FYS-GEO 4500 8 Sep 2009

Before we start:
Questions over the reading?

The problem set

Our syllabus - still subject to change

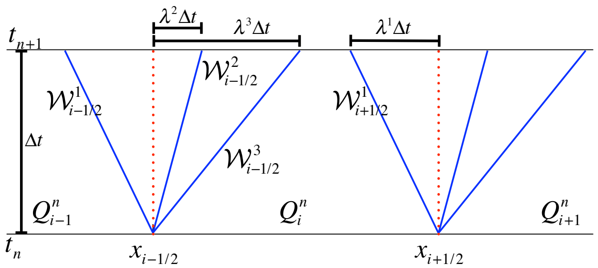
	date		Topic	Chapter in LeVeque
1	17 Aug 2009	Monday 13.15-15.00	introduction to conservation laws, Clawpack	1 & 2 & 5
2	24 Aug 2009	Monday 13.15-15.00	the Riemann problem, characteristics	3
3	**28 Aug 2009**	**Friday 13.15-15.00**	finite volume methods for linear systems	4
4	**8 Sep 2009**	**Tuesday 13.15-15.00**	high resolution methods	6
5	21 Sep 2009	Monday 13.15-15.00	boundary conditions, accuracy, variable coeff.	7,8, **part of 9**
6	**25 Sep 2009**	**Friday 13.15-15.00**	nonlinear conservation laws, traffic flow	11
7	**29 Sep 2009**	**Tuesday 13.15-15.00**	finite volume methods for nonlinear equations	12
8	5 Oct 2009	Monday 13.15-15.00	nonlinear systems, shallow-water equations	13
9	12 Oct 2009	Monday 13.15-15.00	gas dynamics, Euler equation	14
10	19 Oct 2009	Monday 13.15-15.00	finite volume methods for nonlinear systems	15
11	26 Oct 2009	Monday 13.15-15.00	multidimensional hyperbolic problems & methods	18 & 19
12	2 Nov 2009	Monday 13.15-15.00	multidimensional scalar equations & systems	20 & 21
13	**6 Nov 2009**	**Friday 13.15-15.00**	applications: tsunamis, pockmarks, venting, impacts	
14	16 Nov 2009	Monday 13.15-15.00	applications: volcanic jets, pyroclastic flows, lahars	
15	23 Nov 2009	Monday 13.15-15.00	review	
16	30 Nov 2009	Monday 13.15-15.00	discuss progress and problems on projects	
17	7 Dec 2009	Monday 13.15-15.00	FINAL PROJECT REPORTS DUE	

Any problems with the schedule?

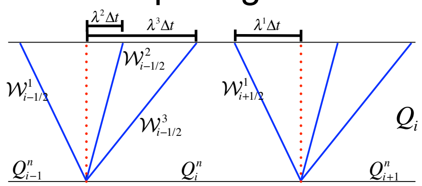
Review: The wave propagation implementation of Godunov's method

A three-equation system has three characteristics. At timestep n, there is a discontinuity at the cell edge between Q_i^n and Q_{i+1}^n. As we evolve the Riemann solution forward to form $\tilde{q}^n(x,t_{n+1})$, this discontinuity splits into three pieces.

We use our knowledge of the splitting to compute the new cell averages.



Review: splitting the discontinuity



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

The wave $\mathcal{W}^2_{i-1/2}$ changes the cell average by $-\frac{\lambda^2\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2}$. The three waves together give us:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\lambda^2\mathcal{W}^2_{i-1/2} + \lambda^3\mathcal{W}^3_{i-1/2} + \lambda^1\mathcal{W}^1_{i+1/2}\right)$$

Defining $\lambda^+ = \max(\lambda,0),\quad \lambda^- = \min(\lambda,0)$, we generalise to m waves:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p_{i-1/2} + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p_{i+1/2}\right]$$

Review: Fluctuations

If $\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}^p$ is the effect of all right-going waves, and

$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i-1/2}^p$ is the effect of all left-going waves from $x_{i-1/2}$,

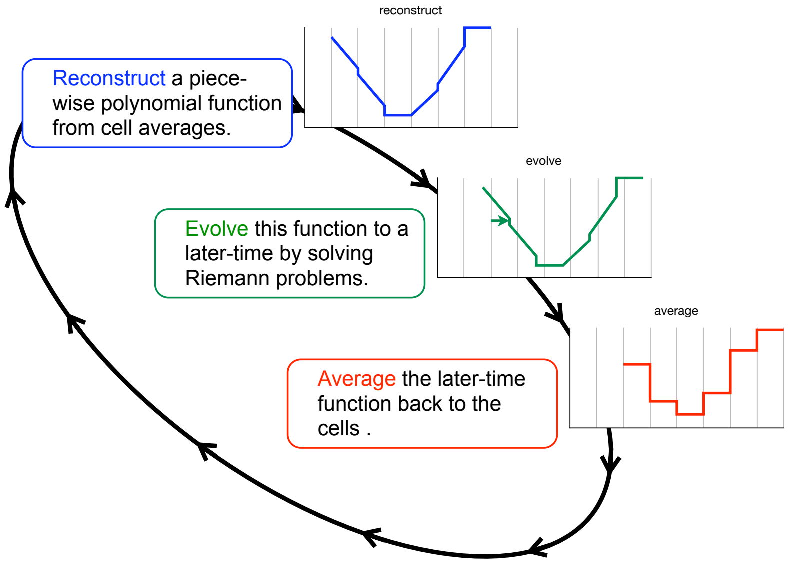
then we can write the update as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

We take the right-going waves from the left interface and the left-going waves from the right interface.

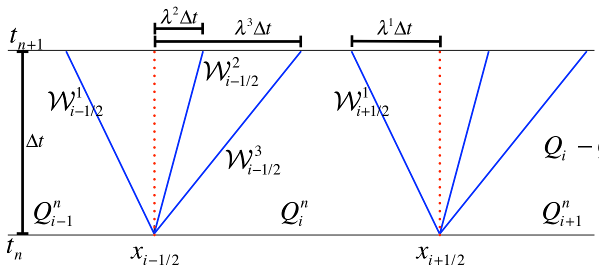
The symbols $\mathcal{A}^\pm\Delta Q_{i\pm1/2}$ are the *fluctuations*.

FYS-GEO4500
High Resolution Methods
(Chapter 6 in Leveque)



Now to the high-resolution implementations of the Godunov REA method.

Extending Godunov's method to high-resolution



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha_{i-1/2}^p r^p \equiv \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p$$

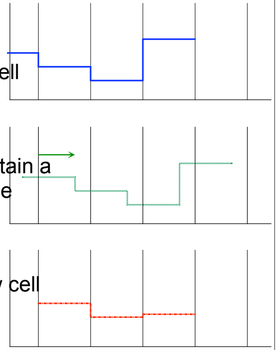
Q_i^n defines a piece-wise *constant* function. The discontinuities at the cell interfaces give rise to Riemann problems

$$F_{i-1/2}^n = f\left(q^{\downarrow}(Q_{i-1}^n, Q_i^n)\right),$$

and the solution at the next time step is obtained from

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right).$$

Godunov's method with piece-wise constant reconstruction is only first order



1. Reconstruct a piece-wise *constant* function from the cell averages.

$$q^n(x,t_n) = Q_i^n \text{ for } x \text{ in cell } i$$

2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces.

$$\tilde{q}^n(x,t_{n+1})$$

3. Average this function over each grid cell to obtain new cell averages.

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

This is done at each time step. The method can be improved by using other interpolation functions, polynomials for example, to improve the accuracy.

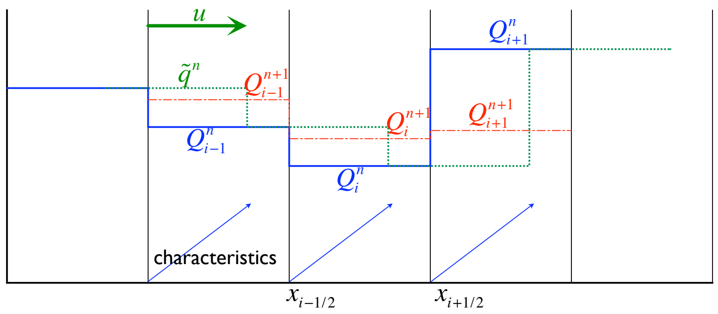
Physics is needed in the second step (evolution stage), as all the characteristics must be known and used in the solution. The first and third steps (projection stages) are entirely numerical (and problem independent).

Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

$$Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



Improvements on the simple system

Recall the update formula developed in chapter 4 that uses the notion of fluctuations:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

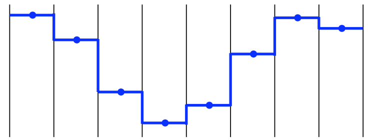
This gives a method that is only first-order accurate. We can improve it by introducing corrections, and writing:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}\right)$$

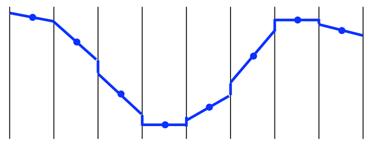
There are several possible techniques, and we illustrate a few here and show how well or how poorly they do.

Piece-wise Linear Reconstruction

Instead of using piece-wise constant reconstruction as in the simple REA update,



We could use a piece-wise linear reconstruction:



We can choose how to do this, subject to the constraint that the cell averages are conserved, and that the slopes somehow reflect the local function behaviour. This is how second-order and high-resolution methods are done.

Second-order methods:

Start with the linear system $q_t + Aq_x = 0$

Write the Taylor series expansion about the present time for the solution q at the advanced time:

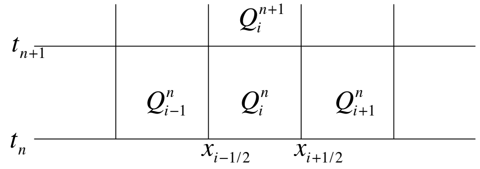
$$q(x,t_{n+1}) = q(x,t_n) + \Delta t q_t(x,t_n) + \frac{1}{2}(\Delta t)^2 q_{tt}(x,t_n) + \ldots$$

The differential equation gives us $q_t = -Aq_x$ and therefore $q_{tt} = A^2 q_{xx}$

so that:

$$q(x,t_{n+1}) = q(x,t_n) - \Delta t A q_x(x,t_n) + \frac{1}{2}(\Delta t)^2 A^2 q_{xx}(x,t_n) + \ldots$$

Lax-Wendroff:



From the first three terms of the Taylor expansion

$$q(x,t_{n+1}) \approx q(x,t_n) - \Delta t A q_x(x,t_n) + \frac{1}{2}(\Delta t)^2 A^2 q_{xx}(x,t_n)$$

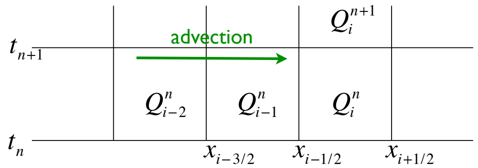
using centred differences:

$$\begin{cases} q_x(x,t_n) \approx \dfrac{1}{2\Delta x}\left(Q_{i+1}^n - Q_{i-1}^n\right) \\ q_{xx}(x,t_n) \approx \left(\dfrac{1}{\Delta x}\right)^2 \left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right) \end{cases}$$

we come to the Lax-Wendroff (1960) formula:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2 \left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

Beam-Warming:



From the first three terms of the Taylor expansion

$$q(x,t_{n+1}) \approx q(x,t_n) - \Delta t A q_x(x,t_n) + \frac{1}{2}(\Delta t)^2 A^2 q_{xx}(x,t_n)$$

Using upwind differences:

$$\begin{cases} q_x(x,t_n) \approx \dfrac{1}{2\Delta x}\left(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n\right) \\ q_{xx}(x,t_n) \approx \left(\dfrac{1}{\Delta x}\right)^2 \left(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n\right) \end{cases}$$

leads to the Beam-Warming (1976) formula for one-sided flows:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2 \left(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n\right)$$

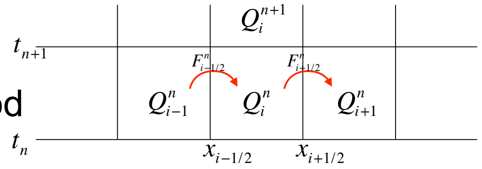
Demonstration of simple methods

?

Clawpack comparing simple methods to high-resolution methods

in $CLAW/book/myConversions/chap6/compareadv

Lax-Wendroff as a finite-volume method



The basic finite-volume update formula is $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

We can put Lax-Wendroff in this form if we write:

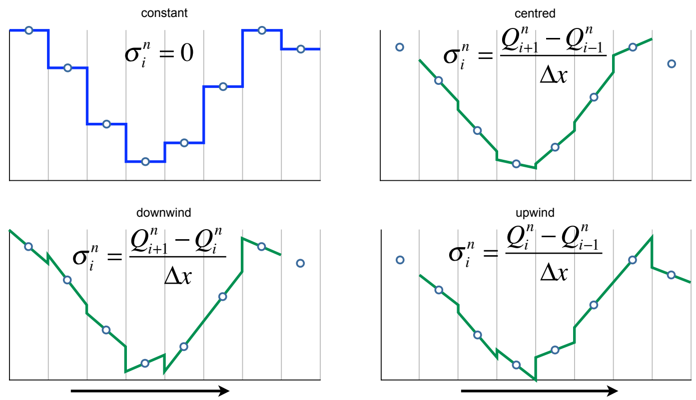
$$F_{i+1/2}^n = \frac{1}{2}A\left(Q_{i+1}^n + Q_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2\left(Q_{i+1}^n - Q_i^n\right)$$

$$F_{i-1/2}^n = \frac{1}{2}A\left(Q_i^n + Q_{i-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2\left(Q_i^n - Q_{i-1}^n\right)$$

then:

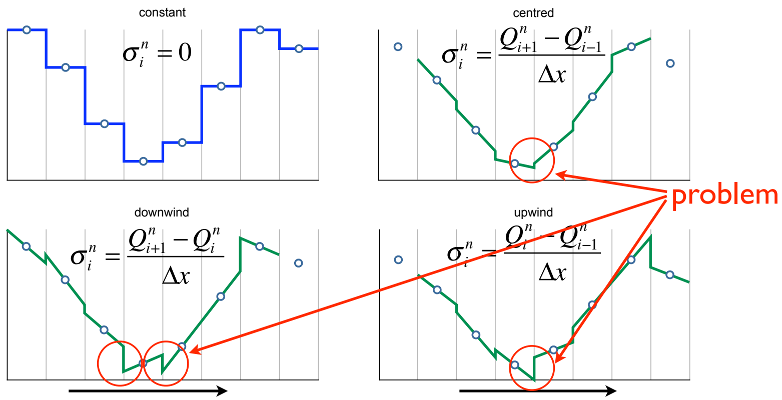
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}A\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

We can choose a variety of slopes for a piecewise linear reconstruction



The aim is to approximate the derivative over the i^{th} cell, for second-order accuracy. The overshoots in these methods cause oscillatory behaviour near discontinuities.

We can choose a variety of slopes for a piecewise linear reconstruction



The aim is to approximate the derivative over the i^{th} cell, for second-order accuracy. The overshoots in these methods cause oscillatory behaviour near discontinuities.

Need for limiters

Second-order methods give good results when the solutions are smooth but generate oscillations where discontinuities occur.

First-order methods give poorer results, but do not generate oscillations near discontinuities. That is, they keep the solution varying *monotonically*.

The idea behind high-resolution methods is to get second-order accuracy when possible, but to keep the solution *monotonic* where the solution is not smooth.

Limiters are introduced to manage this.

The breakthrough work in this area was made by Bram van Leer in a series of papers culminating in 1979.

First we rewrite the Lax-Wendroff flux

$$F^n_{i-1/2} = \frac{1}{2} A\left(Q^n_i + Q^n_{i-1}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A^2\left(Q^n_i - Q^n_{i-1}\right)$$

using

$$\begin{cases} A^{\pm} = R\Lambda^{\pm}R^{-1} \\ A = A^{+} + A^{-} \\ |A| = A^{+} - A^{-} \end{cases} \qquad \Lambda^{\pm} = \begin{bmatrix} (\lambda^1)^{\pm} & & & \\ & (\lambda^2)^{\pm} & & \\ & & \ddots & \\ & & & (\lambda^m)^{\pm} \end{bmatrix}$$

we get

$$F^n_{i-1/2} = \left(A^{-}Q^n_i + A^{+}Q^n_{i-1}\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q^n_i - Q^n_{i-1}\right)$$

which is like the upwind flux with an added (antidiffusive) correction term. We can improve this by limiting the amount of correction actually applied, based on the solution behaviour.

We illustrate how this is done with the scalar advection equation.

How do we choose a slope limiter?

We want to use the slope when the function is smooth to achieve second-order accuracy.

But when the function is not smooth, using the slope results in overshoots, causing oscillatory behaviour.

So we *limit the slope*, based on the local behaviour of the solution.

We write the slope as $\sigma^n_i = \left(\frac{Q^n_{i+1} - Q^n_i}{\Delta x}\right)\phi^n_i$, where ϕ is the *flux limiter function*, and

$\phi = 1$ in the Lax-Wendroff scheme,

$\phi = 0$ in the piecewise-constant upwind scheme.

The REA algorithm suggests ...

that we update the advection equation by

$$Q^{n+1}_i = Q^n_i - \frac{u\Delta t}{\Delta x}\left(Q^n_i - Q^n_{i-1}\right) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma^n_i - \sigma^n_{i-1}\right)$$

where the slope is given by

$$\sigma^n_i = \left(\frac{Q^n_{i+1} - Q^n_i}{\Delta x}\right)\phi^n_i$$

and ϕ is the *flux limiter function*.

How do we choose a slope limiter?

Monotonicity preserving methods:

If a grid function that is initially monotone, i.e. $Q^n_i \geq Q^n_{i-1}$ for all i at step n

remains monotone at the next time: $Q^{n+1}_i \geq Q^{n+1}_{i-1}$ for all i at step $n+1$

then the method is monotonicity preserving.

Total Variation Diminishing (TVD) methods:

Define the total variation of a grid function Q as: $\mathrm{TV}(Q) = \sum_{\text{grid}} |Q_i - Q_{i-1}|$

A method is Total Variation Diminishing if $\mathrm{TV}(Q^{n+1}) \leq \mathrm{TV}(Q^n)$

TVD methods are monotonicity preserving. We chose slope limiters that ensure the method is TVD.

The minmod slope limiter

Define the minmod function: $\operatorname{minmod}(a,b) \equiv \begin{cases} a \text{ if } |a|<|b| \text{ and } ab>0 \\ b \text{ if } |b|<|a| \text{ and } ab>0 \\ 0 \text{ if } ab \le 0 \end{cases}$

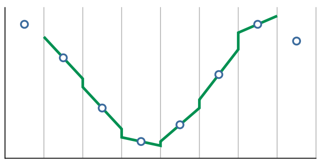
Then choose the slope to use by:

$$\sigma_i^n = \operatorname{minmod}\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}, \frac{Q_{i+1}^n - Q_i^n}{\Delta x} \right)$$

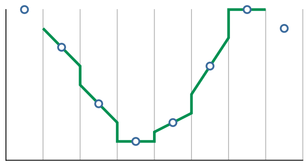
If the slopes have the same sign, the one with the smaller absolute value is chosen; if they have opposite signs, the slope is 0.

Extended to more arguments, the minmod function returns 0 unless all the arguments are the same sign, otherwise it returns the argument with the smallest absolute value.

centred (Lax-Wendroff)



minmod



For generality, we write the slope in terms of the flux-limiter function ϕ

For minmod:

$$\sigma_i^n = \operatorname{minmod}\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}, \frac{Q_{i+1}^n - Q_i^n}{\Delta x} \right)$$

$$= \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x} \right) \phi(\theta_i^n)$$

where $\phi(\theta) = \operatorname{minmod}(\theta, 1)$

and $\theta_i^n = \frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n}$

θ measures the local smoothness of the data. At extrema, θ is negative; if the data are smooth, $\theta \approx 1$ and at discontinuities, θ can be very large.

Widely used flux limiters are:

Linear methods

upwind: $\phi(\theta) = 0$

Lax-Wendroff: $\phi(\theta) = 1$

Beam-Warming: $\phi(\theta) = \theta$

Fromm: $\phi(\theta) = \frac{1}{2}(1+\theta)$

High-resolution methods

minmod: $\phi(\theta) = \operatorname{minmod}(1, \theta)$

superbee: $\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$

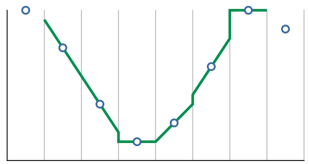
MC: $\phi(\theta) = \max(0, \min((1+\theta)/2, 2, 2\theta))$

vanLeer: $\phi(\theta) = \frac{(\theta + |\theta|)}{(1+|\theta|)}$

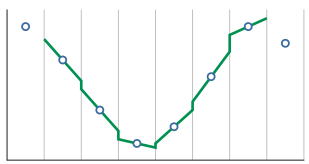
Comparing minmod, superbee, MC and vanLeer limiters

cell	data	left slope	right slope	centred slope	theta	minmod	superbee	MC	vanLeer
0.5	8		-2						
1.5	6	-2	-3	-2.5	0.666667	-2	-3	-2.5	-2.4
2.5	3	-3	-2	-2.5	1.5	-3	-3	-2.5	-2.4
3.5	1	-2	1	-0.5	-2	0	0	0	0
4.5	2	1	3	2	0.333333	1	2	2	1.5
5.5	5	3	3	3	1	3	3	3	3
6.5	8	3	-1	1	-3	3	0	0	0
7.5	7	-1							

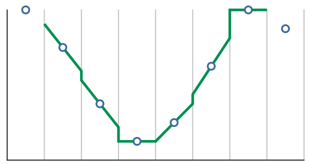
superbee



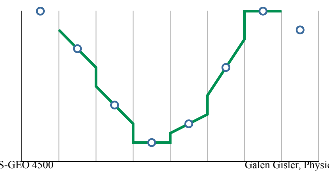
centred



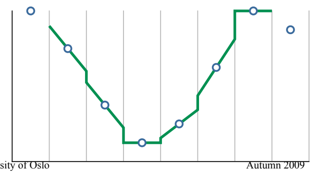
MC



minmod



vanLeer



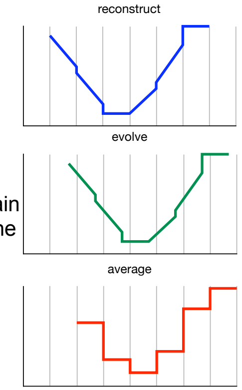
Now we have an REA second order scheme

The steps are identical to the first order REA scheme, except for reconstruction:

1. Reconstruct a piece-wise **linear** function from the cell averages.
$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$
with the property that $\mathrm{TV}(q) \le \mathrm{TV}(Q)$

2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces. $\tilde{q}^n(x,t_{n+1})$

3. Average this function over each grid cell to obtain new cell averages.
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



The reconstruction step depends on the slope limiter that is chosen, and should be subject to TVD constraints. The other two steps do not affect TVD.

Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

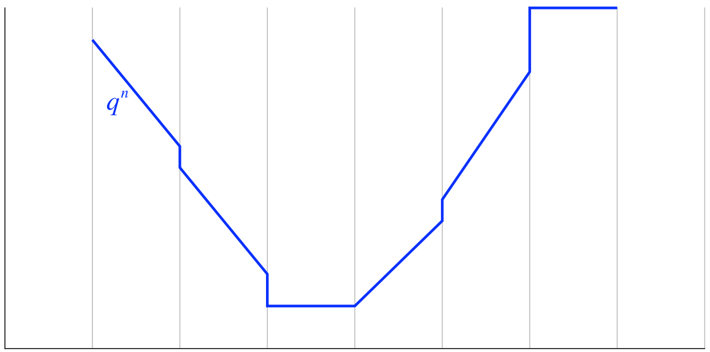
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

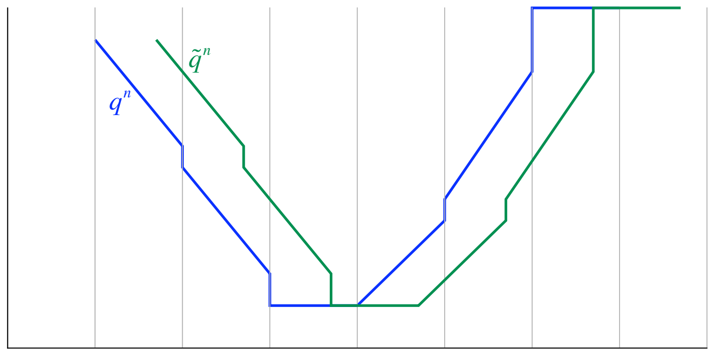


Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

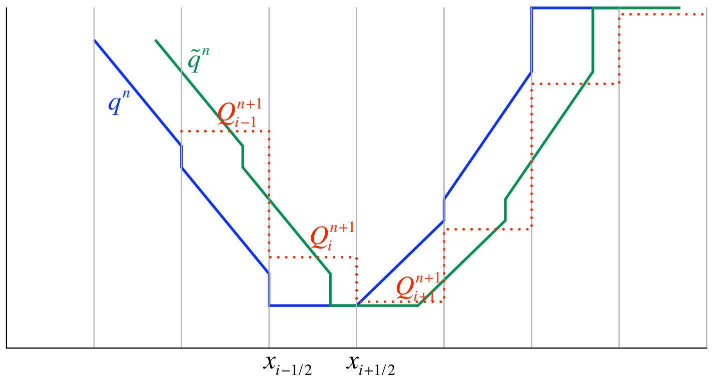


Reconstruct - Evolve - Average

$q^n(x,t_n) = Q_i^n + \sigma_i^n(x - x_i)$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

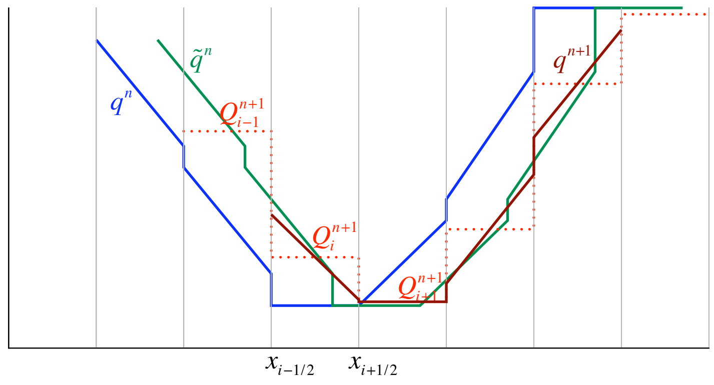


Reconstruct - Evolve - Average

$q^n(x,t_n) = Q_i^n + \sigma_i^n(x - x_i)$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

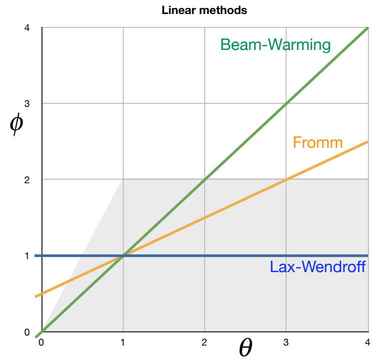
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



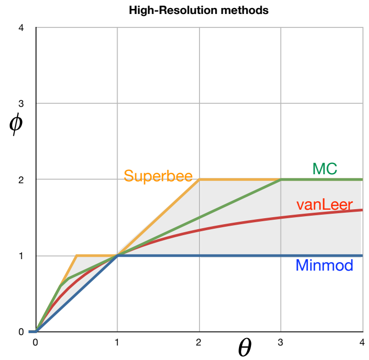
… and then a new piecewise linear reconstruction is done …

How do we make sure we satisfy the Total Variation Diminishing Constraint?

Compare the limiter functions $\phi(\theta)$ where $\theta = \dfrac{\Delta Q_{upwind}}{\Delta Q_{downwind}}$.

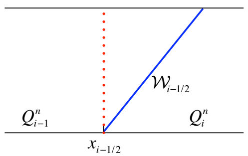


TVD is satisfied when $0 \le \phi(\theta) \le \text{minmod}(2, 2\theta)$



Sweby's region where TVD is satisfied is shaded. Any second-order accurate method must have $\phi(1) = 1$.

Wave limiters



We can think of slope limiters as limiters on the wave strengths. Let $\mathcal{W}_{i-1/2} = Q_i - Q_{i-1}$.

Then the upwind method for the scalar advection equation is $Q_i^{n+1} = Q_i^n - u\dfrac{\Delta t}{\Delta x}\mathcal{W}_{i-1/2}$.

The Lax-Wendroff method is: $Q_i^{n+1} = Q_i^n - u\dfrac{\Delta t}{\Delta x}\mathcal{W}_{i-1/2} - \dfrac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$,

where $\tilde{F}_{i-1/2} = \dfrac{1}{2}\left(1 - \left|u\dfrac{\Delta t}{\Delta x}\right|\right)|u|\mathcal{W}_{i-1/2}$.

For a high-resolution we use $\tilde{F}_{i-1/2} = \dfrac{1}{2}\left(1 - \left|u\dfrac{\Delta t}{\Delta x}\right|\right)|u|\widetilde{\mathcal{W}}_{i-1/2}$,

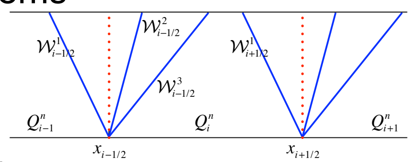
where $\widetilde{\mathcal{W}}_{i-1/2} = \phi_{i-1/2}\mathcal{W}_{i-1/2}$.

Demonstration of better methods

Clawpack comparing simple methods to high-resolution methods

in $CLAW/book/myConversions/chap6/compareadv.2

Extension to linear systems



Approach 1:

Diagonalise the system to $q_t + \Lambda q_x = 0$

Apply the scalar algorithm to each component separately.

Approach 2:

Solve the linear Riemann problem to decompose $Q^n_i - Q^n_{i-1}$ into a number of waves.

Apply a wave limiter to each wave.

These approaches are equivalent, but we'll use the wave propagation method. Note that it is important to apply the limiters to the *waves* rather than to the original variables.

High-resolution methods for systems

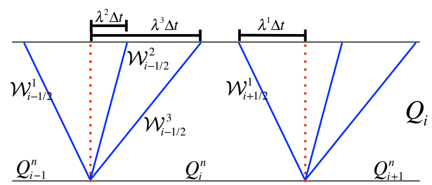
The Lax-Wendroff method in flux difference form had the flux written as:

$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q^n_i - Q^n_{i-1}\right)$$

We need to separate the eigenvectors in order to apply flux limiters, so we rewrite the correction term, using the Godunov-Riemann splititng:

$$\frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q^n_i - Q^n_{i-1}\right) = \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\sum_{p=1}^{m}\alpha^p_{i-1/2} r^p$$

Recall from before that the discontinuity between cells i and i+1 is split into m pieces by the Riemann characteristics:



$$Q_i - Q_{i-1} = \sum_{p=1}^{m}\alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m}\mathcal{W}^p_{i-1/2}$$

High-resolution methods for systems

Now we apply the limiter to the coefficients of the eigenvectors:

$$\tilde{\alpha}^p_{i-1/2} = \alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})$$

$$\theta^p_{i-1/2} = \frac{\alpha^p_{l-1/2}}{\alpha^p_{i-1/2}};\ l = \begin{cases} i-1 \text{ if } \lambda^p > 0 \\ i+1 \text{ if } \lambda^p < 0 \end{cases}$$

Then the flux function is

$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}\sum_{p=1}^{m}\left|\lambda^p\right|\left(1 - \frac{\Delta t}{\Delta x}\left|\lambda^p\right|\right)\alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})r^p.$$

If we write $\widetilde{\mathcal{W}}^p_{i-1/2} = \alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})r^p$ as a limited version of the wave strength, and $s^p_{i-1/2} = \lambda^p$ for a generalised wave speed, we have:

$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}\sum_{p=1}^{m}\left|s^p_{i-1/2}\right|\left(1 - \frac{\Delta t}{\Delta x}\left|s^p_{i-1/2}\right|\right)\widetilde{\mathcal{W}}^p_{i-1/2}$$

Generalisation for Nonlinear Systems

For linear systems, we can rearrange the update into the form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

with

$$\tilde{F}_{i-1/2}^n = \frac{1}{2}\sum_{p=1}^{m}\left|s_{i-1/2}^p\right|\left(1 - \frac{\Delta t}{\Delta x}\left|s_{i-1/2}^p\right|\right)\widetilde{\mathcal{W}}_{i-1/2}^p$$

Generalising to nonlinear systems we can write the update as:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

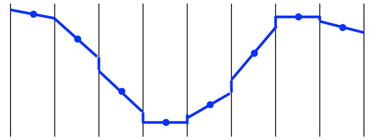
with the fluctuations suitably defined. There are some subtleties we'll get into later, associated with rarefaction waves and entropy conditions.

Review of High-Resolution Methods

We improve the first-order upwind method by introducing corrections, and writing:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}\right)$$

We derive the corrections by considering piece-wise linear (instead of piece-wise constant) reconstructions.



Review of High-Resolution Methods

Taking the basic Lax-Wendroff formula:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}A\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

we re-write it in the flux form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

with

$$F_{i-1/2}^n = \frac{1}{2}A\left(Q_i^n + Q_{i-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2\left(Q_i^n - Q_{i-1}^n\right)$$

Then making use of the divided matrices $A^{\pm}$ we can write this as

$$F_{i-1/2}^n = \left(A^-Q_i^n + A^+Q_{i-1}^n\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q_i^n - Q_{i-1}^n\right)$$

Review of High-Resolution Methods

$$F_{i-1/2}^n = \left(A^-Q_i^n + A^+Q_{i-1}^n\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q_i^n - Q_{i-1}^n\right)$$

This version of the Lax-Wendroff formula has a correction term that can be limited, if we choose, to avoid oscillations around extrema.

For a one-equation system (the advection equation), we can apply a simple functional limiter to the slope:

$$\sigma_i^n = \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\phi_i^n$$

Examples of limiters:

Lax-Wendroff: $\phi(\theta) = 1$

minmod: $\phi(\theta) = \text{minmod}(1,\theta)$

superbee: $\phi(\theta) = \max(0,\min(1,2\theta),\min(2,\theta))$

MC: $\phi(\theta) = \max(0,\min((1+\theta)/2,2,2\theta))$

vanLeer: $\phi(\theta) = \frac{(\theta + |\theta|)}{(1 + |\theta|)}$

Review of High-Resolution Methods

For a system of equations, we use limiters on the waves. The wave-propagation form for a high-resolution version of Lax-Wendroff is:

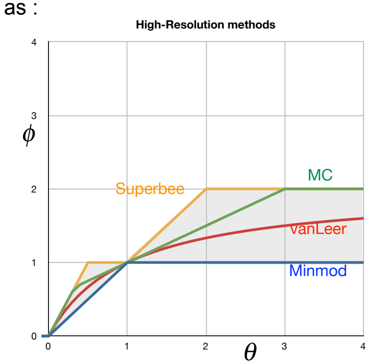
$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}\sum_{p=1}^{m} \left|s^p_{i-1/2}\right| \left(1 - \frac{\Delta t}{\Delta x}\left|s^p_{i-1/2}\right|\right) \widetilde{\mathcal{W}}^p_{i-1/2}$$

with the limited version of the waves defined as :

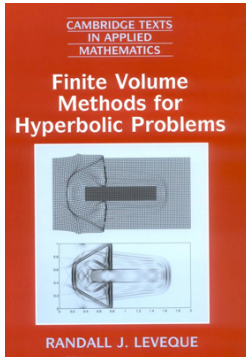
$$\widetilde{\mathcal{W}}^p_{i-1/2} = \alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})r^p$$

and a generalised wave speed

$$s^p_{i-1/2} = \lambda^p$$



Assignment for next time



Read Chapter 6 at least through 6.15.

Run the tests of different linear methods in claw/book/chap6/compareadv as described in Fig 6.1, and 6.2, and in claw/book/chap6/wavepacket in Fig 6.3. Nothing to hand in, but be prepared to discuss your experience.

Work problems 6.1, 6.5, and 6.10. Hand them in to me by Monday 21 September.

Next: Boundary Conditions, Accuracy and Variable Coefficients (Chs 7, 8, part of 9)