FYS-GEO 4500 28 Aug 2009

Before we start:
Questions over the reading?

The problem set

Our syllabus - still subject to change

	date		Topic	Chapter in LeVeque
1	17 Aug 2009	Monday 13.15-15.00	introduction to conservation laws, Clawpack	1 & 2 & 5
2	24 Aug 2009	Monday 13.15-15.00	the Riemann problem, characteristics	3
3	**28 Aug 2009**	**Friday 13.15-15.00**	finite volume methods for linear systems	4
4	**8 Sep 2009**	**Tuesday 13.15-15.00**	high resolution methods	6
5	21 Sep 2009	Monday 13.15-15.00	boundary conditions and accuracy	7 & 8
6	**24 Sep 2009**	**Thursday 13.15-15.00**	nonlinear conservation laws, traffic flow	9 & 11
7	**29 Sep 2009**	**Tuesday 13.15-15.00**	finite volume methods for nonlinear equations	12
8	5 Oct 2009	Monday 13.15-15.00	nonlinear systems, shallow-water equations	13
9	12 Oct 2009	Monday 13.15-15.00	gas dynamics, Euler equation	14
10	19 Oct 2009	Monday 13.15-15.00	finite volume methods for nonlinear systems	15
11	26 Oct 2009	Monday 13.15-15.00	multidimensional hyperbolic problems & methods	18 & 19
12	2 Nov 2009	Monday 13.15-15.00	multidimensional scalar equations & systems	20 & 21
13	**5 Nov 2009**	**Thursday 13.15-15.00**	applications: tsunamis, pockmarks, venting, impacts	
14	16 Nov 2009	Monday 13.15-15.00	applications: volcanic jets, pyroclastic flows, lahars	
15	23 Nov 2009	Monday 13.15-15.00	review	
16	30 Nov 2009	Monday 13.15-15.00	discuss progress and problems on projects	
17	7 Dec 2009	Monday 13.15-15.00	FINAL PROJECT REPORTS DUE	

Any problems with the schedule?

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Any problems with the schedule?

Review of the Riemann problem

The Riemann problem is the original system of equations, $q_t + f(q)_x = 0$ plus the special initial condition consisting of a jump discontinuity:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

In the linear hyperbolic system, we have $q_t + f'(q)q_x = 0$ and the Jacobian can be diagonalised into the form

$$f'(q) = \begin{bmatrix} \frac{\partial f^1}{\partial q^1} & \cdots & \frac{\partial f^1}{\partial q^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial q^1} & \cdots & \frac{\partial f^m}{\partial q^m} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}.$$

with the eigenvalues λ^p , since the system is hyperbolic.

Review of the Riemann problem

The solution vector is resolved or projected onto the eigenvectors r^p ,

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t) r^p$$

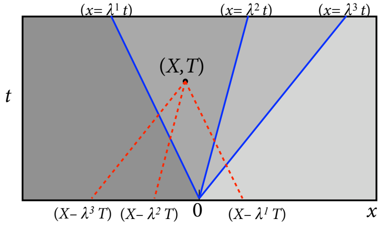
and the system is replaced by the equivalent m advection equations

$$w_t^p + \lambda^p w_x^p = 0,$$

with the solution $w^p(x,t) = w^p(x - \lambda^p t, 0)$. The initial left-right discontinuity is split among the eigenvectors

$$q_l - q_r = \sum_{p=1}^{m} \alpha^p r^p = \sum_{p=1}^{m} \left(w_l^p - w_r^p \right) r^p .$$

The solution at a later time is a mixture of these left and right states, depending on whether x is to the left or the right of the corresponding characteristic.

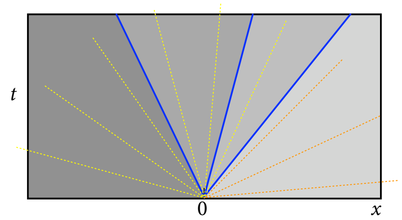


Review of the Riemann problem

If we define the waves $\mathcal{W}^p \equiv \alpha^p r^p = (w_l^p - w_r^p)$ then the solution to the Riemann problem can be written

$$q(x,t) = q_l + \sum_{p=1}^{m} H(x - \lambda^p t) \mathcal{W}^p$$

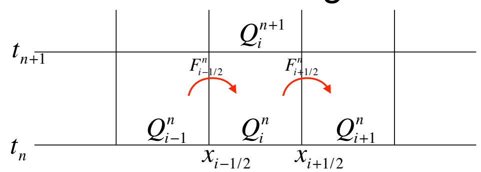
where H is the Heaviside function $H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}.$



The Riemann solution for a linear system is a *similarity solution*: it depends on x/t and not on x or t separately.

FYS-GEO4500
Finite Volume Methods for Linear Systems
(Chapter 4 in Leveque)

Next question: How do we get the fluxes?



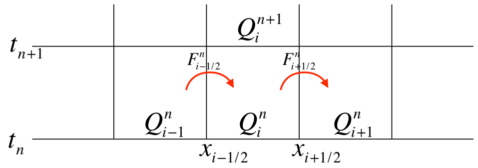
The equation we want to solve is $q_t + f(q)_x = 0$ and we think we know how to do it, from one time step to the next, by solving Riemann problems at each interface.

If it's a linear system we can write $f(q)_x = f'(q) q_x$ and resolve the (constant) Jacobian into its eigenvalues and eigenvectors. But we still need a way to determine the appropriate *numerical* flux that we will use to advance the numerical solution from one time step to the next, using something like:

$$Q_i^{n+1} \approx Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n \right)$$

We'll put aside the Riemann problem for the moment, we'll need it in an hour or so.

Explicit *versus* Implicit

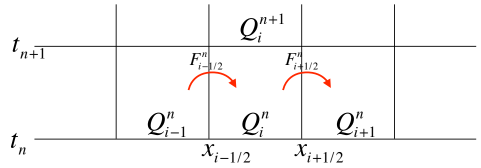


For hyperbolic equations, the domain of dependence is *bounded*, since information propagates with a finite speed.

We can therefore use *explicit* methods, in which the state at the later time is calculated in terms of the state at the present time.

For elliptic and parabolic equations, *implicit* methods, solving an equation involving both the later and present times, are required.

General formulation for conservation laws



In finite volume methods, we divide the problem domain (here one-dimensional) into a grid of *cells*, and form an approximation of the solution value within each cell:

$$Q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)dx, \text{ where } \Delta x = x_{i+1/2} - x_{i-1/2}$$

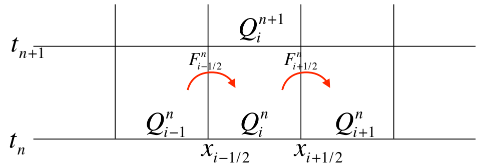
The integral form of the conservation law is

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

Then by integrating over time, we get

$$Q_i^{n+1} \approx Q_i^n - \frac{1}{\Delta x}\left(\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt - \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt\right)$$

Getting the fluxes



If we can find a way to formulate $F_{i+1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt$ in terms of the Q_i^n, then we can write:

$$Q_i^{n+1} \approx Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

This scheme is in conservation form. The fluxes cancel except at the boundaries:

$$\Delta x\sum_{i=1}^{N} Q_i^{n+1} = \Delta x\sum_{i=1}^{N} Q_i^n - \Delta t\left(F_{N+1/2}^n - F_1^n\right)$$

In hyperbolic equations, information propagates at finite speed, so we should formulate the $F_{i+1/2}^n$ from the values Q_i^n, Q_{i+1}^n in neighbouring cells. Then the future Q_i^{n+1} will depend on the three values Q_{i-1}^n, Q_i^n, and Q_{i+1}^n. This is known as a three-point stencil.

Convergence: consistency and stability

The key to finite volume methods is how to approximate the time-integral of the flux from the present time to the future time.

$$F_{i+1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt$$

Everything depends now on how we formulate the flux function $F_{i+1/2}^n$, so we need to define criteria for judging the choice.

The method must be *convergent*, i.e. the numerical solution must approach the true solution as the cell size and time step decrease ($\Delta x, \Delta t \to 0$).

The method must be *consistent* with the system of equations.

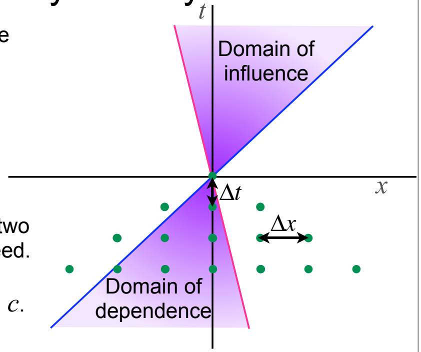
The method must be *stable*, so that small errors don't grow rapidly.

The Courant-Friedrich-Lewy stability limit

The numerical stencil must contain the true domain of influence. This is a *necessary* condition for stability.

Since influence is propagated by the characteristic waves, the true domain of influence depends on the wave speeds.

For a symmetric wave equation there are two waves, but only a single characteristic speed. For acoustics in a stationary medium, the characteristic speed is the speed of sound c.



We define the CFL number $\nu \equiv c\dfrac{\Delta t}{\Delta x}$ and require that $\nu \le 1$ for stability.

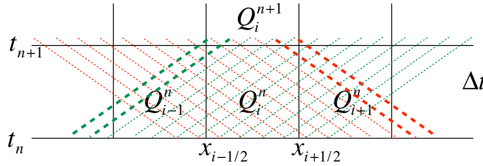
For a hyperbolic system of equations, we can have up to m different wave speeds given by $\lambda^1, \lambda^2, \ldots, \lambda^p$, so the Courant number must be

$$\nu \equiv \frac{\Delta t}{\Delta x}\max\left|\lambda^p\right| \le 1$$

The Courant-Friedrich-Lewy stability limit

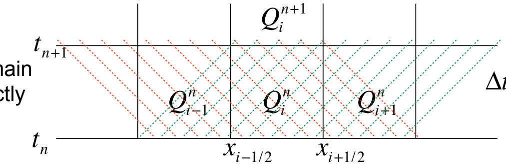
The domain of influence for the symmetric wave equation, wave speed c, three-point stencil.

Unstable, because the domain of influence is larger than the numerical stencil.



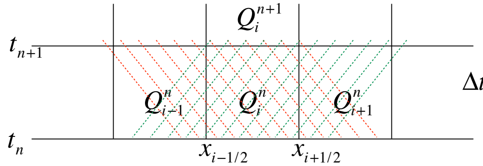
$\Delta t > \dfrac{\Delta x}{c}$

At the limit of stability the domain of influence corresponds exactly to the numerical stencil.



$\Delta t = \dfrac{\Delta x}{c}$

Within the limit of stability the stencil completely contains the domain of influence.



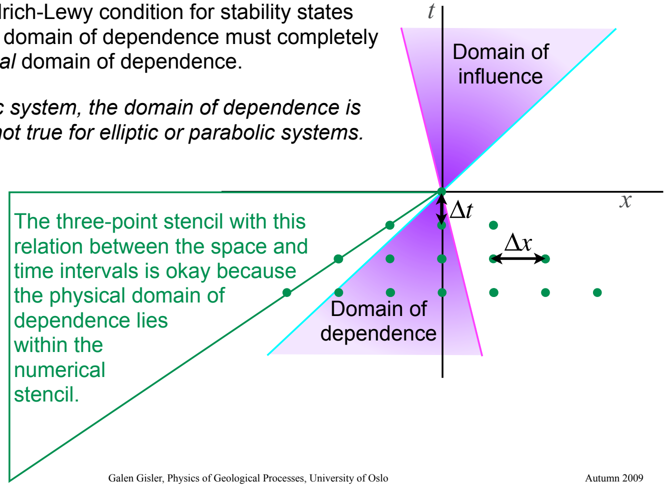
$\Delta t < \dfrac{\Delta x}{c}$

Causal domains in space-time

The domains of influence and dependence depend on the characteristics of the equations.

The Courant-Friedrich-Lewy condition for stability states that the *numerical* domain of dependence must completely contain the *physical* domain of dependence.

For any hyperbolic system, the domain of dependence is bounded. This is not true for elliptic or parabolic systems.



Formulation of the flux function and update rule

$$F_{i+1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt \qquad Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

Here are a few historical choices for **centred** methods:

Formulation of the flux function and update rule

$$F^n_{i+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt \qquad Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$$

Here are a few historical choices for **centred** methods:

$$F^n_{i-1/2} = \frac{1}{2}\left[f(Q^n_{i-1}) + f(Q^n_i)\right]$$
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\left[f(Q^n_{i+1}) - f(Q^n_{i-1})\right]$$

Naive method; unstable

Formulation of the flux function and update rule

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Naive method; unstable

$$F^n_{i-1/2} = \frac{1}{2}\left[f(Q^n_{i-1}) + f(Q^n_i)\right] - \frac{\Delta x}{2\Delta t}\left(Q^n_i - Q^n_{i-1}\right)$$
$$Q_i^{n+1} = \frac{1}{2}\left(Q^n_{i-1} + Q^n_{i+1}\right) - \frac{\Delta t}{2\Delta x}\left[f(Q^n_{i+1}) - f(Q^n_{i-1})\right]$$

Lax-Friedrichs method; stable, but diffusive; first-order accurate

Formulation of the flux function and update rule

$$F^n_{i+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt \qquad Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$$

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Naive method; unstable

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Lax-Friedrichs method; stable, but diffusive; first-order accurate

$$Q^{n+1/2}_{i-1/2} = \frac{1}{2}\left(Q^n_{i-1} + Q^n_i\right) - \frac{\Delta t}{2\Delta x}\left[f(Q^n_i) - f(Q^n_{i-1})\right]$$
$$F^n_{i-1/2} = f(Q^{n+1/2}_{i-1/2})$$
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$$

Two-step Richtmyer-Lax-Wendroff; second-order accurate, but oscillatory

But centred methods do not make the best use of the structure of hyperbolic equations

In hyperbolic equations, the information propagates along characteristics.

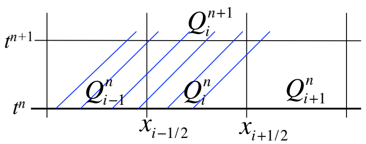
Since we know where the information is coming from, we should make use of that knowledge to formulate the flux function.

For the one-dimensional advection equation, there is only one characteristic, the fluid velocity u. The information comes from the left if u is positive, from the right if u is negative.

So in this simple case, we can use a *one-sided upwind* method, where we decide which side to use from the flow direction.

For systems with characteristics travelling in both directions, we must decide which information to transfer from which side.

A simple upwind method for advection



In the advection problem, the flux is in one direction, $F_{i-1/2}^n = uQ_{i-1}^n$ and the update is

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\left[Q_i^n - Q_{i-1}^n\right]$$

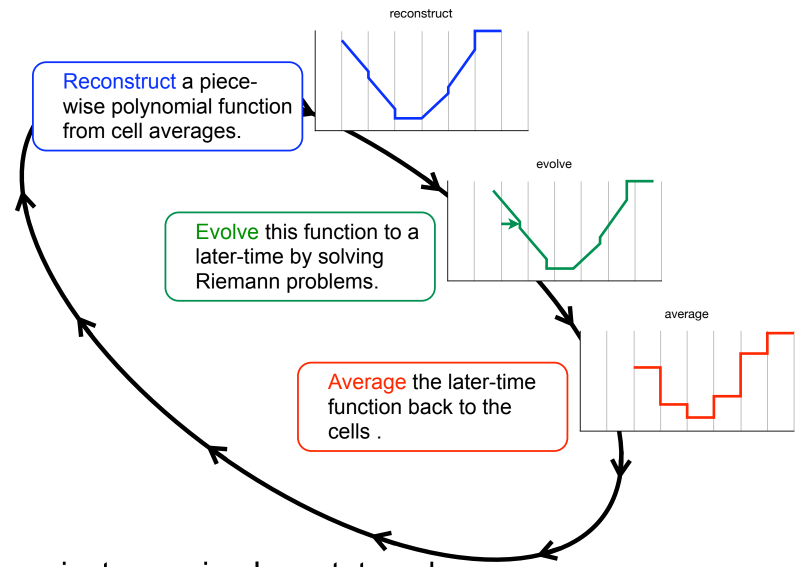
If you don't know *a priori* which direction the flux is, you can use:

$$F_{i-1/2}^n = u^- Q_i^n + u^+ Q_{i-1}^n$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)\right)$$

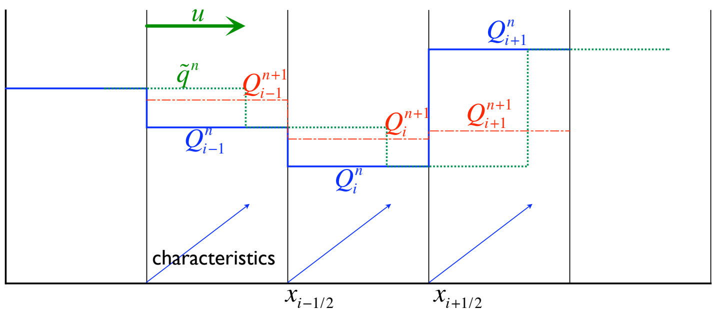
where

$$u^+ = \max(u,0), \quad u^- = \min(u,0).$$



We have just now simply prototyped the Godunov REA method.

How does this work?



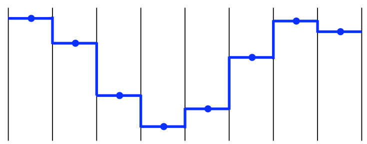
An illustration of the upwind method for CFL number $\nu \equiv u\dfrac{\Delta t}{\Delta x} \simeq 0.7$.

The cell averages Q_i^n are advected by the velocity u from time n to $n+1$,to produce an intermediate value $\tilde{q}^n$. Because the shift is less than a full cell, new cell averages must be computed to obtain the new quantities Q_i^{n+1}.

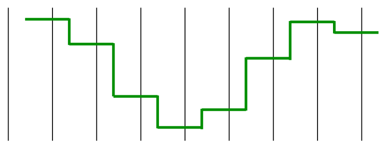
Each cell edge has a discontinuity: we can solve for the new cell value either directly (as we have done), or with the help of the Riemann technique.

First-order upwind for advection problem

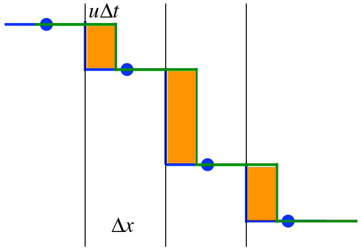
Reconstruct a function from the cell averages: piecewise constant in this case



Evolve the solution: advect it with the characteristic speed



Then compute the new cell averages



The cell average is changed by $$+u\frac{\Delta t}{\Delta x}\left(Q_{i-1}^n - Q_i^n\right)$$

So the upwind method is, as before, simply

$$Q_i^{n+1} = Q_i^n - u\frac{\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right)$$

To generalise, let's write it in wave-propagation form

We write the change in the cell average as

$$u\frac{\Delta t}{\Delta x}\left(Q_{i-1}^n - Q_i^n\right) = -s\frac{\Delta t}{\Delta x}\mathcal{W}_{i-1/2}$$

Where $\mathcal{W}_{i-1/2} = \left(Q_i^n - Q_{i-1}^n\right)$ is the wave strength and s is the wave speed.

At this point, this is only a change in notation, to prepare for the use of the method with *systems* of equations. But this is the same $\mathcal{W}$ we have already encountered in the Riemann problem.

In the advection equation there is (of course) only one upwind direction.

In a system of equations, waves may travel in any direction. We have to handle this somehow.

That's where the Riemann solver comes in.

Generalising the upwind method to systems

The general upwind method for s of either sign for a single wave is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(s^+\mathcal{W}_{i-1/2} + s^-\mathcal{W}_{i+1/2}\right)$$

and as before, we define

$$s^+ = \max(s,0), \quad s^- = \min(s,0).$$

Now recall the Riemann solution for a many-wave problem:

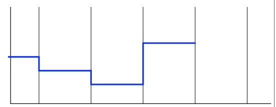
$$q(x,t) = q_l + \sum_{p=1}^{m} H(x-\lambda^p t)\mathcal{W}^p; \quad H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

We just have to put these together.

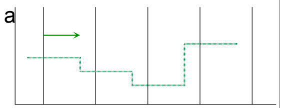
Godunov's method for linear systems

The upwind scheme is representative of REA algorithms, first invented by S.K. Godunov in 1959. REA stands for:

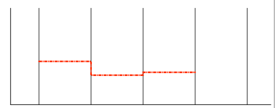
1. Reconstruct a piece-wise constant function from the cell averages.
$$q^n(x,t_n) = Q_i^n \ \text{ for } x \text{ in cell } i$$



2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces.
$$\tilde{q}^n(x,t_{n+1})$$



3. Average this function over each grid cell to obtain new cell averages.
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



This is done at each time step. The method can be improved by using other interpolation functions, polynomials for example, to improve the accuracy.

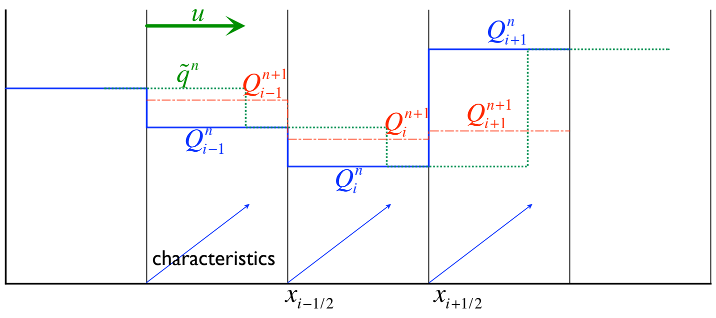
Physics is needed in the second step (evolution stage), as all the characteristics must be known and used in the solution. The first and third steps (projection stages) are entirely numerical (and problem independent).

Reconstruct - Evolve - Average

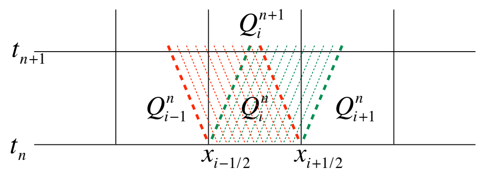
$q^n(x,t_n) = Q_i^n$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

$$Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



Care must be taken with interacting characteristics



In problems where the characteristics travel in both directions, solving the Riemann problem *independently* at each interface requires that the characteristics from neighbouring cell boundaries do not intersect.

This apparently gives a considerably stricter CFL limit: $\nu \equiv u\dfrac{\Delta t}{\Delta x} < \dfrac{1}{2}$.

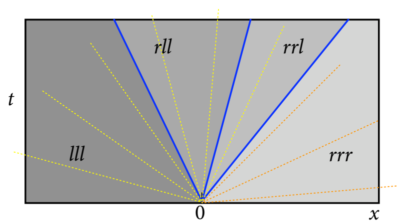
But in fact there are ways of solving the Riemann problem (cooperatively among adjacent cells) that relax this limit.

Numerical flux function in Godunov's method

Recall the formula for the numerical flux: $F_{i+1/2}^n \approx \dfrac{1}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt$

The numerical flux should be the average of the true flux over the time step, but we don't know how the true flux varies.

But if we replace $q^n(x,t)$ by $\tilde{q}^n(x,t)$ we have a tremendous advantage, since the solution to the Riemann problem is a similarity solution, constant along rays from the interface (yellow, orange dashed lines).



Leveque defines a special symbol for $\tilde{q}^n(x_{i-1/2},t)$, namely $q^{\downarrow}(Q_{i-1}^n,Q_i^n)$

and then the flux function is simply

$$F_{i-1/2}^n = f\left(q^{\downarrow}(Q_{i-1}^n,Q_i^n)\right)$$

Godunov's method for a general system

Given a set of cell quantities Q_i^n at time n:

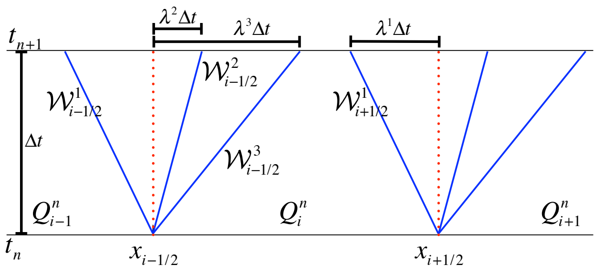
1. Solve the Riemann problem at $x_{i-1/2}$ to obtain $q^{\downarrow}(Q_{i-1}^n,Q_i^n)$
2. Define the flux: $F_{i-1/2}^n = f\left(q^{\downarrow}(Q_{i-1}^n,Q_i^n)\right)$
3. Apply the flux differencing formula: $Q_i^{n+1} = Q_i^n - \dfrac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

This will work for any general system of conservation laws. Only the formulation of the Riemann problem itself changes with the system.

The wave propagation implementation of Godunov's method

For a linear $m \times m$ system $q_t + Aq_x = 0$, the Riemann problem consists of m waves $\mathcal{W}^p$ propagating with constant speed λ^p.

Then $$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p = \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

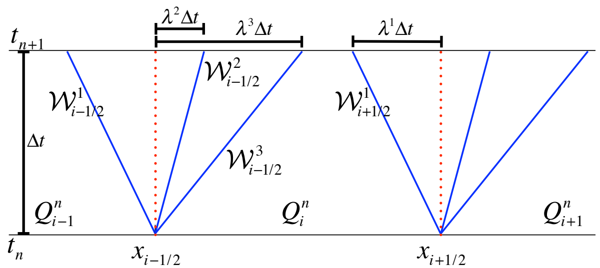


The wave propagation implementation of Godunov's method

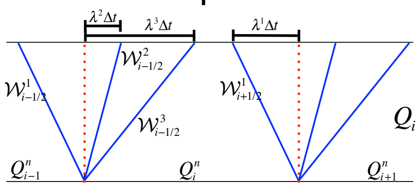
This is analogous to the basic upwind scheme.

A three-equation system has three characteristics. At timestep n, there is a discontinuity at the cell edge between Q^n_i and Q^n_{i+1}. As we evolve the Riemann solution forward to form $\tilde{q}^n(x, t_{n+1})$, this discontinuity splits into three pieces.

We use our knowledge of the splitting to compute the new cell averages.



The waves split the discontinuity



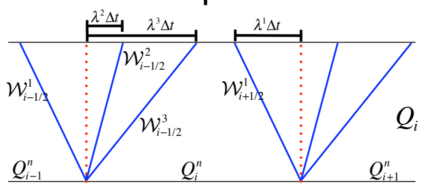
$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

The wave $\mathcal{W}^2_{i-1/2}$ changes the cell average by $-\dfrac{\lambda^2\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2}$. Taking all three waves, keeping track of which direction the information is coming from, we have: $Q^{n+1}_i = Q^n_i - \dfrac{\Delta t}{\Delta x}\left(\lambda^2\mathcal{W}^2_{i-1/2} + \lambda^3\mathcal{W}^3_{i-1/2} + \lambda^1\mathcal{W}^1_{i+1/2}\right)$.

Defining $\lambda^+ = \max(\lambda, 0), \quad \lambda^- = \min(\lambda, 0)$ (as we did for the upwind advection case), we generalise to the $m \times m$ case:

$$Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p_{i-1/2} + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p_{i+1/2}\right]$$

The waves split the discontinuity



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Introduce the notion of fluctuations

If $\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}^p$ is the effect of all right-going waves, and

$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i-1/2}^p$ is the effect of all left-going waves from $x_{i-1/2}$,

then we can write the update as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

Notice that we take the right-going waves from the left interface and the left-going waves from the right interface!

The symbols $\mathcal{A}^\pm \Delta Q_{i\pm 1/2}$ will be referred to as *fluctuations*.

This notation will be useful for nonlinear systems.

What are these fluctuations?

The symbols $\mathcal{A}^\pm \Delta Q_{i\pm 1/2}$ are the *fluctuations*, and we will use these heavily when we get to nonlinear systems.

But for *linear systems,* these are easily resolved into $A^\pm(Q_i^n - Q_{i-1}^n)$ etc.

Here's how…

To resolve the fluctuations in a linear system:

For the **linear** $m \times m$ system $q_t + Aq_x = 0$, remember we had

$$R^{-1}AR = \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}$$

Now we separate this into matrices of positive and negative eigenvalues:

$$\Lambda^+ = \begin{bmatrix} (\lambda^1)^+ & & & \\ & (\lambda^2)^+ & & \\ & & \ddots & \\ & & & (\lambda^m)^+ \end{bmatrix} \qquad \Lambda^- = \begin{bmatrix} (\lambda^1)^- & & & \\ & (\lambda^2)^- & & \\ & & \ddots & \\ & & & (\lambda^m)^- \end{bmatrix}$$

and we define $A^+ = R\Lambda^+R^{-1},\ A^- = R\Lambda^-R^{-1}$ so $\Lambda^+ + \Lambda^- = \Lambda,\ A^+ + A^- = A$

Then $$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right]$$

The fluctuations for a linear system

Recall the solution in terms of waves for the $m \times m$ case

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i+1/2}^p\right]$$

and remember that by our definition of the waves for a linear system:

$$A\left(Q_i - Q_{i-1}\right) = \sum_{p=1}^{m}\lambda^p\alpha_{i-1/2}^p r^p = \sum_{p=1}^{m}\lambda^p\mathcal{W}_{i-1/2}^p$$

so, keeping careful track of where the left-going and right-going waves come from, we have

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right]$$

in analogy with

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

Flux-difference splitting

For the linear system, $\mathcal{W}^p_{i-1/2} = \alpha^p_{i-1/2} r^p$ and since $A^{\pm} = R\Lambda^{\pm}R^{-1}$

then $A^{\pm}\alpha^p_{i-1/2}r^p = (\lambda^p)^{\pm}\alpha^p_{i-1/2}r^p$. From this we get

$$\mathcal{A}^{\pm}\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^{\pm}\mathcal{W}^p_{i-1/2} = A^{\pm}(Q^n_i - Q^n_{i-1})$$

and then the update is

$$Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left(A^{+}(Q^n_i - Q^n_{i-1}) + A^{-}(Q^n_{i+1} - Q^n_i)\right)$$

or, written in terms of the flux, $Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$

with $F^n_{i-1/2} = A^{+}Q_{i-1} + A^{-}Q_i$

Flux-difference splitting

For the more general conservation law, $q_t + f(q)_x = 0$ we define

$$F^n_{i-1/2} = f(Q_{i-1}) + \sum_{p=1}^{m}(\lambda^p)^{-}\mathcal{W}^p_{i-1/2} \equiv f(Q_{i-1}) + \mathcal{A}^{-}\Delta Q_{i-1/2}$$

$$F^n_{i-1/2} = f(Q_i) - \sum_{p=1}^{m}(\lambda^p)^{+}\mathcal{W}^p_{i-1/2} \equiv f(Q_i) - \mathcal{A}^{+}\Delta Q_{i-1/2}$$

These two are equivalent, the same flux through the same cell border, representing either a left-going flux that updates Q_{i-1}or a right-going fluctuation that updates Q_i .

If we subtract one from the other, we have

$$f(Q_i) - f(Q_{i-1}) = \mathcal{A}^{-}\Delta Q_{i-1/2} + \mathcal{A}^{+}\Delta Q_{i-1/2}$$

directly showing the difference in fluxes split into right- and left-going fluctuations.

Generalisation to nonlinear problems

For the nonlinear Riemann problem, the solution is still a *similarity solution*:

$$q(x,t) = q^{*}(x/t)$$

A system of m equations consists of m_W waves propagating at constant speed.

Often $m_w = m$ but not always.

Some waves may be *rarefaction waves* instead of discontinuities (as in the shock tube problem).

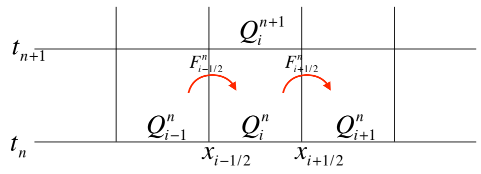
The numerical method is based on an *approximate* Riemann solution with the decomposition

$$Q_i - Q_{i-1} = \sum_{p=1}^{m}\mathcal{W}^p_{i-1/2}$$

where $\mathcal{W}^p_{i-1/2}$ is a wave propagating at some speed $s^p_{i-1/2}$.

We'll get much more of this later ...

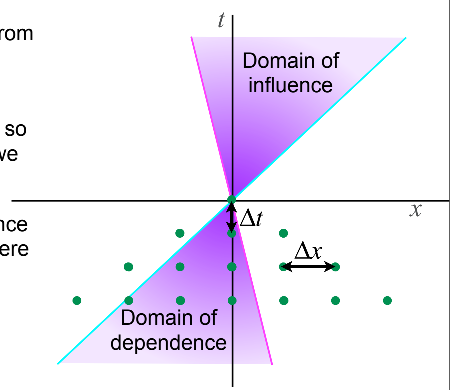
Review of Finite Volume Methods



We need a scheme for obtaining the fluxes from one cell to the next in terms of the available solution values at the present time step.

The conservation law tells us how to do this, so we must ensure that the difference formula we produce is in conservation form.

In hyperbolic problems, the domain of influence is limited; we use this limitation to decide where to take information from.



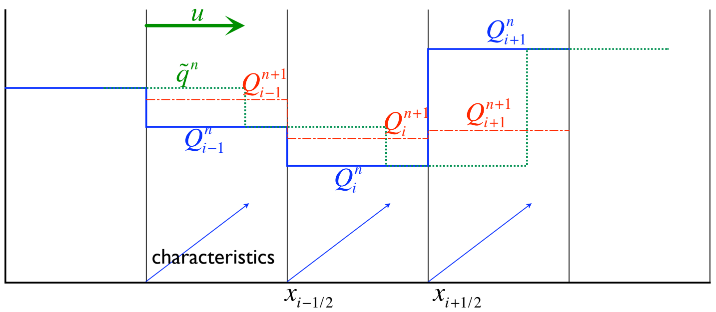
Review of the upwind method

Reconstruct - Evolve - Average

$q^n(x,t_n) = Q_i^n$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

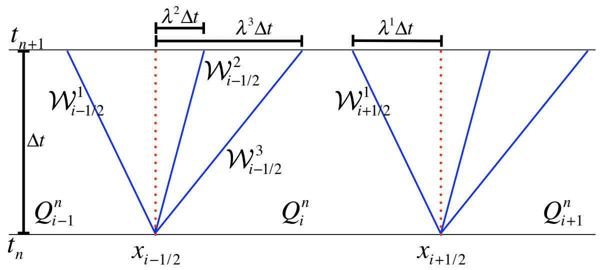
$$Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



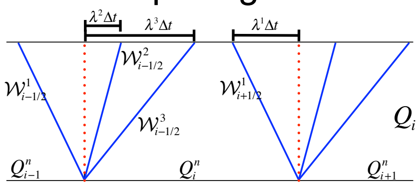
Review: The wave propagation implementation of Godunov's method

A three-equation system has three characteristics. At timestep n, there is a discontinuity at the cell edge between Q_i^n and Q_{i+1}^n. As we evolve the Riemann solution forward to form $\tilde{q}^n(x,t_{n+1})$, this discontinuity splits into three pieces.

We use our knowledge of the splitting to compute the new cell averages.



Review: splitting the discontinuity



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha_{i-1/2}^p r^p \equiv \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p$$

The wave $\mathcal{W}_{i-1/2}^2$ changes the cell average by $-\dfrac{\lambda^2\Delta t}{\Delta x}\mathcal{W}_{i-1/2}^2$. The three waves together give us:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\lambda^2\mathcal{W}_{i-1/2}^2 + \lambda^3\mathcal{W}_{i-1/2}^3 + \lambda^1\mathcal{W}_{i+1/2}^1\right)$$

Defining $\lambda^+ = \max(\lambda,0), \quad \lambda^- = \min(\lambda,0)$, we generalise to m waves:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i+1/2}^p\right]$$

Review: Fluctuations

If $\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p$ is the effect of all right-going waves, and

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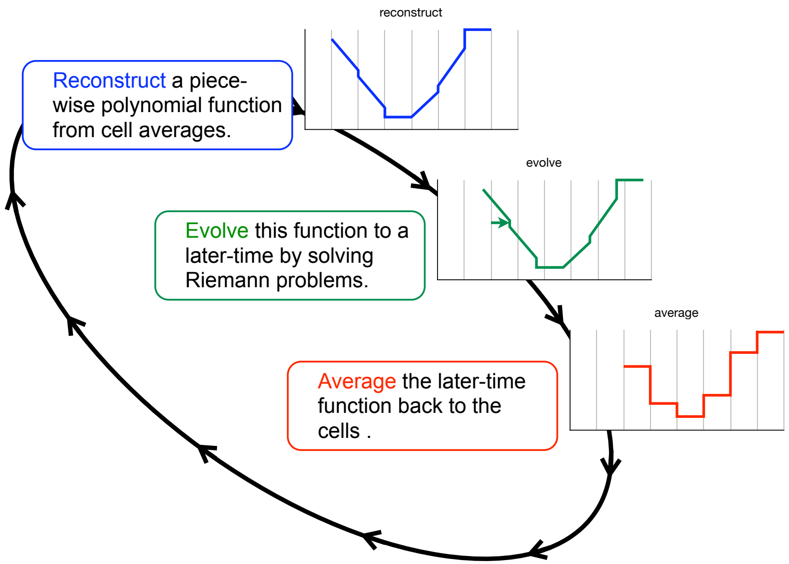
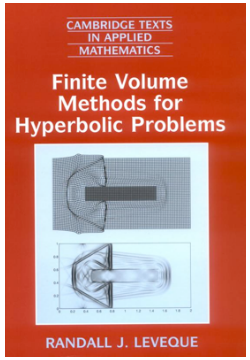
We take the right-going waves from the left interface and the left-going waves from the right interface.

The symbols $\mathcal{A}^{\pm}\Delta Q_{i\pm1/2}$ are the *fluctuations*.



Next we examine high-resolution implementations of the Godunov REA method.

Assignment for next time



Read all of Chapter 4.

Work problems 4.1 and 4.2. Hand them in to me by Tuesday 8 September.

Read all of Chapter 5. Note that there are some differences between the Clawpack 4.4 that you have downloaded and the version described in the book. The bulk of the information is still good, however. The file in your downloaded package called claw43user.pdf is much more complete, and you should start to become familiar with it.

Work problems 5.1, 5.2, and 5.3 using Clawpack. These give you some experience in modifying the data and the code. Take notes on your results (nothing to hand in) and be prepared to discuss them in class on the 8th.

Next: High Resolution Methods (Ch 6)