# Dislokasjoner – line defects

## Stress, deformation, strain

For fluids: have p, T to describe them.

No resistance to change shape

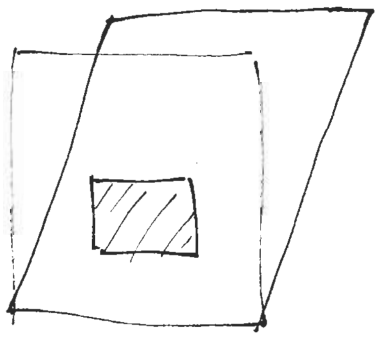
In solids pressure has to be replaced by elastic stresses: tensors.

First: address stress and strain

Let us look at a cube,

perform homogeneous, elastic deformation

- shear
- stretch



If some change remains when forces are unloaded: plastic deformation,

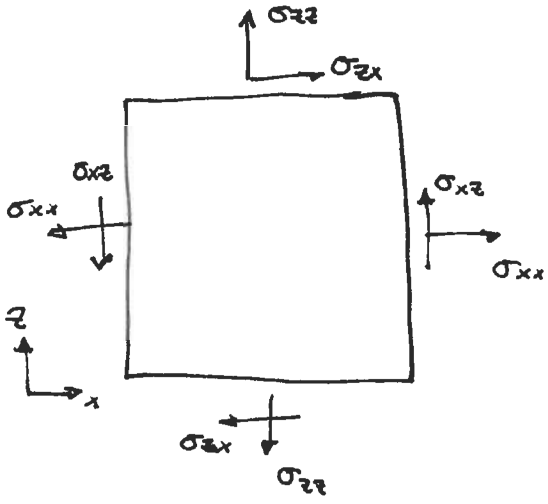
otherwise: elastic deformation: perfectly restores original shape.

Let us cut out a small kube in the deformed material

and keep forces on surface, so that it keeps its shape

Stress : force per area :

orientation of <u>force</u> and <u>area</u>




We use the term

$A\sigma_{ij}$ : force acting on element

with <u>i</u> as normal vector and

j - component of this force

$A\sigma_{zz}$ : tensile force / stress

$A\sigma_{zx}$ : shear force / stress.

<u>Torque:</u>

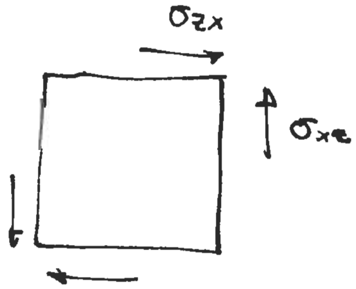
The volume element is <u>not moving</u> : stationary in elastic material

<u>Therefore</u> : no torque

In order for the volume element to be stationary, we require

$$\sigma_{xz} = \sigma_{zx}$$

$\ni$: $\sigma_{ij} = \sigma_{ji}$



This implies that $\sigma_{ij}$ is a symmetric tensor.

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In the dynamic case - the element may have an angular-acceleration

But: moment of inertia goes to zero faster than shear forces

unless the symmetry is fulfilled : will not get finite angular acceleration.

Therefore : general $\sigma_{ij} = \sigma_{ji}$

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This leaves us with 6 independent components

$$\underbrace{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz}}_{\text{tensile components}} \qquad \underbrace{\sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz}}_{\text{shear components}}$$

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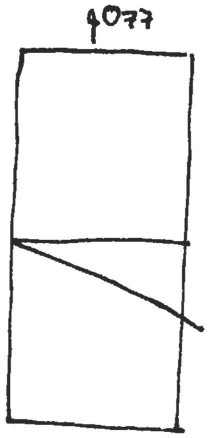
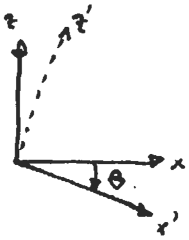
The pressure is

$$p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -\frac{1}{3} \mathrm{Tr}(\sigma_{ij})$$

This is an invariant : not dependent on choice of coord. system!




## Example

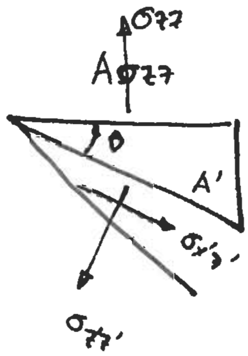
Bar: $\sigma_{zz} \neq 0$

$\sigma_{xx} = \sigma_{xz} = 0$

Force equilibrium for cut bar, areas are

$A$ and $A'$

$A' \cdot \cos\theta = A$




$A' \cdot \cos\theta = A$

### Horizontal

$\sigma_{xx} = 0 \Rightarrow F_x$

$F_x = 0 = \sigma_{x'z'} \cos\theta - \sigma_{z'z'} \sin\theta$

$\sigma_{z'z'} = \cot\theta \cdot \sigma_{x'z'}$

### Vertical

$$F_z = \underset{\substack{\parallel \\ A'\cos\theta}}{A} \cdot \sigma_{zz} = A'(\sigma_{x'z'}\sin\theta + \sigma_{z'z'} \cdot \cos\theta) = A'(\sigma_{x'z'}\sin\theta + \cos\theta \cdot \cot\theta\, \sigma_{x'z'})$$

$$\cancel{A'} \cos\theta \cdot \sigma_{zz} = \cancel{A'} \sigma_{x'z'}(\sin\theta + \cos\theta \cdot \cot\theta)$$

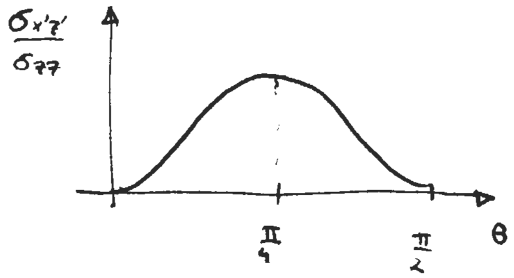
$$\sigma_{x'z'} = \sigma_{zz} \frac{\cos\theta}{\sin\theta + \cos\theta \cdot \cot\theta} = \sigma_{zz} \frac{1}{\tan\theta + \cot\theta}$$

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta} = \frac{1}{\cos\theta \sin\theta}$$

$$\sigma_{x'z'} = \sigma_{zz} \cdot \cos\theta \cdot \sin\theta \qquad \sigma_{z'z'} = \frac{\cos\theta}{\sin\theta} \sigma_{x'z'} = \sigma_{zz} \cdot \cos^2\theta$$

### What is max $\sigma_{x'z'}$?

$$\frac{\partial \sigma_{x'z'}}{\partial \theta} = \sigma_{zz}(\sin^2\theta - \cos^2\theta) = 0 \, ; \; \theta = \frac{\pi}{4}$$




Max shear stress for $\theta = \frac{\pi}{4}$

$$\underline{\sigma_{x'z'} = \sigma_{zz}\left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{2}\right) = \frac{\sigma_{zz}}{2}}$$

$$\sigma_{z'z'} = \frac{\sigma_{zz}}{2}$$

For $\theta = \frac{\pi}{4}$

$\sigma_{x'z'} = \frac{1}{2}\sigma_{zz}$ : maximum resolved shear stress.

Note 1 stresses depend on coordinate system.

Note 2 Because $\sigma_{ij}$ is symmetric:

$$\sigma_{ij} = \sigma_{ji}$$

It is diagonalizable ; i.e. there exists an orthogonal coordinate system in which $\sigma_{ij}$ is diagonal

( Theorem from linear algebra )

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Deformation

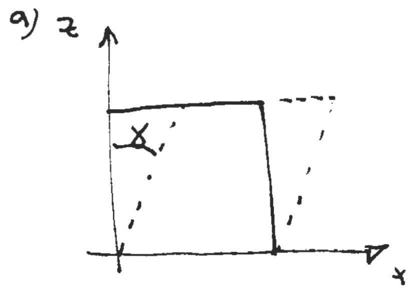
$P \quad \frac{\Delta V}{V}$ → Hydrodynamics

$\updownarrow \quad \updownarrow$

$\sigma_{ij} \quad \epsilon_{ij}$ → Elasticity theory.

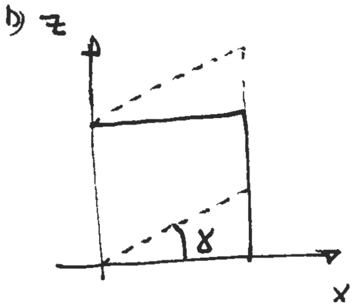
Form a shear deformation

a)




can perform the same deformation this way.

b)




These two states should represent the same deformation only difference is a rotation of the coordinate system.
(Should be Galilean invariant)

Our description of deformation should give the same shear deformation in both cases

$(x, y, z)$ displaced by $u_x, u_y, u_z$ $u_x(x, y, z)$

case a)

$$u_x = \gamma z$$

case b) $u_z = \gamma x$

Define deformation, through changes in $u_x, u_y, u_z$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$$

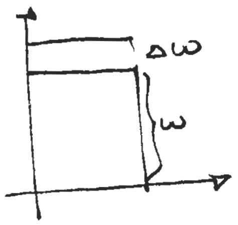
This ensures correct "rotational" transformation property.

This is our definition of shear deformation:

a) $\epsilon_{xz} = \frac{1}{2}\gamma$

b) $\epsilon_{xz} = \frac{1}{2}\gamma$

Form tensile deformation




$$\epsilon_{zz} = \frac{\Delta w}{w} = \frac{\partial u_z}{\partial z} = \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right)$$

General

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

By definition

$$\epsilon_{ij} = \epsilon_{ji}$$

We also observe that the strain $\epsilon_{ij}$ depends on the choice of coordinate systems

We note that

$$\frac{\Delta V}{V} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \quad : \underline{\underline{\text{invariant}}}$$

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The general link between $\sigma_{ij}$ and $\epsilon_{ij}$ in <u>linear</u> elasticity theory is

$$\sigma_{ij} = \sum_{kl} \overbrace{C_{ijkl}}^{\text{81 indekser}} \epsilon_{kl}$$

where

$$\left.\begin{aligned} C_{ijkl} &= C_{jikl} \\ C_{ijkl} &= C_{klij} \end{aligned}\right\} \text{symmetrier}$$

Isotrope legemer :

Same properties in <u>all directions</u>

<u>two</u> independent coefficients

## Cubic crystals:



elasticity may be anisotropic

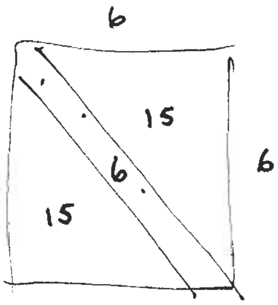
use different, 6 - component - scheme

| $ij$ | 11 | 22 | 33 | 23 | 31 | 12 |
|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |

$$C_{11} = C_{1111}$$

$$C_{12} = C_{1122}$$

$$C_{44} = C_{2323}$$




For cubic crystals, there are $6 + 15$ components in $C$

### Isotropic materials

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}))$$

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}))$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \text{ etc.}$$

$E$ : Young's modulus

$G$ (often $\mu$) : Shear modulus

$\nu$ : Poisson's ratio

Demand that constants are independent of rotation of coordinate system:

$$G = \frac{E}{2(1+\nu)}$$

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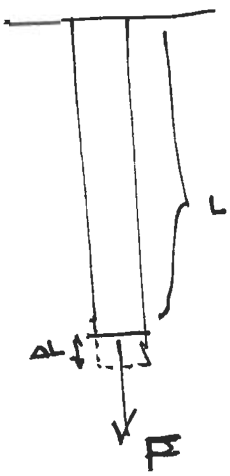
Interpretation of elastic constants




$$\left( u_z = \frac{\Delta L}{L} z \qquad \frac{\partial u_z}{\partial z} = \frac{\Delta L}{L} \right)$$

$$\epsilon_{zz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right) = \frac{\partial u_z}{\partial z}$$

$$= \frac{\Delta L}{L} = \frac{1}{E}(\sigma_{zz} - \nu\overbrace{(\sigma_{xx} + \sigma_{yy})}^{0\,!})$$

$$= \frac{1}{E}\sigma_{zz} = \frac{1}{E}\frac{F}{A}$$

$$\boxed{\frac{F}{A} = E\frac{\Delta L}{L}}$$ Hooke's law

$$F = \underbrace{(E\frac{A}{L})}_{k}\Delta L$$ (Hooke's law for an elastic material)

$k$ → spring constant.

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) = -\frac{\nu}{E}\sigma_{zz}$$

$$\frac{1}{E}\sigma_{zz} = \frac{\Delta L}{L}$$

$$\underline{\epsilon_{xx}} = -\nu\underline{\frac{\Delta L}{L}} = \frac{\Delta x}{x}$$

$$-\frac{\epsilon_{xx}}{\epsilon_{zz}} = \nu$$

"infinitely compressible" "incompressible" : no change in volume.

$-1 < \nu < \frac{1}{2}$

possible values for $\nu$

For almost all materials : $0 < \nu < 0.5$.

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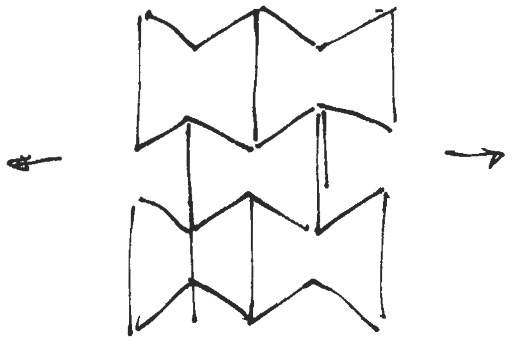
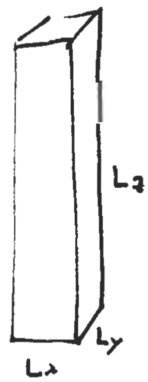
How could a $\nu < 0$ material behave ?

How can you make one?






What does $\nu = \frac{1}{2}$ mean?

$$\epsilon_{xx} = \frac{\Delta L_x}{L_x} = -\nu \frac{\Delta L_z}{L_z} = -\frac{1}{2}\frac{\Delta L_z}{L_z}$$

$$\epsilon_{yy} = -\nu \frac{\Delta L_z}{L_z} = \frac{\Delta L_x}{L_x} = -\frac{1}{2}\frac{\Delta L_z}{L_z}$$

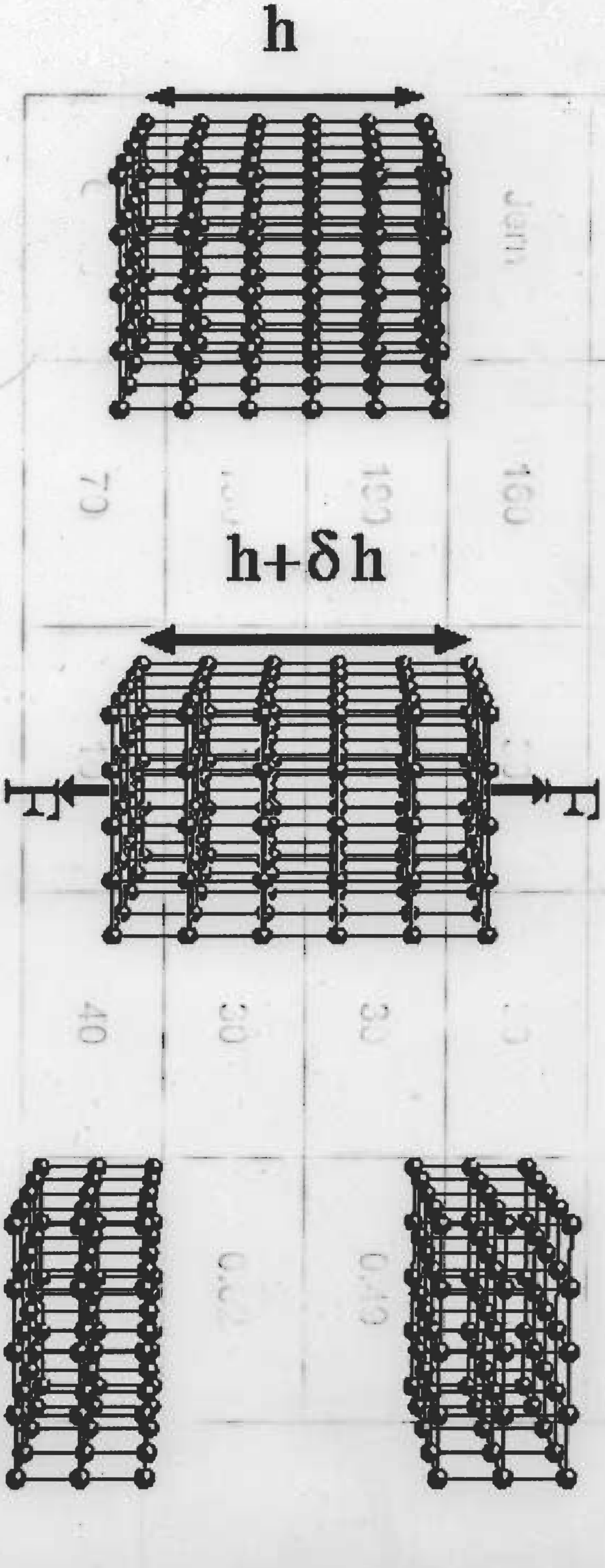
$$\frac{\Delta V}{V} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \quad (1-2\nu)\epsilon_{zz} = 0$$

$\nu = \frac{1}{2}$

| Materiale | E (GPa) | E/5 (GPa) | Teoretisk styrke (GPa) | Praktisk styrke (GPa) |
|---|---|---|---|---|
| Jern | 160 | 30 | 30 | 0.85 |
| Kobber | 190 | 40 | 30 | 0.49 |
| Silisium | 180 | 40 | 30 | 0.62 |
| Glass | 70 | 10 | 40 | 0.02 |

Teoretisk styrke funnet ved realistiske modeller for atomær binding.

Anders Malthe-Sørenssen, 25.02.2004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F
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Plastic crystals – dislocations

Brittle materials

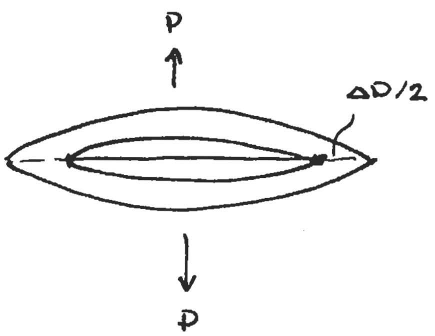
→ fracture formation

→ fracture movement.

Griffith theory, (1920)

Assume there is a crack in the material

The fracture may open under external load



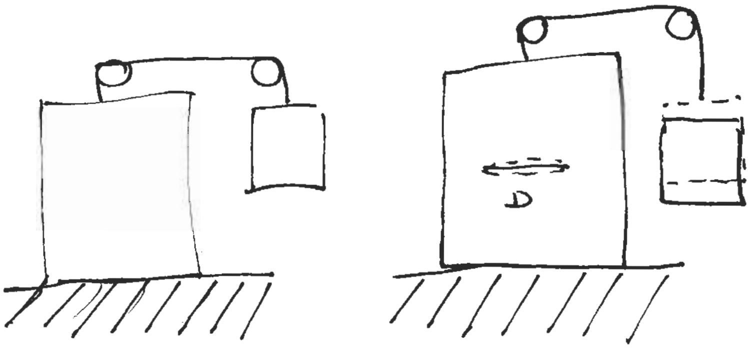

- opens
- form new surface.

Assume change in fracture length is ΔD

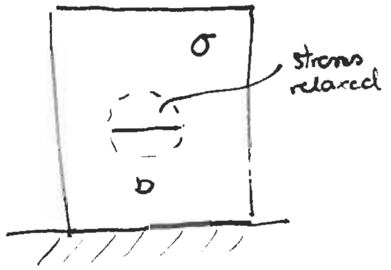
Energy $2\gamma \Delta D - \Delta E_{el}$

reduction of elastic energy.

More detail



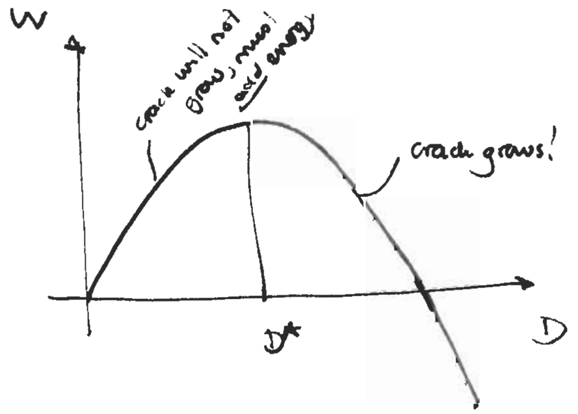
What is the energy difference, corresponding to the work? $W$




$2\gamma D$ : surface energy $\gamma$

$-\dfrac{\sigma^2}{2G} D^2$ relaxation of elastic energy

$$W = 2\gamma D - \frac{\sigma^2}{2G} D^2$$




Critical $D^*$

$$\frac{\partial W}{\partial D} = 0$$

$$2\gamma - \frac{\sigma^2}{G} D^* = 0$$

$$D^* = \frac{2G\gamma}{\sigma^2}$$

or

$$\sigma^* = \sqrt{\frac{2\gamma G}{D}}$$ : critical load, gives length of crack

Griffith criterion

D.

D.

"Justification" of $\dfrac{\sigma^2}{2G} D^2$ term.

$\sigma D$ is force on surface.

$\dfrac{1}{2}\dfrac{\sigma D}{G}$ is distance, because

$$\sigma = G \frac{\partial u}{\partial y}$$

$\dfrac{\sigma}{G} = \dfrac{\partial u}{\partial y}$ ; total displacement at center

$$\frac{\sigma}{G} D = u$$

avg displacement is $\dfrac{1}{2}\dfrac{\sigma}{G} D$

$$W = \frac{1}{2}\frac{\sigma}{G} D \cdot \sigma D = \frac{1}{2}\frac{\sigma^2}{G} D^2$$