FYS-GEO 4500 24 Aug 2009

Before we start:
Questions over the reading?
Problems installing Clawpack?

The problem set

Our syllabus - still subject to change

	date		Topic	Chapter in LeVeque
1	17 Aug 2009	Monday 13.15-15.00	introduction to conservation laws, Clawpack	1 & 2 & 5
2	24 Aug 2009	Monday 13.15-15.00	the Riemann problem, characteristics	3
3	**28 Aug 2009**	**Friday 13.15-15.00**	finite volume methods for linear systems	4
4	**8 Sep 2009**	**Tuesday 13.15-15.00**	high resolution methods	6
5	21 Sep 2009	Monday 13.15-15.00	boundary conditions and accuracy	7 & 8
6	**24 Sep 2009**	**Thursday 13.15-15.00**	nonlinear conservation laws, traffic flow	11
7	28 Sep 2009	Monday 13.15-15.00	finite volume methods for nonlinear equations	12
8	5 Oct 2009	Monday 13.15-15.00	nonlinear systems, shallow-water equations	13
9	12 Oct 2009	Monday 13.15-15.00	gas dynamics, Euler equation	14
10	19 Oct 2009	Monday 13.15-15.00	finite volume methods for nonlinear systems	15
11	26 Oct 2009	Monday 13.15-15.00	multidimensional hyperbolic problems & methods	18 & 19
12	2 Nov 2009	Monday 13.15-15.00	multidimensional scalar equations & systems	20 & 21
13	**5 Nov 2009**	**Thursday 13.15-15.00**	applications: tsunamis, pockmarks, venting, impacts	
14	16 Nov 2009	Monday 13.15-15.00	applications: volcanic jets, pyroclastic flows, lahars	
15	23 Nov 2009	Monday 13.15-15.00	review	
16	30 Nov 2009	Monday 13.15-15.00	discuss progress and problems on projects	
17	7 Dec 2009	Monday 13.15-15.00	FINAL PROJECT REPORTS DUE	

Any problems with the schedule?

Review: conservation law and advection

The fundamental conservation law in one spatial dimension, expressed in differential form, is:

$$q_t(x,t) + f(q(x,t))_x = 0.$$

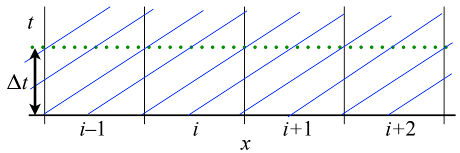
The advection equation, the simplest hyperbolic differential equation,

$$q_t(x,t) + uq_x(x,t) = 0,$$

is a conservation law with the flux function $f(x,t) = uq(x,t)$. Its solution is

$$q(x,t) = q(x-ut,0),$$

and this function is constant along rays in space-time (*characteristics*) with $x - ut$ = constant.



Review: Linear acoustics in a stationary gas

The acoustic equations are:

$$p_t(x,t) + Ku_x(x,t) = 0$$
$$u_t(x,t) + \tfrac{1}{\rho} p_x(x,t) = 0.$$

Expressed in linear form, with matrix notation:

$$q_t(x,t) + Aq_x(x,t) = 0 \qquad q = \begin{bmatrix} p \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & K \\ \frac{1}{\rho_0} & 0 \end{bmatrix}.$$

This can be resolved into the eigensystem $Ar = \lambda r$,

with eigenvalues $\lambda^{1,2} = \pm c = \pm\sqrt{\dfrac{K}{\rho}}$ and eigenvectors $r^{1,2} = \begin{bmatrix} \pm\sqrt{K\rho} \\ 1 \end{bmatrix}.$

The eigenvalues are the wave speeds, and the eigenvectors express relations between the components of the solution q.

FYS-GEO4500
The Riemann Problem
(Chapter 3 in Leveque)

Resolution to the eigensystem is the key to the solution

Our linear hyperbolic system of equations is written as

$$q_t + Aq_x = 0.$$

Since it is hyperbolic, we can resolve it into eigenvalues and eigenvectors

$$Ar^p = \lambda^p r^p \text{ for } p = 1,2,\ldots,m.$$

The next step will be to show that we can form a series of new equations

$$w_t^p + \lambda^p w_x^p = 0 \text{ for } p = 1,2,\ldots,m$$

that are equivalent to the original system, and from which we can assemble the solution vector q.

Notice that these new equations are simply advection equations!

Superposition of waves

But if we are to assemble the solution vector q from the p eigenvalue advection equations, we have to believe that we can superimpose the waves resulting from all of them.

This has to be proven eventually, but first a demonstration in a simple case.

The solution to the acoustic equations in one dimension,

$$p_t(x,t) + Ku_x(x,t) = 0$$
$$u_t(x,t) + \tfrac{1}{\rho} p_x(x,t) = 0,$$

is a pair of sound waves, propagating away from the source with velocity

$$\pm c = \pm\sqrt{\frac{K}{\rho}}.$$

Demonstration of superposition

?

Clawpack with the linear acoustic equations

modified acousimple example from Chapter 3

- second-order Lax-Wendroff with Superbee limiter
- in $CLAW/book/chap3/acousimple

Boundary Conditions for a System

The Initial-Boundary Value Problem for the advection equation required us to set inflow boundary conditions, either at left or right, depending on the sign of the velocity.

For a system with multiple characteristics, some boundary conditions must be set at *left* and some at *right*. In the decoupled advection equations

$$w_t^p + \lambda^p w_x^p = 0,$$

boundary conditions on $w^p\,(x,t)$ are specified on the left if $\lambda^p > 0$, and on the right if $\lambda^p < 0$.

In fact, however, boundary conditions are usually set on the *physical variables* and not on the characteristics. We'll see how this is done later.

The Riemann problem

The Riemann problem is simply the hyperbolic equation being studied, plus special boundary data representing a single jump discontinuity:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

This is fundamental for understanding the theory of hyperbolic equations and fundamental for finite volume solutions of these equations.

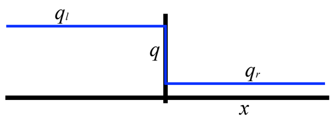
In developing numerical solutions, we will solve the Riemann problem repeatedly, at every cell border, and use these problems to advance the overall solution to the next time step.

Over the course of a full simulation, the Riemann problem may be solved millions or hundreds of millions of times so it is important to do it correctly and efficiently.

The Riemann problem for the advection equation

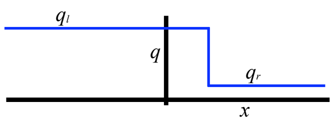
For the advection equation, $q_t + uq_x = 0$, with initial discontinuous data

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$



The solution is

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x > ut \end{cases}$$



The discontinuity simply propagates with speed u. The discontinuity does not diffuse or disperse.

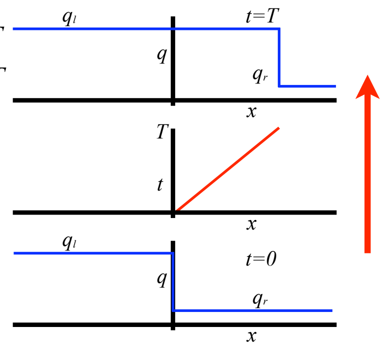
The Riemann problem for the advection equation

The *characteristic* tracks the position x of the discontinuity with time t

$$q(x,T) = q(x-uT,0) = \begin{cases} q_l & \text{if } x < uT \\ q_r & \text{if } x > uT \end{cases}$$

The characteristic:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$



Remember the discontinuity!

Strictly speaking, the Riemann solution is *not* a solution of the partial differential equation $q_t + uq_x = 0$ because the derivatives are infinite at the jump.

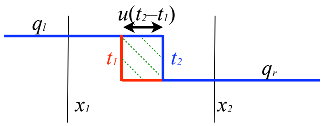
But it *is* a solution of the integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t)$$

Proof: integrate in time to get

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx = \int_{t_1}^{t_2}\big(uq(x_1,t) - uq(x_2,t)\big)\,dt$$

Both sides are zero if the interval does not bridge the jump; both sides are equal to $u(q_l - q_r)(t_2 - t_1)$ if it does.



We can apply the Riemann problem to systems of equations as well…

But first we must do some preliminary work.

You'll see why the advection equation is important!

Characteristics for a system of equations

For the linear $m \times m$ hyperbolic system of equations $q_t + f'(q)q_x = 0$, the Jacobian is

$$A = f'(q) = \begin{bmatrix} \frac{\partial f^1}{\partial q^1} & \cdots & \frac{\partial f^1}{\partial q^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial q^1} & \cdots & \frac{\partial f^m}{\partial q^m} \end{bmatrix}.$$

It has m eigenvectors and eigenvalues found from $Ar^p = \lambda^p r^p$.

The matrix of eigenvectors $R = \left[r^1 \middle| r^2 \middle| \ldots \middle| r^m\right]$ has an inverse R^{-1}

So we can form the matrix

$$R^{-1}AR = \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}$$

Characteristics for a system of equations

With the original Jacobian now in diagonal form,

$$A = \begin{bmatrix} \frac{\partial f^1}{\partial q^1} & \cdots & \frac{\partial f^1}{\partial q^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial q^1} & \cdots & \frac{\partial f^m}{\partial q^m} \end{bmatrix} \qquad R^{-1}AR = \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}$$

and defining $w(x,t) \equiv R^{-1}q(x,t)$, so $Rw(x,t) = q(x,t)$,

we can rewrite the system $q_t + Aq_x = 0$ as $w_t + \Lambda w_x = 0$.

$$Rw_t + ARw_x = 0$$

$$\left(R^{-1}R\right)w_t + \left(R^{-1}AR\right)w_x = 0$$

Characteristics for a system of equations

Since the matrix Λ is diagonal, the system becomes m *independent advection equations*:

$$w_t^p + \lambda^p w_x^p = 0 \quad \text{for } p = 1, \ldots, m$$

The system then has m *distinct* characteristic waves travelling at the speeds given by the eigenvalues λ^p. The system is *strictly hyperbolic* because it has a full set of distinct eigenvalues.

Note we have so far assumed the matrix $A = f'$ is constant. We'll generalise later.

Assembling the solution

Starting with the constant-coefficient system $q_t + Aq_x = 0$, we have found we can write it as

$$w_t + \Lambda w_x = 0,$$

where Λ is the matrix of eigenvalues. The vector w (sometimes called the vector of *characteristic variables*) is found from

$$w(x,t) = R^{-1} q(x,t),$$

where $R = \left[r^1 \middle| r^2 \middle| \ldots \middle| r^m \right]$ is the matrix of right eigenvectors.

Hence the problem is resolved into the m independent advection equations

$$w_t^p + \lambda^p w_x^p = 0 \quad \text{for } p = 1, \ldots, m,$$

each of which has a solution of the form

$$w^p(x,t) = w^p(x - \lambda^p t, 0).$$

Assembling the solution

To get the solution to the full Riemann problem, we simply superimpose the waves

$$w^p(x,t) = w^p(x - \lambda^p t, 0),$$

and the full solution is therefore

$$q(x,t) = Rw(x,t) = \sum_{p=1}^{m} w^p(x,t) r^p.$$

p-characteristics, superposition of waves

The solution to the Riemann problem for a linear $m \times m$ system of equations is

$$q(x,t) = Rw(x,t) = \sum_{p=1}^{m} w^p(x,t) r^p,$$

a superposition of waves, each of strength w^p and moving at speed λ^p.

The functions $w^p(x,t)$ are called *characteristic variables*, whose initial values $w^p(x,0)$ are simply advected at speed λ^p along the curves

$$X(t) = x_0 + \lambda^p t.$$

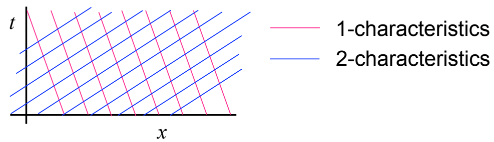
Each such curve is called a p-characteristic.

Conventionally the eigenvalues and their characteristics are ordered in increasing value of the speed λ^p and labelled with the index p.

The characteristics cover space-time

Every point in the x–t plane is crossed by *all* the characteristics, if the problem is strictly hyperbolic.

In this diagram for a 2x2 system, the red lines are characteristics of the p=1 family, the blue of the p=2 family.



So the exact solution, *everywhere*, consists of a superposition of right states moving to the left along the red lines and left states moving to the right along the blue lines. The solution is defined in all of space-time by simply adding the appropriate right and left states. This can be extended to any $m \times m$ system, and to multiple dimensions as well.

It's easy! Now we'll go over it again, slightly differently…

The Riemann problem for a system of equations

The Riemann problem is simply the hyperbolic equation being studied, plus special boundary data, piecewise constant, with a single jump discontinuity:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

This discontinuity will propagate along the characteristic curves. But note that q will now be considered to be a vector.

We can solve the Riemann problem for a linear $m \times m$ system of equations using the mathematics we've already developed.

For a nonlinear system, the solution will have a similar structure, but we defer that discussion for later.

We start by writing $q_l = \sum_{p=1}^{m} w_l^p r^p$ and $q_r = \sum_{p=1}^{m} w_r^p r^p$

Right and Left Eigenvectors

We construct the matrix R from the eigenvectors of the Jacobian of the PDE system. These are the *right eigenvectors* of the system:

$$R = \left[r^1 \middle| r^2 \middle| \ldots \middle| r^m \right] \qquad A r^p = \lambda^p r^p$$

The rows of the matrix inverse of R form the *left eigenvectors*:

$$L = R^{-1} = \begin{bmatrix} l^1 \\ l^2 \\ \vdots \\ l^m \end{bmatrix} \qquad l^p A = \lambda^p l^p$$

We can therefore rewrite our w vector as

$$w(x,t) = R^{-1} q(x,t) = L q(x,t)$$
$$w^p(x,t) = l^p q(x,t)$$

This vector satisfies the advection equation: $w_t + \Lambda w_x = 0$ with Λ the diagonal matrix of eigenvalues.

The solution to the system of equations

We obtained the m advection equations $w_t^p + \lambda^p w_x^p = 0$

whose solutions are $w^p(x,t) = w^p(x - \lambda^p t, 0)$.

Now we combine all the w^p into the vector w and write the solution to the original problem:

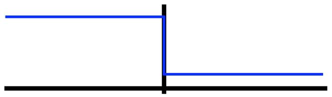
$$\begin{aligned} q(x,t) &= R w(x,t) \\ &= \sum_{p=1}^{m} w^p(x,t) r^p \\ &= \sum_{p=1}^{m} \left[l^p q(x - \lambda^p t, 0) \right] r^p \end{aligned}$$

The solution is a superposition of m waves, each moving at its own characteristic speed.

Solving the Riemann problem

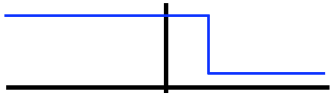
Then each advection equation has initial (Riemann) data:

$$w^p(x,0)=\begin{cases} w_l^p & \text{if } x<0 \\ w_r^p & \text{if } x>0 \end{cases}$$



And the discontinuity in each component propagates with its own speed λ^p:

$$w^p(x,t)=\begin{cases} w_l^p & \text{if } x-\lambda^p t<0 \\ w_r^p & \text{if } x-\lambda^p t>0 \end{cases}$$



The solution

$$q(x,t)=\sum_{p=1}^{m} w^p(x,t)r^p$$

is then a mixture of left and right states, the mixture changing with time and space because the speeds λ^p are different.

Solving the Riemann problem

We assume that the eigenvalues λ^p have been ordered with increasing (positive) speeds at higher index p, and separate the sum into two pieces according as λ^p is less than or greater than x/t.

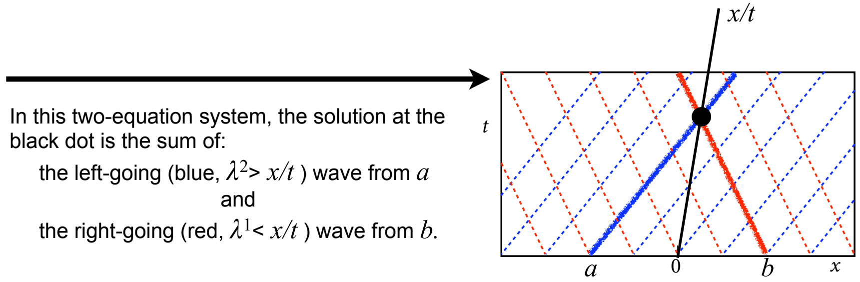
$$q(x,t)=\sum_{p:\lambda^p<x/t} w_r^p r^p + \sum_{p:\lambda^p>x/t} w_l^p r^p$$

In this two-equation system, the solution at the black dot is the sum of:

the left-going (blue, $\lambda^2 > x/t$) wave from a

and

the right-going (red, $\lambda^1 < x/t$) wave from b.

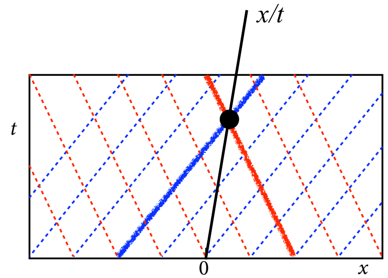


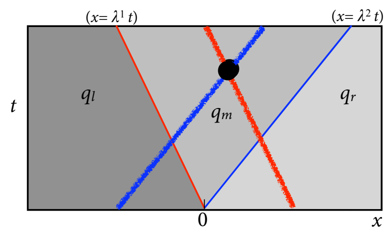
Look at it from the origin:

Because there are two waves, a simple discontinuity at the origin divides to produce two new discontinuities.

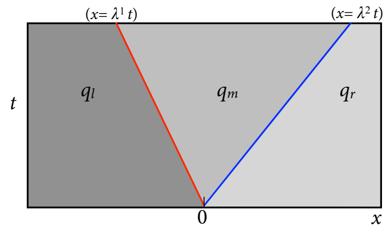
The left and right states persist on the left and right sides of the characteristics from the origin and a new intermediate state develops between them.

The state at the black dot is the intermediate state, in common with other points in the region.





Riemann diagram for a two-equation system



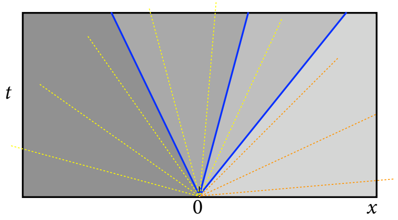
For a linear two-equation Riemann problem with left and right states q_l and q_r, the discontinuity at the origin divides. Two waves (characteristics) propagate away from the origin with constant speeds λ^1 and λ^2.

As the waves separate, a new constant state develops in the middle with

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

At any later time, there are two discontinuities, each smaller than the original one.

Similarity solutions



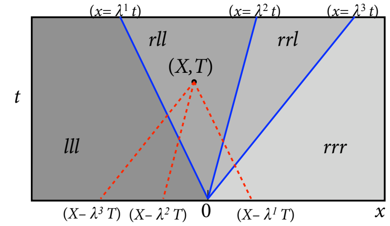
Riemann diagram valid for a *constant-coefficient linear* 3 equation system

The Riemann problem for a linear system results in *self-similar* solutions: the solution depends on $\frac{x}{t}$ and not on x or t separately. The solution is thus constant within the wedges defined by the characteristics.

And remember:

For any hyperbolic system, the domain of dependence is bounded.

Constructing the solution for a 3×3 system



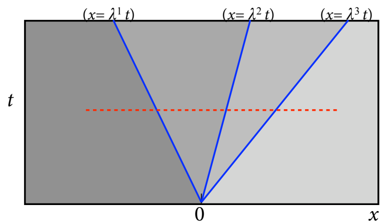
The red dashed lines connect the points that influence the point (X,T); the blue solid lines connect the points affected by the origin.

In the wedge where point (X,T) sits, the solution can be denoted rll, short for

$$q(X,T) = w_r^1 r^1 + w_l^2 r^2 + w_l^3 r^3$$

and so on for the other wedges. Across each characteristic, the solution has a jump discontinuity, and the solution is constant within each wedge.

Decomposing the jump



Generalise to an $m \times m$ system.

Along the red dashed line (i.e. at any time after t=0), the original jump discontinuity has been broken up into a linear combination of the eigenvectors of the system matrix A.

$$q_l - q_r = \alpha^1 r^1 + \alpha^2 r^2 + \ldots + \alpha^m r^m \quad \text{where} \quad \alpha^p = (w_r^p - w_l^p)$$

We solve for the jump coefficients α by:

$$R\alpha = q_l - q_r$$

$$\alpha = R^{-1}(q_l - q_r)$$

$$\alpha^p = l^p (q_l - q_r)$$

The solution for $q(x,t)$ can then be written

$$q(x,t) = q_l + \sum_{p=1}^{m} H(x - \lambda^p t)\alpha^p r^p \quad \text{where} \quad H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

The wave notation

A notation that will be useful later on is to denote the jump in q across the p^{th} wave in the Riemann solution as $\mathcal{W}^p$ where

$$\mathcal{W}^p = \alpha^p r^p$$

These will be called *waves*.

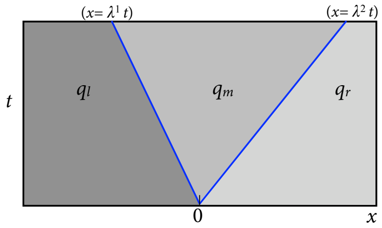
Then the solution to the Riemann problem can be written

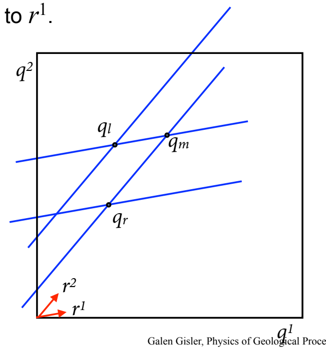
$$q(x,t) = q_l + \sum_{p=1}^{m} H(x - \lambda^p t)\mathcal{W}^p$$

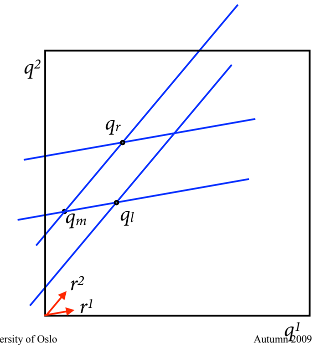
where H is the Heaviside function $H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$

Phase plane for the two-equation system

In a two-equation system, one can construct a phase plane of the components of $q=(q^1,q^2)$ (below). The initial data are placed in this plane. The solution vector q can move in this plane only parallel to the eigenvectors r^1 and r^2. Hence the middle state q_m can be found by construction. Since $\lambda^1 < \lambda^2$ the move from q_l to q_m must go parallel to r^1.







Some examples: Burger's Equation

The simplest nonlinear partial differential equation is Burger's equation:

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

$$u_t + uu_x = 0.$$

As the second form explicitly shows, it is in conservation form, and it is everywhere hyperbolic, with variable eigenvalue u, though nonlinear.

This is the simplest differential equation which demonstrates the development of discontinuities and so proves the differential form inadequate!

Demonstration of Burger's Equation

Clawpack with Burger's equation

modified example from Chapter 11

second-order Lax-Wendroff with VanLeer limiter

in $CLAW/book/chap11/burgers/_plots/_Plotindex.html

Example: the Euler equations of gas dynamics

Recall the equations of continuity and momentum for the motion of a fluid:

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0$$

To these we add an equation for the conservation of energy E:

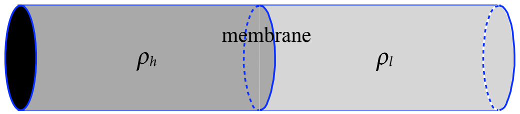
$$E_t + \left(u(E+p)\right)_x = 0$$

And we must supplement with an equation of state, $p = P(\rho, E)$, but we won't worry about the details for now.

Here it is sufficient to recognise that this system of 3 equations gives rise to 3 distinct characteristic waves. It is a *nonlinear* system, however.

We'll see how this works in a one-dimensional shock tube.

The shock tube:



The shock tube is a closed tube filled with gas, separated by a membrane into sections with different densities.

The membrane is suddenly removed, and the gas is now free to move from one section to the other.

What happens?

How many waves are there, and which way do they propagate?

Demonstration of shock tube

?

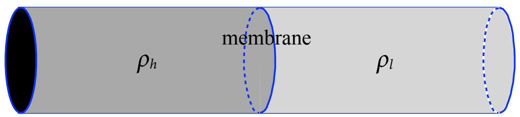
Clawpack with the Euler equations

modified example from Chapter 15

second-order Lax-Wendroff with Superbee limiter

in $CLAW/book/myConversions/chap15/shocktube/_plots/_PlotIndex.html

The shock tube



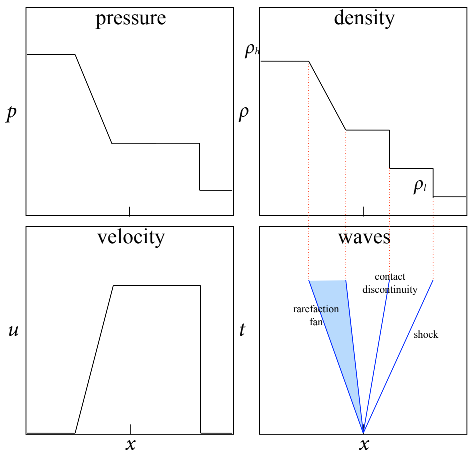
A closed tube filled with gas, separated by a membrane into sections with different densities.

The membrane is suddenly removed, and the gas starts moving from the high-density region into the lower density region.

Three waves develop: a *shock wave*, a *contact discontinuity*, and a *rarefaction wave* (or *fan*). The first two travel to the right, the third to the left.

At the shock, velocity, pressure and density are all discontinuous. At the contact, only density is discontinuous. In the rarefaction fan, all variables are continuous, but their derivatives are not.

The third wave is not a sharp discontinuity because of the problem's nonlinearity.



Review of the Riemann problem

The Riemann problem is the original system of equations, $q_t + f(q)_x = 0$ plus the special initial condition consisting of a jump discontinuity:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

In the linear hyperbolic system, we have $q_t + f'(q)q_x = 0$ and the Jacobian can be diagonalised into the form

$$f'(q) = \begin{bmatrix} \frac{\partial f^1}{\partial q^1} & \cdots & \frac{\partial f^1}{\partial q^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial q^1} & \cdots & \frac{\partial f^m}{\partial q^m} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}.$$

with the eigenvalues λ^p , since the system is hyperbolic.

Review of the Riemann problem

The solution vector is resolved or projected onto the eigenvectors r^p ,

$$q(x,t)=\sum_{p=1}^{m} w^p(x,t)r^p$$

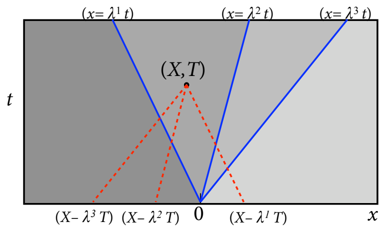
and the system is replaced by the equivalent m advection equations

$$w_t^p+\lambda^p w_x^p=0,$$

with the solution $w^p(x,t)=w^p(x-\lambda^p t,0)$. The initial left-right discontinuity is split among the eigenvectors

$$q_l-q_r=\sum_{p=1}^{m}\alpha^p r^p=\sum_{p=1}^{m}\left(w_l^p-w_r^p\right)r^p.$$

The solution at a later time is a mixture of these left and right states, depending on whether x is to the left or the right of the corresponding characteristic.

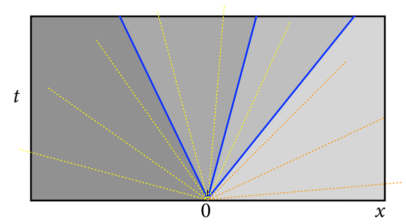


Review of the Riemann problem

If we define the waves $\mathcal{W}^p\equiv\alpha^p r^p=(w_l^p-w_r^p)$ then the solution to the Riemann problem can be written

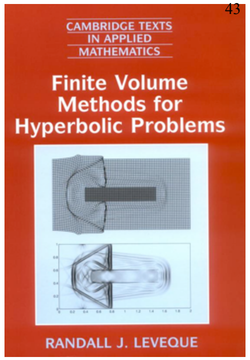
$$q(x,t)=q_l+\sum_{p=1}^{m}H(x-\lambda^p t)\mathcal{W}^p$$

where H is the Heaviside function $H(x)=\begin{cases}0 & \text{if } x\le 0\\ 1 & \text{if } x>0\end{cases}.$



The Riemann solution for a linear system is a *similarity solution*: it depends on x/t and not on x or t separately.

Assignment for next time



Read all of Chapter 3.

Pay careful attention to the examples 3.1, 3.2, 3.3, and 3.4.

Work problems 3.1, 3.4, and 3.7. Hand them in to me by this Friday (the 28th)

Next: Finite Volume Methods for Linear Systems (Ch 4)