i b_i \cdot g_i^2 - M_i \cdot 2g_i \cdot \frac{1}{16\pi}g_i^3 b_i}{g_i^4} = 0. \tag{6.22}$$

In other words, R does not change with scale t.

If we now use that the coupling constants unify at the GUT scale to the coupling g_u, and assume that the soft-masses are the same at that scale $m_{1/2} = M_1(\mu_{\text{GUT}}) = M_2(\mu_{\text{GUT}}) = M_3(\mu_{\text{GUT}})$,[16] it follows that

$$\frac{M_1}{g_1^2} = \frac{M_2}{g_2^2} = \frac{M_3}{g_3^2} = \frac{m_{1/2}}{g_u^2}, \tag{6.23}$$

at all scales![17] This is a very powerful and predictive assumption. Because of the relationship

[16] Again, not unreasonable if the spontaneous symmetry breaking mechanism acts uniformly for all the gauginos.

[17] At one-loop level.

between the electroweak couplings and the electric charge, $e = g' \cos\theta_W = g \sin\theta_W$, it leads to the following relation

$$M_3 = \frac{\alpha_s}{\alpha} \sin^2\theta_W M_2 = \frac{3}{5}\frac{\alpha_s}{\alpha} \cos^2\theta_W M_1, \tag{6.24}$$

which, inserting values for the fine structure constant, the strong coupling, and the Weinberg angle, numerically predicts

$$M_3 : M_2 : M_1 \simeq 6 : 2 : 1$$

at the electroweak scale. We will return to the implications of this when discussing the gauginos in Sec. 6.12.

In Fig. 6.4, again taken from Martin [15], we show the running of all the types of soft parameters in the MSSM. We assume unified soft-mass parameters $m_{1/2}$ for the gauginos and m_0 for the Higgs and sfermions at the GUT scale. Shown is the gaugino mass parameters M_i (solid black), the Higgs mass parameters $m^2_{H_{d/u}}$ (dot-dashed green), the third generation sfermion soft terms m_{d_3}, m_{Q_3}, m_{u_3}, m_{L_3} and m_{e_3} (dashed red and blue, listed from top to bottom), and the corresponding first and second generation terms (solid red and blue lines).

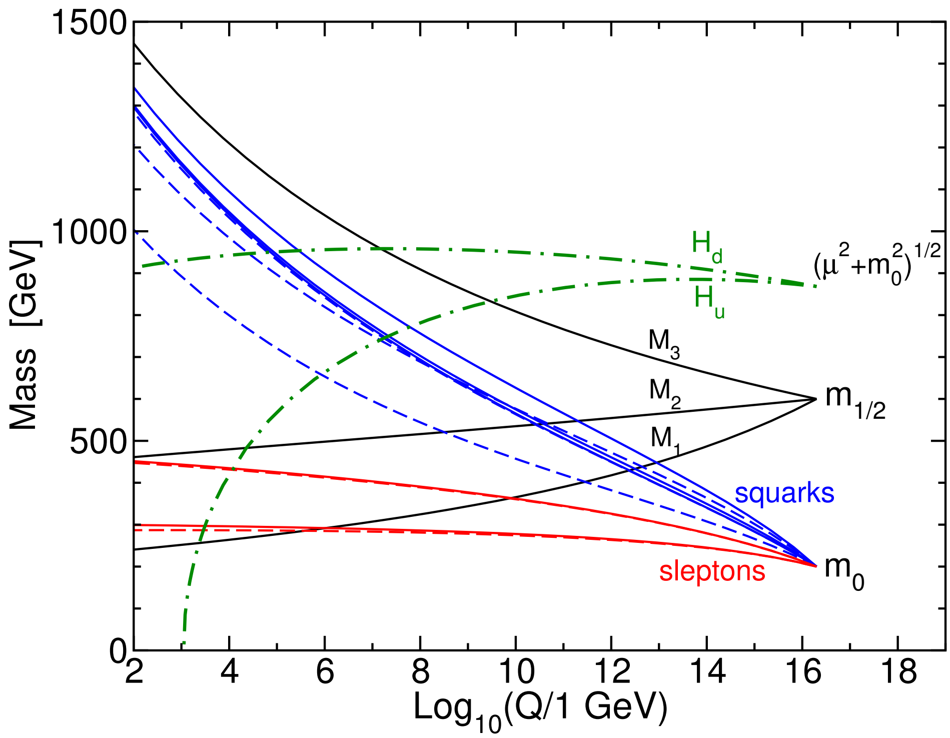

Figure 6.4: The RGE evolution of soft-mass parameters in the MSSM with typical minimal supergravity-inspired boundary conditions imposed at $2 \cdot 10^{16}$ GeV. The parameter values used for this illustration were $m_0 = 200$ GeV, $m_{1/2} = -A_0 = 600$ GeV, $\tan\beta = 10$, and $\mathrm{sgn}(\mu) = +$.

6.9 Radiative Electroweak Symmetry Breaking

In the Standard Model the gauge symmetries prevent mass terms for both vector bosons and fermions. The W and Z bosons, and all the fermions are given mass by spontaneously electroweak symmetry breaking (EWSB), which is induced by the shape of the scalar potential for the Higgs field H, which is an $SU(2)_L$ doublet of scalar fields:

$$V(H) = \mu^2|H|^2 + \lambda|H|^4, \tag{6.25}$$

with $|H|^2 = H^\dagger H$.

Here, the requirement for successful EWSB is that $\lambda > 0$ and $\mu^2 < 0$.[18] The first of these is a consistency requirement that ensures that the potential is **bounded from below**, *i.e.* that in the limit of large field values the potential does not turn to negative infinity. The second ensures that the minimum of the potential, the vacuum, is not given by zero field values, *i.e.* that the Higgs field has a **vacuum expectation value** (vev).

In supersymmetry we found the following general scalar potential in Eq. (4.24) for unbroken supersymmetry,

$$V(A, A^*) = \sum_i \left|\frac{\partial W}{\partial A_i}\right|^2 + \frac{1}{2}\sum_a g^2(A^*T^aA)^2 > 0, \tag{6.26}$$

where the first part is due to the elimination of the auxiliary F-fields in the scalar superfields, while the second part is due to the elimination of the auxiliary D-fields in the vector superfields. In addition we have to add all terms containing only relevant scalar fields from the soft breaking terms in Eq. 5.9.

For the scalar Higgs component fields in the MSSM this gives the potential

$$\begin{aligned}
V(H_u, H_d) \;=\; & |\mu|^2(|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2) && \text{(from F-terms)}\\
&+\frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 && \text{(from D-terms)}\\
&+\frac{1}{2}g^2|H_u^+H_d^{0*} + H_u^0H_d^{-*}|^2 &&\\
&+m_{H_u}^2(|H_u^0|^2 + |H_u^+|^2) + m_{H_d}^2(|H_d^0|^2 + |H_d^-|^2) && \text{(from soft breaking terms)}\\
&+[b(H_u^+H_d^- - H_u^0H_d^0) + \text{c.c.}] && (6.27)
\end{aligned}$$

This potential has 8 d.o.f. from 4 complex scalar fields H_u^+, H_u^0, H_d^0 and H_d^-. At the same time it has 6 parameters: the two Standard Model gauge couplings g and g', the magnitude of the supersymmetric parameter μ, and the three soft-breaking parameters b, $m_{H_u}^2$ and $m_{H_d}^2$. Notice how if $b = m_{H_u}^2 = m_{H_d}^2 = 0$, meaning no supersymmetry breaking terms, the Higgs potential has a global minimum at $V = 0$ for zero values of all the Higgs fields. In this case there is no EWSB, so EWSB is intimately connected to supersymmetry breaking in the MSSM.

We now want to do as in the Standard Model and break $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$ in order to give masses to gauge bosons and SM fermions.[19] To do this we need to show that, and under which conditions, Eq. (6.27) has: i) a minimum for finite, *i.e.* non-zero, field values, ii) that this minimum has a remaining $U(1)_{\text{em}}$ symmetry and iii) that the potential is bounded from below. We will here restrict our analysis to tree level, ignoring loop effects on the potential.

[18] This is called the **Mexican hat** or **wine bottle potential**, depending on preferences.

[19] You may ask why we can not use the soft-terms from the spontaneous breaking of *supersymmetry* to do this. However, the soft-terms are unable to effectively provide masses to vector bosons and fermions because they deal (mostly) with scalar fields.

We start by using our $SU(2)_L$ gauge freedom, picking a gauge so that we rotate away any field value for H_u^+ at the minimum of the potential. So without loss of generality we can use that $H_u^+ = 0$ at the minimum in what follows. At the minimum we must also have $\partial V/\partial H_u^+ = 0$ since it is a minimum, and by explicit differentiation of the potential one can show that $H_u^+ = 0$ then also leads to $H_d^- = 0$. This is good and proper since it guarantees our item ii), that $U(1)_{\rm em}$ is a symmetry for the minimum of the potential, since the charged fields then have no vevs.

We are now left with the following potential only in terms of the uncharged Higgs fields H_u^0 and H_d^0 (after the $SU(2)_L$ gauge choice and at the minimum):

$$
\begin{aligned}
V(H_u^0, H_d^0) &= (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 \\
&\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2 - (bH_u^0 H_d^0 + \text{c.c.})
\end{aligned}
\tag{6.28}
$$

We can now absorb a complex phase in H_u^0 or H_d^0, in order to take b to be real and positive. This does not affect other terms because they are protected by absolute values. The minimum must also have the total phase of $H_u^0 H_d^0$ real and positive, to get an as large as possible negative contribution from the b term, which is the only term that can be negative. Thus the vevs $v_u \equiv \langle H_u^0 \rangle$ and $v_d \equiv \langle H_d^0 \rangle$ must have opposite phases. By the remaining $U(1)_Y$ gauge symmetry of the potential, which is effectively a phase rotation, and the fact that H_u^0 and H_d^0 have opposite hypercharge, we can transform v_u and v_d so that they are real and have the same sign. For the potential to have a negative mass term, and thus fulfill point i) above, we must then have

$$
b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2). \tag{6.29}
$$

Since we have broken supersymmetry we must also check that the potential is actually bounded from below, our point iii), which was guaranteed for a supersymmetric vacuum. For large $|H_u^0|$ or $|H_d^0|$ the quartic gauge terms in (6.28) blows up to save the potential, except for $|H_u^0| = |H_d^0|$, the so-called D-flat directions. This means that we must also require

$$
2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2. \tag{6.30}
$$

To summarise what we have learnt so far: at the minimum of the Higgs potential we know that there exists a gauge choice so that the expectation values of the charged Higgs component fields are zero, $\langle H_u^+ \rangle = 0$ and $\langle H_d^- \rangle = 0$, and we fulfil the condition for the existence of an extremal point in the neutral Higgs component fields

$$
\frac{\partial V}{\partial H_u^0} = \frac{\partial V}{\partial H_d^0} = 0. \tag{6.31}
$$

In addition, for the minimum to have non-zero field values that break EWSB, and for the potential to be bounded from below, the parameters of the potential must simultaneously fulfil the inequalities

$$
\begin{aligned}
b^2 &> (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2), \\
2b &< 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2.
\end{aligned}
$$

The resulting non-zero expectation values at the minimum for the neutral Higgs component fields are denoted v_u and v_d.

To satisfy (6.29) and (6.30), a negative value for $m_{H_u}^2$ (or $m_{H_d}^2$) can help, in particular if $|\mu|^2 + m_{H_u}^2 < 0$, as that automatically fulfils (6.29). Such a negative value is indeed perfectly allowed as a parameter in the Lagrangian. No negative particle masses will result.

In fact, if we assume that $m_{H_d} = m_{H_u}$ at some scale μ, for example the GUT scale, then (6.29) and (6.30) cannot be simultaneously be satisfied *at that scale*. However, to 1-loop the RGE running of these mass parameters is:

$$16\pi^2 \beta_{m_{H_u}^2} \equiv 16\pi^2 \frac{dm_{H_u}^2}{dt} = 6|y_t|^2(m_{H_u}^2 + m_{Q_3}^2 + m_{u_3}^2) + ...$$

$$16\pi^2 \beta_{m_{H_d}^2} \equiv 16\pi^2 \frac{dm_{H_d}^2}{dt} = 6|y_b|^2(m_{H_d}^2 + m_{Q_3}^2 + m_{d_3}^2) + ...,$$

where y_t and y_b are the top and bottom quark Yukawa couplings, and $m_{Q_3} = m_{33}^Q$, $m_{u_3} = m_{33}^u$, and $m_{d_3} = m_{33}^d$, in a simplification of our previous notation for the soft-masses in Sec. 6.6. Because $y_t \gg y_b$, $m_{H_u}^2$ runs much faster with scale than $m_{H_d}^2$. If the parameters start out the same at some high scale, say from some universal supersymmetry breaking effect, as we go down to the electroweak scale $m_{H_u}^2$ becomes significantly smaller than $m_{H_d}^2$, and may become negative, fulfilling the EWSB criteria. For an illustration, see Fig. 6.4, where the running of $\mu^2 + m_{H_u}^2$ and $\mu^2 + m_{H_u}^2$ is shown. It is this property of starting our from some universal Higgs parameters at a high scale, which then by RGE effects break electroweak symmetry at lower scales, that is termed **radiative EWSB** (REWSB). Thus, in the MSSM with universal soft terms at a high scale there is an explanation why EWSB happens, it is not put in by hand in the potential as it is in the Standard Model.

Following EWSB, to get the familiar vector boson masses measured by experiment, the vevs need to satisfy the constraint from the electroweak parameters:

$$v_u^2 + v_d^2 \equiv v^2 = \frac{2m_Z^2}{g^2 + g'^2} \approx (174 \text{ GeV})^2. \tag{6.32}$$

Thus we have one free parameter coming from the two Higgs vevs in the MSSM. We can write this as

$$\tan\beta \equiv \frac{v_u}{v_d},$$

where by convention $0 < \beta < \pi/2$, so that $0 < \tan\beta < \infty$.

Using the condition for the existence of an extremal point in (6.31), the two non-SM parameters b and $|\mu|$ can be eliminated as free parameters from the model, however, not the sign of μ. Alternatively, we can choose to eliminate $m_{H_u}^2$ and $m_{H_d}^2$. You can look at this as giving away the freedom of these parameters to the vevs, and then fixing one vev by the electroweak constraint, and using $\tan\beta$ for the other.

Let us make a little remark here on the parameter μ. Given the criteria for REWSB above we have what is called the $\boldsymbol{\mu}$ **problem**. The soft terms all get their scale from some common mechanism at some common high energy scale, it is assumed, setting the parameters b, $m_{H_u}^2$ and $m_{H_d}^2$ in the Higgs potential. However, μ is a mass term in the superpotential (the only one in fact) and could *a priori* take *any* value, even M_P. Why is μ then of the order of the soft terms, which is what allows us to achieve REWSB, when a much larger value would prevent us from fulfilling the criteria in (6.29) and (6.30)?[20]

[20]This problem can be solved in extensions of the MSSM such as the Next-to-Minimal Supersymmetric Standard Model (NMSSM).

6.10 Higgs boson properties

Of the eight d.o.f. in the scalar potential for the Higgs component fields three are Goldstone bosons that get eaten by Z and $W^\pm$ to give them masses. The remaining five d.o.f. form two neutral scalars h, H, two charged scalars $H^\pm$ and one neutral pseudo-scalar (CP-odd) A.[21] At tree level one can show that these have the masses:

$$\begin{aligned} m_A^2 &= \frac{2b}{\sin 2\beta} = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2, && (6.33)\\ m_{h,H}^2 &= \frac{1}{2}\left(m_A^2 + m_Z^2 \mp \sqrt{(m_A^2 - m_Z^2)^2 + 4m_Z^2 m_A^2 \sin^2 2\beta}\right), && (6.34)\\ m_{H^\pm}^2 &= m_A^2 + m_W^2. && (6.35)\end{aligned}$$

As a consequence m_A and $\tan\beta$ can be used to parametrise the masses of the Higgs sector (at tree level), and H, $H^\pm$ and A are in principle unbounded in mass since they grow as $b/\sin 2\beta$. However, at tree level the lightest Higgs boson is restricted to

$$m_h < m_Z|\cos 2\beta| < 91.2 \text{ GeV}. \tag{6.36}$$

In contrast we have the current best measurement of the Higgs boson mass of $m_h = 125.10 \pm 0.14\,\text{GeV}$, combining results from the LHC [16].

Fortunately, there are large loop-corrections or the MSSM would have been excluded already.[22] Because of the size of the Yukawa couplings the largest corrections to the mass of the lightest Higgs comes from loops with top quarks and its supersymmetric partners, the scalar top quarks, or stops, $\tilde{t}_L$ and $\tilde{t}_R$. See Fig. 4.1 for the relevant Feynman diagrams. In the limit where the mass of the stop quarks are larger than the top, $m_{\tilde{t}_R}, m_{\tilde{t}_L} \gg m_t$, and with stop mass eigenstates close to the chiral eigenstates (more on this later), we get the dominant loop correction

$$\Delta m_h^2 = \frac{3}{4\pi^2}\cos^2\alpha\, y_t^2 m_t^2 \ln\left(\frac{m_{\tilde{t}_L} m_{\tilde{t}_R}}{m_t^2}\right), \tag{6.37}$$

where α is a mixing angle for h and H with respect to the superfield component fields H_u^0 and H_d^0, given by

$$\frac{\sin\alpha}{\sin\beta} = -\frac{m_H^2 + m_h^2}{m_H^2 - m_h^2}, \tag{6.38}$$

at tree level.

With this and other corrections the upper bound on the lightest Higgs boson mass is weaker:

$$m_h \leq 135 \text{ GeV},$$

assuming a common sparticle mass scale of around $m_{\text{SUSY}} \leq 1\,\text{TeV}$. Higher values for the sparticle masses give large fine-tuning and weaken the bound very little because of the logarithm in Eq. (6.37). The bound can be further weakened by adding extra field content to the MSSM, *e.g.* as in the NMSSM, but there is an upper perturbative limit of $m_h \approx 150\,\text{GeV}$.

[21] In addition to the scalars, we know that the Higgs supermultiplets contain four fermions, $\tilde{H}_u^0$, $\tilde{H}_d^0$, $\tilde{H}_u^+$ and $\tilde{H}_d^-$ (higgsinos). We will see later that these mix with the fermion partners of the gauge bosons (gauginos).

[22] It is worth pointing out here that the MSSM, despite its many parameters, is a falsifiable theory. For example, had the Higgs boson mass been $\sim 15\,\text{GeV}$ higher, which is perfectly allowed in the Standard Model, the MSSM would have been excluded.

It is very interesting to discuss what the Higgs discovery actually implies for low-energy supersymmetry. As can be seen from the above numbers it requires rather large squark masses even in the favourable scenario with $\tan\beta \gg 1$ where the tree level mass is $m_h \sim 90$ GeV. A naive estimate from Eq. (6.37) gives $m_{\tilde{t}} > 1$ TeV. However, this does not take into account possible negative contributions to the Higgs mass from heavy gauginos (fermions in the vector superfields), and possible increases in the stop contribution due to tuning of the mixing of the chiral eigenstates $\tilde{t}_L$ and $\tilde{t}_R$, in the mass eigenstates $\tilde{t}_1$ and $\tilde{t}_2$.

Since the lightest stop quark is expected to be the lightest squark in scenarios with universal soft masses at some high scale – the reasoning here is the large downward RGE running of m^Q_{33} from a common squark soft mass at some high scale due to the large top Yukawa coupling – the expected sparticle spectrum lies mostly above 1 TeV, with the possible exception of gauginos/higgsinos. This points to so-called **Split-SUSY** scenarios with heavy scalars and light gauginos, and a relatively large degree of fine-tuning. If one can live with this little hierarchy problem, it will explain why no signs of supersymemtry have been seen yet at the LHC.

If you are willing to accept fine-tuning of the stop mixing instead, or come up with a good reason for why the mixing should be just-so to give a maximal Higgs mass, you can keep fairly light stop quarks. With the addition of light higgsinos and a light gluino the model is then technically natural, these scenarios are called **Natural SUSY** and could be within the current or near future reach of the LHC. The problem with these models, as we shall see, is that the higgsinos are degenerate, and thus difficult to detect.

To do calculations with the Higgs bosons in the MSSM we need the Feynman rules that result from the relevant Lagrangian terms. Since these have been listed elsewhere we will not repeat them here, but recommend in particular the PhD-thesis of Peter Richardson [17], where they can be found in Appendix A.6, including all interactions with fermions and sfermions. These can also be found, together with all gauge and self-interactions, in the classic paper by Gunion and Haber [18]. Note that in this paper a complex Higgs singlet appears in some interactions because they perform their calculations in the NMSSM, but this can safely be ignored and all other results carry over into the MSSM.

6.11 The gluino

The fermion partner of the Standard Model gluon g is called the **gluino** $\tilde{g}$, and as the gluon it is a colour octet Majorana fermion. We usually talk about the gluino as being one particle, however, as an adjoint representation of $SU(3)$ there are actually eight (thus octet) distinct gluons, and we write $\tilde{g}^a$ when we want to make the distinction. As a colour octet it has nothing to mix with in the MSSM – this is still true even if we allow for R-parity violation – and at tree level the mass is given by the soft term M_3. Since it lives in the same superfield as the massless gluon it would otherwise had zero mass.

The one complication for the gluino is that it is strongly interacting so $M_3(\mu)$ runs, relatively speaking, quickly with energy scale μ. It is useful to instead talk about the scale-independent **pole-mass** $m_{\tilde{g}}$, meaning the pole of the renormalised propagator,

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})},$$

where m_0 is the Lagrangian mass, Σ is the self-energy, and the pole mass is the solution $\not{p} = m$ to the equation $\not{p} - m_0 - \Sigma(\not{p}) = 0$. For the gluino, including all one-loop effects in the self-energy due to gluon exchange and squark loops, see Fig. 6.5, in the $\overline{DR}$ renormalisation scheme we get:[23]

$$m_{\tilde{g}} \simeq M_3(\mu)\left[1+\frac{\alpha_s}{4\pi}\left(15+6\ln\frac{\mu}{M_3}+\sum_{\text{all}\,\tilde{q}}A_{\tilde{q}}\right)\right],$$

where the squark loop contribution $A_{\tilde{q}}$ for a given squark depends on the squark $m_{\tilde{q}}$ and corresponding quark m_q masses, and are given by

$$A_{\tilde{q}} = \int_0^1 dx\, x\ln\left(x\frac{m_{\tilde{q}}^2}{M_3^2}+(1-x)\frac{m_q^2}{M_3^2}-x(1-x)-i\epsilon\right).$$

Due to the 15-factor the correction can be significant (colour factor).

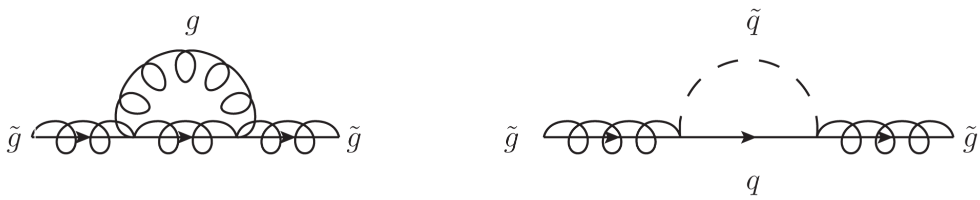

Figure 6.5: One loop contributions to the gluino mass.

Complete Feynman rules for gluinos can be found in Appendix C of the classic MSSM reference paper of Haber & Kane [19]. A more comprehensible alternative may be Appendix A.3 from the PhD-thesis of Bolz [20]. This thesis also provides a diagramatic prescription of how to handle clashing fermion lines that can appear with Majorana fermions such as the gluino.

6.12 Neutralinos & Charginos

In the MSSM we have a bunch of fermion fields that can mix when electroweak symmetry is broken and we do not have to care about the $SU(2)_L \times U(1)_Y$ charges of the fields, only the charges under the remaining $U(1)_{\text{em}}$ symmetry matter. The candidates for mixing are the (Majorana) fermions from the $U(1)$ and $SU(2)$ vector superfields B^0 and W^a called the **gauginos**:

$\tilde{B}^0$ (**bino**), $\tilde{W}^0$ (neutral **wino**), $\tilde{W}^\pm$ (charged **wino**),

and the fermions from the Higgs superfields H_u and H_d, called **higgsinos**:

$$\tilde{H}_u^+,\quad \tilde{H}_u^0,\quad \tilde{H}_d^- \quad\text{and}\quad \tilde{H}_d^0.$$

Together these are called the **electroweakinos**.

[23]Note that the right-hand side here *is* μ dependent since the expression is only to finite order.

In the Standard Model the neutral gauge fields B^0_μ and W^0_μ mix into the photon γ and the Z-boson. The neutral gauginos can mix the same way into the states

$$\tilde{\gamma} = N'_{11}\tilde{B}^0 + N'_{12}\tilde{W}^0 \qquad (\textbf{photino}), \tag{6.39}$$

$$\tilde{Z} = N'_{21}\tilde{B}^0 + N'_{22}\tilde{W}^0 \qquad (\textbf{zino}). \tag{6.40}$$

However, more generally, they also mix with the neutral higgsinos to form the four **neutralinos**:[24]

$$\tilde{\chi}^0_i = N_{i1}\tilde{B}^0 + N_{i2}\tilde{W}^0 + N_{i3}\tilde{H}^0_d + N_{i4}\tilde{H}^0_u, \quad i = 1, 2, 3, 4, \tag{6.41}$$

where N_{ij} indicates size of the component of each of the fields in the **gauge eigenstate basis**

$$\tilde{\psi}^{0T} = \left(\tilde{B}^0,\, \tilde{W}^0,\, \tilde{H}^0_d,\, \tilde{H}^0_u\right). \tag{6.42}$$

In the gauge eigenstate basis the neutralino mass term can be written as

$$\mathcal{L}_{\chi-\text{mass}} = -\frac{1}{2}\tilde{\psi}^{0T} M_{\tilde{\chi}} \tilde{\psi}^0 + \text{c.c.},$$

where the mass matrix $M_{\tilde{\chi}}$ is found from the bilinear terms in the Lagrangian with gauge eigenstates to be

$$M_{\tilde{\chi}} = \begin{bmatrix} M_1 & 0 & -\frac{1}{\sqrt{2}}g'v_d & \frac{1}{\sqrt{2}}g'v_u \\ 0 & M_2 & \frac{1}{\sqrt{2}}gv_d & -\frac{1}{\sqrt{2}}gv_u \\ -\frac{1}{\sqrt{2}}g'v_d & \frac{1}{\sqrt{2}}gv_d & 0 & -\mu \\ \frac{1}{\sqrt{2}}g'v_u & -\frac{1}{\sqrt{2}}gv_u & -\mu & 0 \end{bmatrix}. \tag{6.43}$$

In this matrix, the upper left diagonal part comes from the soft terms for the $\tilde{B}^0$ and the $\tilde{W}^0$, the lower right off diagonal matrix comes from the superpotential term $\mu H_u H_d$, while the remaining entries come from Higgs-higgsino-gaugino terms from the kinetic part of the Lagrangian, *e.g.* $H_u^\dagger e^{\frac{1}{2}g\sigma W + g'B} H_u$, which become mass terms when one of the neutral Higgs component fields acquires a vev. With the Z-mass condition on the vevs (6.32) we can also write

$$\frac{1}{\sqrt{2}}g'v_d = \cos\beta\sin\theta_W m_Z, \tag{6.44}$$

$$\frac{1}{\sqrt{2}}g'v_u = \sin\beta\sin\theta_W m_Z, \tag{6.45}$$

$$\frac{1}{\sqrt{2}}gv_d = \cos\beta\cos\theta_W m_Z, \tag{6.46}$$

$$\frac{1}{\sqrt{2}}gv_u = \sin\beta\cos\theta_W m_Z. \tag{6.47}$$

The mass matrix can now be diagonalised to find the $\tilde{\chi}^0_i$ masses. If N is a unitary diagonalisation matrix for $M_{\tilde{\chi}}$, we can write[25]

$$\mathcal{L}_{\chi-\text{mass}} = -\frac{1}{2}\tilde{\psi}^{0T} N^T N^* M_{\tilde{\chi}} N^\dagger N \tilde{\psi}^0 + \text{c.c.}, = -\frac{1}{2}\tilde{\chi}^{0T} D \tilde{\chi}^0 + \text{c.c.},$$

[24] The neutral higgsinos are also Majorana fermions despite coming from scalar superfields. Unlike the (s)fermion superfields the Higgs superfields have no $\bar{H}$ chiral partners to supply the left–right Weyl spinor combinations required for Dirac fermions. Thus the neutralinos are Majorana fermions.

[25] A symmetric matrix is always unitary diagonalisable.

where $D = N^* M_{\tilde{\chi}} N^\dagger$ is diagonal and contains the real and non-negative neutralino masses $m_{\tilde{\chi}_i^0} \geq 0$ that are the eigenvalues of $M_{\tilde{\chi}}$. We also see that N gives the mixing of the gauge eigenstates $\tilde{\psi}^0$ into the mass eigenstates $\tilde{\chi}^0 = N\tilde{\psi}^0$. We number the neutralino mass eigenstates in (6.41), or, equivalently, sort the mass eigenvalues after diagonalisation, so that the neutralinos are numbered from lightest to heaviest. The neutralinos also have loop corrections to their masses coming from the self energies, however, since the coupling is weak – in the technical sense – these are usually significantly smaller than for the gluino.

The mass parameters of the neutralino mass matrix may in general be complex, leading to complex entries in N. Redefinition of fields can rotate away either the M_1 or M_2 phase, to make the parameter real and positive, but not both of these, and not the μ-phase. These phases give rise to problematic CP-violation that can easily be in contradiction with experiments. Therefore, M_1, M_2 and μ are often just assumed to be real in order not to violate experimental bounds. In this case a diagonalisation matrix N can be found that is orthogonal, meaning with only real entries, which simplifies calculations. In this case the diagonal mass values in D are not guaranteed to be positive. This does not imply negative fermion masses, but instead indicates a phase factor that must be incorporated into Feynman rules for the interactions of the mass eigenstates.

One particularly interesting solution to the diagonalisation is in the limit where EWSB is a small effect, $m_Z \ll |\mu \pm M_1|$, $|\mu \pm M_2|$, and when we have the hierarchy $M_1 < M_2 \ll |\mu|$. The mass eigenvalues scale with the size of the supersymmetric parameters, which makes the lightest neutralino bino-like, $\tilde{\chi}_1^0 \approx \tilde{B}^0$, the next-to-lightest wino like, $\tilde{\chi}_2^0 \approx \tilde{W}^0$, and $\tilde{\chi}_{3,4}^0 \approx \frac{1}{\sqrt{2}}(\tilde{H}_d^0 \pm \tilde{H}_u^0)$, and the masses are to first order in $1/\mu$:

$$m_{\tilde{\chi}_1^0} = M_1 + \frac{m_Z^2 \sin^2\theta_W \sin 2\beta}{\mu} + \ldots \tag{6.48}$$

$$m_{\tilde{\chi}_2^0} = M_2 - \frac{m_W^2 \sin 2\beta}{\mu} + \ldots \tag{6.49}$$

$$m_{\tilde{\chi}_{3,4}^0} = |\mu| + \frac{m_Z^2}{2\mu}(\operatorname{sgn}\mu \mp \sin 2\beta) + \ldots \tag{6.50}$$

Since the LSP is stable in R-parity conserving theories the lightest neutralino is an excellent candidate for dark matter. In particular since a neutralino with mass around 100 GeV has a natural relic density close to the measured dark matter density of the Universe. We will return to this issue in Chapter 8.

From the charged electroweakinos we can make **charginos** $\tilde{\chi}_i^\pm$ that are Dirac fermions with mass term

$$\mathcal{L}_{\chi^\pm-\text{mass}} = -\frac{1}{2}\tilde{\psi}^{\pm T} M_{\chi^\pm} \tilde{\psi}^\pm + \text{c.c.},$$

where the gauge eigenstate basis is $\tilde{\psi}^{\pm T} = (\tilde{W}^+, \tilde{H}_u^+, \tilde{W}^-, \tilde{H}_d^-)$, and the mass matrix is given by

$$M_{\tilde{\chi}^\pm} = \begin{bmatrix} 0 & 0 & M_2 & gv_d \\ 0 & 0 & gv_u & \mu \\ M_2 & gv_u & 0 & 0 \\ gv_d & \mu & 0 & 0 \end{bmatrix}.$$

Here the M_2 terms come from the soft terms for the charged winos $\tilde{W}^\pm$, the μ terms come from the superpotential as above, while the remaining terms come from the kinetic terms.

We can here re-write

$$\begin{aligned} gv_d &= \sqrt{2}\cos\beta\, m_W, && (6.51)\\ gv_u &= \sqrt{2}\sin\beta\, m_W. && (6.52)\end{aligned}$$

Diagonalising this mass matrix gives the mass eigenstates $\tilde{\chi}_i^{\pm}$, $i = 1, 2$. The eigenvalues are doubly degenerate, giving the same masses to the $\tilde{\chi}_i^+$ and $\tilde{\chi}_i^-$ particle and anti-particle pairs, and are explicitly given as:

$$m_{\tilde{\chi}^{\pm}_{1,2}} = \frac{1}{2}\left(|M_2|^2 + |\mu|^2 + 2m_W^2 \mp \sqrt{(|M_2|^2 + |\mu|^2 + 2m_W^2)^2 - 4|\mu M_2 - m_W^2 \sin^2\beta|^2}\right).$$

In the same limit of small EWSB effects discussed above we have a wino-like lightest chargino, $\tilde{\chi}_1^{\pm} \approx \tilde{W}^{\pm}$, and a higgsino-like heavy chargino, $\tilde{\chi}_2^{\pm} \approx \tilde{H}_u^+/\tilde{H}_d^-$, with masses

$$\begin{aligned} m_{\tilde{\chi}_1^{\pm}} &= M_2 - \frac{m_W^2}{\mu}\sin 2\beta + \ldots, && (6.53)\\ m_{\tilde{\chi}_2^{\pm}} &= |\mu| + \frac{m_W^2}{\mu}\mathrm{sgn}\,\mu + \ldots. && (6.54)\end{aligned}$$

Note that in this limit $m_{\tilde{\chi}_2^0} \simeq m_{\tilde{\chi}_1^{\pm}}$ since they are both wino-like and governed by the M_2 soft mass.

We saw earlier that the soft-mass ratio

$$M_3 : M_2 : M_1 \simeq 6 : 2 : 1,$$

appears at a scale of around $\mu = 1\,\mathrm{TeV}$ if the same soft-masses unify at the GUT-scale. From our above discussion, as long as $|\mu| \gg M_1, M_2$, this gives the very predictive mass relationships $m_{\tilde{g}} \simeq 6m_{\tilde{\chi}_1^0}$, $m_{\tilde{\chi}_2^0} \simeq m_{\tilde{\chi}_1^{\pm}} \simeq 2m_{\tilde{\chi}_1^0}$. However, it is important to remember that this often used relationship is based on the *conjecture* of gauge coupling unification, and the unification of gaugino soft masses!

We should mention that some authors prefer other symbols for the neutralinos and charginos. Common examples are $\tilde{N}_i$ or $\tilde{Z}_i$ for neutralinos, and $\tilde{C}_i$ or $\tilde{W}_i$ (again!) for the charginos.

Feynman rules for charginos & neutralinos can again be found in Haber & Kane [19].

6.13 Sleptons & Squarks

The **sfermions**, the scalar partners of the Standard Model fermions, the quarks and leptons, consists of the **squarks** and the **sleptons**. These inherit the interactions of their partner fermions since they live in the same superfields.

For their masses, reading of from the MSSM Lagrangian, including the possible soft-breaking terms, there are multiple tree-level contributions to the sfermion masses. In the following discussion $\tilde{F}_i$ represents a generic $SU(2)_L$ doublet of sfermions with generation index i, for example $\tilde{Q}_i = (\tilde{u}_{iL}, \tilde{d}_{iL})$, while $\tilde{f}_{iR}$ represents a singlet, for example $\tilde{u}_{iR}$.

We can make the following list of mass terms:

i) Under the reasonable assumption that soft masses are (close to) diagonal[26] the sfermions get contributions $-m_{F_i}^2 \tilde{F}_i^\dagger \tilde{F}_i$ and $-m_{f_i}^2 \tilde{f}_{iR}^* \tilde{f}_{iR}$ from the soft terms in (6.13). These are typically dominant.

ii) There are F-term contributions that come from Yukawa terms in the superpotential of the form $y_f F H \bar{K}$, where F and $\bar{K}$ are two scalar superfields with sfermions, and H is one of the two Higgs superfields. From the contribution $\sum |W_i|^2$ to the scalar potential these give Lagrangian terms $y_f^2 H^{0*} H^0 \tilde{f}_{iL}^* \tilde{f}_{iL}$ and $y_f^2 H^{0*} H^0 \tilde{f}_{iR}^* \tilde{f}_{iR}$. After EWSB when the Higgs field gets a vev we then get the mass terms $m_f^2 \tilde{f}_{iL}^* \tilde{f}_{iL}$ and $m_f^2 \tilde{f}_{iR}^* \tilde{f}_{iR}$, where $m_f = v_{u/d}\, y_f$. These are only significant for large Yukawa coupling y_f, and give the same mass as their Standard Model fermion partner gets from the same Yukawa terms.

iii) There are also so-called **hyperfine** terms that come from D-terms $\sum g^2 (A^* T^a A)^2$ in the scalar potential that give Lagrangian terms of the form (sfermion)^2(Higgs)^2 when one of the scalar fields A is a neutral Higgs field, and the other is a sfermion. Under EWSB, when the Higgs field gets a vev these become mass terms. They contribute with a mass

$$\Delta_F = (T_{3F} g^2 - Y_F g'^2)(v_d^2 - v_u^2) = (T_{3F} - Q_F \sin^2\theta_W) \cos 2\beta\, m_Z^2,$$

where the weak isospin, T_3, hypercharge, Y, and electric charge, Q, are for the left-handed supermultiplet F to which the sfermion belongs. However, these contributions are usually quite small.

iv) Furthermore, there are also F-terms that combine scalars from the $\mu H_u H_d$ term and Yukawa terms $y_f F H \bar{K}$ in the superpotential. These give Lagrangian terms $-\mu^* H^{0*} y_f \tilde{f}_L \tilde{f}_R^*$. With a Higgs vev this gives mass terms $-\mu^* v_{u/d}\, y_f \tilde{f}_R^* \tilde{f}_L + \text{c.c.}$

v) Finally, the soft Yukawa terms of the form $a_f \tilde{F} H \tilde{f}_R^*$ with a Higgs vev give mass terms $a_f v_{u/d} \tilde{f}_L \tilde{f}_R^* + \text{c.c.}$[27]

For the first two generations of sfermions, terms of type ii), iv) and v) are small due to small Yukawa couplings. Then the sfermion masses are for example

$$m_{\tilde{u}_L}^2 = m_{Q_1}^2 + \Delta\tilde{u}_L, \tag{6.55}$$
$$m_{\tilde{d}_L}^2 = m_{Q_1}^2 + \Delta\tilde{d}_L, \tag{6.56}$$
$$m_{\tilde{c}_L}^2 = m_{Q_2}^2 + \Delta\tilde{c}_L, \tag{6.57}$$
$$m_{\tilde{s}_L}^2 = m_{Q_2}^2 + \Delta\tilde{s}_L, \tag{6.58}$$
$$m_{\tilde{u}_R}^2 = m_{u_1}^2 + \Delta\tilde{u}_R \tag{6.59}$$
$$m_{\tilde{d}_R}^2 = m_{d_1}^2 + \Delta\tilde{d}_R \tag{6.60}$$
$$m_{\tilde{s}_R}^2 = m_{u_2}^2 + \Delta\tilde{u}_R. \tag{6.61}$$

Mass splitting between same generation slepton/squark is then given by the hyperfine splitting

$$m_{\tilde{e}_L}^2 - m_{\tilde{\nu}_{eL}}^2 = m_{\tilde{d}_L}^2 - m_{\tilde{u}_L}^2 = -\frac{1}{2} g^2 (v_d^2 - v_u^2) = -\cos 2\beta\, m_W^2,$$

[26] This assumption is of course made to avoid flavour changing neutral currents (FCNCs). However, it is also reasonable in that if the soft masses are diagonal, or even all the same, at a high scale, the RGE running will not create large off-diagonal terms.

[27] We often assume that $a_f = A_0 y_f$ in order to further reduce the FCNC, meaning that there is a global constant A_0 with unit mass relating the Yukawa couplings and the trilinear A-term couplings.

since they have the same hypercharge, see Table 6.1. For $\tan\beta > 1$ this gives the definite prediction $m^2_{\tilde{e}_L} > m^2_{\tilde{\nu}_{eL}}$ and $m^2_{\tilde{d}_L} > m^2_{\tilde{u}_L}$.

The **third generation sfermions** $\tilde{t}$, $\tilde{b}$ and $\tilde{\tau}$ have a more complicated mass matrix structure, *e.g.* in the gauge eigenstate basis $(\tilde{t}_L, \tilde{t}_R)$ for stop quarks the mass term is

$$\mathcal{L}_{\text{stop}} = -\begin{pmatrix}\tilde{t}_L^* & \tilde{t}_R^*\end{pmatrix} m^2_{\tilde{t}} \begin{pmatrix}\tilde{t}_L \\ \tilde{t}_R\end{pmatrix},$$

where the mass matrix is given by

$$m^2_{\tilde{t}} = \begin{bmatrix} m^2_{Q_3} + m_t^2 + \Delta\tilde{u}_L & v(a_t^*\sin\beta - \mu y_t\cos\beta) \\ v(a_t\sin\beta - \mu^* y_t\cos\beta) & m^2_{u_3} + m_t^2 + \Delta\tilde{u}_R \end{bmatrix}. \tag{6.62}$$

Here the diagonal elements come from i), ii) and iii), while the off-diagonal elements come from iv) and v).

To find the particle masses, we must diagonalise this matrix, writing it in terms of the mass eigenstates $\tilde{t}_1$ and $\tilde{t}_2$, acquiring also here a unitary mixing matrix for the mass eigenstates in terms of the gauge eigenstates $\tilde{t}_L$ and $\tilde{t}_R$:

$$\begin{pmatrix}\tilde{t}_1 \\ \tilde{t}_2\end{pmatrix} = \begin{bmatrix} c_{\tilde{t}} & -s^*_{\tilde{t}} \\ s_{\tilde{t}} & c_{\tilde{t}} \end{bmatrix} \begin{pmatrix}\tilde{t}_L \\ \tilde{t}_R\end{pmatrix}, \tag{6.63}$$

where the matrix entries are related by $|c_{\tilde{t}}|^2 + |s_{\tilde{t}}|^2 = 1$ and $m^2_{\tilde{t}_1} < m^2_{\tilde{t}_2}$ are the eigenvalues of (6.62). The suggestive form of the mixing matrix indicates that if the off-diagonal elements of the original mass matrix has only real elements, this mixing matrix can be written as an element in $SO(2)$, using sine and cosine of a mixing angle $0 \leq \theta_{\tilde{t}} < \pi$, $c_{\tilde{t}} = \cos\theta_{\tilde{t}}$ and $s_{\tilde{t}} = \sin\theta_{\tilde{t}}$. The matrices for $\tilde{b}$ and $\tilde{t}$ have the same structure.

Since the third generation sneutrino $\tilde{\nu}_{eL}$ does not have a corresponding right-handed state in the MSSM, there is no mixing, and it has the same mass term as the first and second generation sneutrinos.

A good source for sfermion interaction Feynman rules is the PhD-thesis of Richardson [17].

6.14 Excercises

Exercise 6.1

Using the explicit form of the $SU(3)_C$ transformations with the Gell-Mann matrices, show that with our definition of the superpotential term $\bar{U}_i\bar{D}_j\bar{D}_k$ this is invariant under $SU(3)_C$.

Exercise 6.2

Show how you can eliminate the parameters $|\mu|$ and b by using the properties of the minimum of the potential in Eq. (6.28).

Exercise 6.3

Show Eqs. (6.44)–(6.47).

Chapter 7

Sparticle phenomenology

In this chapter we discuss the phenomenology of supersymmetric models and how to search for low-energy particle realisations of supersymmetry in experiments. We begin by returning to supersymmetry breaking in order to define some reasonable and (partially) motivated subsets of the 124 MSSM parameters which can be used to define more constrained models. We then discuss supersymmetry at lepton and hadron colliders, and finally look at precision measurements that are indirectly sensitive to the existence of sparticles.

7.1 Supersymmetry at lepton colliders

High-energy lepton colliders have traditionally been circular e^+e^--colliders. These have the advantage of a well known centre-of-mass (CoM) energy given by the total energy of the electron–positron pair, clean final states, and periodic application of the accelerating gradient. The challenge is to reach high energies and high luminosities (collision rates). Since the electrons are light they radiate a lot of bremsstrahlung photons when bent in orbits. The highest energy so-far at an e^+e^--collider was the 209 GeV CoM-energy at LEP2 in 2000. Plans are being made both for linear e^+e^--colliders, and a muon collider where there is less bremsstrahlung because of the higher muon mass, meaning that higher energies can be reached. However, there are significant technical challenges ahead for both. The linear colliders need very long installation tunnels and a very high accelerating gradient, while the muon collider must be able to produce and store the unstable muons.

Most supersymmetry searches at lepton colliders rely on the pair production of oppositely charged sparticles with electroweak couplings from $e^+e^- \to \gamma^*/Z^* \to S\bar{S}$, where S symbolises a generic sparticle. Due to R-parity conservation these sparticles both decay to lighter sparticles and Standard Model decay products, until they, potentially after several successive decays known as a **cascade decay**, leave only the stable LSP. This LSP must be electrically (and most likely colour) neutral due to experimental constraints on massive long-lived charged particles that would bind to atomic nuclei. The neutrality of the LSP means that it escapes detectors unseen.

The search for supersymmetry thus focuses on events with the Standard Model decay products of sparticles and an imbalance in momentum conservation due to the two missing LSPs. By the measured sum of the momenta of all the visible decay products the sum of the total momenta for the invisible particles can be inferred as going in the opposite direction. This is known as a **missing energy** signature.

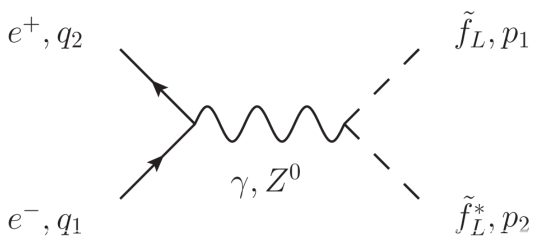

Figure 7.1: Feynman diagram for the pair production of left-handed sfermions in the s-channel at an e^+e^--collider.

In practice such missing energy measurements are challenging experimentally due to the absence of detectors near the incoming beams in the longitudinal direction, where particles that are in principle visible may escape undetected and create an artificial momentum imbalance in the longitudinal direction. At high energies this is exacerbated by the increase of collinear bremsstrahlung from the incoming electrons at the interaction point due to interaction of the beams. This will be a particularly difficult for a future $0.5 - 3.0$ TeV CoM International Linear Collider (ILC) or the Compact LInear Collider (CLIC) project.[1]

We can now discuss more specifically the search for sfermions, neutralinos & charginos, and Higgs bosons at lepton colliders.

7.1.1 Sfermions

We can estimate the leading order amplitude of the s-channel sfermion pair production process shown in Fig. 7.1. We being by writing down the matrix element with an intermediary γ as:

$$\mathcal{M} = \overline{v} i e \gamma^\mu u \frac{-i g_{\mu\nu}}{k^2 + i\epsilon}[-ie \cdot e_f (p_1 - p_2)^\nu], \tag{7.1}$$

which gives a squared matrix element of, assuming that the CoM s is much greater than m_Z and taking into account both the photon and the Z,

$$|\mathcal{M}|^2 \simeq \frac{g^4 e_f^2}{8 \cos\theta_W} \frac{st + (m_{\tilde{f}}^2 - t)^2}{s^2} \times (1 + (4\sin^2\theta_W - 1)^2). \tag{7.2}$$

Here, we take safely take $(1 + (4\sin^2\theta_W - 1)^2) \simeq 1$. The complete differential cross-section is then:

$$\frac{d\sigma}{dt} = \frac{1}{32\pi}\frac{1}{s^2}|\mathcal{M}|^2. \tag{7.3}$$

This cross section is relatively small due to the electroweak coupling factor g^4 and sfermion mass suppression, so sfermion production events will be rare. We show an example of the slepton pair production cross section including the Z-resonance, and the t-channel contributions from neutralinos (see below) in Fig. 7.2.

After production sfermions will decay, typically to a lighter neutralino or chargino and a Standard Model fermion f, in processes of the type $\tilde{f} \to f\tilde{\chi}_i^0$ or $\tilde{f} \to f'\tilde{\chi}_i^\pm$. Since the energy

[1]For more information on these projects see the websites for the International Linear Collider `http://www.linearcollider.org/` and the Compact LInear Collider `https://clic.cern/`

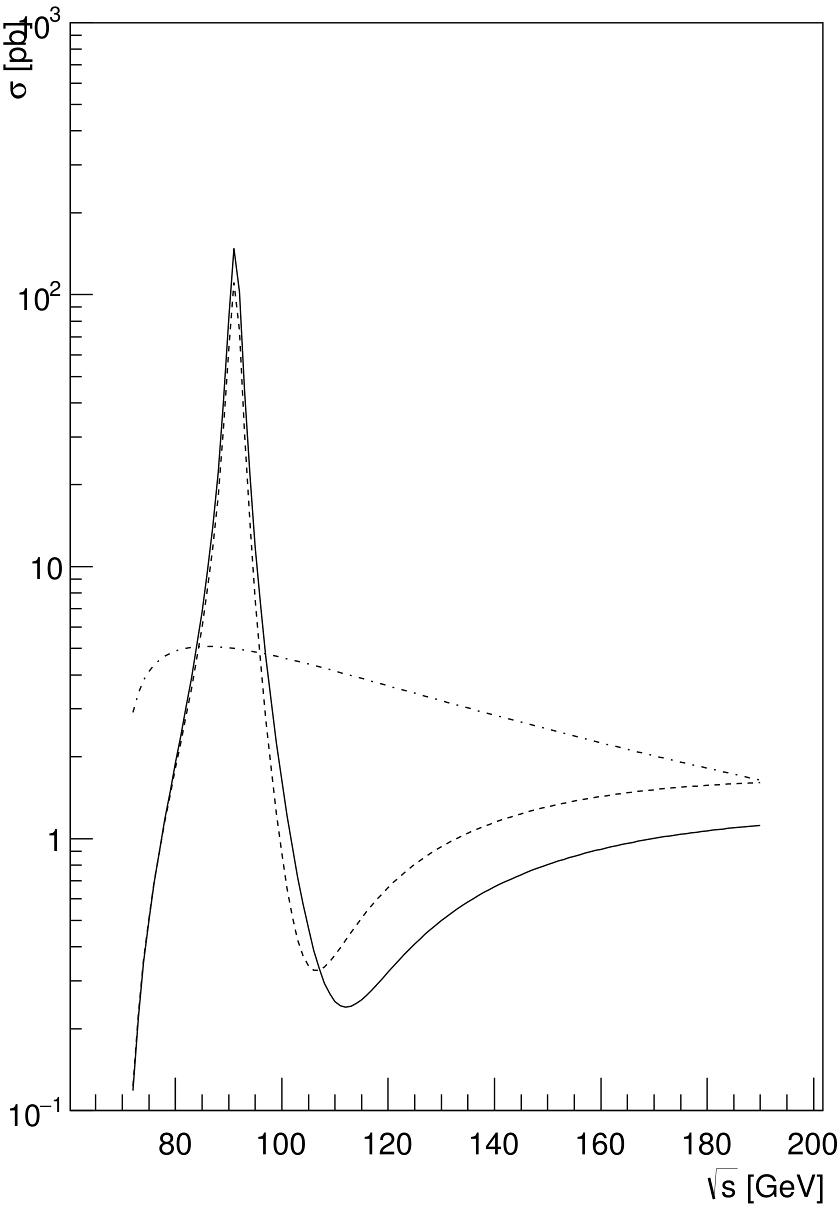

Figure 7.2: Cross sections for selectron pair production as a function of CoM energy. Cross sections for $\tilde{e}^*_L\tilde{e}_L$ (solid line), $\tilde{e}^*_R\tilde{e}_R$ (dashed line), and $\tilde{e}^*_L\tilde{e}_R$ (dashed dotted line) are shown separately. The particular model point has a common slepton mass of $m_{\tilde{e}_{L/R}} = 35$ GeV.

reach is limited, only the very lightest sfermions are likely to be producible, which are usually the sleptons, and these are likely to be near (but above) the mass of the lightest neutralino. See Fig. 6.4 for some expectations of the masses in an mSUGRA context. This means that the most expected decay is $\tilde{\ell} \to \ell\tilde{\chi}_1^0$, and the total process is then $e^+e^- \to \tilde{\ell}^+\tilde{\ell}^- \to \ell^+\ell^-\tilde{\chi}_1^0\tilde{\chi}_1^0$, giving two oppositely charged Standard Model leptons and missing energy as the experimental signature.

Such a (weak) signal then needs to be discriminated from Standard Model backgrounds from for example W^+W^- pair production, which can lead to the final state $\ell^+\nu_\ell\ell^-\bar{\nu}_\ell$, featuring the same leptons and missing energy from the neutrinos. Since backgrounds are usually well under control at lepton colliders, the limit to searches is the ability to produce the sparticles at all. With a CoM energy $\sqrt{s}$ sparticles up to $\sqrt{s}/2$ are energetically possible, and a rule of thumb is that the reach approaches $\sqrt{s}/2$ from below given sufficient data, never quite getting to $\sqrt{s}/2$.

Should some excess be discovered, we need a smoking duck in order to confirm that this is indeed supersymmetry. We would like to identify and measure the masses of as many new particles as possible, and hopefully also their spin. The properties of the sparticles can be measured through the inferred cross section, and the kinematical distribution of the final state products. The ability of a lepton collider to easily change the CoM energy $\sqrt{s}$, allows for a so-called **threshold scan** of the cross section where the cross section is measured as a function of $\sqrt{s}$ around where it becomes zero. This in turn allows for a very precise measurement of the mass of the pair produced particle. As an example of a kinematic distribution, for the process discussed here, it can be shown that the energy distribution for the final state leptons is a uniform distribution between $E_{\rm min}$ and $E_{\rm max}$ where

$$E_{\rm max/min} = \frac{\sqrt{s}}{4}\left(1 - \frac{m_{\tilde{\chi}_1^0}^2}{m_{\tilde{l}}^2}\right)\left(1 \pm \left(1 - \frac{4m_{\tilde{l}}^2}{s}\right)^{1/2}\right), \tag{7.4}$$

which, with a known slepton mass, also gives a handle on the LSP mass $m_{\tilde{\chi}_1^0}$ even though it is undetected.

7.1.2 Neutralinos & charginos

For charginos and neutralinos the production cross section depends on their wino, bino and higgsino components. You would be forgiven to think that pair production of the lightest neutralino $e^+e^- \to \tilde{\chi}_1^0\tilde{\chi}_1^0$ would be the natural sparticle to search for, however, this has some significant problems. Since it is usually the LSP it does not decay, and there is nothing in the event that can actually be measured. We cannot use the missing energy as that requires an *imbalance* in momentum. Given sufficiently hard (energetic) radiation from either the initial electron or positron, a single photon recoiling against missing energy could potentially be measured, and this, so-called **mono-photon search**, was indeed a search channel for dark matter production at LEP. However, for neutralino dark matter this does not work all that well for other reasons. The $Z\tilde{\chi}_i^0\tilde{\chi}_j^0$ vertex shown in Fig. 7.3 has the Feynman rule

$$\frac{ig}{2\cos\theta_W}\gamma^\mu\left[\left(N_{i3}N_{j3}^* - N_{i4}N_{j4}^*\right)P_L - \left(N_{i3}^*N_{j3} - N_{i4}^*N_{j4}\right)P_R\right], \tag{7.5}$$

which depends only on the higgsino components of the neutralinos, N_{i3} and N_{i4}. This can be understood from the fact that there are no ZZZ or $Z\gamma\gamma$ vertices in the Standard Model that can be supersymmetrised, only a Zhh vertex. For the photon there is no tree level coupling to the neutralinos at all since there are no direct couplings between the Higgs and the photon in the Standard Model. Thus, only neutralinos with significant higgsino components can be produced this way. To top it off, a light higgsino with a mass dominated by the μ parameter would have very similar values of N_{i3} and N_{i4}, thus canceling the coupling.

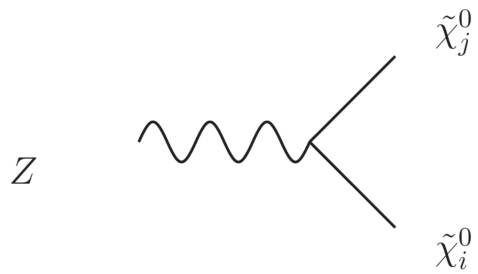

Figure 7.3: Vertex for $Z\tilde{\chi}_i^0\tilde{\chi}_j^0$.

The selectron and electron sneutrino have a special rôle for e^+e^- colliders due to the resulting t-channel diagrams. Figure 7.4 shows the t-channel diagrams that are important in pair production at a e^+e^- collider. Neutralino pair production with t-channel selectron exchange does not suffer from the same problems as neutralino pair production in the s-channel. However, the process depends on the selectron mass as $m_{\tilde{e}}^{-4}$ for large mass values and is rapidly suppressed.

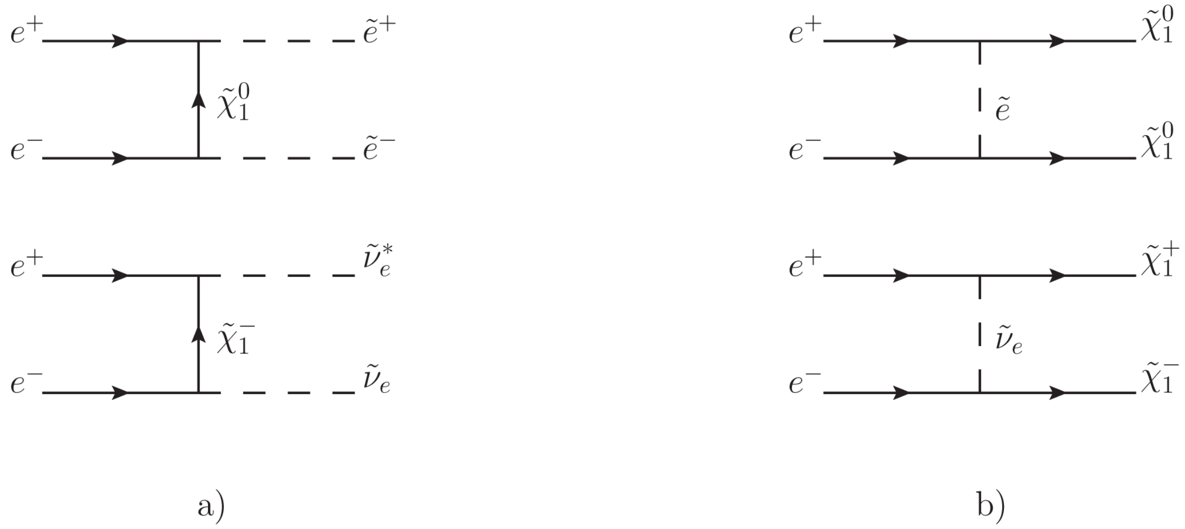

Figure 7.4: The t-channel diagrams for pair production of a) selectrons and electron sneutrinos, and b) neutralinos and charginos.

The consequences of the above discussion is that it is the production of the second lightest neutralino and the lightest chargino that is typically searched for in the combinations $\tilde{\chi}_2^0\tilde{\chi}_1^0$, $\tilde{\chi}_2^0\tilde{\chi}_2^0$, and $\tilde{\chi}_1^+\tilde{\chi}_1^-$. These decay – unless there is a slepton with a mass intermediary to the produced particle and the LSP – to the LSP and a possibly off-shell vector boson, $\tilde{\chi}_2^0 \to Z^{(*)}\tilde{\chi}_1^0$ and $\tilde{\chi}_1^\pm \to W^{(*)}\tilde{\chi}_1^0$. Thus the expected experimental signature is the fermionic decay products of the vector bosons with additional missing energy from the LSPs.

If the mass relevant differences between the neutralinos and charginos are small then the resulting decay products of the off-shell vector bosons will have little energy – the technical term for this is that they are **soft**, as opposed to very energetic particles that are **hard**. Soft particles are difficult to reconstruct in detectors, and events dominated by soft particle production are easily missed, meaning that degenerate scenarios are difficult to discover, and sparticles mass bounds become very poor in this case. This is particularly unfortunate since a scenario where $\mu \ll M_1, M_2$ is well motivated on theoretical grounds, and with μ controlling two neutralino masses and one chargino, the lightest two neutralinos and the lightest chargino indeed become degenerate.

7.1.3 Higgs bosons

Because of the very small Yukawa coupling of electrons, Higgs bosons are not expected to have significant direct production cross sections at a e^+e^- collider. The most realistic production channels are Higgs-strahlung where the Higgs boson is attached to a vector boson: $e^+e^- \to Zh$, $e^+e^- \to ZH$, $e^+e^- \to Ah$ and $e^+e^- \to AH$, charged Higgs pair production $e^+e^- \to H^+H-$, and W^+W^- vector boson fusion into Higgs bosons $e^+e^- \to \nu_e\bar{\nu}_e h(H)$.

7.1.4 Current bounds at lepton colliders

The below bounds are all from the LEP (Large Electron Positron) collider, running from 1989 until 2000, which outdated all previous bounds with a top energy of $\sqrt{s} = 209$ GeV, recording an integrated luminosity (amount of data) of 233 pb^{-1} above 204 GeV. Results exist from all four LEP experiments ALEPH, DELPHI, L3 and OPAL.[2] The numbers below are all taken from the 2014 PDG (Particle Data Group) review [21], but the conclusions from the LEP data has changed little since then.

While these bounds often come from pair-production of the relevant sparticles, and thus are less model dependent than the hadron collider bounds presented in the next section, there remains some model dependence in many results, which, unfortunately, is sometimes ignored in the literature. Complicating matters is a reliance by the LEP experiments on theoretical assumptions such as GUT-scale coupling and gaugino mass unification in many searches.

- Selectron: The strongest limit given is $m_{\tilde{e}_L} > 107$ GeV and $m_{\tilde{e}_R} > 73$ GeV (ALEPH 2002) in searches for acoplanar di-electrons.[3] However, the limit is the result of a scan over MSSM parameter space simplified by assuming a common m_0 and $m_{1/2}$ at GUT scale. Interpreted in the even more constrained mSUGRA with $A_0 = 0$ the bounds are even stronger, 152 GeV and 95 GeV, respectively. In contrast, due to strict limits on the precisely measured Z-width, there is a fully model independent limit of $m_{\tilde{e}_{L/R}} > 40$ GeV.[4]

- Smuon: $m_{\tilde{\mu}_R} > 94$ GeV (DELPHI 2003). The limit is obtained as in the MSSM scenario for the selectron.

- Stau: $m_{\tilde{\tau}_1} > 81.9$ GeV (DELPHI 2003) assuming exclusive $\tilde{\tau}_1 \to \tau\tilde{\chi}_1^0$ decays and $m_{\tilde{\tau}_1} - m_{\tilde{\chi}_1^0} > 15$ GeV.

[2] Most of which are silly acronyms of course.

[3] The observant reader will notice that two electrons are always in the same plane, however, when experimentalists say acoplanar, they mean not in one plane with the beam axis.

[4] Similar model independent limits around half the Z-mass exists for all sparticles that couple to the Z.

- Sneutrinos: From LEP we have an indirect limit of $m_{\tilde{\nu}} > 94$ GeV (DELPHI 2003) in neutralino & slepton searches. This assumes $m_{\tilde{e}_R} - m_{\tilde{\chi}_1^0} > 10$ GeV. From the Z-width we can obtain the model independent limit $m_{\tilde{\nu}} > 44.7$ GeV.

- Neutralino: The limit $m_{\tilde{\chi}_1^0} > 46$ GeV (DELPHI 2003) has been derived from the direct searches for $\tilde{\chi}_1^0\tilde{\chi}_2^0$ and $\tilde{\chi}_2^0\tilde{\chi}_2^0$. This assumes gauge coupling unification and a common gaugino mass $m_{1/2}$ at GUT scale. Even in the Z-decays, the contribution to the width depends on the higgsino component of the lightest neutralino, so $m_{\tilde{\chi}_1^0} \simeq 0$ GeV is still in principle allowed [22].

- Here we have $m_{\tilde{\chi}_1^\pm} \geq 94$ GeV (DELPHI 2003), assuming GUT scale universality of m_0 and $m_{1/2}$ and using multiple direct search channels from production of charginos, neutralinos and sleptons. It also assumes either no third generation sfermion mixing or $m_{\tilde{\chi}_1^\pm} - m_{\tilde{\chi}_1^0} > 6$ GeV. From the Z-width we can extract a strict limit of $m_{\tilde{\chi}_1^\pm} \geq 45$ GeV.

7.2 Supersymmetry at hadron colliders

Since the particles collided at hadron colliders (protons and other nuclei) are heavier they are not as susceptible to loosing energy through bremsstrahlung as the light leptons. Thus they can more readily be accelerated in a circular collider with periodic acceleration. However, this mass both means that linear colliders are inappropriate since the acceleration is smaller for the same applied fields, and that it is more difficult to steer the hadrons in a circular orbit. Thus the energy available at a hadron collider is limited by the strength of the bending magnets.

Up to 2009 the highest energy hadron collider was the Tevatron based at Fermilab outside of Chicago, that collided protons and anti-protons at $\sqrt{s} = 1.96$ TeV. The current record is held by the Large Hadron Collider (LHC) at CERN, colliding protons at a top energy of $\sqrt{s} = 13$ TeV.[5]

Since a proton–proton collider mostly collides partons, the quarks and gluons inside the protons, and since the strong coupling is large, this means that we can get large cross sections and potentially many events for QCD charged sparticles, *i.e.* squarks and gluinos, provided their masses are low enough. Cross sections for particles that have only electroweak charges (sleptons, electroweakinos, Higgs bosons) are expected to be considerably smaller, however, these sparticles are also expected to be lighter. The balance between these factors means that both types of sparticles are searched for.

As discussed earlier, with R-parity conservation sparticles are produced in pairs and both decay to the LSP. In Fig. 7.5 we show the sparticle mass spectrum for a particular mSUGRA benchmark point called SPS1a that has $m_0 = 100$ GeV, $m_{1/2} = 250$ GeV, $A_0 = -100$ GeV, $\tan\beta = 10$ and $\operatorname{sgn}\mu = +$ [23].[6] By producing the squarks or gluinos near the top of the mass spectrum in a collision we see that we can get many different quite long decay chains, called cascades. We illustrate such an event in Fig. 7.6.

In hadron collisions the momenta of the incoming partons is unknown, so no complete momentum balance can be made. However, the sum of the momenta in the direction transverse

[5] The LHC was designed to operate at $\sqrt{s} = 14$ TeV, but has had problems fully reaching that energy. In the next round of the LHC to start in 2022, called Run III, the plan is to go to $\sqrt{s} = 13.5$ TeV.

[6] This benchmark point is now quite old and clearly excluded by searches, but it still serves as a nice illustration of the kind of sparticle spectrum expected in mSUGRA models.

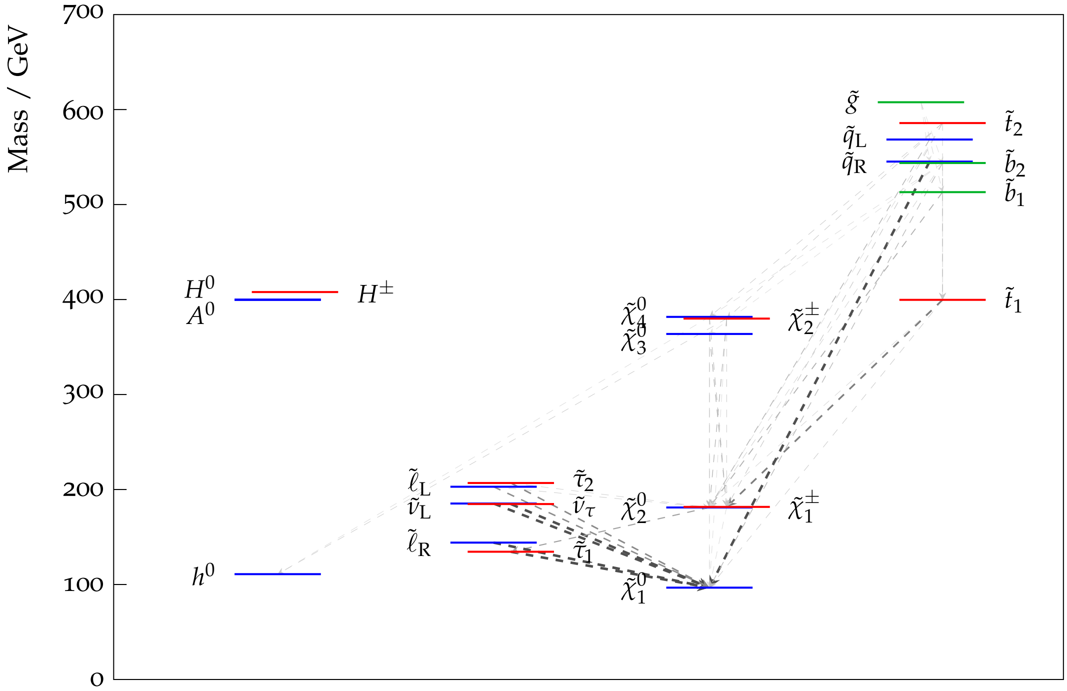

Figure 7.5: Supersymmetric particle mass spectrum (coloured solid lines) and possible decay channels (dashed lines) for the SPS1a benchmark point. Only decays with branching rations above 5% are shown. The line opacity indicates relative branching ratios. This plot was generated using `PySLHA 3.0.1` [24].

to the beam direction is zero. Thus with escaping LSPs we can observe **missing transverse energy**, $\not{E}_T$ or E_T^{miss}, *i.e.* an imbalance in the directional sum of all energy deposits transverse to the beam direction. The rest of the signal depends on which sparticles are searched for. In broad terms this means that if searching for gluino of squark production then at some point the cascade decay must shed colour charge to arrive at the LSP, resulting in Standard Model quarks or gluons that hadronise and produce jets in the detector (hadronic showers). If instead searching for electroweakly produced sparticles or Higgs bosons, one usually assumes an absence of significant jet activity, and instead looks for leptons from the decays of the sleptons or electroweakinos, possibly through vector bosons.

At the LHC the Standard Model backgrounds are much more significant and problematic. For example, at design luminosity the LHC produces about 10 pairs of top quarks per second. Due to the top quark decay $t \to bW^+$, resulting in charged leptons and potential missing energy from neutrinos in W decays, as well as jets produced in hadronic W decays and from the b-quarks, this alone is a significant background to many supersymmetry searches. Figure 7.7 shows the expected backgrounds and signals produced in different channels at the $\sqrt{s} = 14$ TeV LHC for different assumed particle masses. One can see that even the largest supersymmetric cross sections that we get for squark and gluino production are orders of magnitude below the Standard Model backgrounds for masses beyond 500 GeV.

To cope with this challenge, the searches define kinematical variables that are designed to separate supersymmetry events from Standard Model events. For example, searching for the production of squarks and/or gluinos decaying to the LSP, one expects jets and missing

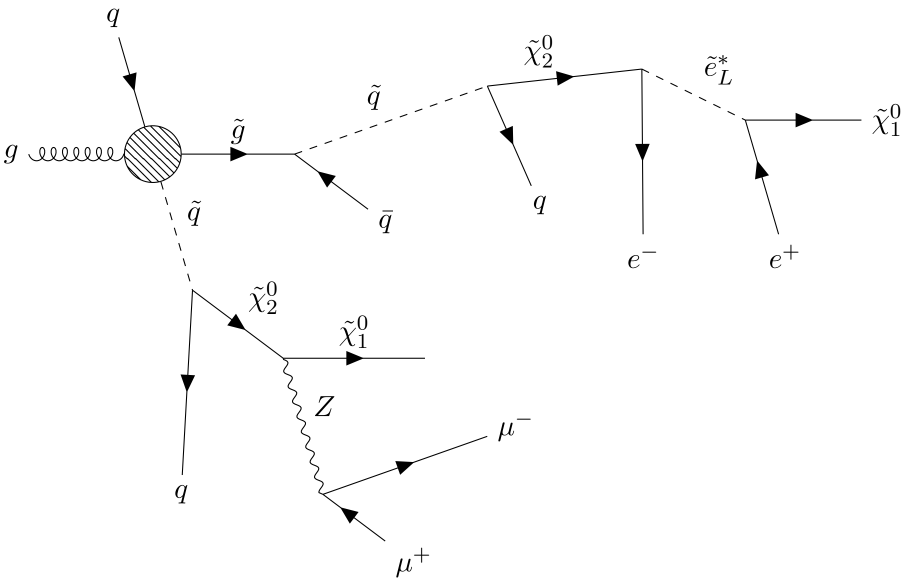

Figure 7.6: Cascade decays starting with the production of a gluino and squark pair from a gluon–quark interaction.

energy from two LSPs. One can then define the **effective mass**

$$M_{\text{eff}} = \sum p_T^{\text{jet}} + \not{E}_T, \tag{7.6}$$

from the sum of the transverse momentum of the jets in the event, p_T, and the missing energy. The expectation is then that this is more significant in squark and gluino production events than in Standard Model events, so that a high requirement on M_{eff} reduces the number of background events without significantly removing events with sparticles. This helps for example to combat $t\bar{t}$ production because the jet p_Ts in those events are limited by the energy in the top quark mass, while the more massive squarks and gluinos will have higher jet p_Ts.

However, there are also models where this kind of approach is ineffective. Imagine a scenario where only the lightest stop $\tilde{t}_1$ is light enough to be copiously produced. If $m_{\tilde{t}_1} - m_{\tilde{\chi}_1^0} < m_W$ then the stop will dominantly decay as $\tilde{t}_1 \to c\tilde{\chi}_1^0$ or $\tilde{t}_1 \to bl\nu\tilde{\chi}_1^0$, where all final state particles have rather low energy (p_T), so-called **soft particles**. As for the degenerate neutralino scenario discussed above, this is very difficult to discover with standard techniques.

Another example of the use of kinematics is the distribution of invariant masses. In Fig. 7.8 we show an example of two sequential two-body decays. Even if particle A here is invisible, for example the LSP, we can use two visible decay products a and b to form the invariant mass m_{ab}. One can show that the distribution for m_{ab} has a triangular shape with a sharp endpoint at the maximum

$$(m_{ab}^{\max})^2 = \frac{\left(m_C^2 - m_B^2\right)\left(m_B^2 - m_A^2\right)}{m_B^2}, \tag{7.7}$$

where we have assumed that a and b are massless.[7] This can be used both to select events that potentially have supersymmetric particles in them depending on the value of the invariant

[7] A more complicated expression covers the massive case.

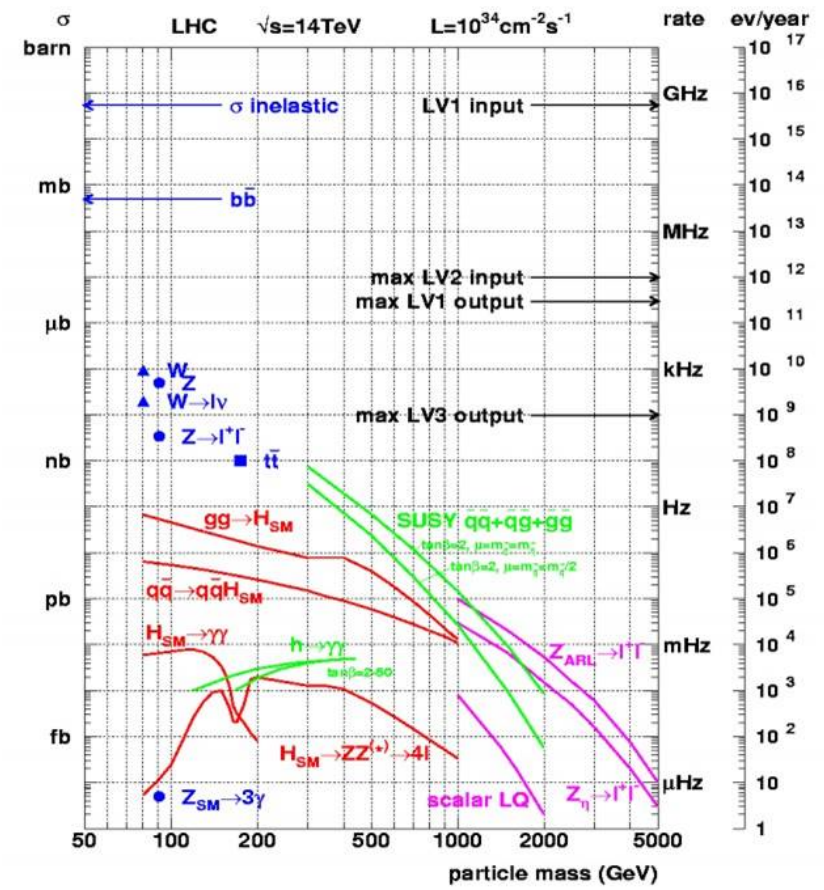

Figure 7.7: Plot of the expected cross sections and rates for various processes at the $\sqrt{s} =$ 14 TeV LHC plotted against the mass of the particles, assuming the design final luminosity of the LHC (rate of data taking). The current Run II of the LHC has collected around 140 fb^{-1} of data, so the lowest point on the cross section axis indicates that $\mathcal{O}(1)$ events are expected to have been produced by now.

mass, and to determine relationships between the sparticle masses. In Fig. 7.9 we show a simulation of the invariant mass distribution of two opposite-sign same-flavour (OSSF) leptons $m_{\ell\ell}$ from the production of $\tilde{\chi}_2^0$ and its decay chain $\tilde{\chi}_2^0 \to \ell^\pm \tilde{\ell}_R^\mp \to \ell^\pm \ell^\mp \tilde{\chi}_1^0$.

As alternatives to these standard searches for pair produced sparticles with missing energy there are ongoing searches for decaying LSPs when R-parity is violated, or the production of single sparticles.[8] There is also the possibility of **massive metastable charged particles** (MMCPs), typically in scenarios with a gravitino LSP, where the next-to-lightest supersymmetric particle (NLSP) is charged and long-lived because the decay to the gravitino is via a very weak gravitational coupling. The latter also includes so-called **R-hadrons** if the NLSP has colour charge, which means that it will hadronise after production and be a short-lived but very massive meson or baryon. We should also mention the searches for the extra Higgs

[8]Single sparticle production at the LHC requires rather large R-parity violating couplings for the $LQ\bar{D}$ or $\bar{U}\bar{D}\bar{D}$ operators, of the order of $\lambda > 10^{-2}$.

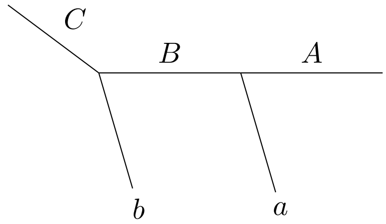

Figure 7.8: A generic illustration of two successive two-body decays $C \to bB$ and $B \to aA$.

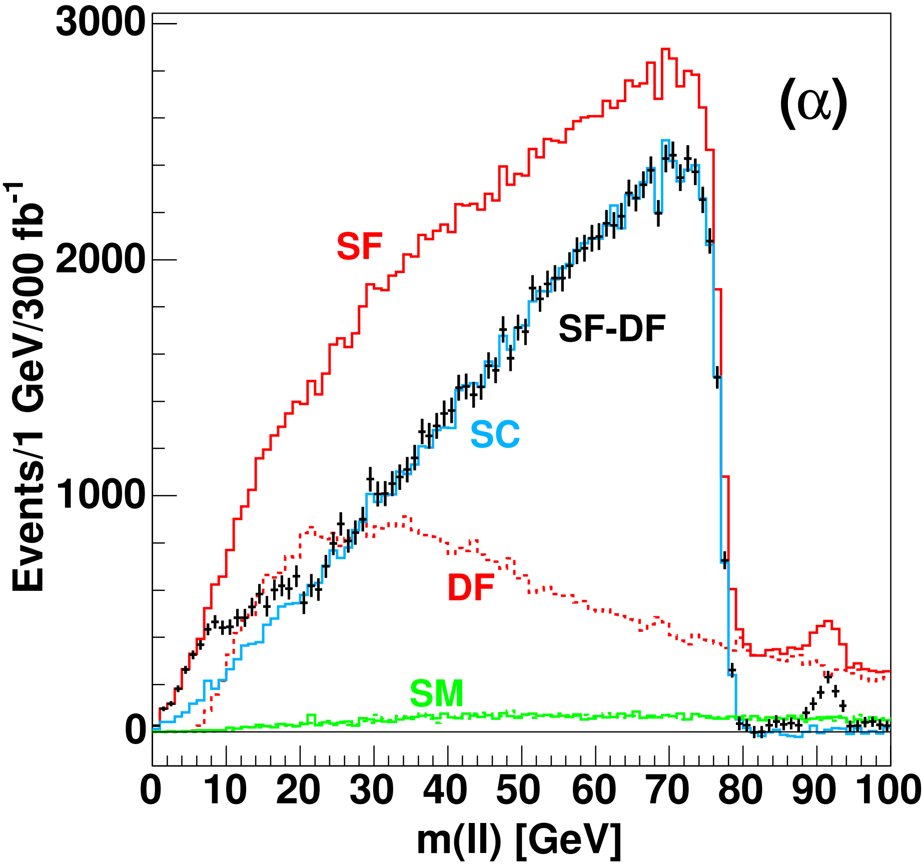

Figure 7.9: Invariant mass distribution of opposite sign same flavour (OSSF) dileptons for the mSUGRA benchmark model point SPS1a [25].

states predicted in the MSSM.[9]

7.3 Current bounds on sparticle masses

The LHC has finished its Run II and collected a total of around 140 fb^{-1} of data per experiment at $\sqrt{s} = 13$ TeV of energy. It is currently preparing for Run III to start in 2022. Direct bounds from the LHC experiments ATLAS and CMS now supersede bounds from other colliders (Tevatron and LEP) in almost all channels, with some exceptions for models

[9] But we don't really have time.

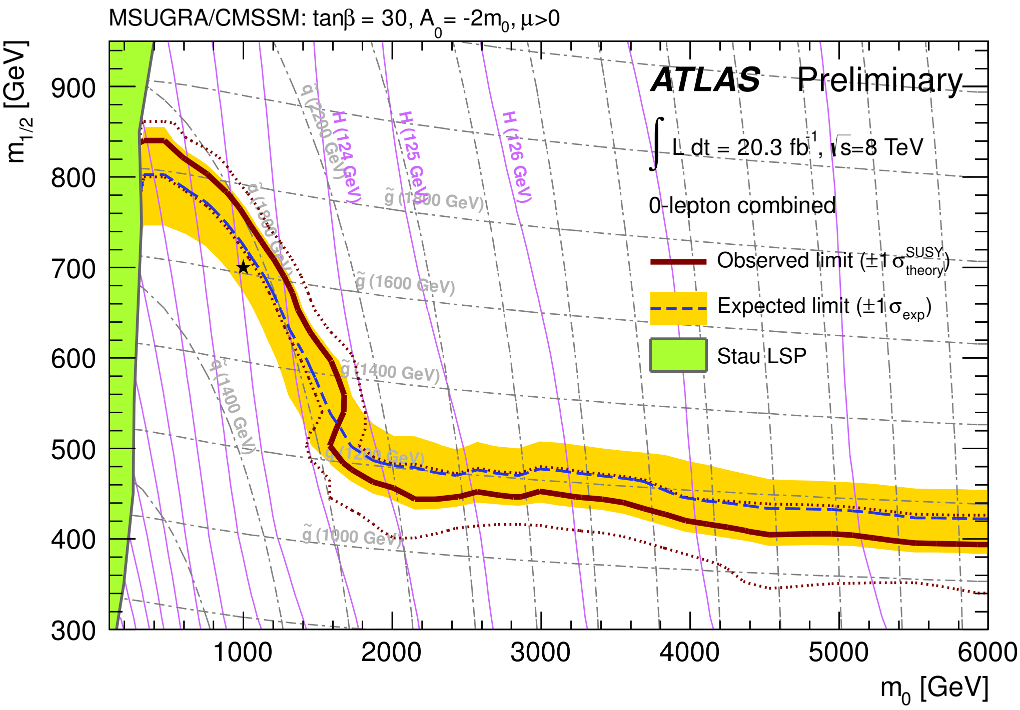

Figure 7.10: Plot of the excluded area in the $m_{1/2}$-m_0 plane of the mSUGRA parameter space for $\tan\beta = 30$, $A_0 = -2m_0$ and $\mu > 0$ for searches using missing energy and jets. The limit is the red line. The green area is theoretically forbidden because it has a charged LSP (the stau) [26].

with degenerate masses. The below limits are mostly limits with the full Run II dataset and represent the current state-of-art in sparticle searches. We mostly use examples from ATLAS, the corresponding plots from CMS are very similar.

The strongest current limits in terms of mass are on the gluino and squarks simply because of the large production cross sections. Significant bounds on electroweakinos and sleptons exist, but these are either model dependent (depend on squark/gluino mass assumptions and cascade decays), or weaker if they rely only on electroweak production.

7.3.1 Squarks and gluinos

In Fig. 7.10 we show the limits from the ATLAS experiment on the mSUGRA model using searches for jets plus missing energy with all the data collected at 8 TeV. The mSUGRA parameters $\tan\beta$ and A_0 have been chosen in order to give relatively large Higgs masses for small values of $m_{1/2}$ and m_0. The figure also shows the corresponding first and second generation squark masses, the gluino mass (both dot-dashed lines), and the Higgs mass (purple) for these parameter values. We then have the following approximate bounds in mSUGRA: $m_{\tilde{q}} > 1600\,\text{GeV}$ and $m_{\tilde{g}} > 1100\,\text{GeV}$. Bounds on the mSUGRA space directly have not been updated since this plot.

Notice that in the figure the squark mass bound is more or less equivalent to the mass required for a sufficiently heavy Higgs, thus the direct search does not constrain the squarks

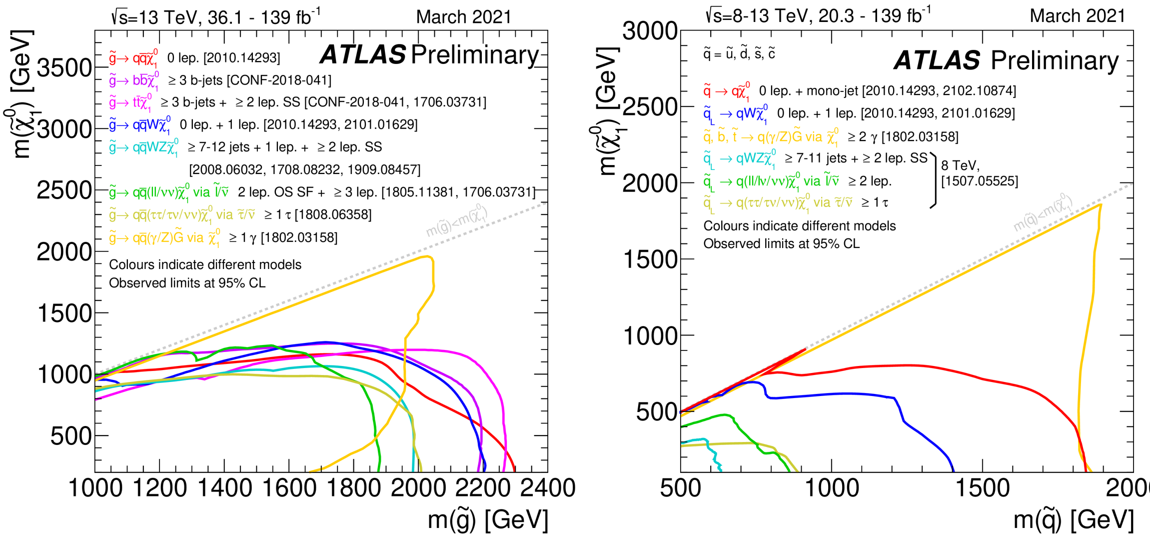

Figure 7.11: Excluded regions in the $(m_{\tilde{g}}, m_{\tilde{\chi}_1^0})$ (left) and $(m_{\tilde{q}}, m_{\tilde{\chi}_1^0})$ (right) mass planes under different assumptions on the decays of the gluino and squarks [27]. References for the individual analysis given in the figure.

masses significantly more here than the indirect constraint from the Higgs mass.

An important question is how these bounds change as we move away from the mSUGRA assumptions. In mSUGRA the resulting gluino and squark decays are very much constrained by the model. For more general models the limits depend on what decay chains are dominant. In Fig. 7.11 we show current limits in the gluino–lightest neutralino and squark–lightest neutralino mass planes under various assumptions on the decay chain shown in different colours.

For the gluino the most minimal assumption we can make is that the gluino sheds its adjoint colour charge in the decay to two quarks of the first or second generation and a neutralino, $\tilde{g} \to q\bar{q}\tilde{\chi}_1^0$, either by a direct three-body decay or via an intermediary squark (red line). We then get limits up to $m_{\tilde{g}} > 2300\,\text{GeV}$ under the assumption of a very light neutralino. This limit gradually weakens as the neutralino mass increases, and disappears beyond $m_{\tilde{\chi}_1^0} > 1000\,\text{GeV}$ for $m_{\tilde{g}} > 1000\,\text{GeV}$. For $m_{\tilde{g}} < 1000\,\text{GeV}$ (not visible on the plot) most of the parameter space is excluded, with the exception that a small sliver remains when $m_{\tilde{g}} \simeq m_{\tilde{\chi}_1^0}$ in the degenerate scenario.

For the first two generations of squarks the most minimal decay is $\tilde{q} \to q\tilde{\chi}_1^0$. Here the bound (red) reaches to around $m_{\tilde{q}} > 1800\,\text{GeV}$ for a light neutralino, while for heavier neutralinos the bound disappears beyond $m_{\tilde{\chi}_1^0} > 700 - 800\,\text{GeV}$ for $m_{\tilde{q}} > 800\,\text{GeV}$. Below a squark mass of around 800 GeV most of the parameter space is excluded, but we again lose sensitivity when there is degeneracy between the squark and the neutralino. Here a search for mono-jets has some impact in closing the gap, which shows as a spike in the excluded region between 800 and 900 GeV, however, despite poor visibility in the plot, a very degenerate squark–neutralino pair is still allowed below 800 GeV.

An additional assumption made in this plot is that all the eight squarks of the first two-generations are degenerate in mass and adding to the cross section. Should one squark

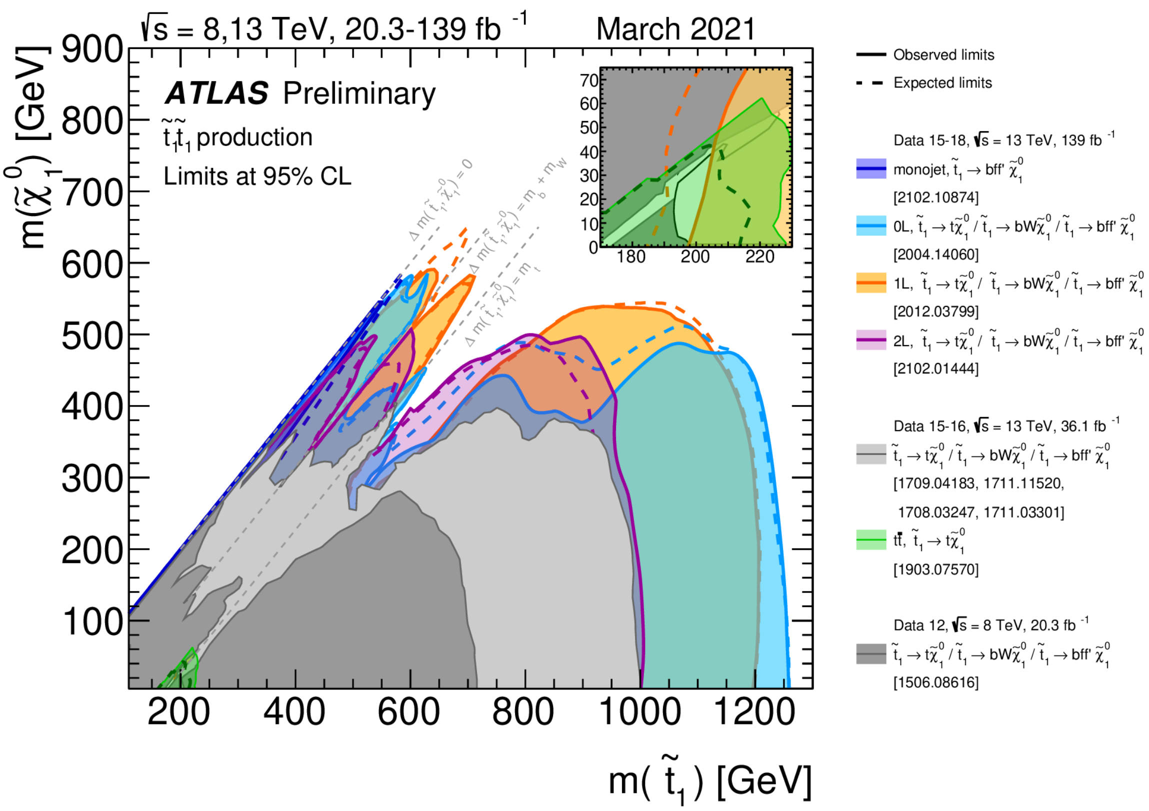

Figure 7.12: Plot of the excluded area in the ($m_{\tilde{t}_1}$,$m_{\tilde{\chi}_1^0}$) mass plane [27]. References for the individual analysis given in the figure.

generation or flavour be significantly lighter than the others this means a further reduction in the production cross section and thus a weaker bound. It is also fairly clear that removing R-parity, meaning that the LSP decays, also weakens the above conclusions due to the possible absence of significant missing energy.

7.3.2 Stop

For the stop there are many possible competing decay channels, meaning that limits set are rather model dependent. The two main decay categories for the lightest stop are decays via the chargino, $\tilde{t}_1 \to b\tilde{\chi}_1^+$, a supersymmetrised version of the Standard Model $t \to bW^+$, and decays directly to the neutralino $\tilde{t}_1 \to t\tilde{\chi}_1^0/bW\tilde{\chi}_1^0/bff'\tilde{\chi}_1^0$, where ff' represents the fermions in a W decay, and where the dominant decay mode depends on the stop–neutralino mass difference. Since the chargino decay is also typically $\tilde{\chi}_1^+ \to ff'\tilde{\chi}_1^0$, the experimental signature of stop production is $bff'\tilde{\chi}_1^0$, a mixture of missing energy, fermions that may either be leptons or result in jets, and a b-quark that forms a jet with distinct properties, called a b-jet.

A summary of (the many) ATLAS searches for the stop is found in Fig. 7.12 showing limits in the $(m_{\tilde{t}_1}, m_{\tilde{\chi}_1^0})$ mass plane. The dashed grey lines show the regions where the different stop decays are kinematically possible. We observe that for light neutralinos the limit goes all the

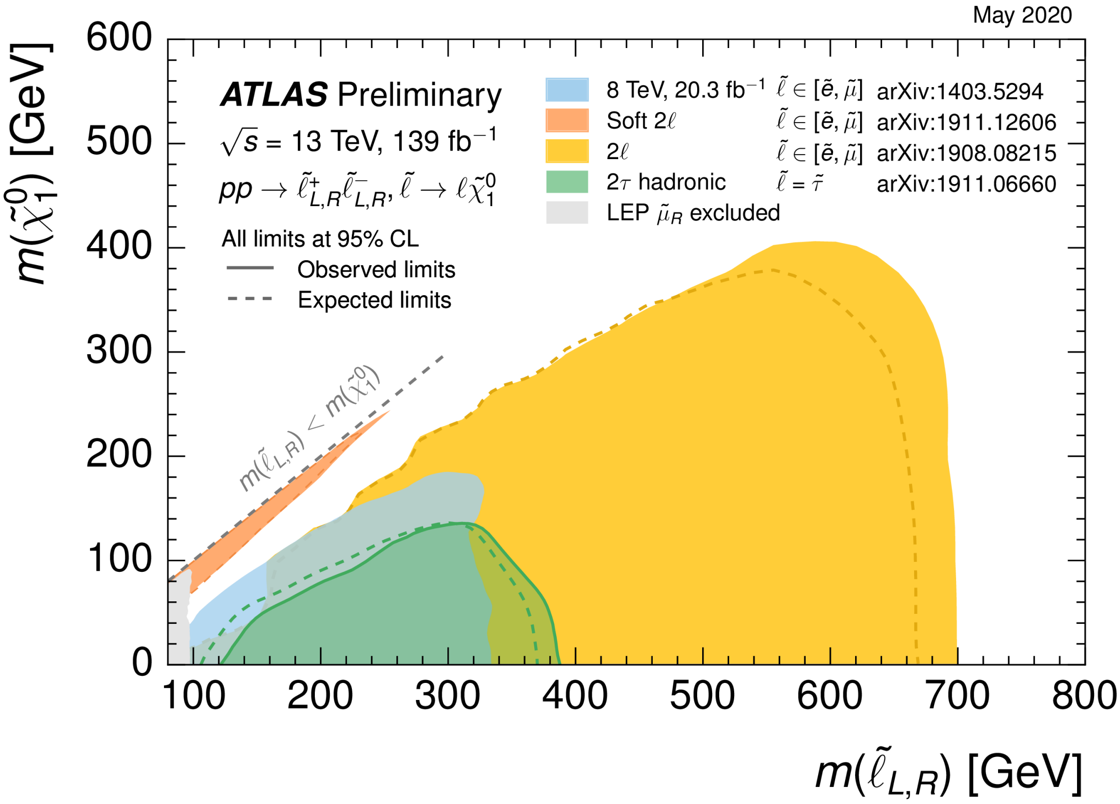

Figure 7.13: Plot of the excluded area in the $(m_{\tilde{l}_{L,R}}, m_{\tilde{\chi}^0_1})$ plane for mass degenerate right- and left-handed sleptons of different flavours [27].

way up to $m_{\tilde{t}_1} > 1250\,\text{GeV}$, while there are essentially no limits for $m_{\tilde{\chi}^0_1} > 580\,\text{GeV}$. Again degenerate scenarios have weaker bounds. When the stop approaches the neutralino plus top mass stop masses down to 600 GeV are allowed. And, again not very visible, for a degenerate stop–neutralino the limit is virtually non-existent. As an additional complication, if the stop–neutralino mass difference is below the bottom quark mass, the standard stop decays no longer work and the flavour changing neutral current decay $\tilde{t}_1 \to c\tilde{\chi}^0_1$ becomes dominant. This decay has multiple still open theoretical questions, both about the potentially long lifetime of the stop in this scenario, and exactly where in parameter space this transition occurs.

7.3.3 Sleptons

The mass bounds on sleptons will be very dependent on the assumed production mechanism. If the sleptons are produced indirectly in cascade decays from heavier squarks or gluinos they could have large cross sections, however, the most model independent bounds come from assuming only direct electroweak pair production through a virtual photon or Z.

The result for degenerate right- and left-handed charged sleptons from electroweak production, assuming decays with 100% branching ratio to the lightest neutralino, are shown in Fig. 7.13. These limits separate between selectron and smuon production, and stau production. The former are assumed degenerate, and have stricter bounds than the stau, which is harder to reconstruct due to the many different possible tau decays involving extra neutri-

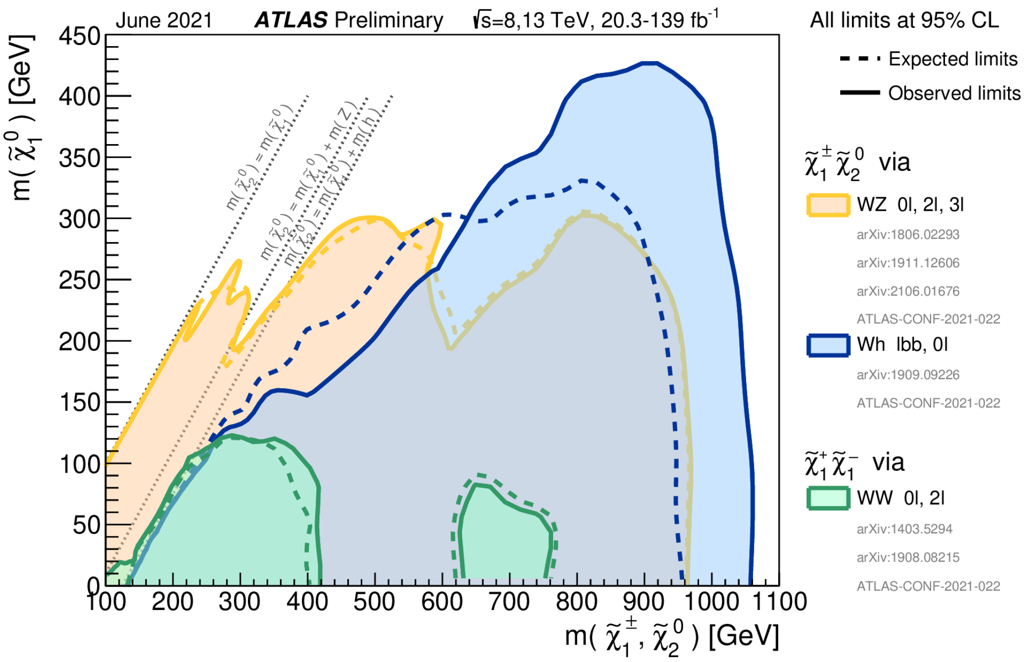

Figure 7.14: Plot of the excluded area in the $(m_{\tilde{\chi}_1^\pm,\tilde{\chi}_2^0}, m_{\tilde{\chi}_1^0})$ mass plane for different search signatures [27]. This plot assumes a degenerate wino-like $\tilde{\chi}_2^0$ and $\tilde{\chi}_1^\pm$ and a 100% branching ratio to the given decay channels.

nos. For all slepton flavours we see that there is a gap in the mass plane down to very low masses where no sleptons are excluded, all the way down to masses below 100 GeV where the LEP bound applies. This is yet another example of the problems with degeneracy. If the slepton–neutralino mass difference is around 50 GeV and lower, it becomes difficult to reliably reconstruct the soft leptons from the slepton decay and to separate them from the ordinary Standard Model pair production of lepton pairs.

7.3.4 Charginos and neutralinos

As for the sleptons, bounds are dependent on the production process assumed. The search for direct electroweak pair production of the lightest neutralino, $\tilde{\chi}_1^0\tilde{\chi}_1^0$, has the same problem as at LEP, the coupling is vanishingly small for a non-higgsino $\tilde{\chi}_1^0$. This means that it is the heavier neutralinos and charginos that are typically searched for. Here some considerations on the possible masses hierarchies come into play. If the lightest neutralino is dominantly bino and the next-to-lightest dominantly wino ($M_1 < M_2 < |\mu|$) then the $\tilde{\chi}_2^0$ and $\tilde{\chi}_1^\pm$ states are degenerate in mass and the most important searches will be for $\tilde{\chi}_2^0\tilde{\chi}_1^\pm$ and $\tilde{\chi}_1^+\tilde{\chi}_1^-$ production.[10] These then decay as $\tilde{\chi}_2^0 \to Z\tilde{\chi}_1^0$ or $\tilde{\chi}_2^0 \to h\tilde{\chi}_1^0$, and $\tilde{\chi}_1^\pm \to W\tilde{\chi}_1^0$, where the bosons may be off-shell if the mass difference is small. This is the scenario usually considered by the experiments, in part because winos give the largest electroweak production cross sections.

[10] Again because of the coupling in Eq. (7.5) the production of a pair of wino-like $\tilde{\chi}_2^0\tilde{\chi}_2^0$ is suppressed and the cross section is small.

We show the current limits assuming degenerate wino-like states $\tilde{\chi}_2^0$ and $\tilde{\chi}_1^\pm$ in Fig. 7.14. We can see that limits up to 1050 GeV can be set for $\tilde{\chi}_2^0$ and $\tilde{\chi}_1^\pm$ when the lightest neutralino is below around 200 GeV. When the bosons in the decay go off-shell we see that the sensitivity of the searches go down significantly. If we move away from the wino-production scenario the production cross sections drop, and so does the reach of the experiments in the mass plane.

An additional important complication appears if the lightest neutralino is wino ($M_2 < M_1, |\mu|$) or higgsino ($|\mu| < M_1, M_2$). Then the lightest neutralino and chargino, $\tilde{\chi}_1^0$ and $\tilde{\chi}_1^\pm$ (for higgsinos also $\tilde{\chi}_2^0$), are degenerate, and the crucial question becomes how degenerate they are. If the mass difference is small but above around 300 MeV, then the particles are very difficult to discover at all, despite potentially having very large cross sections if they are light. The decays of the produced $\tilde{\chi}_1^\pm$ (and for higgsinos $\tilde{\chi}_2^0$) into $\tilde{\chi}_1^0$ lead to very soft decay products, either leptons in $\tilde{\chi}_1^\pm \to \ell^\pm \nu_\ell \tilde{\chi}_1^0$ (and $\tilde{\chi}_2^0 \to \ell^+\ell^-\tilde{\chi}_1^0$), or decays to pions (the lightest hadronic states) $\tilde{\chi}_1^\pm \to \pi^\pm \tilde{\chi}_1^0$ (and $\tilde{\chi}_2^0 \to \pi^0 \tilde{\chi}_1^0$), that are unobservable above the backgrounds of a hadron collider.

If instead the mass difference becomes of the order off or less than the pion mass the width of the chargino can become so small that the state is sufficiently long lived to decay somewhere inside the detector, and not, effectively, at the interaction point. This would lead to charged tracks from the charginos that disappear somewhere inside the detector. Such disappearing track signatures are readily observable, and in fact the limits on very degenerate neutralinos and charginos are stronger than the limits on only somewhat degenerate ones.

The consequences of this complication is that it is very difficult to exclude near massless wino- or higgsino-like charginos and neutralinos at the LHC. However, since the chargino mass is constrained by the Z-width measured at LEP, there is an absolute lower limit of 45 GeV for both in this scenario. To improve significantly on this limit one may have to wait for a new lepton collider.

7.4 Precision observables

A different way to search for the signs of supersymmetric particles is their indirect effect on very accurately measured Standard Model particle properties and processes, so-called **precision observables**, mostly through loop diagrams with sparticles. We will here discuss four different types of probes: the electroweak precision observables, the value of the anomalous magnetic moment of the muon $(g-2)_\mu$, the flavour changing neutral current (FCNC) process $b \to s\gamma$ and the very rare (and FCNC) process $B_s \to \mu\mu$.

7.4.1 Electroweak precision observables

When we talk about **electroweak precision observables**, we study particle properties such as M_W (or M_Z), Γ_W, Γ_Z, m_t and $\sin\theta_W$, as well as the Higgs mass m_h and the properties of the Higgs such as its couplings to all the other particles (gauge and Yukawa couplings) and its self-coupling.

Up to 2012 we could study all of these as *functions* of the unknown Higgs mass, looking for deviations that could be a sign of for example supersymmetry. We show a fit of the Higgs mass to all available electroweak data and direct exclusion bounds in Fig. 7.15, made by the Gfitter Collaboration in 2011 just before the LHC started taking significant amounts of data [28], a fit pretty much indicating that the most probable Standard Model Higgs mass

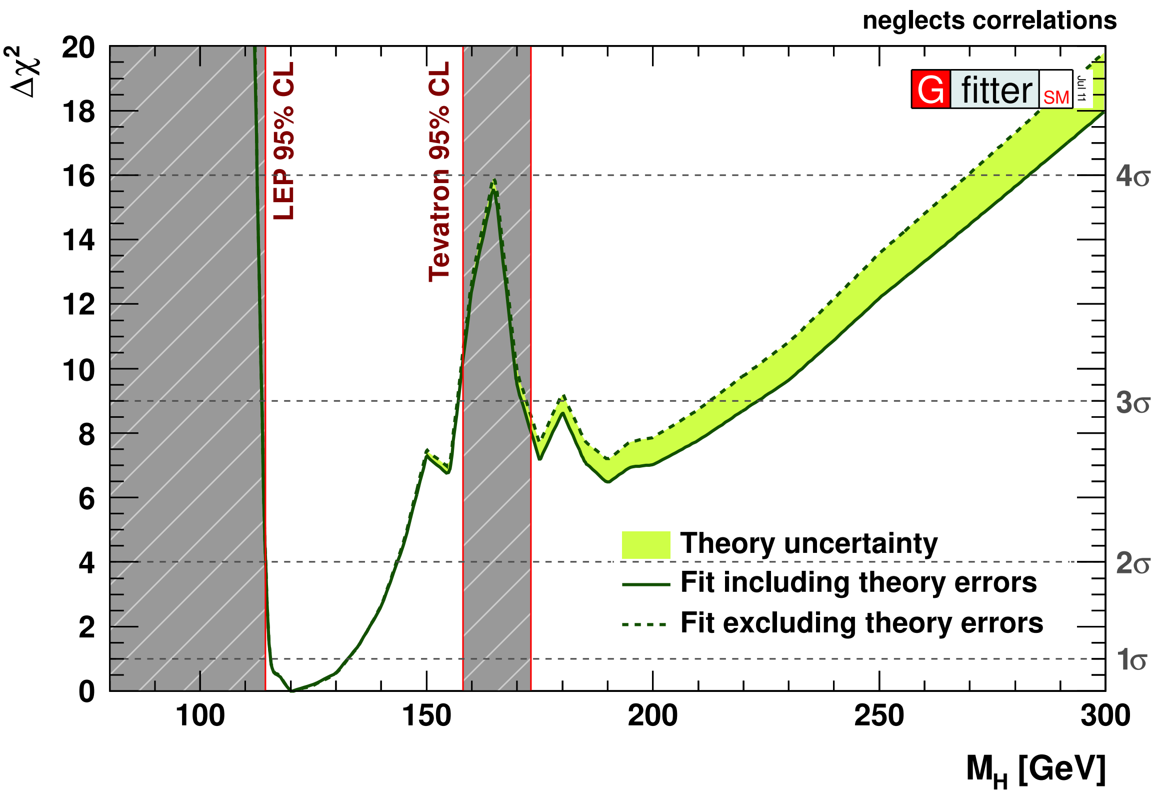

Figure 7.15: Plot of the total $\Delta\chi^2$ from all precision observable measurements and the direct exclusions bounds for the Standard Model Higgs set by LEP and the Tevatron, as a function of the Higgs mass [28].

was around 120 GeV. Here,

$$\Delta\chi^2 = \chi^2(m_h) - \min_{m_h} \chi^2(m_h),$$

is the difference in the χ^2 from the best fit point as a function of the Higgs mass, which has a standard interpretation in terms of σs of significance for a one dimensional fit.

Figure 7.16 shows a similar plot for mSUGRA. At that time the absolute minimum of the fit, even taking into account the different number of parameters, gave a better fit for mSUGRA, $\min \chi^2_{\text{mSUGRA}} < \min \chi^2_{\text{SM}}$, but this changed quickly when the Higgs was found because of the position of the two minima.

Now all the parameters of the Standard Model – neutrinos excepted – have been determined to some precision. Thus the Standard Model is a completely constrained system. If we now do a electroweak fit the situation looks like that in Fig. 7.17, where we show the global fit to all measurements *except* the top and W masses, compared to the measured values of the W and top masses. Clearly what we are seeing here is (still?) consistent with the Standard Model.

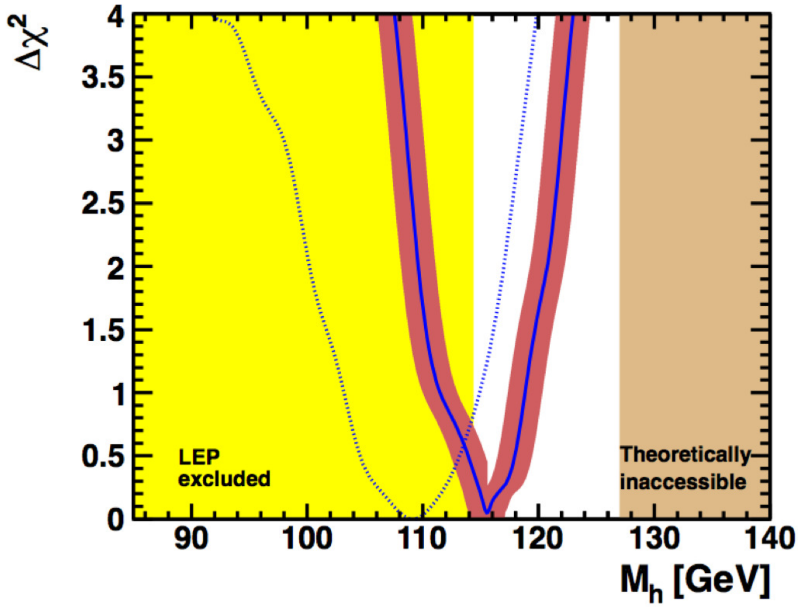

Figure 7.16: Plot of the $\Delta\chi^2$ from all precision observable measurements for mSUGRA as a function of the Higgs mass. The yellow area shows the area excluded by LEP searches (*not* included in the fit), while the brown shows the theoretically inaccessible area.

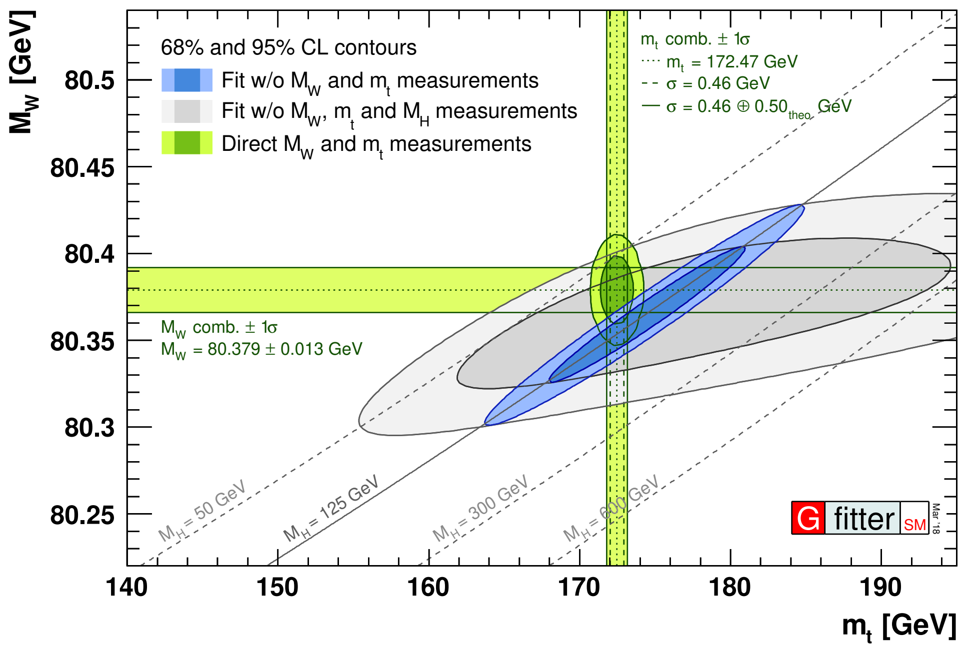

Figure 7.17: Electroweak fit excluding M_W and m_t (blue), compared to their measured values (green) [29].

7.4.2 $(g-2)_\mu$

The **anomalous magnetic moment** of the muon, g_μ, in effect the interaction of a muon with an external electromagnetic field shown in Fig. 7.18 a), was calculated by Dirac in relativistic quantum mechanics as $g_\mu = 2$. However, in quantum field theory we have higher order corrections from loop-process such as shown in Fig. 7.18 b), that makes g_μ deviate from two.

It was very precisely measured by the E821 experiment at BNL [30] to be:

$$g_\mu^{\text{exp}} = 2.00116592089(63),$$

or, in terms of the deviation $a_\mu = g_\mu - 2$,

$$a_\mu^{\text{exp}} = 11659208.9(6.3) \cdot 10^{-10},$$

where the parenthesis indicates the uncertainty on the last digits.

In the Standard Model we find the prediction

$$a_\mu^{\text{SM}} = 11659183.0(5.1) \cdot 10^{-10},$$

giving a difference with respect to the experimental value of

$$\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (25.9 \pm 8.1) \cdot 10^{-10},$$

a value which is 3.2σ away from zero. Recently, the Muon g-2 Collaboration at Fermilab made a new measurement [31] confirming this discrepancy with the value

$$\Delta a_\mu = 25.1 \pm 5.9 \cdot 10^{-10},$$

corresponding to 4.2σ. The Muon g-2 Collaboration is set to make further improvements on this number, expecting to reduce the uncertainty by a factor of $2-4$.

However, we should be aware that some of the Standard Model contributions involve hadronic loops, *e.g.* the so-called **hadronic vacuum polarisation** shown in Figure 7.18 b), where one has to rely on experimental information from low energy $e^+e^- \to \gamma^* \to$ hadrons in order to estimate a contribution of $a_\mu^{\text{HVP}} = 10.5(2.6) \cdot 10^{-10}$, which is of the same order of magnitude as the discrepancy, and may be prone to errors in the interpretation.

In the MSSM we can have contributions to a_μ at the one-loop level, either by the exchange of one of the new Higgs bosons across the muon line, or by loops containing a smuon $\tilde{\mu}$ or muon-sneutrino $\tilde{\nu}_\mu$, together with a neutralino or chargino. The diagrams for the latter two processes are shown in Figure 7.18 c) and d). These contribute opposite sign terms $a_\mu(\tilde{\chi}^0)$ and $a_\mu(\tilde{\chi}^-)$. A thorough analysis shows that we need $\mu > 0$ in order to give a positive contribution that will close the gap between the experimnetal value and the prediction. In order to get a sufficiently large contribution the loop masses must be less than $500-600\,\text{GeV}$ for $\tan\beta = 40-50$ and $200-300\,\text{GeV}$ for $\tan\beta \simeq 10$. However, as we saw in Sec. 7.3 this is not implausible.

7.4.3 $b \to s\gamma$

The quark level process $b \to s\gamma$ is a Flavour Changing Neutral Current (FCNC) process which must proceed through loops in the Standard Model since there are no tree-level FCNC

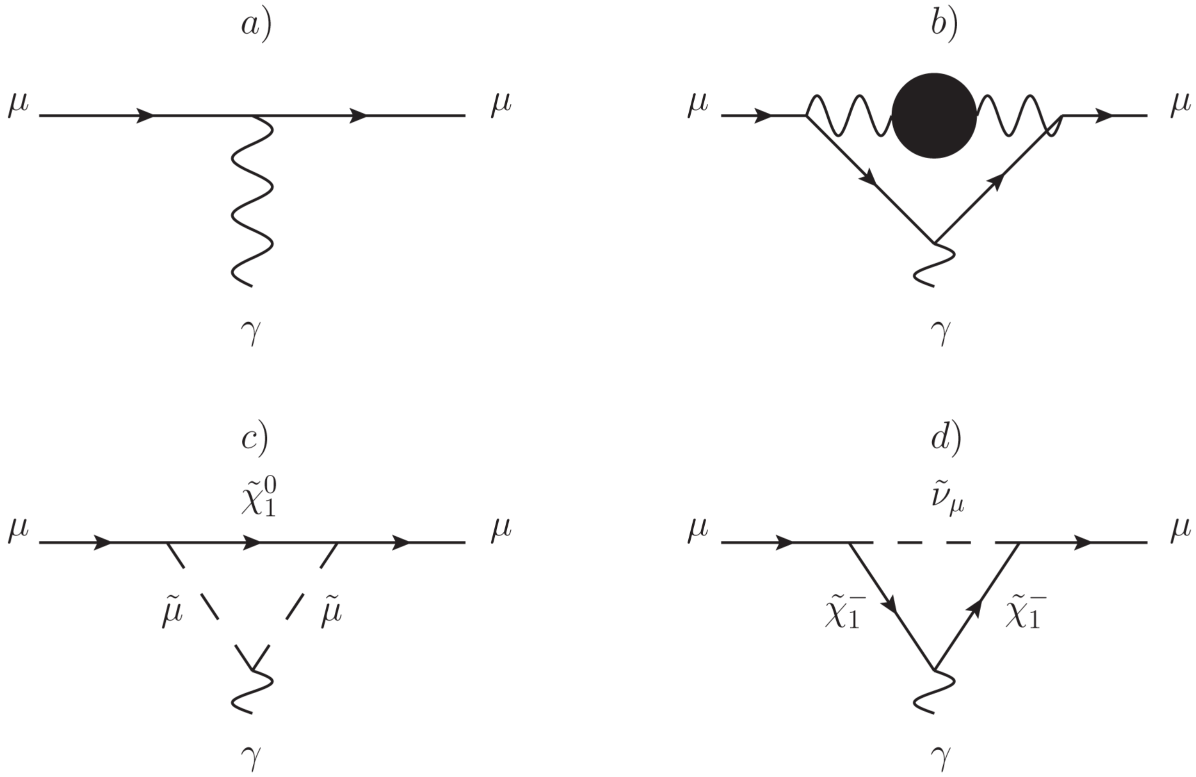

Figure 7.18: Diagrams for muon interaction with an electromagnetic field. Loop corrections to the tree level diagram a) give the value of a_μ. Diagram b) shows hadronic vacuum polarization where the blob contains QCD fields. Diagrams c) and d) show the lowest order MSSM contributions to a_μ.

interactions there. At meson level it leads to measurable decays of the type $B \to X_s\gamma$, where X_s is some meson with a strange quark, *e.g.* the decay $B \to K\gamma$. Figure 7.19 a) shows the one-loop Standard Model contribution with a virtual W and up-type quarks. This contribution is suppressed by the smallness of the CKM entries in the W-vertices which favours diagrams with a top quark in the loop, and the large masses, M_W and m_t, in the loop.

The process has been calculated at next-to-next-to-leading order (NNLO) in the Standard Model to be $\mathrm{Br}(B \to X_s\gamma)_{\mathrm{SM}} = (3.36 \pm 0.23) \cdot 10^{-4}$ for $E_\gamma \geq 1.6\,\mathrm{GeV}$ [32, 33], inclusively summing over all meson final states.[11]

Supersymmetric particles may contribute to this process, for example with off diagonal soft mass terms in diagrams such as Fig. 7.19 b) and d), where in b) a $m_{bs}^2\tilde{b}^*\tilde{s}$ soft-breaking off-diagonal Lagrangian mass term changes a $\tilde{b}_1$ squark to a $\tilde{s}$ squark, and in d) where a $m_{tc}^2\tilde{t}^*\tilde{c}$ term changes a $\tilde{t}_1$ to a $\tilde{c}$. Charged Higgs bosons with flavour non-diagonal couplings can also give important contributions in diagrams such as Fig. 7.19 c).

The main MSSM contributions are expected to come from the chargino–stop[12] and charged Higgs–top loops, as shown in Figs. 7.19 c) and d), respectively. However, there is little room

[11] For the process $b \to d\gamma$ the Standard Model calculation yields $\mathrm{BR}(B \to X_d\gamma) = 1.73^{+0.12}_{-0.22} \cdot 10^{-5}$.

[12] We usually expect higher generation off-diagonal soft terms to be larger due to RGE running controlled by Yukawa couplings.

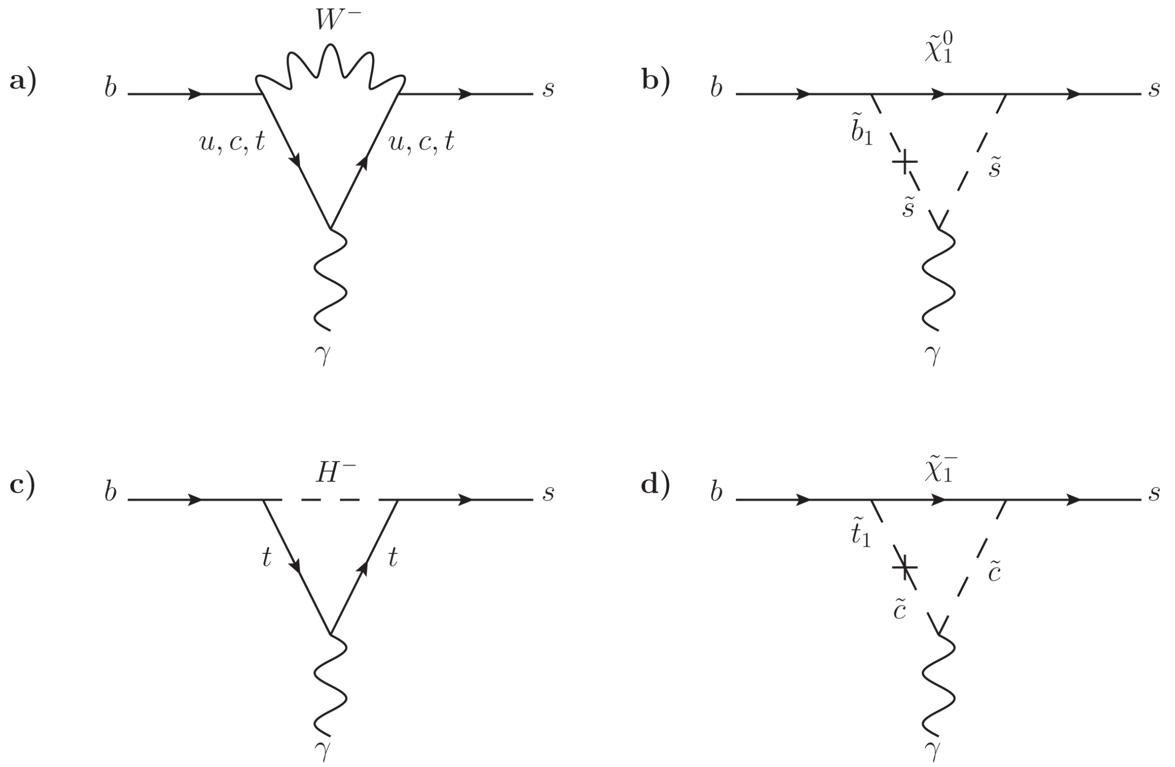

Figure 7.19: Diagrams for the process $b \to s\gamma$. a) shows the SM diagram while b), c) and d) show MSSM contributions.

for effects from supersymmetry since the current experimental world average is $\text{Br}(B \to X_s\gamma) = (3.32 \pm 0.15) \cdot 10^{-4}$ [34]. This means that either the charged Higgs is heavy enough and the stop-scharm soft mass term small enough not to contribute significantly, or that there are cancellations between the two contributions. This is both an example of a process that puts significant constraints on off-diagonal soft terms, and a process that is important in constraining the possible values of the charged Higgs mass.

7.4.4 $B_s \to \mu^+\mu^-$

The process $B_s \to \mu^+\mu^-$ is another FCNC process as either the bottom or the strange quark must change flavour in order to get a coupling to the muons. The Standard Model process is shown in Fig. 7.20 a), where the CKM factor in the W vertex allows one of the quarks to change flavour, but also suppresses the decay rate. For strange historical reasons involving darts and illegal substances this process is known as a **penguin diagram** [35].

On top of the small CKM factor, the process also suffers from what is called **helicity suppression** in the Standard Model. The Z-boson is spin-1, while the starting point meson B_s is spin-0 (pseudoscalar), meaning that the spins of the quarks in the meson are opposite. At some point in the diagram the helicity (chirality) must "flip". This introduces an extra suppression proportional to $m_\mu^2/M_{B_s}^2$, making the expected rate extremely small and sensitive to supersymmetry contributions. We get a similarly suppressed process for B_d with a $\bar{d}$-quark instead of the $\bar{s}$ in the initial state.

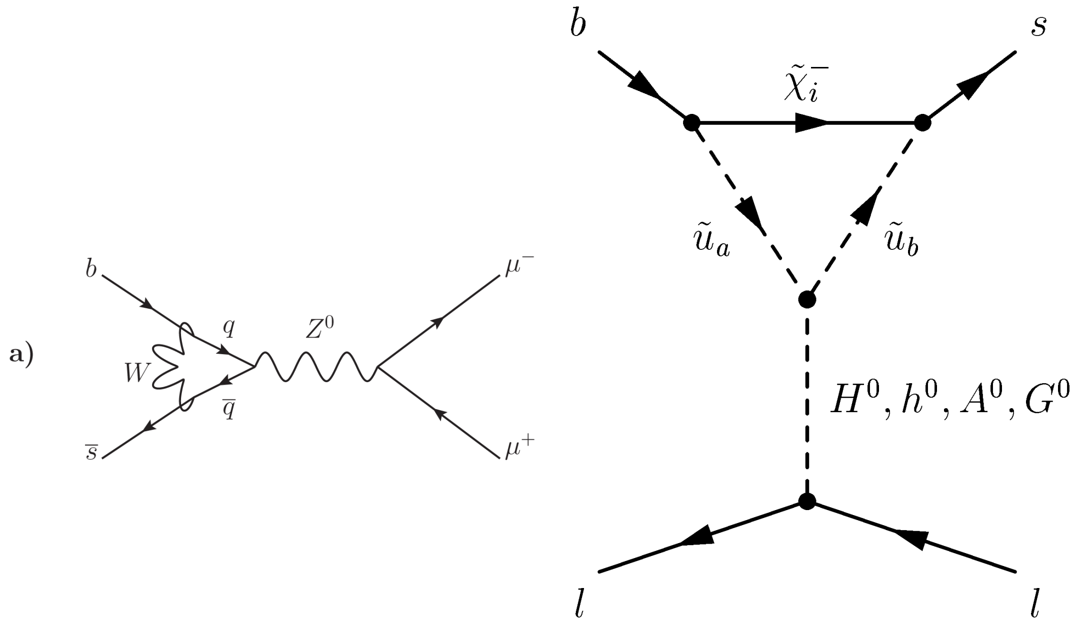

Figure 7.20: Diagrams for the process $B_s \to \mu^+\mu^-$. Diagram a) shows one of the leading SM contributions, while b) shows one contribution from the MSSM taken from [36].

The predicted SM branching ratios for these processes are [37]:

$$\mathrm{Br}(B_s \to \mu^+\mu^-) = (3.65 \pm 0.23) \cdot 10^{-9}, \tag{7.8}$$
$$\mathrm{Br}(B_d \to \mu^+\mu^-) = (1.06 \pm 0.09) \cdot 10^{-10}. \tag{7.9}$$

First evidence for the $B_s \to \mu^+\mu^-$ decay was shown by the LHCb Collaboration in 2012. The current value for the branching ratio from combined measurements at LHCb, ATLAS and CMS is [38]

$$\mathrm{Br}(B_s \to \mu^+\mu^-) = 2.69^{+0.37}_{-0.35} \cdot 10^{-9}, \tag{7.10}$$
$$\mathrm{Br}(B_d \to \mu^+\mu^-) < 1.6 \cdot 10^{-10}, \tag{7.11}$$

where the value for B_d is an upper limit at the 90% confidence level. It is interesting to observe that the measured value is lower than the Standard Model prediction at around the 2σ level, this has lead to speculations of destructive interference from new physics.

If we limit ourselves to scenarios with only diagonal soft terms the dominant contribution in the MSSM is from process such as shown in Fig. 7.20 b), with a chargino and up-type squark loop. These contributions are proportional to $\tan^6\beta$, which makes the decay process highly sensitive to scenarios with large $\tan\beta$. To see this dependence, notice that μ couples to the mediating heavy higgses H/A^0 through the Yukawa term $y^l_{22} L_2 H_d \overline{E}_2$ in the superpotential, and the Yukawa constant in this term, $y^l_{22} = y_\mu$, is connected to the fermion mass through $m_\mu = y_\mu v \cos\beta$. Thus this vertex is proportional to $1/\cos\beta$ or $\tan\beta$, giving a factor $\tan^2\beta$

in the amplitude squared.[13]

Furthermore, a chargino(higgsino)–stop loop can couple the strange and bottom quarks to the Higgs. These couplings are proportional to the bottom Yukawa coupling y_b, from the superpotential terms $y^d_{33}Q_3H_d\bar{D}_3$, which appears in the stop–chargino–bottom vertex, and the $y^d_{32}Q_3H_d\bar{D}_2$, which appears in the strange–chargino–stop vertex. Both these Yukawa couplings are proportional to y_b and thus to $1/\cos\beta$, giving a further factor of $\tan^4\beta$ in the amplitude squared. This $\tan\beta$ dependence makes $B_s \to \mu^+\mu^-$ an excellent channel for discovering supersymmetry, and puts very stringent bounds on the sparticle masses in large $\tan\beta$ scenarios.

7.5 Excercises

Exercise 7.1

From relativistic kinematics, show Eq. (7.7). *Hint:* the choice of rest frame is very important in order to simplify the calculation.

Exercise 7.2

Find the total cross section for the process $q\bar{q} \to \tilde{q}\tilde{q}^*$ via an s-channel gluon shown in Fig. 7.21.

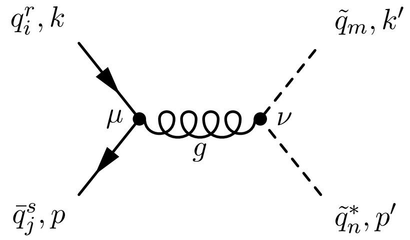

Figure 7.21: Strong production of a squark–anti-squark pair through a gluon.

[13] Remember that in the limit of large $\tan\beta$

$$\cos\beta = \pm\frac{1}{\sqrt{1+\tan^2\beta}} = \pm\frac{1}{\tan\beta\sqrt{1+\frac{1}{\tan^2\beta}}} \simeq \pm\frac{1}{\tan\beta}. \tag{7.12}$$

Chapter 8

Supersymmetric dark matter

We have a standard model also for cosmology, the Lambda Cold Dark Matter model (ΛCDM). This models the observed universe on three main ingredients: the known Standard Model matter (dominantly the baryonic matter of the atoms, radiation and neutrinos), dark energy (Λ) and cold dark matter (CDM), where the cold signifies that this ingredient is non-relativistic. In this chapter we will look closer at candidates for dark matter (DM) in supersymmetric models.

8.1 Evidence for dark matter

The idea of dark matter goes back quite a long way. Today we have evidence for the existence of dark matter through several effects where we observe its gravitational influence on ordinary matter. We list the evidence below:

1) Kinematics (Zwicky 1933 [40]): The motion of galaxies (velocity dispersion) cannot be explained by the visible matter. This has also been observed on the scales of galaxies in their rotation curves (Rubin 1970 [41]).

2) Gravitational lensing (Tyson 1996 [42]). First observed in galactic clusters. Clusters show evidence of lensing not explained by luminous matter. Dark matter dynamics (non-interacting) are demonstrated by the Bullet cluster (Clowe 2006 [43]).

3) Large scale structures (clusters, superclusters, filaments and voids): The structures observed in the 2dFGRS (2-degree Field Galaxy Redshift survey Colles 2001 [44]) and the SDSS (Sloan Digital Sky Survey Tegmark 2004 [45]) imply a relative matter density of $\Omega_m \equiv \frac{\rho_m}{\rho_c} \simeq 0.29$ where $\rho_c = 1.05 \cdot 10^{-5} h^2\,\mathrm{GeV/cm^3}$ is the critical energy density for a flat universe.[1] They also imply that the majority of DM must be **cold** (non-relativistic), because warm DM would suppress clustering.

4) Big-Bang Nucleosynthesis (BBN): The formation of light elements in the period $t = 1 - 1000\,\mathrm{s}$ after the Big Bang. Measurements of Early Universe abundance of light elements, mainly D and He, points to a baryonic matter density of $\Omega_b \approx 0.04$. This gives $\Omega_{\mathrm{leftover}} = \Omega_{\mathrm{DM}} \approx 0.25$.

[1] h is defined through the Hubble constant H_0 as $H_0 = 100h\,\mathrm{km/Mpc/s}$.

5) Supernovae (Riess 1998 [46] and Perlmutter 1999 [47]): Measurements of type Ia supernovae (SNe Ia) were used as standard candles to show an accelerated expansion of the Universe. This fixes $\Omega_\Lambda - \Omega_m \simeq k$ where Ω_Λ is the energy density of dark energy/cosmological constant.

6) Cosmic Microwave Background (CMB) (Penzias & Wilson 1965 [48]): The temperature variation of the CMB over the sky of the order of 0.0002 K is sensitive to all cosmological parameters, and gives $\Omega_\Lambda + \Omega_m \simeq k$, where k is some constant.

The evidence above can be used to constrain the ΛCDM concordance model of cosmology, which has just a handful of ingredients such as baryonic and dark matter, radiation (photons) and dark energy. In Fig. 8.1 we show the effects of the SNe, CMB and large scale structure data (BAO) on this model in the plane of matter density Ω_m and dark energy density Ω_Λ.

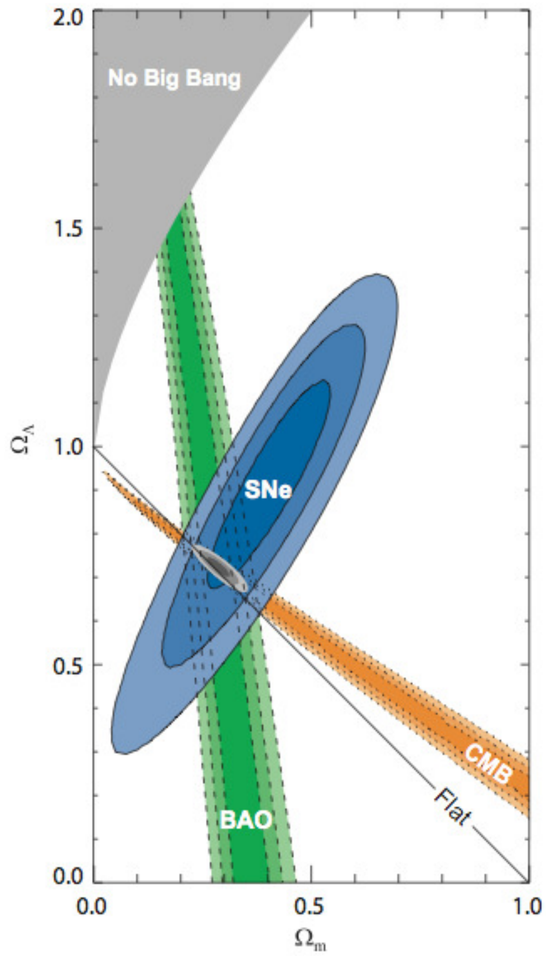

Figure 8.1: Limits from different observational sources on the dark energy density, Ω_Λ, compared to the total mass density in the universe, Ω_m.

A maximum likelihood fit to a selected subset of the measurements by the Planck Collaboration [49] gives the parameters for the model shown in Table 8.1. This means that the non-baryonic matter constitutes some 85% of the matter in the Universe, and that its relative energy density is quite well determined, at the percent level.

Parameter	Ω_Λ	$\Omega_{\mathrm{m}} h^2$	$\Omega_b h^2$	H_0 [km/Mpc/s]	t_0 [Gy]
Value	$0.685^{+0.018}_{-0.016}$	0.1426 ± 0.0025	0.02205 ± 0.00028	67.3 ± 1.2	13.817 ± 0.048

Table 8.1: Measured values for cosmological parameters [49].

8.2 WIMP magic

The very existence of a stable Weakly[2] Interacting Massive Particle (WIMP) χ automatically gives an additional component to the total energy density of the Universe. WIMPs are found in a number of theories, for example the lightest neutralino of the MSSM, the lightest Kaluza-Klein particle of a theory with extra dimensions or an inert (no vev) Higgs boson.

This is due to the in equilibrium **thermal production** of the WIMP through the process $SM\ SM \to \chi\chi$, where SM are some Standard Model particles, and the reverse annihilation process $\chi\chi \to SM\ SM$, in the early hot Universe ($T \gg m_\chi$). As the temperature decreases to $T < m_\chi$ and there is not enough energy in an average collision for the production of χ to occur, only the reverse process can take place, and the comoving density[3] falls with the temperature of the Universe.

The WIMPs then experience what is called a **chemical decoupling**, or loss of chemical equilibrium, due to the expansion of the Universe. This is when the WIMPs become so dilute, because of the expansion, that they in effect no longer interact inelastically, and this roughly happens when the expansion rate becomes larger than the rate of annihilation. The WIMPs then get a constant (comoving) density, we say that they experience a **freeze-out** at this temperature T_c. With weak-scale masses and couplings the freeze-out happens at $T_c \approx 0.05 m_\chi$, however, this is before or at the same time as **kinetic decoupling** where the WIMPs effectively lose elastic interactions with the other matter, meaning that χ freezes-out with non-relativistic velocities and become cold dark matter.

The exact time (temperature) of freeze-out is controlled by the annihilation cross section of χ, larger cross sections keep chemical equilibrium for longer, in turn resulting in lower dark matter relic abundance. This abundance, in number density, can be found from the Boltzmann equation

$$\frac{dn_\chi}{dt} = -3Hn_\chi - \langle \sigma v \rangle (n_\chi^2 - n_\chi^{eq2}), \tag{8.1}$$

where n_χ^{eq} and n_χ are the chemical equilibrium and actual comoving number densities, H is Hubble's constant for the expansion rate, and $\langle \sigma v \rangle$ the velocity averaged annihilation cross section for $\chi\chi \to SM\ SM$. Solutions to this equation for increasing $\langle \sigma v \rangle$ are shown in Fig. 8.2.

In practice one must also often take into account co-annihilation with other particles with mass within 10%−20% of the χ, and numerical codes such as `DarkSUSY` [50] or `MicrOMEGAs` [51, 52] are typically used to solve the Boltzmann equation. For weak scale particles a rough approximation to the resulting dark matter density is

$$\Omega_\chi h^2 = 0.1 \times \frac{3 \times 10^{-26}\,\mathrm{cm^3 s^{-1}}}{\langle \sigma v \rangle},$$

[2] Weak as in electro-weak, meaning on the same scale as the weak force.

[3] Taking the expansion of the universe into account by looking at the number of particles in a volume expanding at the same rate.

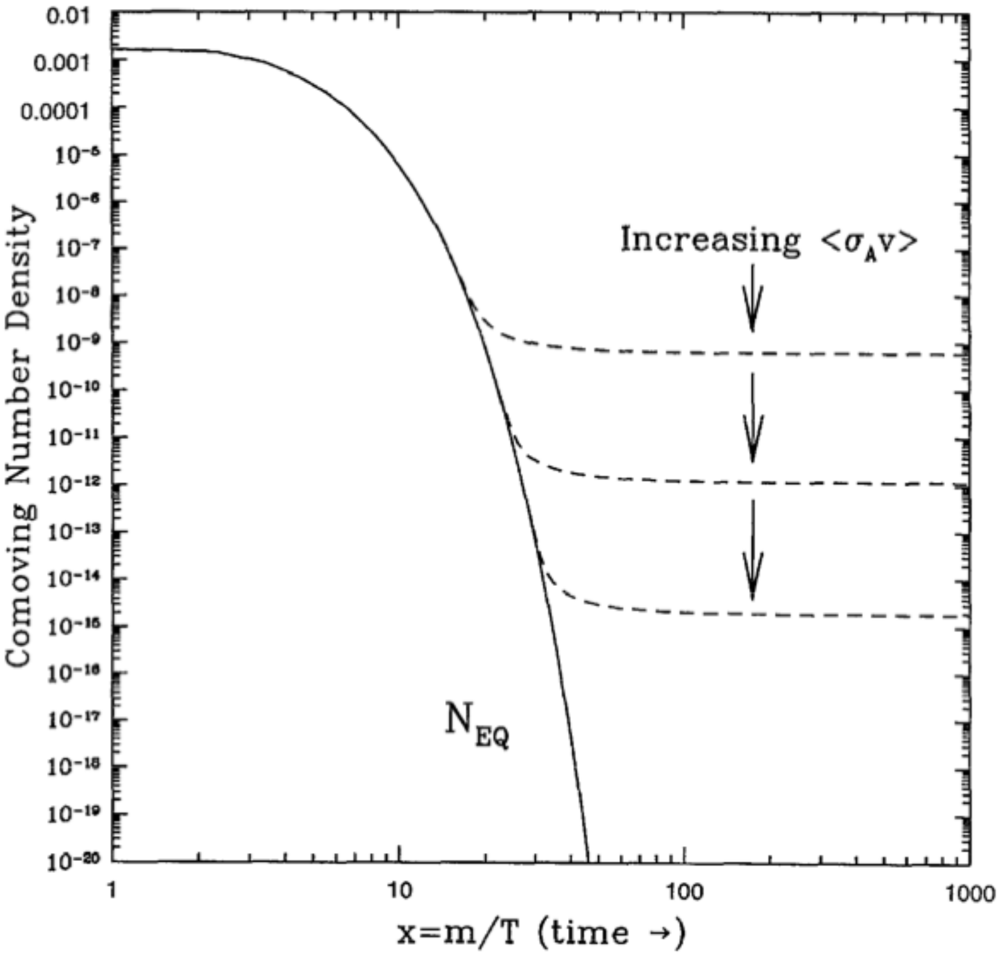

Figure 8.2: Illustration of the freeze-out of the comoving number density of a WIMP as a function of time, where the black line represents a model without chemical decoupling, and the dotted lines represent different freeze out temperatures for different velocity averaged annihilation cross sections.

and since the annihilation cross section can be shown to be

$$\langle \sigma v \rangle \approx \frac{\alpha_{weak}^2}{m_{weak}^2} \approx 10^{-25}\,\mathrm{cm}^3\mathrm{s}^{-1}, \tag{8.2}$$

the predicted dark matter density is

$$\Omega_\chi h^2 \approx 0.1 \times \left(\frac{g_{weak}}{g_\chi}\right)^4 \left(\frac{m_\chi}{m_{weak}}\right)^2.$$

When compared to the values in Table 8.1 this is called the **WIMP-miracle** since it supplies just about the correct missing energy density for the WIMP as long as it has a weak scale mass and a weak scale coupling to the Standard Model. The strong sensitivity of the predicted dark matter density to both the mass and the coupling, and the precisely measured experimental value, means that the WIMP scenario is very predictive and testable.

For a more detailed discussion of the WIMP miracle, see the standard cosmology book by Kolb and Turner [53].

8.3 Dark matter candidates in supersymmetry

With R-parity conservation in place we have seen that the LSP is stable and any neutral LSP can then in principle constitute all or part of the dark matter. Without R-parity only super-weakly coupling particles like gravitinos and axinos are candidates. Below we briefly discuss the various possibilities.

8.3.1 Neutralino

As soon as you have a stable neutralino LSP, you usually get into trouble trying to explain why there is so little dark matter. For example, the standard mSUGRA bino-like $\tilde{\chi}_1^0$ scenario prefers LSP masses around 100 GeV or lower not to overproduce dark matter. Due to current lower bounds on the $\tilde{\chi}_1^0$ mass and the measured Higgs mass this tends to be difficult to realise in practise. This scenario with a low mass bino-like neutralino is often called the **bulk region** scenario, which can be seen in Fig. 8.3 in the lower left corner.

Alternatives to the bulk region scenario use co-annihilation or resonant annihilation to increase $\langle\sigma v\rangle$ and thus decrease the dark matter density. The **stau-coannihilation region**, where $\tilde{\tau}_1\tilde{\chi}_1^0 \to SM\,SM$ is an efficient annihilation process reducing the dark matter density, exists for $m_{\tilde{\tau}_1} - m_{\tilde{\chi}_1^0} \leq 10\,\text{GeV}$. This makes the scenario difficult to discover at collider experiments due to the production of soft (low-energy) taus. In mSUGRA the stau-coannihilation region can be found for small m_0, and is shown as the lower strip in Fig. 8.3 which follows the lower theoretical bound (brown) where the stau becomes the LSP. Outside of mSUGRA similar regions can also be found for smuons and selectrons, with large mass degeneracy between the slepton and the LSP.

The **stop-coannihilation region**, where $\tilde{t}_1\tilde{\chi}_1^0 \to SM\,SM$ is efficient typically has $m_{\tilde{t}_1} - m_{\tilde{\chi}_1^0} \leq 25\,\text{GeV}$. In mSUGRA this exists for large values of $|A_0|$, small m_0 and $m_{1/2}$. Again, this is difficult to discover because of the soft decay products of the stop.

The **Higgs funnel region** is found for $2m_{\tilde{\chi}_1^0} \simeq m_{A,H}$ and large $\tan\beta$, where the neutralino has resonant annihilation through a heavy Higgs boson. For mSUGRA, this is shown in Figure 8.3 as the two diagonal structures roughly in the middle of the plot, rising as a funnel upwards.

The **chargino-coannihilation** region can be found when we have higgsino or wino LSPs, where a chargino is automatically degenerate with the LSP resulting in a lower dark matter density typically below experimental bounds. In mSUGRA this region is called the **focus point region**, and is found for large m_0 and low μ, leading to so-called split-SUSY, as the sfermion masses need to be pressed up quite a bit, and there is a large mass difference between gauginos and sfermions. The focus point region can be seen in Fig. 8.3 following the upper theoretical bound where EWSB breaks down.

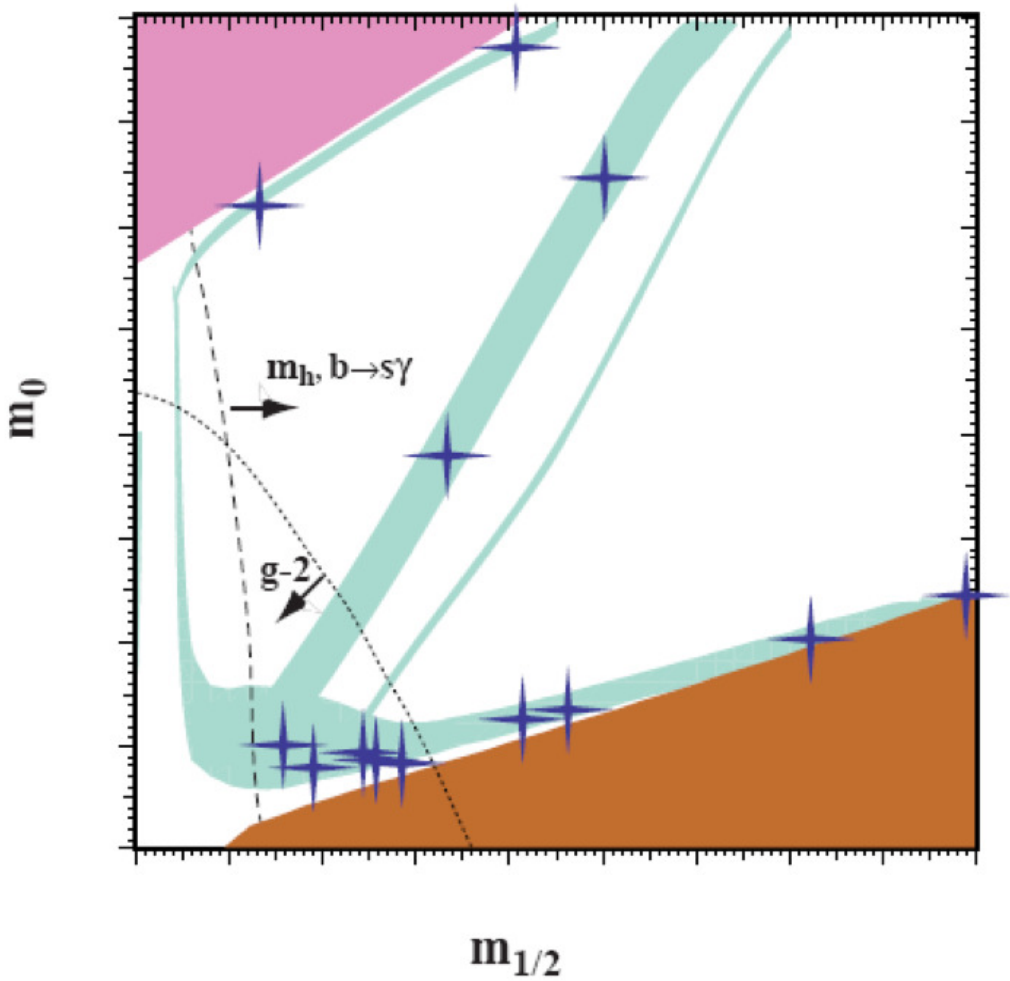

Figure 8.3: Generic illustration of the allowed neutralino dark matter regions (puke green) in the $(m_0, m_{1/2})$-plane for mSUGRA. Except for the low m_0 and $m_{1/2}$ regions the area outside of the allowed green region gives too much dark matter. The dashed line shows the Higgs mass limit which pushes towards larger values of $m_{1/2}$, while the dotted line represents the limit from the anomalous magnetic moment of the muon.

8.3.2 Sneutrinos

The left handed sneutrino $\tilde{\nu}_L$ is happily excluded as a potential dark matter candidate due to the large cross section for $\tilde{\nu}_L q \to \tilde{\nu}_L q$ via Z-exchange.[4] The large cross means that it should already have been seen by direct detection experiments, see Sec 8.4. It is also problematic to get $m_{\tilde{\nu}_L} < m_{\tilde{l}_L}$ due to hyperfine-splitting. However, the right-handed sneutrino $\tilde{\nu}_R$ couples very weakly and is still a viable candidate. This would necessitate adding right-handed

[4] For the neutralinos this problem only exists for a higgsino $\tilde{\chi}_1^0$ LSP, as the wino and bino do not couple to the Z.

neutrinos and sneutrino superfields to the MSSM.

8.3.3 Gravitino

The **gravitino** is the massive spin-3/2 supersymmetric partner of the graviton in models with supergravity. With only gravitational strength interactions, the gravitino is *not* a WIMP as it is never in chemical equilibrium. If it is the LSP it can be created from the decays of the next-to-lightest supersymmetric particle (NLSP), giving, in R-parity conserving scenarios,

$$\Omega_{\tilde{G}} = \frac{m_{\tilde{G}}}{m_{\mathrm{NLSP}}} \Omega_{\mathrm{NLSP}}.$$

However, these scenarios are problematic because the NLSP is long-lived due to the extreamly weak interaction and creates potential trouble in BBN by injecting energy that changes the production of light elements.

Alternatively, the gravitino can be created in non-thermal production processes in the period of reheating after inflation. One potential process $gg \to \tilde{g}\tilde{G}$ is shown in Fig. 8.4. The reverse process $\tilde{g}\tilde{G} \to gg$ is not efficient as the density of gravitinos is never high enough given the small cross section. This type of dark matter creation process is often called **freeze-in**. For the gravitino this gives a new magic formula:

$$\Omega_{\tilde{G}} h^2 \approx 0.5 \cdot \left(\frac{T_R}{10^{10}\,\mathrm{GeV}} \right) \left(\frac{100\,\mathrm{GeV}}{m_{\tilde{G}}} \right) \left(\frac{m_{\tilde{g}}}{1\,\mathrm{TeV}} \right)^2, \tag{8.3}$$

where T_R is the unknown and very weakly constrained reheating temperature. This production mechanism is also valid for models with R-parity violation. There the small gravitino coupling $\propto \frac{1}{M_P}$ makes the gravitinos very long-lived, with lifetimes longer than the age of the Universe, but not absolutely stable because of the R-parity violating operators.

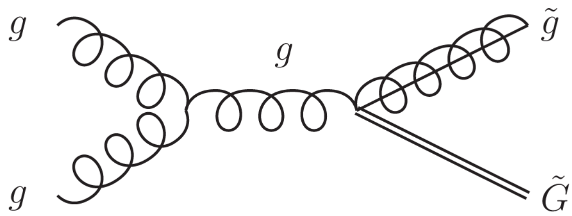

Figure 8.4: One possible diagram for the non-thermal production of gravitinos from the scattering of gluons.

8.3.4 Others

One could even imagine colour charged supersymmetric particles as dark matter, in particular the gluino, which, if stable, after hadronisation with Standard Model quarks and/or gluons would form so-called **R-hadrons**. These have very strict limits from searches, but these limits are somewhat obfuscated by complications in modelling R-hadron scattering.

8.4 Direct detection

In addition to the direct production of dark matter at colliders and the corresponding searches for missing energy, there are two other main ways to search for dark matter, direct and indirect detection. Here we briefly discuss **direct detection**.

Direct detection seeks to make weak and thus rare dark matter interactions with standard model matter visible by very low background searches in large volumes, using local galactic halo dark matter interacting with ordinary matter on Earth. The feasibility of direct detection is very dependent on the χ scattering cross section on nucleons (quarks), which can in principle be calculated in a given model,[5] but also on the dark matter halo density distribution and velocity distribution, which have large uncertainties.

The rate of interaction between dark matter and a particular type of nucleons in a sample can be expressed in the differential scattering rate with respect to the recoil energy E_r of a scattered nucleon with mass M as

$$\frac{dN}{dE_r} = \frac{\sigma \rho_{DM}}{2\mu^2 m_\chi} |F(q)|^2 \int_{v_{min}}^{v_{esc}} \frac{f(\vec{v})}{v} d^3v, \tag{8.4}$$

where σ is the dark matter scattering cross section off the nucleus in question, ρ_{DM} is the dark matter halo density at Earth, $\mu = m_\chi M/(m_\chi + M)$ is the dark matter and nucleus reduced mass, $F(q)$ is a nuclear form factor dependent on the scattering momentum transfer $q = \sqrt{2ME_r}$ and the nucleon type, $f(\vec{v})$ is the velocity distribution of the dark matter in the halo, $v_{min} = \sqrt{ME_r/2\mu^2}$ is the minimal velocity that gives a recoil energy E_r and v_{esc} is the escape velocity from the halo.

In general direct detection relies heavily on suppressing Standard Model backgrounds which would be the overwhelmingly dominant cause of observed recoils. This is done by moving the experiments deep under ground where they are shielded from cosmic rays, and by specifically shielding the detector samples against local radioactivity. Beyond this, there are two main tactics followed in order to try to directly detect dark matter:

- Suppress (almost) all backgrounds and monitor as large a volume as possible. This used in experiments such as XENON, PANDAS and CDMS.
- Look for an annular modulation in the event rate – due to the Earth's movement in the galactic rest frame and thus in the dark matter halo, changing the velocity integral in Eq. (8.4) – to observe a small dark matter signal on top of a constant background, such as used in the DAMA and CoGeNT experiments.

Figure 8.5 shows results from the current most important direct detection experiments as limits on the dark matter–nucleon cross section as a function of dark matter mass. Observe that while DAMA claims a detection (closed regions), these are already excluded by many other experiments. For reference, a neutralino WIMP could reasonably be expected to have a scattering cross section in the range $10^{-45} - 10^{-50}$ cm^2 for masses in the range 100–1000 GeV [54]. The dashed yellow line shows what is called the **neutrino floor**. This is the cross section that can be reached before the experiments would be saturated with recoils due to neutrinos scattering off the experiment sample. Going below the neutrino floor would need

[5] However, while the quark–dark matter cross section is relatively easy to calculate, the fact that the quark is bound in a nucleon, and the nucleon bound to a nucleus makes this more difficult in practice, involving also estimates from nuclear physics.

new experimental techniques, fundamentally, the direction of the particle creating the recoil would need to be reconstructed.

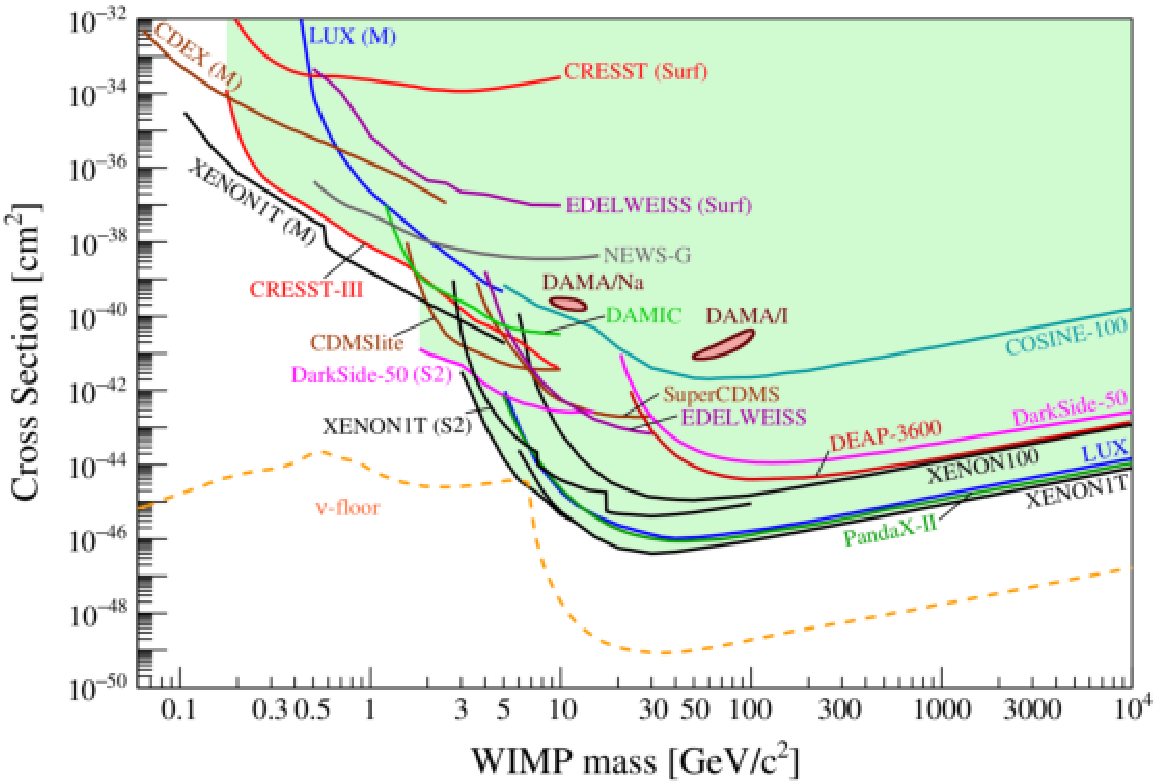

Figure 8.5: Plot of different exclusion and detection results for direct detection of dark matter in the WIMP mass versus WIMP–nucleon cross section plane [55]. The dashed yellow line shows the neutrino floor.

8.5 Indirect detection

In **indirect detection** we look for the annihilation, $\chi\chi \to SM\,SM$, or decay products of dark matter in multiple final (messenger) states in cosmic rays. The viable search channels must be stable Standard Model particles, so that they can reach the Earth (or satellites in orbit). The messengers should also have as low backgrounds from ordinary astrophysical processes as possible, this makes searches with electrons and protons extra challenging. The remaining candidates are photons, neutrinos, positrons, antiprotons and antideuterons.[6] We now discuss the properties of each of these in a little more detail.

Photons

These can either come from direct production processes such as the annihilation channels $\chi\chi \to \gamma\gamma, Z\gamma$, which is relatively easy to detect because the spectrum is a sharp line at

[6]Potentially, even heavier antinuclei than antideuterons could be used, but they would be even rarer.

exactly the mass of the dark matter, or they could be γ from bremsstrahlung or pion decays in the annihilation decay products, which creates a broad spectrum. This is harder to detect because of the large background of bog-standard astrophysical photons, but is expected to make up the majority of photons from dark matter. Photons from dark matter have the advantage that they point to the source, so we can focus our searches on areas in our galaxy or nearby galaxies with large ρ_{DM}, where annihilation or decay is more probable, and thus reducing potential backgrounds relative to the signal. We can also look for photons that are extragalactic in origin, but then we have to account for the red-shifting of the spectrum.

Dark matter annihilating in our own galaxy into photons should result in a differential flux at Earth in terms of energy E and solid angle Ω given by

$$\frac{d\Phi}{dEd\Omega} = \frac{1}{8\pi m_\chi^2}\frac{dN_\gamma}{dE}\langle\sigma v\rangle\int_{\text{l.o.s.}}\rho_{DM}^2(l)dl, \tag{8.5}$$

where $N_\gamma(E)$ gives the number of photons with energy E in a single annihilation event and the integral is over the dark matter density in the line of sight. We see that the flux depends on the square of the dark matter density since annihilation requires two particles to be present. For decaying dark matter the corresponding expression is proportional to ρ_{DM}.

There have been some indications of an excess of photons above expected backgrounds from the galactic centre (a.k.a. the Hooperon [56]), however, no unambiguous dark matter signal has been confirmed. Current limits from the Fermi-LAT experiment seems to rule out most possible models for a dark matter explanation for this excess, and, more importantly, sets a cross section limit for dark matter annihilation close, and for some masses beyond, to the canonical limit of

$$\langle\sigma v\rangle = 3\times 10^{-26}\,\text{cm}^3\text{s}^{-1},$$

which would give the correct dark matter density, see Fig. 8.6.[7] This limit is some of the strongest evidence today against WIMP dark glitter.

Neutrinos

These also point to their source since since they are electrically neutral, and can be extragalactic in origin just like the photons, still reaching the Earth. The same flux calculation can be used, starting from the neutrino spectrum from dark matter annihilation. While the astrophysical background is much smaller, the neutrino signal is difficult to detect due to the weak matter interaction. The current leading experiment in detecting neutrinos from dark matter annihilation or decay is the IceCube experiment at the South Pole that has instrumented a cubic kilometre of the South Pole ice with photodetectors a kilometre below the surface.[8]

One interesting alternative possibility is that dark matter scatters on ordinary matter sufficiently strongly that the dark matter accumulates at the centre of the Sun (or possibly the Earth), because it scatters of the Suns atoms, loses energy and becomes gravitationally bound, ultimately being further downscattered in energy in multiple interactions, settling at the Suns centre. When these DM particles annihilate the only decay products that can escape the Sun's interior are neutrinos. These could then potentially be detected, and the fact that

[7] Keep in mind though, that the velocity distribution of any dark matter today would not be the same as during freeze-out at a much higher temperature, so the cross section numbers are not exactly translatable.

[8] This should, on galactic scales, get some sort of prize for the geniusly nutscase idea.

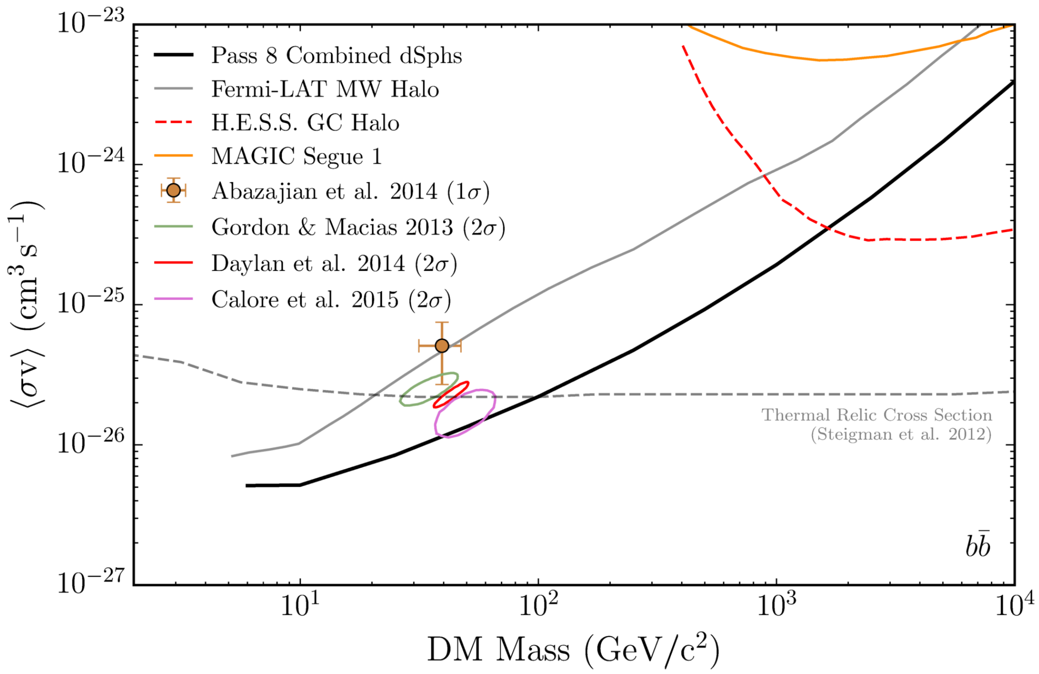

Figure 8.6: Results from Fermi-LAT indirect gamma-ray searches in the $\chi\chi \to b\bar{b}$ channel. Grey line shows limit from Milky Way halo search, black line from Milky Way dwarf spheroidal galaxy search with six years of data [57].

high energy neutrinos would be coming directly from the Sun would reduce astrophysical backgrounds significantly.

Positrons

Charged particles propagate in a complicated way through the galactic magnetic field, and they are therefore impossible to track back to the source. At the same time sources outside of our own Galaxy cannot contribute significantly to the flux at Earth. Positrons also have relatively large astrophysical backgrounds, for example pulsars are expected to produce a significant amount of lower energy positrons, so experiments search for small excesses on large backgrounds, mostly at high energies. Some potential excess has been seen in the past by Fermi-LAT and PAMELA, but this has not been conclusive.

Antiprotons

Just as the positrons these propagate in a complicated manner, but the backgrounds are under better control due to the lack of significant astrophysical antiproton production, except from the scattering of very high energy cosmic rays on the interstellar medium. The current best limits here come from the AMS-02 experiment.

Antideuterons

These have very low backgrounds because the production of antideuterons in astrophysical processes is extremely rare, in particular at low energies, due to the conservation of baryon number and the required high energy of cosmic rays to produce two antibaryons (at least two baryons must be produced in the same interaction). This means that the detection of an antideuteron should be a smoking duck signal of dark matter annihilation or decay. However, the physics of the formation of antideuterons from dark matter is quite complicated and hard to reliably calculate, brining in extra uncertainty. Current searches set upper bounds on the flux of antideuterons at the Earth. It is hoped that AMS-02 will provide new better data on antideuterons soon.[9]

8.6 Excercises

Exercise 8.1

Show that $\chi\chi \to Z \to f\bar{f}$ gives

$$\sigma v \approx \frac{g^4 E_\chi^2}{128\pi m_Z^2}, \tag{8.6}$$

which in the low-velocity limit can be shown to be

$$\langle \sigma v \rangle_0 \approx 10^{-25}\,\mathrm{cm}^3\mathrm{s}^{-1}.$$

[9] This has been hoped for a very long time.

Appendix A

Solutions to exercises

Exercise 1.1
Given the group multiplication definition the identity element must be $e = T_0 = 1$ since $T_a \circ T_0 = T_{a+0} = T_a$. Let $T_a^{-1} = T_b$. Since $T_a \circ T_b = T_{a+b} = 1$, we find $b = -a$. (This does not show that the inverse is unique, but this is not a requirement.) We have now demonstrated the required existence of an identity element (iii) and an inverse (iv) for $T(1)$ as required by the group definition (the order of operations can obviously be reversed). The closure of the multiplication operation (i) is true by its definition. The associativity (ii) can be demonstrated as follows $(T_a \circ T_b) \circ T_c = T_{a+b} \circ T_c = T_{a+b+c} = T_a \circ T_{b+c} = T_a \circ (T_b \circ T_c)$.

Exercise 1.2
We need to demonstrate the four group requirements. We have closure since the product of two such unitary matrices U_1 and U_2 is a unitary matrix: $(U_1U_2)^\dagger U_1U_2 = U_2^\dagger U_1^\dagger U_1U_2 = U_2^\dagger U_2 = I$. Matrix multiplication is always associative. There exists an identity among the matrices since the identity matrix is unitary: $I^\dagger I = II = I$. There exist an inverse $U^{-1} = U^\dagger$ for every U, since the matrix $U^\dagger$ is unitary if U is: $(U^\dagger)^\dagger U^\dagger = UU^\dagger = I$.

Exercise 1.3
Since $O(1)$ consists of 1×1 matrices, over $\mathbb{R}$ we suppose, also known as real numbers, the orthogonality requirement means that the transpose of that number (which is the number itself) multiplied by the number should be 1. There are two such reals, 1 and -1. The group thus has two elements. Since matrix multiplication in this case is just normal multiplication we have re-found the group $\mathbb{Z}_2$.

Exercise 1.4
An $n \times n$ matrix U over $\mathbb{C}$ has n^2 complex or $2n^2$ real parameters. The hermitian matrix $M = U^\dagger U$ has n real elements on the diagonal and $(n^2 - n)/2$ complex elements above and below the diagonal. Because $M^\dagger = M$ all the $(n^2-n)/2$ elements below the diagonal are given by the complex conjugate the corresponding elements above the diagonal. Since U is unitary $M = I$. That all the n real diagonal elements of M is equal to one gives n restrictions (n real equations) for the elements of U. Further, all $(n^2-n)/2$ complex elements above the diagonal are zero, which gives $(n^2-n)/2$ complex equations, which means n^2-n real equations. For the

terms below the diagonal we do not obtain any new equations. As a result the free parameters for the $U(n)$ matrices are $2n^2 - n - (n^2 - n) = n^2$. For $SU(n)$ we must use the additional requirement $\det U = 1$. Because $1 = \det M = \det(U^\dagger U) = \det U^\dagger \det U = |\det U|^2$, we see that $\det U$ must be a phase factor, and that the requirement $\det U = 1$ only gives one new condition (equation). Thus there are $n^2 - 1$ independent real parameters in $SU(n)$.

Exercise 1.5

A matrix in $SU(2)$ has $2^2 - 1 = 3$ free parameters. Let us start with a generic 2×2 complex valued matrix

$$U = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

For this to be unitary we must have

$$U^\dagger U = \begin{bmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\gamma|^2 & \alpha^*\beta + \gamma^*\delta \\ \beta^*\alpha + \delta^*\gamma & |\beta|^2 + |\delta|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then from the off-diagonal, $\delta^*\gamma = -\beta^*\alpha$ and $\gamma^*\delta = -\alpha^*\beta$ which gives $|\delta|^2|\gamma|^2 = |\alpha|^2|\beta|^2$, and inserting from the diagonal we have $(1 - |\beta|^2)|\gamma|^2 = (1 - |\gamma|^2)|\beta|^2$ which can be solved for $|\gamma|^2 = |\beta|^2$. Similarly from the diagonal $|\delta|^2 = |\alpha|^2$. We can then generically write $\gamma = e^{i\theta}\beta$ and $\delta = e^{i\phi}\alpha$, where θ and ϕ are two phases. The diagonal requirement is then completely fulfilled if $|\alpha|^2 + |\beta|^2 = 1$ and the matrix is

$$U = \begin{bmatrix} \alpha & \beta \\ e^{i\theta}\beta & e^{i\phi}\alpha \end{bmatrix}.$$

The determinant then requires $\alpha e^{i\phi}\alpha - \beta e^{i\theta}\beta = 1$ which is fulfilled for a phase ϕ that rotates α to α^* and a phase θ that rotates β to $-\beta^*$. Thus we can write

$$U = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}.$$

Exercise 1.6

Exercise 1.7

Since H is a subgroup of G then by point ii) in the definition since $h_j \in H$ we must also have $h_j^{-1} \in H$. With $h_i, h_j^{-1} \in H$ by point i) in definition $h_i \circ h_j^{-1} \in H$.

Exercise 1.8

Let $U_i, U_j \in SU(n)$, then

$$\det(U_i U_j^{-1}) = \det(U_i)\det(U_j^{-1}) = 1.$$

This means that $U_i U_j^{-1} \in SU(n)$. In other words, $SU(n)$ is a proper subgroup of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V)\det(U)\det(V^{-1}) = \frac{\det(V)}{\det(V)}\det(U) = 1.$$

In other words, $SU(n)$ is also a normal subgroup of $U(n)$, thus $U(n)$ is not simple.

Exercise 1.9

Let H be a subgroup of G and $g \in G$. For a member of the left coset $f \in gH$ then $gH = fH$ because $f \in gH$ implies that there must exist an $x \in H$ so that $gx = f$. Thus $fH = (gx)H = g(xH)$. Since H is a group this means $xH = H$. Thus every element in G belong to exactly one left coset. The argument of right cosets is identical.

Exercise 1.14

Split the representation space $V = V_1 \oplus V_2$, where we assume V_1 is a closed subspace of V under the unitary representation $\rho(g)$ (the reducible part). Let $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$. Since V_1 is closed $\rho(g)\mathbf{v}_1 \in V_1$ and thus the inner product $(\rho(g)\mathbf{v}_1, \mathbf{v}_2) = 0$. By unitarity the inner product is also $(\rho(g)\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, \rho(g)^{-1}\mathbf{v}_2)$, thus $\rho(g)^{-1}V_2 \subset V_2$ and both subspaces are closed, and the space thus completely reducible.

Exercise 1.18

A member U of $SU(2)$ fulfils $UU^\dagger = I$ and has $\det U = 1$ and can be written as

$$U = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$ and $a^2+b^2+c^2+d^2 = 1$. Using a composition function here it is tempting to solve for $a = \sqrt{1-b^2-c^2-d^2}$ and let a, b, c be the three free parameters if $SU(2)$. Note that solving for b, c or d is not really an option as a requirement for deriving the generators from the composition function is that the zero parameters, here $b = c = d = 0$, give the identity element of the group, which is the identity matrix. However, there is a potential issue here that choosing the sign on the square root restricts the parameterisation to apply for only group members with positive real components in the upper left element, meaning that this composition function only works for part of the group. This is another example of a local description of the group. Other parameterisations exist, however, they may have other problems such as a non-unique zero element (meaning no inverse of the composition function exists).

Fortunately, this parameterisation is enough to give all of the generators of the whole group:

$$\begin{aligned} X_1 &= \left.\frac{\partial U}{\partial d}\right|_{b=c=d=0} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \\ X_2 &= \left.\frac{\partial U}{\partial c}\right|_{b=c=d=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ X_3 &= \left.\frac{\partial U}{\partial b}\right|_{b=c=d=0} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \end{aligned}$$

While these are not the Pauli matrices we might be expecting, they are related to the Pauli matrices by a simple multiplicative factor $\sigma_1 = -iX_1$, $\sigma_2 = -iX_2$, and $\sigma_3 = -iX_3$. In fact the matrices we have found are typically used in mathematical texts as the generators of $SU(2)$, but then the exponential map has no factor of i. Notice that these matrices are anti-hermitian, while the Pauli matrices are hermitian.

The infinitesimal approach writes the group member as $U = e^{ida_i X_i} \simeq I + ida_i X_i$, and applying the unitarity requirement

$$UU^\dagger \simeq (I + ida_i X_i)(I + ida_i X_i)^\dagger = I + ida_i(X_i - X_i^\dagger) + \dots,$$

means that a necessary condition on the generators X_i is that they are hermitian $X_i = X_i^\dagger$. The general form of the generators is thus restricted to

$$X = \begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix},$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$. The property of the determinant can be found as follows (see Exercise **??**):

$$\det U = \det e^{ida_i X_i} = e^{\mathrm{Tr}[ida_i X_i]} = e^{ida_i \, \mathrm{Tr}[X_i]}.$$

This means that for the determinant to be one, each trace must be zero, meaning that the generators are traceless so that the general form of the generators is restricted to

$$X = \begin{bmatrix} \alpha & \gamma \\ \gamma^* & -\alpha \end{bmatrix}.$$

From the requirement $\alpha^2 + |\gamma|^2 = 1$ we can recreate the three Pauli matrices by the three choices: 1) $\alpha = 1$, $\gamma = 0$, 2) $\alpha = 0$, $\gamma = 1$, and 3) $\alpha = 0$, $\gamma = -i$. Naturally, there is a continuum of equivalent expressions for the generators from other compatible choices.

This approach emphasises two important properties of the generators. Firstly that the generators are hermitian. Since the generators will function as operators in a QM of QFT setting, this is very desirable. Second, the generators are traceless. This will often be a calculational advantage.

Exercise 2.1

Since

$$J_+J_-|m\rangle = (J_-J_+ + J_3)|m\rangle = (J_-J_+ + m)|m\rangle = J_-N_m^+|m+1\rangle + m|m\rangle = (N_{m+1}^- N_m^+ + m)|m\rangle,$$

and

$$J_+J_-|m\rangle = J_+N_m^-|m-1\rangle = N_{m-1}^+ N_m^-|m\rangle,$$

we get by comparison $N_{m+1}^- N_m^+ + m = N_{m-1}^+ N_m^-$. Now $N_j^+ = 0$ since $|j\rangle$ is the highest weight state, and thus $N_{j-1}^+ N_j^- = j$.

Exercise 2.3

A boost in the x-direction can be parameterised in terms of the rapidity η as $\beta = \tanh\eta$ and $\gamma = \cosh\eta$,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$$

The generator is then

$$iK_1 = \frac{\partial \Lambda}{\partial \eta}\bigg|_{\eta=0} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad K_1 = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Copying this for the y- and z-directions gives

$$K_2 = i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad K_3 = i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since rotations do not change time the corresponding matrices for the rotations must consist of the generators for $SO(3)$ from Eq. (1.7) inserted into the lower 3×3 part of a 4×4 matrix which is otherwise zero. We also append them with a factor i to consistently make them Hermitian to be consistent with our definition of generators. This gives

$$J_1 = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_2 = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad J_3 = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By explicit calculation we then have $[K_1, K_2] = -iJ_3$, $[K_1, K_3] = iJ_2$ and $[K_2, K_3] = -iJ_1$, which can be more elegantly written as $[K_i, K_j] = -i\epsilon_{ijk}J_k$. Similarly, we find $[K_1, J_2] = iK_3$, $[K_1, J_3] = -iK_2$, and $[K_2, J_3] = iK_1$, with all other commutators of rotation and boost generators zero, which can be summarised as $[K_i, J_j] = i\epsilon_{ijk}K_k$.

Exercise 2.4

The generators for the rotation part of the group have already been found in (1.7). Since $SO(3)$ is a subgroup there are no changes to these. In terms of M they give

$$\begin{aligned} M_{ij} &= \epsilon_{ijk}J_k = \epsilon_{ijk}L_k = \frac{1}{2}\epsilon_{ijk}\epsilon_{klm}i(x_l\partial_m - x_m\partial_l) \\ &= \frac{1}{2}i(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(x_l\partial_m - x_m\partial_l) \\ &= i(x_i\partial_j - x_j\partial_i). \end{aligned}$$

The boost generators are found by looking first at a boost in the x-direction parameterised in terms of the rapidity η as $\beta = \tanh\eta$ and $\gamma = \cosh\eta$. The transformation of the domain of functions defined on four-vectors is given explicitly by the Lorentz transformations

$$t' = f_0(x) = \gamma(t - vx) = \cosh\eta(t - \tanh\eta\, x),$$

and

$$x' = f_1(x) = \gamma(x - vt) = \cosh\eta(x - \tanh\eta\, t).$$

This gives the following generator

$$\begin{aligned}
iK_1 &= \left.\frac{\partial f_0}{\partial \eta}\frac{\partial}{\partial t}\right|_{\eta=0} + \left.\frac{\partial f_1}{\partial \eta}\frac{\partial}{\partial x}\right|_{\eta=0} \\
&= \left.\left(\sinh\eta(t-\tanh\eta\, x) - \frac{x}{\cosh\eta}\right)\frac{\partial}{\partial t}\right|_{\eta=0} + \left.\left(\sinh\eta(x-\tanh\eta\, t) - \frac{t}{\cosh\eta}\right)\frac{\partial}{\partial x}\right|_{\eta=0} \\
&= -x\frac{\partial}{\partial t} - t\frac{\partial}{\partial x} = -(x_1\partial_0 - x_0\partial_1),
\end{aligned}$$

or $K_1 = i(x_1\partial_0 - x_0\partial_1)$. Similarly for boost in the y- and z-direction we get $K_2 = i(x_2\partial_0 - x_0\partial_2)$ and $K_3 = i(x_3\partial_0 - x_0\partial_3)$. This is consistent with $M_{i0} = K_i = i(x_i\partial_0 - x_0\partial_i)$.

Exercise 2.5

For a Lorentz boost in the x-direction all parameters are zero except $\omega_{10} = -\omega_{01} = \eta$. This gives

$$\Lambda = \exp(i\omega^{10}M_{10}) = \exp(i\eta K_1) = \cosh(i\eta K_1) + \sinh(i\eta K_1).$$

where we have used the relationship between the formal power series for the exponential and hyperbolic functions. Now the hyperbolic cosine holds the terms even in K_1 and hyperbolic sine the odd terms. From

$$iK_1 = -\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \quad (iK_1)^2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \quad (iK_1)^3 = -\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

we can write

$$\Lambda = I + (\cosh\eta - 1)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \sinh\eta\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} = \begin{bmatrix}\cosh\eta & -\sinh\eta & 0 & 0\\ -\sinh\eta & \cosh\eta & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{bmatrix}.$$

Thus we are left with the standard matrix for Lorentz transformations in the x-direction (using rapidity).

Exercise 2.8

The first relation follows trivially from the commutation of P_μ with P_ν. To show the second we first use that

$$[M_{\mu\nu}, P_\rho P^\rho] = [M_{\mu\nu}, P_\rho]P^\rho + P_\rho[M_{\mu\nu}, P^\rho],$$

and Eq. (2.9) to get:

$$[M_{\mu\nu}, P_\rho P^\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu)P^\rho - iP_\rho({g_\mu}^\rho P_\nu - {g_\nu}^\rho P_\mu),$$

thus

$$[M_{\mu\nu}, P_\rho P^\rho] = -2i[P_\mu, P_\nu] = 0.$$

Exercise 2.9
From the definition of W_ν we have

$$\begin{aligned}[P_\mu, W_\nu] &= \frac{1}{2}\epsilon_{\nu\rho\sigma\tau}[P_\mu, P^\rho M^{\sigma\tau}] \\ &= \frac{1}{2}\epsilon_{\nu\rho\sigma\tau}g_{\mu\gamma}([P^\gamma, P^\rho]M^{\sigma\tau} + P^\rho[P^\gamma, M^{\sigma\tau}]) \\ &= \frac{i}{2}\epsilon_{\nu\rho\sigma\tau}g_{\mu\gamma}P^\rho(g^{\sigma\gamma}P^\tau - g^{\tau\gamma}P^\sigma) \\ &= \frac{i}{2}\epsilon_{\nu\rho\sigma\tau}g_{\mu\gamma}P^\rho g^{\sigma\gamma}P^\tau - \frac{i}{2}\epsilon_{\nu\rho\sigma\tau}g_{\mu\gamma}P^\rho g^{\tau\gamma}P^\sigma \\ &= \frac{i}{2}\epsilon_{\nu\rho\mu\tau}P^\rho P^\tau - \frac{i}{2}\epsilon_{\nu\rho\sigma\mu}P^\rho P^\sigma \\ &= 0,\end{aligned}$$

using the Poincaré algebra properties from (2.9).

Exercise 2.12

$$[X_\mu, P_\nu] = \frac{1}{2}[\bar{Q}\gamma_\mu\gamma^5 Q, P_\nu] = \frac{1}{2}(\bar{Q}\gamma_\mu\gamma^5)_a[Q_a, P_\nu] + \frac{1}{2}[\bar{Q}_a, P_\nu](\gamma_\mu\gamma^5 Q)_a = 0$$

Exercise 3.1

$$\begin{aligned}\int d^2\theta\,\theta\theta &= -\frac{1}{4}\int d\theta^A d\theta_A\,\theta^A\theta_A \\ &= \frac{1}{2}\int d\theta^A d\theta_A\,\theta_1\theta_2 = -\int d\theta_1 d\theta_2\,\theta_1\theta_2 = \int d\theta_1\theta_1\int d\theta_2\theta_2 = 1\end{aligned}$$

Exercise 3.3
We start from

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),$$

and $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. This gives

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= A(x+i\theta\sigma\bar{\theta}) + \sqrt{2}\theta\psi(x+i\theta\sigma\bar{\theta}) + \theta\theta F(x+i\theta\sigma\bar{\theta}) \\ &= A(x) + \partial_\mu A(x)i\theta\sigma^\mu\bar{\theta} + \frac{1}{2}\partial_\mu\partial_\nu A(x)(i\theta\sigma^\mu\bar{\theta})(i\theta\sigma^\nu\bar{\theta}) + \sqrt{2}\theta\psi(x) \\ &\quad +\sqrt{2}\partial_\mu\theta\psi(x)i\theta\sigma^\mu\bar{\theta} + \theta\theta F(x) \\ &= A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) - \frac{1}{2}\partial_\mu\partial_\nu A(x)\frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta} + \sqrt{2}\theta\psi(x) \\ &\quad +i\sqrt{2}\theta\partial_\mu\psi(x)\theta\sigma^\mu\bar{\theta} + \theta\theta F(x) \\ &= A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) - \frac{1}{4}\Box A(x)\theta\theta\bar{\theta}\bar{\theta} + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta\theta F(x),\end{aligned}$$

where we have used that $(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}$ and $\theta\partial_\mu\psi\theta\sigma^\mu\bar{\theta} = \frac{1}{2}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta}$.

Exercise 4.1
The Lagrangian for a single scalar superfield Φ is

$$\mathcal{L} = \Phi^\dagger\Phi + \delta^2(\bar{\theta})(g\Phi + m\Phi\Phi + \lambda\Phi\Phi\Phi) + \delta^2(\theta)(g\Phi^\dagger + m\Phi^\dagger\Phi^\dagger + \lambda\Phi^\dagger\Phi^\dagger\Phi^\dagger). \tag{4.37}$$

Using the following expressions for the scalar superfield in terms on component fields

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= A(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} \\
&\quad +\theta\theta F(x). \\
\Phi^\dagger(x,\theta,\bar{\theta}) &= A^*(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) \\
&\quad +\bar{\theta}\bar{\theta}F^*(x).
\end{aligned}$$

we have for the kinetic term,

$$\begin{aligned}
\int d^4\theta\,\Phi^\dagger\Phi &= -\frac{1}{4}A\Box A^* - \frac{1}{4}A^*\Box A + |F|^2 \\
&\quad + \int d^4\theta\left\{(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(\theta\sigma^\nu\bar{\theta})\partial_\nu A + i\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}\theta\psi - i\bar{\theta}r\psi\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta}\right\},
\end{aligned}$$

where we have removed the factor $\theta\theta\bar{\theta}\bar{\theta}$ by integration. Using the identities

$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}, \tag{4.38}$$

$$\theta\sigma^\mu\partial_\mu\bar{\psi}\theta\psi = -\frac{1}{2}\psi\sigma^\mu\partial_\mu\bar{\psi}\theta\theta, \tag{4.39}$$

$$\partial_\mu\psi\sigma^\mu\bar{\theta}\bar{\theta}\bar{\psi} = -\frac{1}{2}\partial_\mu\psi\sigma^\mu\bar{\psi}\bar{\theta}\bar{\theta}, \tag{4.40}$$

gives

$$\begin{aligned}
\int d^4\theta\,\Phi^\dagger\Phi &= -\frac{1}{4}A\Box A^* - \frac{1}{4}A^*\Box A + \frac{1}{2}\partial^\mu A\partial_\mu A^* + |F|^2 \\
&\quad -\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi}.
\end{aligned}$$

Now, since

$$\frac{1}{2}\partial^\mu A\partial_\mu A^* = \frac{1}{2}\partial^\mu(A\partial_\mu A^*) - \frac{1}{2}A\Box A^*, \tag{4.41}$$

and

$$\begin{aligned}
-\frac{1}{4}A\Box A^* &= -\frac{1}{4}\partial_\mu(A\partial^\mu A^*) + \frac{1}{4}\partial_\mu A\partial^\mu A^* \\
&= -\frac{1}{4}\partial_\mu(A\partial^\mu A^*) + \frac{1}{4}\partial^\mu((\partial_\mu A)A^*) - \frac{1}{4}A^*\Box A,
\end{aligned}$$

we can write

$$-\frac{1}{4}A\Box A^* - \frac{1}{4}A^*\Box A + \frac{1}{2}\partial^\mu A\partial_\mu A^* = -A^*\Box A + \text{total derivatives}. \tag{4.42}$$

Using (2.29) we can similarly write

$$\begin{aligned} -\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} &= \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \\ &= i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\partial_\mu(\bar{\psi}\bar{\sigma}^\mu\psi). \end{aligned} \tag{4.43}$$

Removing the terms with total derivatives the kinetic term is

$$\int d^4\theta\, \Phi^\dagger\Phi = \int d^4x \left\{-A^*(x)\Box A(x) + |F(x)|^2 + i\partial_\mu\psi(x)\bar{\sigma}^\mu\psi(x)\right\}. \tag{4.44}$$

The (left-handed) superpotential terms are

$$\begin{aligned} \int d^4\theta\, \bar{\theta}\bar{\theta}\Phi &= F, \\ \int d^4\theta\, \bar{\theta}\bar{\theta}\Phi\Phi &= 2AF + \psi\psi, \\ \int d^4\theta\, \bar{\theta}\bar{\theta}\Phi\Phi\Phi &= 3A^2F + 3A\psi\psi. \end{aligned}$$

where we have used (2.28) to rewrite the terms with ψ. The total action is then

$$\begin{aligned} S &= \int d^4x \left\{-A^*(x)\Box A(x) + |F(x)|^2 + i\partial_\mu\bar{\psi}(x)\bar{\sigma}^\mu\psi(x)\right. \\ &+ \left. gF(x) + 2mA(x)F(x) + m\psi(x)\psi(x) + 3\lambda A(x)^2F(x) + 3\lambda A(x)\psi(x)\psi(x) + \text{h.c.}\right\} \end{aligned} \tag{4.45}$$

The equation of motion for F is then

$$\frac{\partial\mathcal{L}}{\partial F^*} = F + g + 2mA^* + 3\lambda A^{*2} = 0, \tag{4.46}$$

which we can solve for F. With $\frac{\partial W}{\partial A} = g + 2mA + 3\lambda A^2$, $F = -\frac{\partial W^*}{\partial A^*}$ and the action is

$$\begin{aligned} S &= \int d^4x \{-A^*(x)\Box A(x) + i\partial_\mu\psi(x)\bar{\sigma}^\mu\psi(x) + m\psi(x)\psi(x) + 3\lambda A(x)\psi(x)\psi(x) \\ &\quad + m\bar{\psi}(x)\bar{\psi}(x) + 3\lambda A^*(x)\bar{\psi}(x)\bar{\psi}(x) - \left|\frac{\partial W}{\partial A}\right|^2 \Bigg\}. \end{aligned} \tag{4.47}$$

Using the fermionic mass matrices in (4.23) this can be written as

$$\begin{aligned} S &= \int d^4x \Bigg\{-A^*(x)\Box A(x) + i\partial_\mu\bar{\psi}(x)\bar{\sigma}^\mu\psi(x) + \frac{1}{2}\frac{\partial^2 W}{\partial A^2}\psi(x)\psi(x) \\ &\quad + \frac{1}{2}\frac{\partial^2 W^*}{\partial A^{*2}}\bar{\psi}(x)\bar{\psi}(x) - \left|\frac{\partial W}{\partial A}\right|^2 \Bigg\}. \end{aligned} \tag{4.48}$$

Exercise 4.2
To calculate the field strength in Wess–Zumino gauge we start from the vector superfield itself

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}D(x), \tag{4.51}$$

and make the coordinate change $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. This gives

$$\begin{aligned} V_{WZ}(x,\theta,\bar{\theta}) &= \theta\sigma^\mu\bar{\theta}V_\mu(y) - i\theta\sigma^\nu\bar{\theta}\partial_\nu\theta\sigma^\mu\bar{\theta}V_\mu(y) + \theta\theta\bar{\theta}\bar{\lambda}(y) + \bar{\theta}\bar{\theta}\theta\lambda(y) + \theta\theta\bar{\theta}\bar{\theta}D(y) \\ &= \theta\sigma^\mu\bar{\theta}V_\mu(y) + \theta\theta\bar{\theta}\bar{\lambda}(y) + \bar{\theta}\bar{\theta}\theta\lambda(y) + \theta\theta\bar{\theta}\bar{\theta}[D(y) - \frac{i}{2}\partial^\mu V_\mu(y)], \end{aligned} \tag{4.52}$$

where we have used (2.31).

Since, in this coordinate system, $D_A = \partial_A + 2i(\sigma^\mu\bar{\theta})_A\partial_\mu$ and $\bar{D}_{\dot{A}} = -\partial_{\dot{A}}$, so that $\bar{D}\bar{D} = \partial_{\dot{A}}\partial^{\dot{A}}$, we can then find the field strength as

$$\begin{aligned} W_A &= -\frac{1}{4}\bar{D}\bar{D}D_A V_{WZ}(y,\theta,\bar{\theta}) \\ &= -\frac{1}{4}\partial_{\dot{A}}\partial^{\dot{A}}\left[(\sigma^\mu\bar{\theta})_A V_\mu(y) + i\theta_A\bar{\theta}\bar{\theta}\partial^\mu V_\mu(y) + 2\theta_A\bar{\theta}\bar{\lambda}(y) + \bar{\theta}\bar{\theta}\lambda_A(y) + 2\theta_A\bar{\theta}\bar{\theta}D(y)\right. \\ &\quad \left.+2i(\sigma^\nu\bar{\theta})_A\partial_\nu\left(\theta\sigma^\mu\bar{\theta}V_\mu(y) + \theta\theta\bar{\theta}\bar{\lambda}(y)\right)\right] \\ &= -\frac{1}{4}\partial_{\dot{A}}\partial^{\dot{A}}\left[(\sigma^\mu\bar{\theta})_A V_\mu(y) + i\theta_A\bar{\theta}\bar{\theta}\partial^\mu V_\mu(y) + 2\theta_A\bar{\theta}\bar{\lambda}(y) + \bar{\theta}\bar{\theta}\lambda_A(y) + 2\theta_A\bar{\theta}\bar{\theta}D(y)\right. \\ &\quad \left.+2i(\sigma^\nu\bar{\theta})_A\theta\sigma^\mu\bar{\theta}\partial_\nu V_\mu(y) - i\theta\theta\bar{\theta}\bar{\theta}(\sigma^\nu\partial_\nu\bar{\lambda}(y))_A\right] \\ &= i\theta_A\partial^\mu V_\mu(y) + \lambda_A(y) + 2\theta_A D(y) \\ &\quad +2i\partial_{\dot{A}}\partial^{\dot{A}}(\sigma^\nu\bar{\theta})_A\theta\sigma^\mu\bar{\theta}\partial_\nu V_\mu(y) - i\theta\theta(\sigma^\nu\partial_\nu\bar{\lambda}(y))_A \\ &= \lambda_A(y) + 2\theta_A D(y) + (\sigma^{\mu\nu}\theta)_A F_{\mu\nu}(y) - i\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(y))_A \end{aligned} \tag{4.53}$$

where we have used $\partial_{\dot{A}}\partial^{\dot{A}}(\bar{\theta}\bar{\theta}) = 4$ and $2i(\sigma^\nu\bar{\theta})_A\theta\theta\bar{\theta}\partial_\nu\bar{\lambda}(y) = -i\theta\theta\bar{\theta}\bar{\theta}(\sigma^\nu\partial_\nu\bar{\lambda}(y))_A$, and where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the field strength for the field V_μ.

Then the terms with a $\theta\theta$ factor in $W^A W_A$ are

$$\begin{aligned} W^A W_A|_{\theta\theta} &= -i\theta\theta\lambda(y)\sigma^\mu\partial_\mu\bar{\lambda}(y) - i\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(y))^A\lambda_A(y) + 4\theta\theta D^2(y) \\ &\quad +2D(y)\theta\sigma^{\mu\nu}\theta F_{\mu\nu}(y) + 2(\sigma^{\mu\nu}\theta)^A F_{\mu\nu}(y)\theta_A D(y) + (\sigma^{\mu\nu}\theta)^A F_{\mu\nu}(y)(\sigma^{\rho\sigma}\theta)_A F_{\rho\sigma}(y) \\ &= -2i\theta\theta\lambda(y)\sigma^\mu\partial_\mu\bar{\lambda}(y) + 4\theta\theta D^2(y) \\ &\quad +4\theta\sigma^{\mu\nu}\theta D(y)F_{\mu\nu}(y) + (\sigma^{\mu\nu}\theta)^A F_{\mu\nu}(y)(\sigma^{\rho\sigma}\theta)_A F_{\rho\sigma}(y) \\ &= -2i\theta\theta\lambda(y)\sigma^\mu\partial_\mu\bar{\lambda}(y) + 4\theta\theta D^2(y) - \theta\theta\frac{1}{2}F_{\mu\nu}(y)F^{\mu\nu}(y) - \theta\theta\frac{i}{2}F_{\mu\nu}(y)\tilde{F}^{\mu\nu}(y) \end{aligned} \tag{4.54}$$

where we have used that from (2.34) $\theta\sigma^{\mu\nu}\theta = 0$ to remove a term, and rewritten

$$(\sigma^{\mu\nu}\theta)^A F_{\mu\nu}(y)(\sigma^{\rho\sigma}\theta)_A F_{\rho\sigma}(y) = -\theta\theta\frac{1}{2}F_{\mu\nu}(y)F^{\mu\nu}(y) - \theta\theta\frac{i}{2}F_{\mu\nu}(y)\tilde{F}^{\mu\nu}(y), \tag{4.55}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength tensor.

Since these terms are invariant under the change of coordinates $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ – because all terms beyond the first order disappear in the expansion due to too many θ-factors – and since no other terms can survive the $\int d^4\theta$-integration in the Lagrangian, we are left with

$$\int d^4\theta\,\bar{\theta}\bar{\theta}W^A W_A = -2i\lambda(x)\sigma^\mu\partial_\mu\bar{\lambda}(x) + 4D^2(x) - \frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{i}{2}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x). \tag{4.56}$$

We see that the component field Lagrangian, as expected and as promised in Sec. 4.4, contains a field strength term for the vector component field. At this point we may worry about the appearance of the dual field strength tensor which we do not have in the Standard Model. However, we can show that the term in question can be written as a total derivative,

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \\ &= 2\epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)(\partial_\rho A_\sigma) \\ &= 2\epsilon^{\mu\nu\rho\sigma}(\partial_\rho\partial_\mu A_\nu)A_\sigma - 2\epsilon^{\mu\nu\rho\sigma}\partial_\rho(\partial_\mu A_\nu A_\sigma) \end{aligned} \tag{4.57}$$

where the first term disappears due to the asymmetry of the Levi-Civita symbol. That means that the term with the dual field strength tensor disappears if the fields fall off rapidly enough towards infinity. This will always be true for abelian field theories, but is not necessarily so for non-abelian theories, where we can have so-called instanton effects.

Exercise 4.3

We should note that in this case $W = 0$ as we can not construct gauge invariant terms in the superpotential with only one superfield that is charged under the gauge group. This also implies that $F(x) \equiv 0$.

For an abelian gauge group the kinetic term in Wess–Zumino gauge is

$$\mathcal{L}_{\text{kin}} = \int d^4\theta\, \Phi^\dagger e^{qV_{WZ}}\Phi = \int d^4\theta \left[\Phi^\dagger\Phi + q\Phi^\dagger\Phi V_{WZ} + \frac{1}{2}q^2\Phi^\dagger\Phi V_{WZ}^2\right]. \tag{4.59}$$

The first term in the integral we have calculated already in Exercise 1. Inserting Eqs. (3.35), (3.36) and (3.48) the second term is

$$\begin{aligned} \int d^4\theta\, \Phi^\dagger\Phi V_{WZ} &= \int d^4\theta\, \Big\{[iA^*(\theta\sigma^\nu\bar\theta)\partial_\nu A - iA(\theta\sigma^\nu\bar\theta)\partial_\nu A^* + 2\bar\theta\bar\psi\theta\psi](\theta\sigma^\mu\bar\theta)V_\mu \\ &\quad + \sqrt{2}A\bar\theta\bar\psi\theta\theta\bar\theta\bar\lambda + \sqrt{2}A^*\theta\psi\bar\theta\bar\theta\theta\lambda + \theta\theta\bar\theta\bar\theta|A|^2D\Big\} \\ &= \int d^4\theta\, \Big\{\frac{i}{2}A^*\theta\theta\bar\theta\bar\theta\partial^\mu A V_\mu - \frac{i}{2}A\theta\theta\bar\theta\bar\theta\partial^\mu A^* V_\mu - \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\bar\psi\bar\sigma^\mu\psi V_\mu \\ &\quad + \frac{1}{\sqrt{2}}A\theta\theta\bar\theta\bar\theta\bar\psi\bar\lambda + \frac{1}{\sqrt{2}}A^*\theta\theta\bar\theta\bar\theta\psi\lambda + \theta\theta\bar\theta\bar\theta|A|^2D\Big\} \\ &= \frac{i}{2}(A^*\partial^\mu A - A\partial^\mu A^*)V_\mu - \frac{1}{2}\bar\psi\bar\sigma^\mu\psi V_\mu \\ &\quad + \frac{1}{\sqrt{2}}A\bar\psi\bar\lambda + \frac{1}{\sqrt{2}}A^*\psi\lambda + |A|^2D, \end{aligned} \tag{4.60}$$

where we have used Eqs. (2.28), (2.31), and (4.58) for the second equality.

The third term is found from Eqs. (3.35), (3.36) and (3.49), with only one possible surviving term

$$\int d^4\theta\, \frac{1}{2}\Phi^\dagger\Phi V_{WZ}^2 = \int d^4\theta\, \frac{1}{4}|A|^2\theta\theta\bar\theta\bar\theta V^\mu V_\mu = \frac{1}{4}|A|^2V^\mu V_\mu. \tag{4.61}$$

Summing up these three terms the total contribution from the kinetic terms is then

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= -A^*\Box A + i\partial_\mu\bar\psi\bar\sigma^\mu\psi + \frac{i}{2}q(A^*\partial^\mu A - A\partial^\mu A^*)V_\mu - \frac{1}{2}q\bar\psi\bar\sigma^\mu\psi V_\mu \\ &\quad + \frac{1}{\sqrt{2}}qA\bar\psi\bar\lambda + \frac{1}{\sqrt{2}}qA^*\psi\lambda + q|A|^2D + \frac{1}{4}q^2|A|^2V^\mu V_\mu. \end{aligned} \tag{4.62}$$

We can simplify this expression by defining a covariant derivative from the vector component field $D_\mu \equiv \partial_\mu - i\frac{q}{2}V_\mu$.[20] We immediately see that

$$i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{1}{2}q\bar{\psi}\bar{\sigma}^\mu\psi V_\mu = iD^*_\mu\bar{\psi}\bar{\sigma}^\mu\psi, \tag{4.63}$$

and

$$\begin{aligned}|D_\mu A|^2 &\equiv D^*_\mu A^* D^\mu A = (\partial_\mu A^* + i\frac{q}{2}V_\mu A^*)(\partial^\mu A - i\frac{q}{2}V^\mu A)\\ &= |\partial_\mu A|^2 + \frac{q^2}{4}V_\mu V^\mu |A|^2 + i\frac{q}{2}(A^*\partial^\mu A - \partial^\mu A^* A)V_\mu,\end{aligned} \tag{4.64}$$

so that we can write, using $-A^*\Box A = |\partial_\mu A|^2$,

$$\mathcal{L}_{\mathrm{kin}} = \frac{1}{\sqrt{2}}qA\bar{\psi}\bar{\lambda} + \frac{1}{\sqrt{2}}qA^*\psi\lambda + q|A|^2D + iD^*_\mu\bar{\psi}\bar{\sigma}^\mu\psi + |D_\mu A|^2$$

Adding the field strength terms from Exercise 2 the complete Lagrangian is

$$\begin{aligned}\mathcal{L} = {} & iD^*_\mu\bar{\psi}(x)\bar{\sigma}^\mu\psi(x) + |D_\mu A(x)|^2 - i\lambda(x)\sigma^\mu\partial_\mu\bar{\lambda}(x)\\ &+\frac{1}{\sqrt{2}}qA(x)\bar{\psi}(x)\bar{\lambda}(x) + \frac{1}{\sqrt{2}}qA^*(x)\psi(x)\lambda(x) + q|A(x)|^2D(x)\\ &+2D^2(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x).\end{aligned} \tag{4.65}$$

Exercise 4.4

In QED we have three particles, the electron, positron and the photon. As a Dirac fermion the electron needs two scalar superfields to provide the two different left- and right-handed Weyl-spinor components. However, then the positron is already given by the hermitian conjugates of these spinors. For the photon we need an abelian vector superfield for the $U(1)_{\mathrm{em}}$ gauge group. Thus the minimum field content consists of two scalar superfields L and $\bar{E}$, and a vector superfield V.

The supergauge transformations for the scalar superfields are $L \to L' = \exp(-ie\Lambda)L$ and $\bar{E} \to \bar{E}' = \exp(ie\Lambda)\bar{E}$, where e is the elementary electrical charge, and the difference in sign signifies the opposite charges of the electron and positron. The general form of the superpotential must then be $W = mLE$ with a single (mass) parameter m. No tadpole terms in L and $\bar{E}$ can survive since these are not gauge singlets, and only the LE mass term survives since the two fields in the mass term must have opposite charges under the gauge group. Similarly, no Yukawa term can fulfil the gauge invariance criterium.

Solving for the auxiliary F-fields we get

$$\begin{aligned}F^*_L &= -W_L = -\frac{\partial W[A_L, A_E]}{\partial A_L} = -mA_E,\\ F^*_e &= -W_E = -\frac{\partial W[A_L, A_E]}{\partial A_E} = -mA_L.\end{aligned}$$

[20] The unusual factor of 2 here compared to the usual abelian gauge covariant derivative is due to our earlier choices in how to write down the vector superfield in terms of component fields. If we want we can absorb this by redefining V_μ.

Generalising the discussion in Exercise 1, the contribution to the Lagrangian from the superpotential in terms of component fields is

$$\mathcal{L}_{\text{superpot}} = m\psi_L\psi_E + m^*\bar{\psi}_L\bar{\psi}_E - |m|^2(|A_L|^2 + |A_E|^2), \tag{4.66}$$

where we must reasonably identify m with the electron mass, see also that, as expected the scalar fields A_E and A_L obtain the same mass.

From the field content we know that the kinetic terms take the form $L^\dagger e^{eV} L$ and $\bar{E}^\dagger e^{-eV}\bar{E}$ where the sign ensures the gauge invariance under the supergauge transformations of the scalar and vector superfields together. Taking results from Exercise 3 the Lagrangian contribution from the kinetic terms is

$$\begin{aligned}\mathcal{L}_{\text{kin}} \;=\;& \frac{1}{\sqrt{2}}eA_L\bar{\psi}_L\bar{\lambda} + \frac{1}{\sqrt{2}}eA_L^*\psi_L\lambda + e|A_L|^2 D + iD_\mu^*\bar{\psi}_L\bar{\sigma}^\mu\psi_L + |D_\mu A_L|^2 \\ & -\frac{1}{\sqrt{2}}eA_E\bar{\psi}_E\bar{\lambda} - \frac{1}{\sqrt{2}}eA_E^*\psi_E\lambda - e|A_E|^2 D + iD_\mu^*\bar{\psi}_E\bar{\sigma}^\mu\psi_E + |D_\mu A_E|^2,\end{aligned}$$

where λ is the Weyl fermion from the vector superfield V, D is the auxiliary field from V, and the covariant gauge derivative is $D_\mu = \partial_\mu - \frac{ie}{2}A_\mu$, where A_μ is the electromagnetic gauge field.

In Exercise 2 we saw that the contribution from the abelian field strength term is

$$\mathcal{L}_{\text{field strength}} = -i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + 2D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.67}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Solving for the auxiliary D-field we get

$$D = \frac{e}{4}(|A_E|^2 - |A_L|^2). \tag{4.68}$$

This gives a total Lagrangian of

$$\begin{aligned}\mathcal{L} \;=\;& iD_\mu^*\bar{\psi}_L\bar{\sigma}^\mu\psi_L + |D_\mu A_L|^2 + iD_\mu^*\bar{\psi}_E\bar{\sigma}^\mu\psi_E + |D_\mu A_E|^2 - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ & + m\psi_L\psi_E + m^*\bar{\psi}_L\bar{\psi}_E - |m|^2(|A_L|^2 + |A_E|^2) \\ & + \frac{1}{\sqrt{2}}eA_L\bar{\psi}_L\bar{\lambda} + \frac{1}{\sqrt{2}}eA_L^*\psi_L\lambda - \frac{1}{\sqrt{2}}eA_E\bar{\psi}_E\bar{\lambda} - \frac{1}{\sqrt{2}}eA_E^*\psi_E\lambda - \frac{e^2}{8}(|A_E|^2 - |A_L|^2)^2.\end{aligned}$$

Exercise 7.2

We will in the following take all outgoing momenta to go out of the vertex. The indices $abcd$ are gluon indices $(1, ..., 8)$, rs are spin indices $(1, 2)$, and $ijmn$ are colour indices $(1, 2, 3)$. The relevant Feynman rules are as follows:

- Incoming quark: $u^r(k)$.
- Incoming antiquark: $\bar{v}^s(p)$.
- Gluon propagator: $-i\frac{g_{\mu\nu}}{s}\delta^{ab}$.
- Vertex $q\bar{q}g$: $-it^a_{ij}\gamma^\mu g_s$.

- Vertex $\tilde{q}\tilde{q}^* g$: $-it^b_{mn}(k'-p')^\nu g_s$.

We will assume the SM particles to have negligible mass compared to the squarks.

The matrix element is then given as

$$\mathcal{M} = -\frac{g_s^2}{s} t^a_{ij} t^b_{mn} \delta^{ab} \bar{v}^s \gamma^\mu u^r (k'-p')_\mu.$$

In the squared amplitude we average over all incoming spin and colour, and sum over the outgoing:

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}\cdot\frac{1}{9}\frac{g_s^4}{s^2}\sum_{ab}\sum_{cd}\sum_{rs}\sum_{ijmn} t^a_{ij}t^b_{mn}t^c_{ij}t^d_{mn}\delta^{ab}\delta^{cd}\bar{v}^s\gamma^\mu u^r \bar{u}^r\gamma^\nu v^s (k'-p')_\mu (k'-p')_\nu \\
&= \frac{1}{4}\cdot\frac{1}{9}\frac{g_s^4}{s^2}\underbrace{\sum_{ijmn}\left(\sum_a t^a_{ij}t^a_{mn}\sum_c t^c_{ji}t^c_{nm}\right)}_{\equiv C_f}\sum_{rs}\bar{v}^s\gamma^\mu u^r\bar{u}^r\gamma^\nu v^s(k'-p')_\mu(k'-p')_\nu \\
&= \frac{1}{4}\cdot\frac{1}{9}\frac{g_s^4}{s^2}C_f p_\alpha k_\beta TR(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu)(k'-p')_\mu(k'-p')_\nu \\
&= \frac{1}{9}\frac{g_s^4}{s^2}C_f p_\alpha k_\beta (p^\mu k^\nu - p^\beta k_\beta \eta^{\mu\nu} + p^\nu k^\mu)(k'-p')_\mu(k'-p')_\nu \\
&= \frac{1}{9}\frac{g_s^4}{s^2}C_f(2p\cdot(k'-p')k\cdot(k'-p') - p\cdot k(k'-p')\cdot(k'-p')),
\end{aligned}$$

where we have isolated the colour factors into the coefficient C_f.

In the centre of mass frame, we have $p=(E,\vec{p})$, $k=(E,-\vec{p})$, $p'=(E,\vec{p'})$ and $k'=(E,-\vec{p'})$. From this we find

$$\begin{aligned}
2p\cdot(k'-p')k\cdot(k'-p') - p\cdot k(k'-p')\cdot(k'-p') &= 2\cdot 4|\vec{p}|^2|\vec{p'}|^2(1-\cos^2\theta) \\
&= 2s|\vec{p'}|^2(1-\cos^2\theta),
\end{aligned}$$

where $\theta$is the acute angle between the incoming quarks, and $s=(p+k)^2=4E^2=4|\vec{p}|^2$. This gives the squared averaged amplitude

$$|\bar{\mathcal{M}}|^2 = \frac{2}{9}\frac{g_s^4}{s}C_f\cdot|\vec{p'}|^2(1-\cos^2\theta),$$

and the differential cross section

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{|\vec{p'}|^2}{32\pi^2\sqrt{s}s}|\bar{\mathcal{M}}|^2 \\
&= \frac{1}{144}\frac{g_s^4}{\pi^2\sqrt{s}s^2}C_f|\vec{p'}|^3(1-\cos^2\theta).
\end{aligned}$$

Integrating over the solid angle then gives the total cross section

$$\begin{aligned}\sigma &= \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega \\ &= 2\pi \frac{1}{144} \frac{g_s^4}{\pi^2\sqrt{s}s^2} C_f |\vec{p'}|^3 \int_{-1}^{1} (1-\cos^2\theta) d(\cos\theta) \\ &= \frac{4}{3} 2\pi \frac{1}{144} \frac{g_s^4}{\pi^2\sqrt{s}s^2} C_f |\vec{p'}|^3 \\ &= \frac{1}{54\pi} \frac{g_s^4}{\sqrt{s}s^2} C_f |\vec{p'}|^3.\end{aligned}$$

We can rewrite $|\vec{p'}|^3$ by noticing that

$$|\vec{p'}| = \sqrt{E^2 - m^2} = \frac{1}{2}\sqrt{4E^2 - 4m^2} = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}.$$

The colour factor is calculated below and found to be $C_f = 2$, so the total cross section is

$$\sigma = \frac{g_s^4}{216\pi s}\sqrt{\left(1 - \frac{4m^2}{s}\right)^3}.$$

Using $\alpha_s = \frac{g_s^2}{4\pi}$, and assuming that both the left-, and right-handed squarks have the same mass, we arrive at the final expression

$$\sigma = \frac{4}{27}\frac{\pi\alpha_s^2}{s}\sqrt{\left(1 - \frac{4m^2}{s}\right)^3}.$$

To calculate the colour factor C_f we use that the sum over the generators t is given (see for example [39]) as:

$$\sum_a t^a_{ij} t^a_{mn} = \frac{1}{2}\left(\delta_{in}\delta_{jm} - \frac{1}{N_C}\delta_{ij}\delta_{mn}\right),$$

and using that $\delta_{ij} = \delta_{ji}$, we have

$$\begin{aligned}C_f &= \frac{1}{4}\sum_{ijmn}\left(\delta_{in}\delta_{jm} - \frac{1}{N_C}\delta_{ij}\delta_{mn}\right)\left(\delta_{jm}\delta_{in} - \frac{1}{N_C}\delta_{ji}\delta_{nm}\right) \\ &= \frac{1}{4}\sum_{ijmn}\left(\delta_{in}\delta_{jm}\delta_{jm}\delta_{in} - \frac{2}{N_C}\delta_{ij}\delta_{mn}\delta_{jm}\delta_{in} + \frac{1}{N_C^2}\delta_{ij}\delta_{mn}\delta_{ji}\delta_{nm}\right) \\ &= \frac{1}{4}\sum_{ijmn}\left(\delta_{in}\delta_{in}\delta_{jm}\delta_{jm} - \frac{2}{N_C}\delta_{ij}\delta_{mn}\delta_{jm}\delta_{in} + \frac{1}{N_C^2}\delta_{ij}\delta_{ij}\delta_{mn}\delta_{mn}\right) \\ &= \frac{1}{4}\left(N_C^2 - \frac{2}{N_C}N_C + \frac{1}{N_C^2}N_C^2\right) \\ &= \frac{1}{4}(N_C^2 - 1) = 2.\end{aligned}$$

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