Exercise 7.2 (*gravitational stresses and Janssen's law*): We study a freely standing square column of cross-sectional area $A = a^2$ and height h, described by isotropic linear elasticity with Young's modulus E and Poisson's ratio ν. The density of the material is ρ, and we assume that the density does not change significantly due to gravity.

a) Show that the vertical stress, σ_{zz}, inside the column as a function of height z is given as $\sigma_{zz} = \rho g z$ (starting at $z = 0$ at the top of the column).

b) What is the vertical strain, ϵ_{zz}, as a function of distance z to the top of the column?

c) What is the horizontal strain, $\epsilon_{xx} = \epsilon_{yy}$ as a function of z?

d) Assume the column is placed inside a rigid container so that it cannot expand horizontally. Show that the horizontal stress, σ_{xx}, can be written as:

$$\sigma_{xx} = \frac{\nu}{1-\nu} \cdot \sigma_{zz}$$

e) We assume there is a friction acting from the rigid container on the column with a coefficient of friction μ. Show that the vertical stress can be described by the following differential equation:

$$\frac{d\sigma_{zz}}{dz} = -\rho g + \frac{4\mu}{a}\sigma_{xx}$$

f) Show that the vertical stress as a function of the distance z from the top surface of the column can be written the following form:

$$\sigma_{zz} = \rho g \xi (1 - e^{z/\xi})$$

where ξ is a characteristic length, and z is zero at the surface and negative below the surface. Find the dependence of the characteristic length on the the column width a and the coefficient of friction μ.

Proof:

a) We study the force balance on an element of height dz at a position z. The forces on this element are from the stresses at each end, i.e. at positions z and $z + dz$, and from the weight of the block:

$$F = \rho g A dz - \sigma_{zz}(z)A + \sigma_z zz + dzA = 0$$

This gives

$$\frac{d\sigma_{zz}}{dz} = \rho g$$

and

$$\sigma_{zz} = \rho g z,$$

where the top of the surface is set at $z = 0$, and z¡0. Note that compressive stresses are negative in the representation we have used.

b) The vertical strain is found from

$$\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})$$

where $\sigma_{xx} = \sigma_{yy} = 0$. Therefore

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} = \frac{\rho g}{E} z$$

Note: the strain is also negative, corresponding to a compression.

c) The horizontal strain is found from

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})$$

Giving:

$$\epsilon_{xx} = -\frac{\nu}{E}\sigma_{zz} = -\nu\frac{\rho g}{E} z$$

This strain is positive, the column is expanding.

d) For a restricted column, the horizontal strain is zero, but the horizontal stresses are not zero. That is

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) = 0$$

and

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})$$

summing the two equations produce:

$$\sigma_{xx} + \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy} + 2\sigma_{zz}) = 0$$

Giving

$$\sigma_{xx} + \sigma_{yy} = \frac{2\nu}{1-\nu}\sigma_{zz}$$

Since the column is square, we expect $\sigma_{xx} = \sigma_{yy}$, and

$$\sigma_{xx} = \frac{\nu}{1-\nu}\sigma_{zz}$$

where $\sigma_{zz} = \rho g z$, we get

$$\sigma_{xx}\frac{\nu}{1-\nu}\rho g z$$

which is also generally a negative stress, corresponding to a compression.

e) For a volume element of height dz at position z, there are now additional forces due to friction. For each of the four sidewalls, there is a normal force $N = \sigma_{xx} a dz$ which is $N = \frac{\nu}{1-\nu}\sigma_{zz} a dz$ The force balance for the whole volume element is:

$$F = \rho g A dz - \sigma_{zz}(z)A + \sigma_{zz}(z+dz)A - 4adz\mu\frac{\nu}{1-\nu}\sigma_{zz} = 0$$

This produces the following differential equation:

$$\frac{d\sigma_{zz}}{dz} = -\rho g + \frac{4\mu}{a}\frac{\nu}{1-\nu}\sigma_{zz}$$

we introduce the length $\xi = fraca4\mu\frac{1-\nu}{\nu}$, and the equation reduces to

$$\frac{d\sigma_{zz}}{dz} = -\rho g + \frac{1}{\xi}\sigma_{zz}$$

f) We can simplify this equation to

$$\frac{d\sigma_{zz}}{dz} = \frac{1}{\xi}(\sigma_{zz} - -\rho g\xi)$$

We introduce a transformation $s(z) = \sigma_{zz0} - rhog\xi$, and the equation simplifies to

$$\frac{ds}{dz} = \frac{1}{\xi}s$$

With the solution $s(z) = s_0 e^{z/\xi}$. We find σ_{zz} from $s(z)$ and get $\sigma_{zz} - \rho g \xi = s_0 e^{z/\xi}$ and therefore $\sigma_{zz} = \rho g \xi + s_0 e^{z/\xi}$

The boundary condition is that $\sigma_{zz}(0) = 0$, giving $\sigma_{zz} = \rho g \xi (1 - e^{z/\xi})$

`Note 1:` z is negative.
`Note 2:` for this case, the stress and therefore the pressure, reaches a constant value after a distance proportional to ξ which is proportional to the width a of the column. This corresponds to Janssen's law. It gives a good explanation for why the pressure inside a granular material in a container does not increase with depth after a distance proportional to the width of the container. The pressure is taken up by friction with the wall of the container.

Exercise 8.1 (*rubber band*): For simple elongations, rubber can be described as a linear elastic material for both small and intermediate deformation (up to 50% elongation). We assume that rubber has a Young's modulus of 1 MPa. We study a rubber band of 5 mm width, 1 mm thickness, and 10 cm length.

(a) What is the length of the band if we hang a 0.25 kg weight in it?

(b) We assume that the critical stress, σ, for critical growth of a fracture is

$$\sigma = \sqrt{\frac{E\gamma}{D}},$$

where D is the length of a crack in the band. For rubber, $\gamma = 0.05$ N/m. If we push a needle with diameter 1mm through the broadest side of the band, what is the maximum weight the band can withstand before breaking.

Proof:

a) The band can be described using elasticity theory. The force is $F = mg$, and the elastic deformation is given by

$$\frac{F}{A} = E\frac{\Delta y}{y}$$

which gives

$$\Delta y = \frac{mgy}{AE}$$

where $A = ab = 5\ rmmm \times 1$ mm is the area of the cross-section of the band, and y is the length of the band. Inserting the values, gives

$$\Delta y = \frac{mgy}{abE} = 0.049 \text{ m}$$

The total length is therefore approximately 15 cm.

b) Using the formula supplied we find that

$$\sigma = \sqrt{\frac{E\gamma}{D}} = \sqrt{\frac{1 \text{ MN/m}^2 \cdot 0.05 \text{ N/m}}{0.001 \text{ m}}} = \sqrt{50 \times 10^6 \text{ N}^2/\text{m}^4} = 7071 \text{ N/m}^2$$

This corresponds to a weight mg, giving rise to the following deformation:

$$\sigma = \frac{F}{A} = \frac{mg}{A}$$

The maximum mass is therefore:

$$m = \frac{\sigma ab}{g} = \frac{7071 \text{ N/m}^2 \times 0.005 \text{ m} \times 0.001 \text{ m}}{9.81 \text{ N/kg}} = 0.34 \text{ kg}$$

This corresponds to an elongation of about 6.7 cm.