FYS-GEO 4500

Lecture Notes #3
Finite-Volume Methods and High Resolution

Where we are today

	date	Topic	Chapter in LeVeque
1	1.Sep 2011	introduction to conservation laws, Clawpack	1 & 2
2	15.Sep 2011	the Riemann problem, characteristics	3 & 5
3	22.Sep 2011	finite volume methods for linear systems, high resolution	4 & 6
4	29.Sep 2011	boundary conditions, accuracy, variable coeff.	7,8, part 9
5	6.Oct 2011	nonlinear conservation laws, finite volume methods	11 & 12
6	13.Oct 2011	nonlinear equations & systems	13 & 14
7	20.Oct 2011	finite volume methods for nonlinear systems	14 & 15
8	27.Oct 2011	source terms and multidimensions	16,17,18,19
9	3.Nov 2011	multidimensional systems	20 & 21
	10.Nov 2011	no lecture	
*10	17.Nov 2011	capacity functions, source terms, project plans	
11	24.Nov 2011	other topics and project plans	
12	1.Dec 2011	student presentations	
	8.Dec 2011	no lecture	
*13	15.Dec 2011	FINAL REPORTS DUE	

Review of the Riemann problem

The Riemann problem is the original system of equations, $q_t + f(q)_x = 0$ plus the special initial condition consisting of a jump discontinuity:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

In the linear hyperbolic system, we have $q_t + f'(q)q_x = 0$ and the Jacobian is diagonalised into the form

$$f'(q) = \begin{bmatrix} \frac{\partial f^1}{\partial q^1} & \cdots & \frac{\partial f^1}{\partial q^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial q^1} & \cdots & \frac{\partial f^m}{\partial q^m} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}.$$

with the eigenvalues λ^p , since the system is hyperbolic.

Review of the Riemann problem

The solution vector is resolved or projected onto the eigenvectors r^p ,

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t) r^p$$

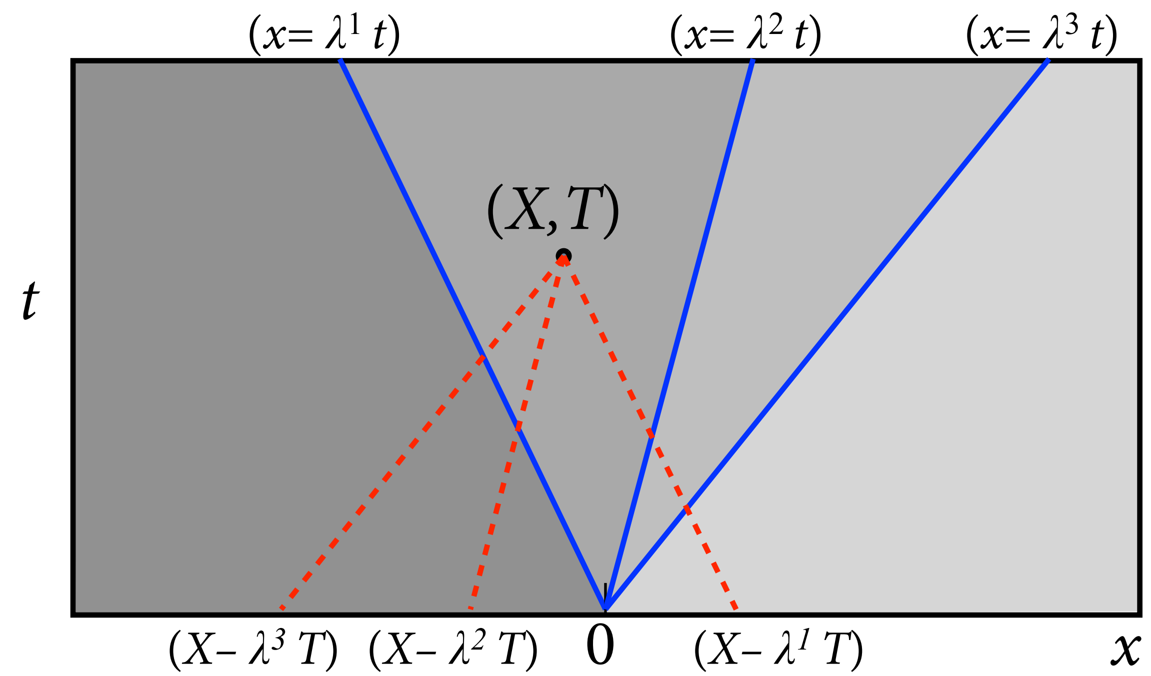
and the system is replaced by the equivalent m advection equations

$$w_t^p + \lambda^p w_x^p = 0,$$

with the solution $w^p(x,t) = w^p(x - \lambda^p t, 0)$. The initial left-right discontinuity is split among the eigenvectors

$$q_l - q_r = \sum_{p=1}^{m} \alpha^p r^p = \sum_{p=1}^{m} \left(w_l^p - w_r^p \right) r^p .$$

The solution at a later time is a mixture of these left and right states, depending on whether x is to the left or the right of the corresponding characteristic.

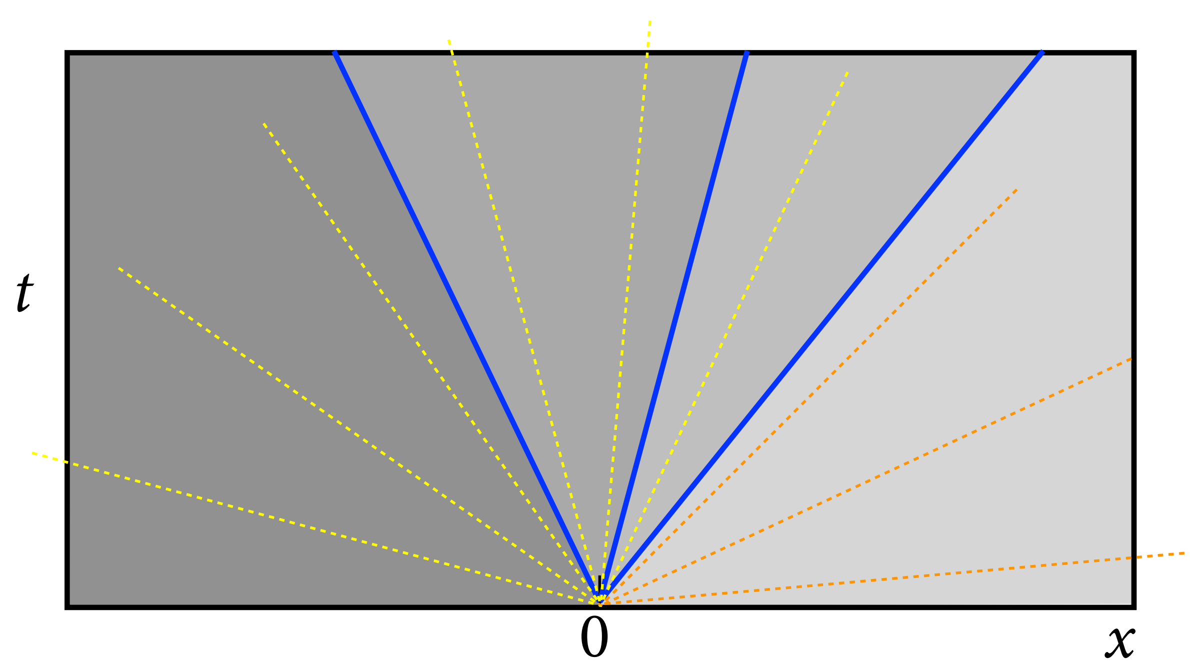


Review of the Riemann problem

If we define the waves $\mathcal{W}^p \equiv \alpha^p r^p = (w_l^p - w_r^p)$ then the solution to the Riemann problem can be written

$$q(x,t) = q_l + \sum_{p=1}^{m} H(x - \lambda^p t)\mathcal{W}^p$$

where H is the Heaviside function $H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}.$



The Riemann solution for a linear system is a *similarity solution*: it depends on x/t and not on x or t separately.

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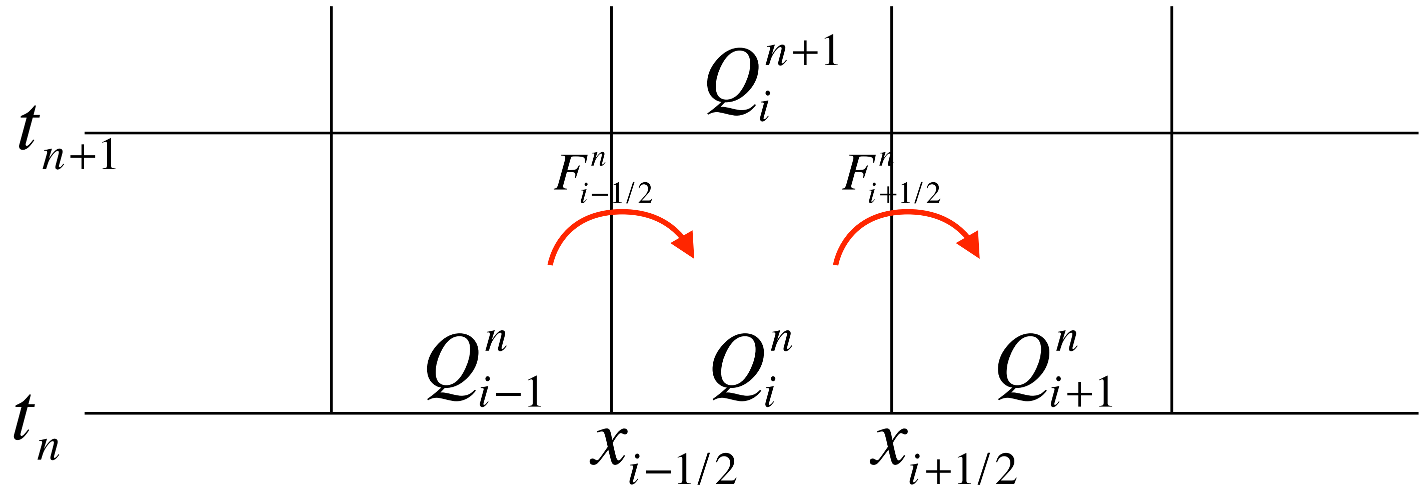
Finite Volume Methods for Linear Systems
(Chapter 4 in Leveque)

Next question: How do we get the fluxes?



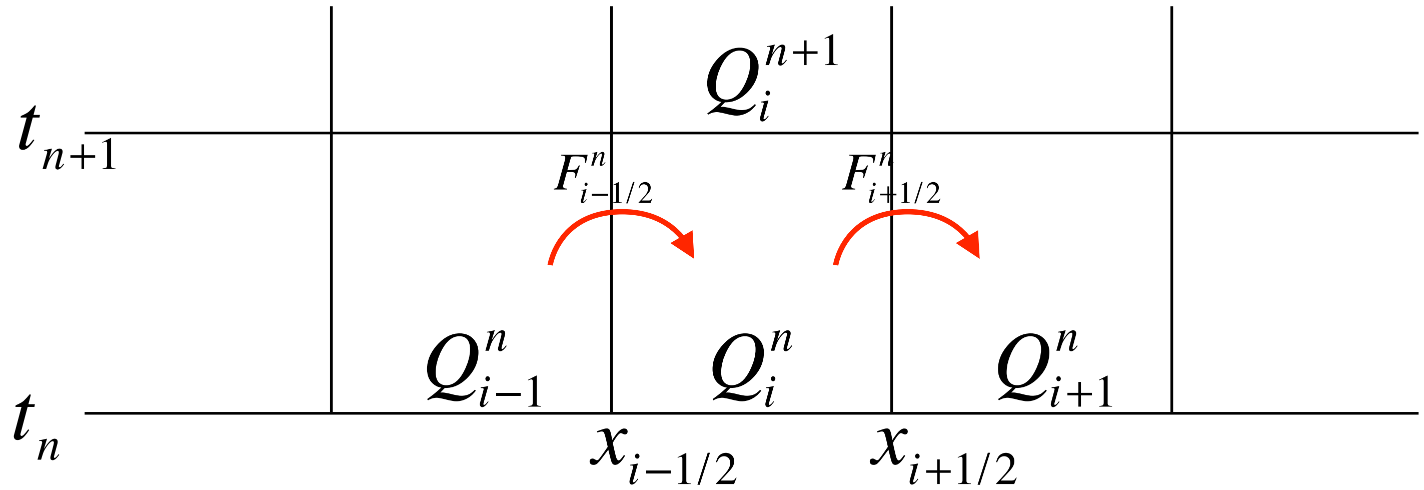
The equation we want to solve is $q_t + f(q)_x = 0$ and we think we know how to do it, from one time step to the next, by solving Riemann problems at each interface.

If it's a linear system we can write $f(q)_x = f'(q)q_x$ and resolve the (constant) Jacobian into its eigenvalues and eigenvectors. But we still need a way to determine the appropriate *numerical* flux that we will use to advance the numerical solution from one time step to the next, using something like:

$$Q_i^{n+1} \approx Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

We'll put aside the Riemann problem for the moment, we'll need it in an hour or so.

Explicit *versus* Implicit

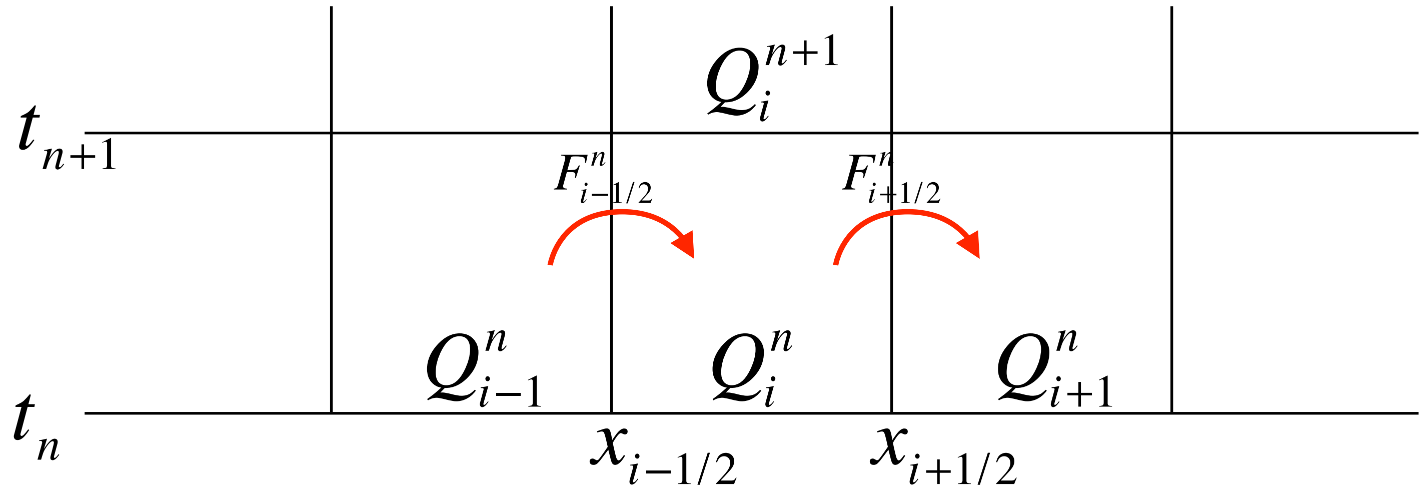


For hyperbolic equations, the domain of dependence is *bounded*, since information propagates with a finite speed.

We can therefore use *explicit* methods, in which the state at the later time is calculated in terms of the state at the present time.

For elliptic and parabolic equations, *implicit* methods, solving an equation involving both the later and present times, are required.

General formulation for conservation laws



In finite volume methods, we divide the problem domain (here one-dimensional) into a grid of *cells*, and form an approximation of the solution value within each cell:

$$Q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)dx, \text{ where } \Delta x = x_{i+1/2} - x_{i-1/2}$$

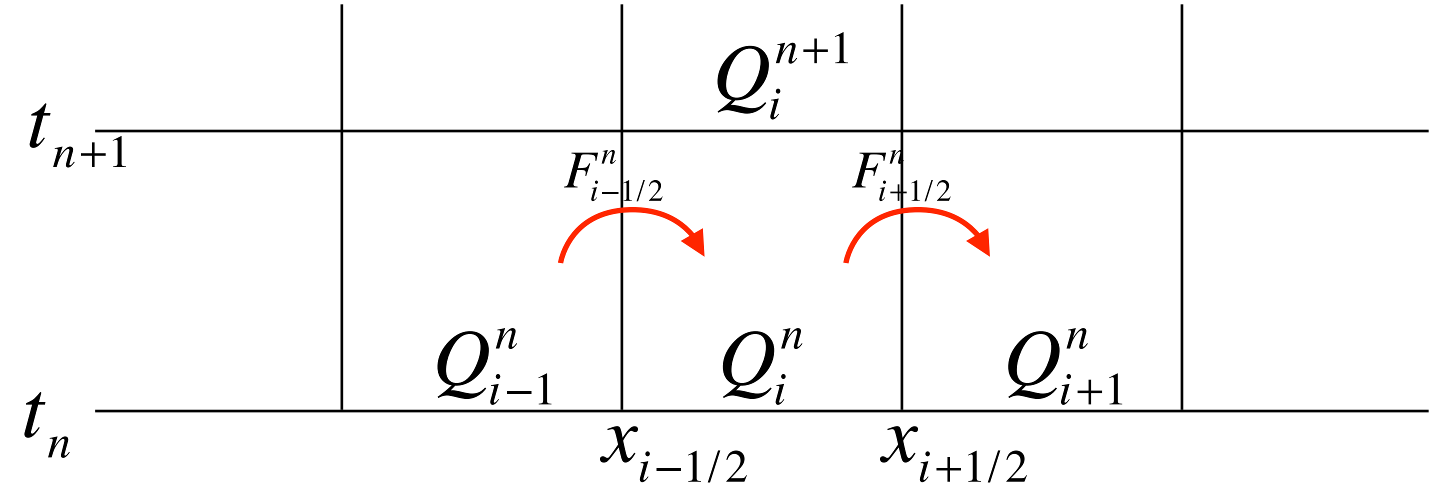
The integral form of the conservation law is

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

Then by integrating over time, we get

$$Q_i^{n+1} \approx Q_i^n - \frac{1}{\Delta x}\left(\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt - \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt\right)$$

Getting the fluxes



If we can find a way to formulate $F^n_{i+1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt$ in terms of the Q_i^n, then we can write:

$$Q_i^{n+1} \approx Q_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$$

This scheme is in conservation form. The fluxes cancel except at the boundaries:

$$\Delta x \sum_{i=1}^{N} Q_i^{n+1} = \Delta x \sum_{i=1}^{N} Q_i^n - \Delta t\left(F^n_{N+1/2} - F^n_1\right)$$

In hyperbolic equations, information propagates at finite speed, so we should formulate the $F^n_{i+1/2}$ from the values Q_i^n, Q_{i+1}^n in neighbouring cells. Then the future Q_i^{n+1} will depend on the three values Q^n_{i-1}, Q^n_i, and Q^n_{i+1}. This is known as a three-point stencil.

Convergence: consistency and stability

The key to finite volume methods is how to approximate the time-integral of the flux from the present time to the future time.

$$F_{i+1/2}^{n} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt$$

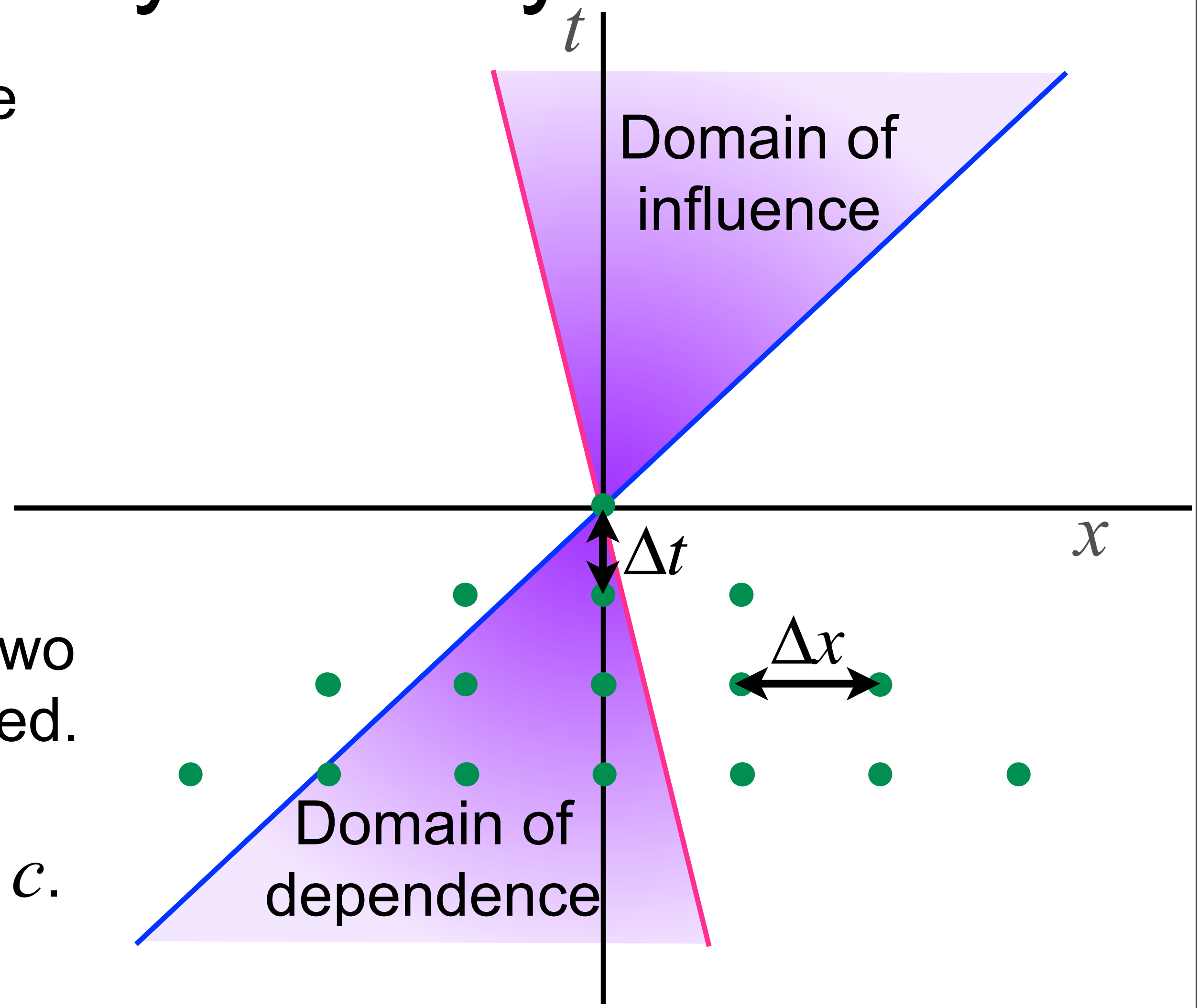
Everything depends now on how we formulate the flux function $F_{i+1/2}^{n}$, so we need to define criteria for judging the choice.

The method must be *convergent*, i.e. the numerical solution must approach the true solution as the cell size and time step decrease ($\Delta x, \Delta t \rightarrow 0$).

The method must be *consistent* with the system of equations.

The method must be *stable*, so that small errors don't grow rapidly.

The Courant-Friedrich-Lewy stability limit



The numerical stencil must contain the true domain of influence. This is a *necessary* condition for stability.

Since influence is propagated by the characteristic waves, the true domain of influence depends on the wave speeds.

For a symmetric wave equation there are two waves, but only a single characteristic speed. For acoustics in a stationary medium, the characteristic speed is the speed of sound c.

We define the CFL number $\nu \equiv c\dfrac{\Delta t}{\Delta x}$ and require that $\nu \leq 1$ for stability.

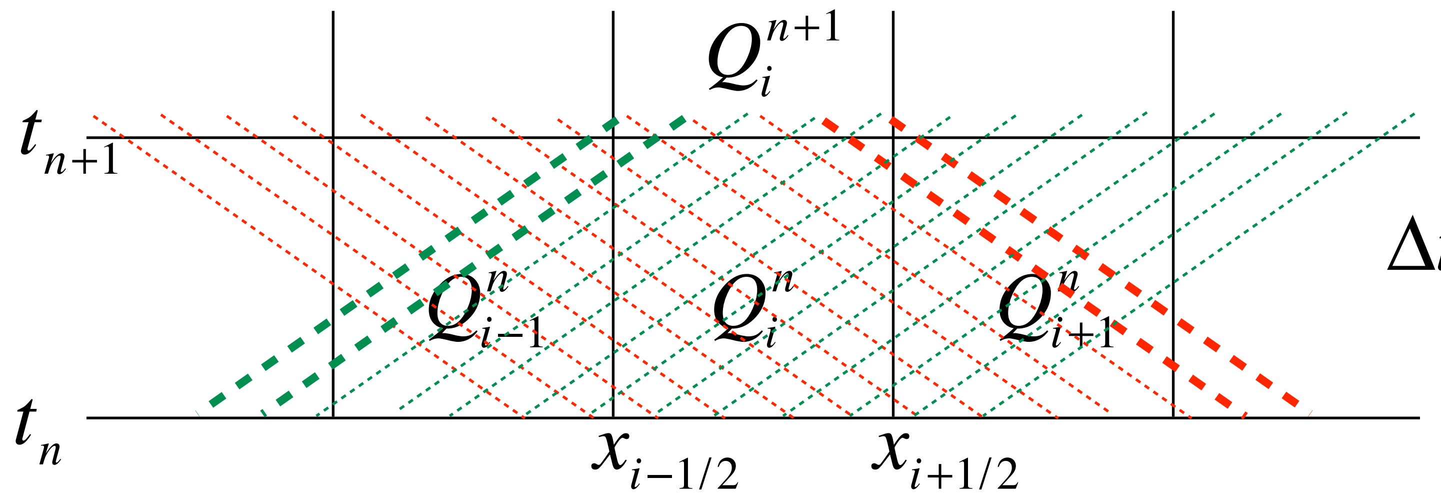
For a hyperbolic system of equations, we can have up to m different wave speeds given by $\lambda^1, \lambda^2, \ldots, \lambda^p$, so the Courant number must be

$$\nu \equiv \frac{\Delta t}{\Delta x} \max\left|\lambda^p\right| \leq 1$$

The Courant-Friedrich-Lewy stability limit

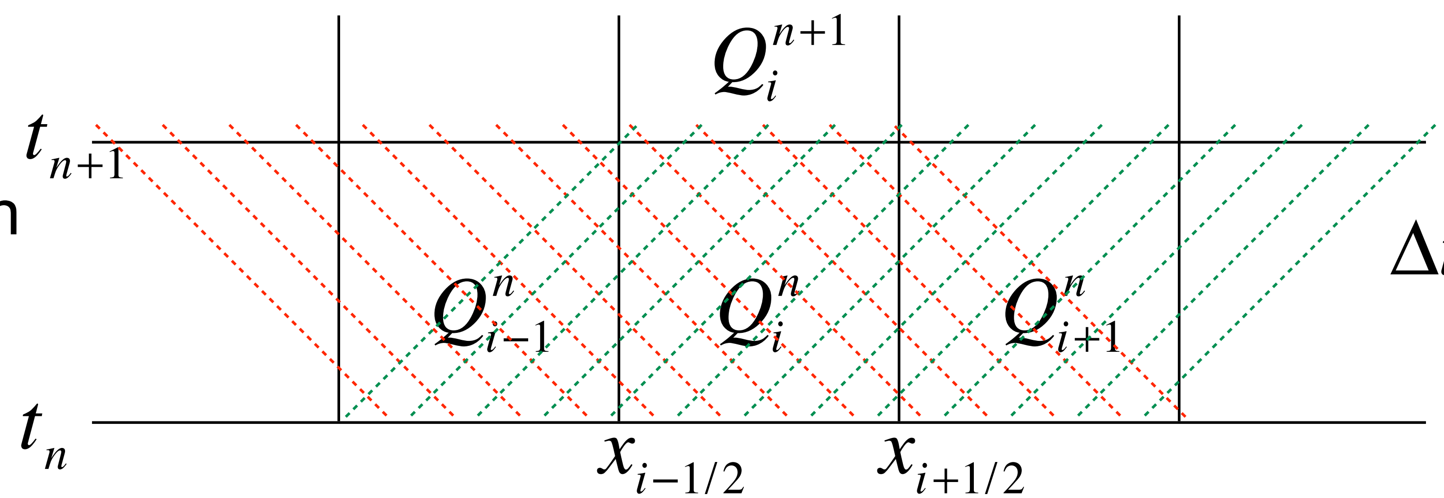
The domain of influence for the symmetric wave equation, wave speed c, three-point stencil.

Unstable, because the domain of influence is larger than the numerical stencil.



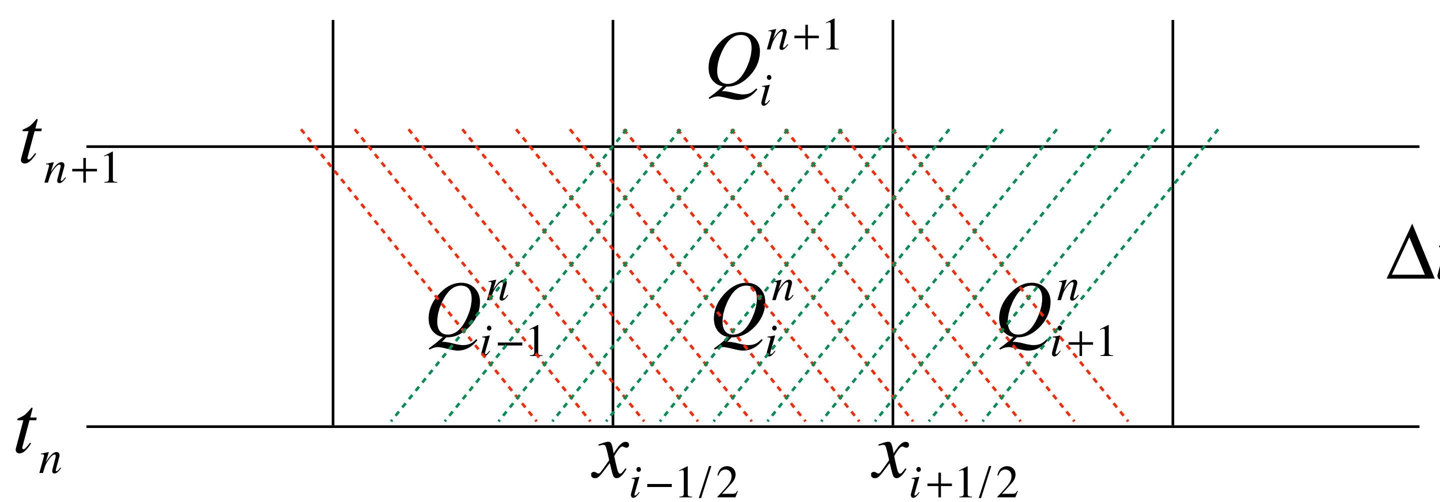
$$\Delta t > \frac{\Delta x}{c}$$

At the limit of stability the domain of influence corresponds exactly to the numerical stencil.



$$\Delta t = \frac{\Delta x}{c}$$

Within the limit of stability the stencil completely contains the domain of influence.



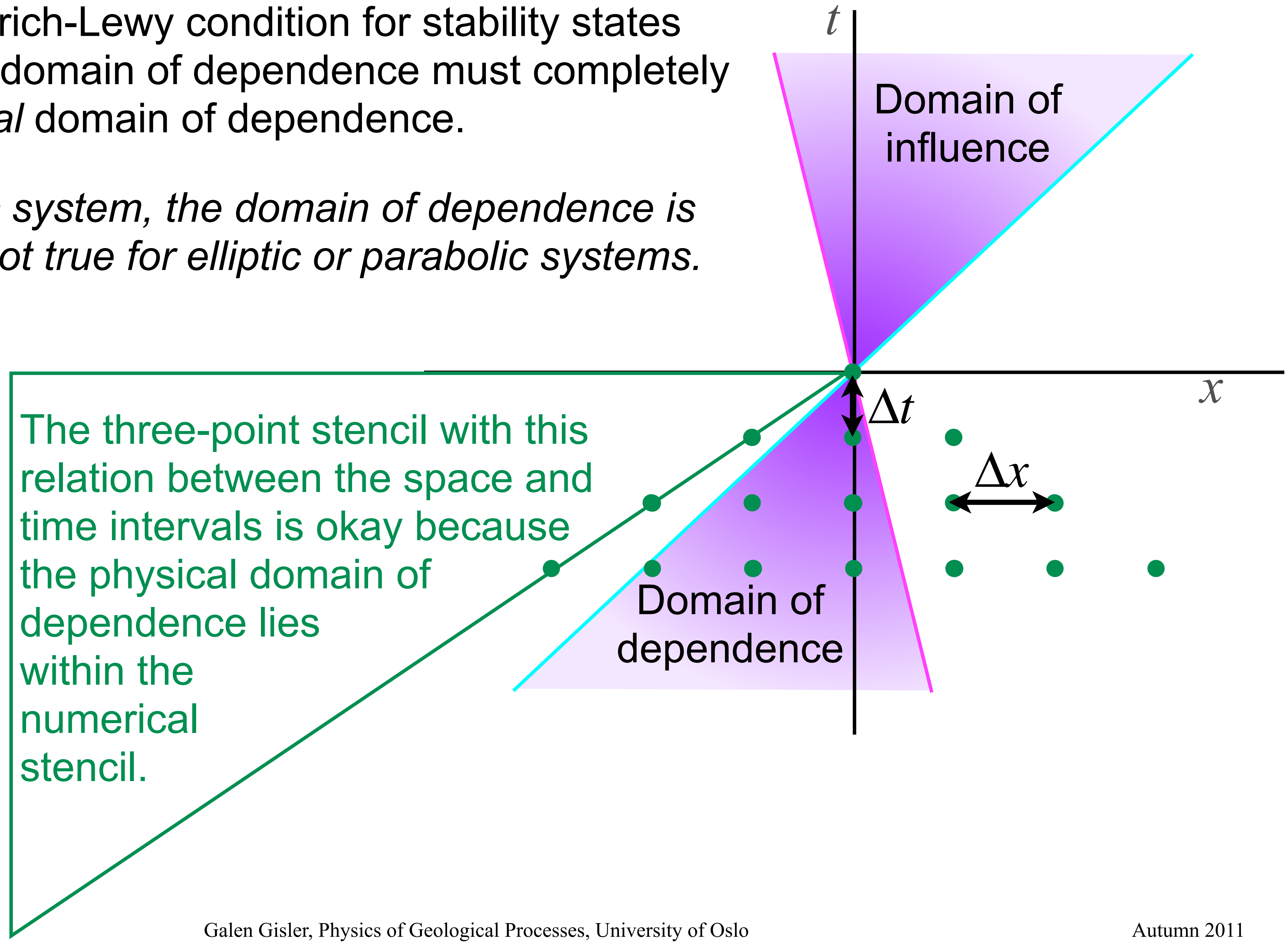
$$\Delta t < \frac{\Delta x}{c}$$

Causal domains in space-time

The domains of influence and dependence depend on the characteristics of the equations.

The Courant-Friedrich-Lewy condition for stability states that the *numerical* domain of dependence must completely contain the *physical* domain of dependence.

For any hyperbolic system, the domain of dependence is bounded. This is not true for elliptic or parabolic systems.



Formulation of the flux function and update rule

$$F_{i+1/2}^{n} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt \qquad Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

Here are a few historical choices for **centred** methods:

Formulation of the flux function and update rule

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Here are a few historical choices for **centred** methods:

$$F_{i-1/2}^n = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right]$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\left[f(Q_{i+1}^n) - f(Q_{i-1}^n)\right]$$

Naive method; unstable

Formulation of the flux function and update rule

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$$F^n_{i-1/2} = \frac{1}{2}\left[f(Q^n_{i-1}) + f(Q^n_i)\right] - \frac{\Delta x}{2\Delta t}\left(Q^n_i - Q^n_{i-1}\right)$$
$$Q_i^{n+1} = \frac{1}{2}\left(Q^n_{i-1} + Q^n_{i+1}\right) - \frac{\Delta t}{2\Delta x}\left[f(Q^n_{i+1}) - f(Q^n_{i-1})\right]$$

Lax-Friedrichs method; stable, but diffusive; first-order accurate

Formulation of the flux function and update rule

$$F_{i+1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt \qquad Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

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Naive method;
unstable

$$F_{i-1/2}^n = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right] - \frac{\Delta x}{2\Delta t}\left(Q_i^n - Q_{i-1}^n\right)$$

$$Q_i^{n+1} = \frac{1}{2}\left(Q_{i-1}^n + Q_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left[f(Q_{i+1}^n) - f(Q_{i-1}^n)\right]$$

Lax-Friedrichs method;
stable, but diffusive;
first-order accurate

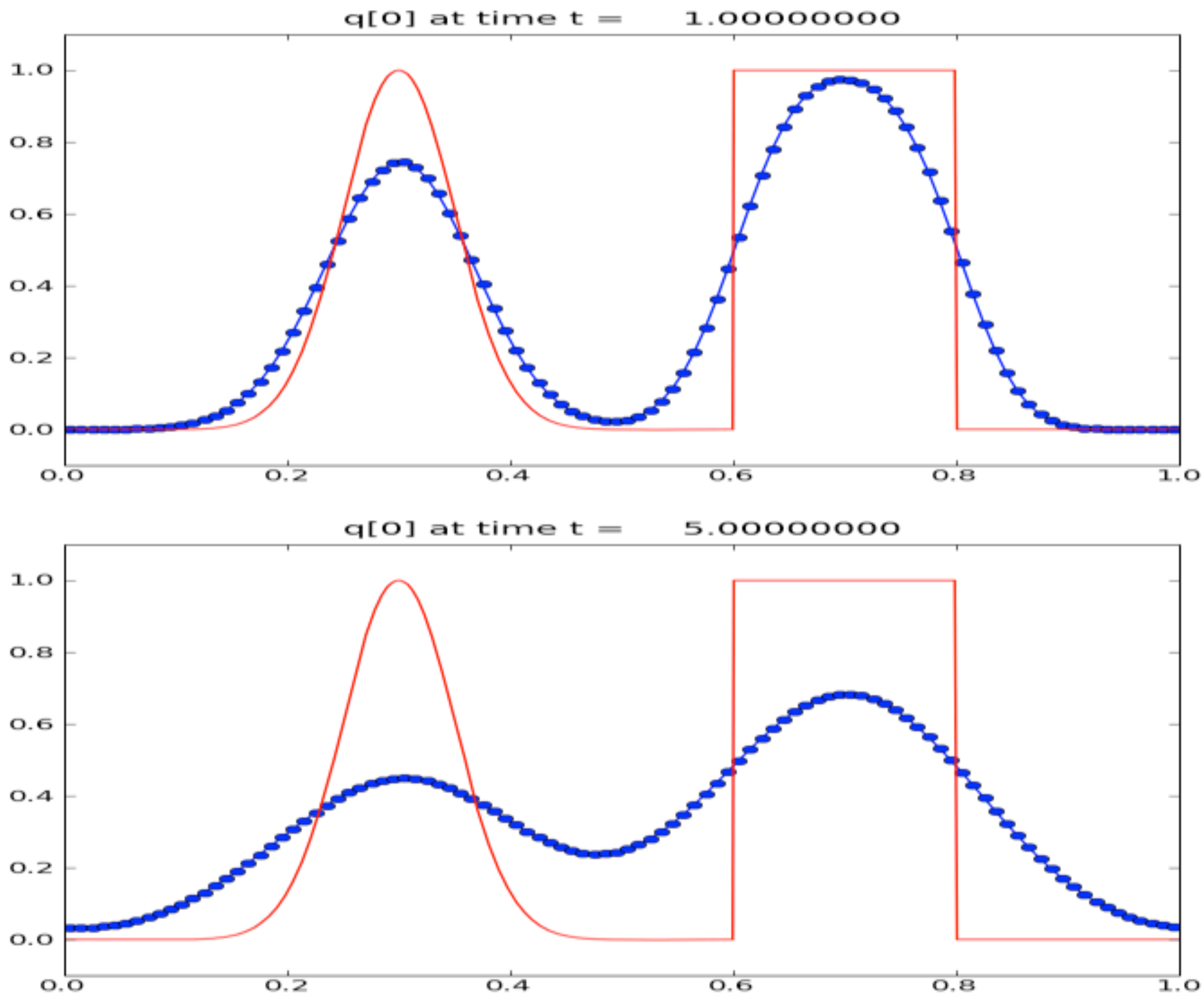
$$Q_{i-1/2}^{n+1/2} = \frac{1}{2}\left(Q_{i-1}^n + Q_i^n\right) - \frac{\Delta t}{2\Delta x}\left[f(Q_i^n) - f(Q_{i-1}^n)\right]$$

$$F_{i-1/2}^n = f(Q_{i-1/2}^{n+1/2})$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

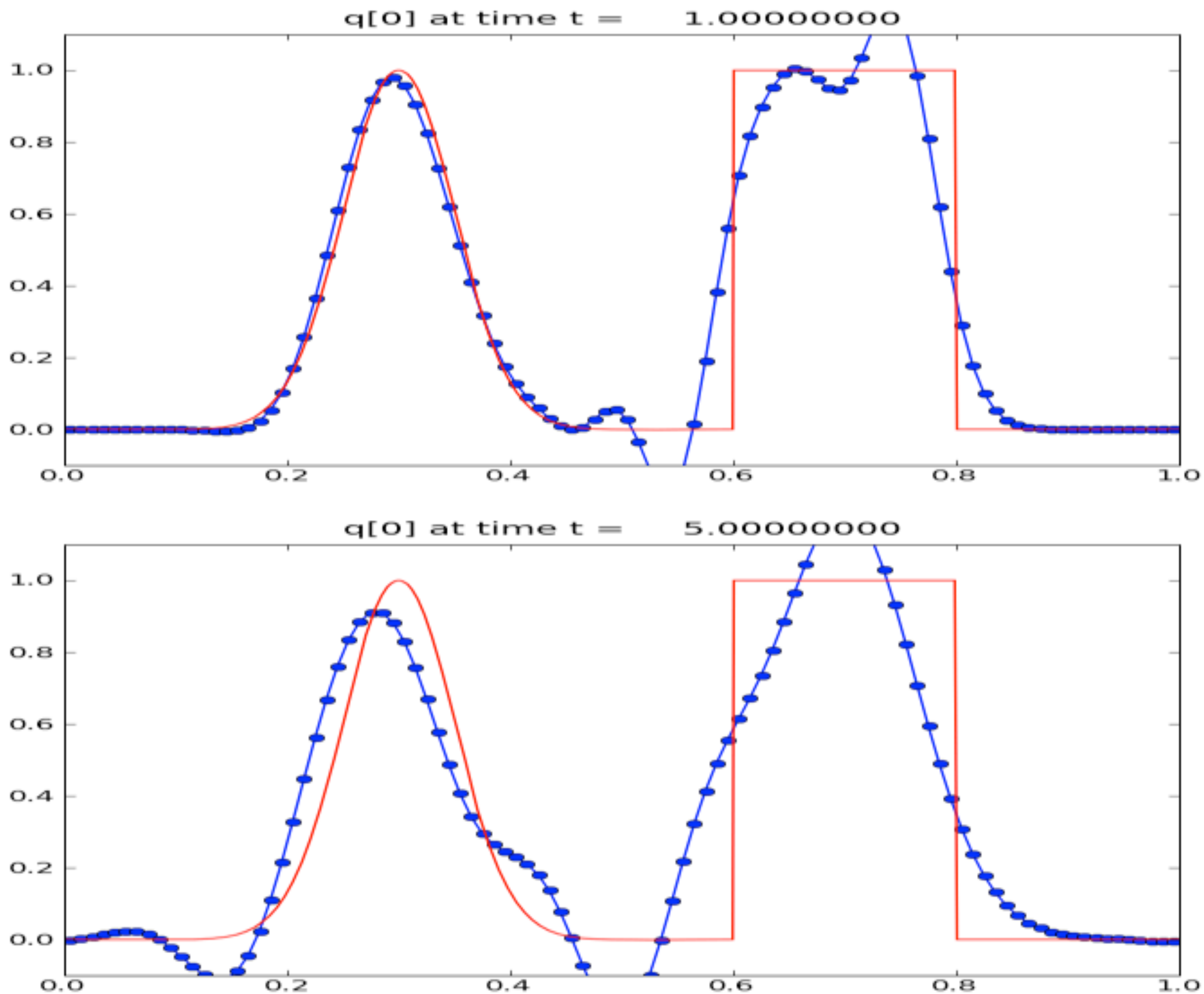
Two-step
Richtmyer-Lax-Wendroff;
second-order accurate,
but oscillatory

Advection using simple methods



1st Order Godunov (Lax-Friedrichs)

Advection using simple methods



2nd Order Lax-Wendroff

But centred methods do not make the best use of the structure of hyperbolic equations

In hyperbolic equations, the information propagates along characteristics.

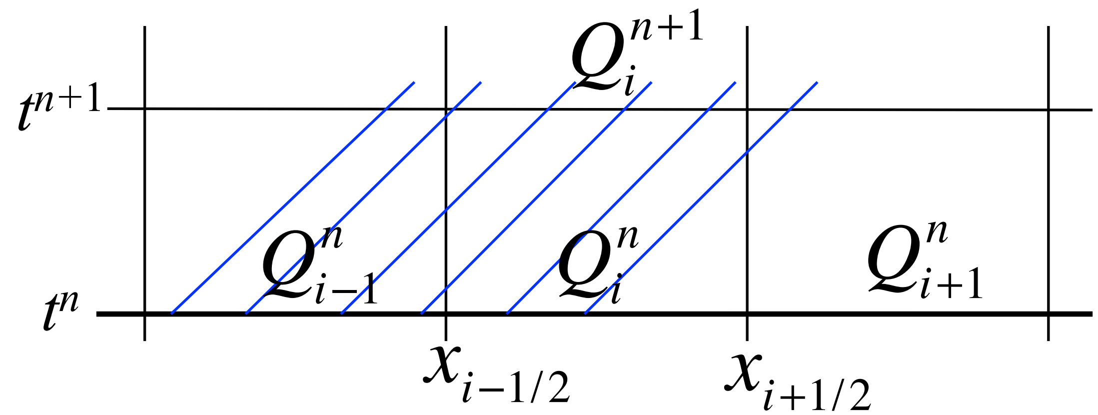
Since we **know** where the information is coming from, we should make use of that knowledge to formulate the flux function.

For the one-dimensional advection equation, there is only one characteristic, the fluid velocity u. The information comes from the left if u is positive, from the right if u is negative.

So in this simple case, we can use a *one-sided upwind* method, where we decide which side to use from the flow direction.

For systems with characteristics travelling in both directions, we must transfer some information from each side. How do we do this?

A simple upwind method for advection



In the advection problem, the flux is in one direction, $F_{i-1/2}^n = uQ_{i-1}^n$ and the update is

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\left[Q_i^n - Q_{i-1}^n\right]$$

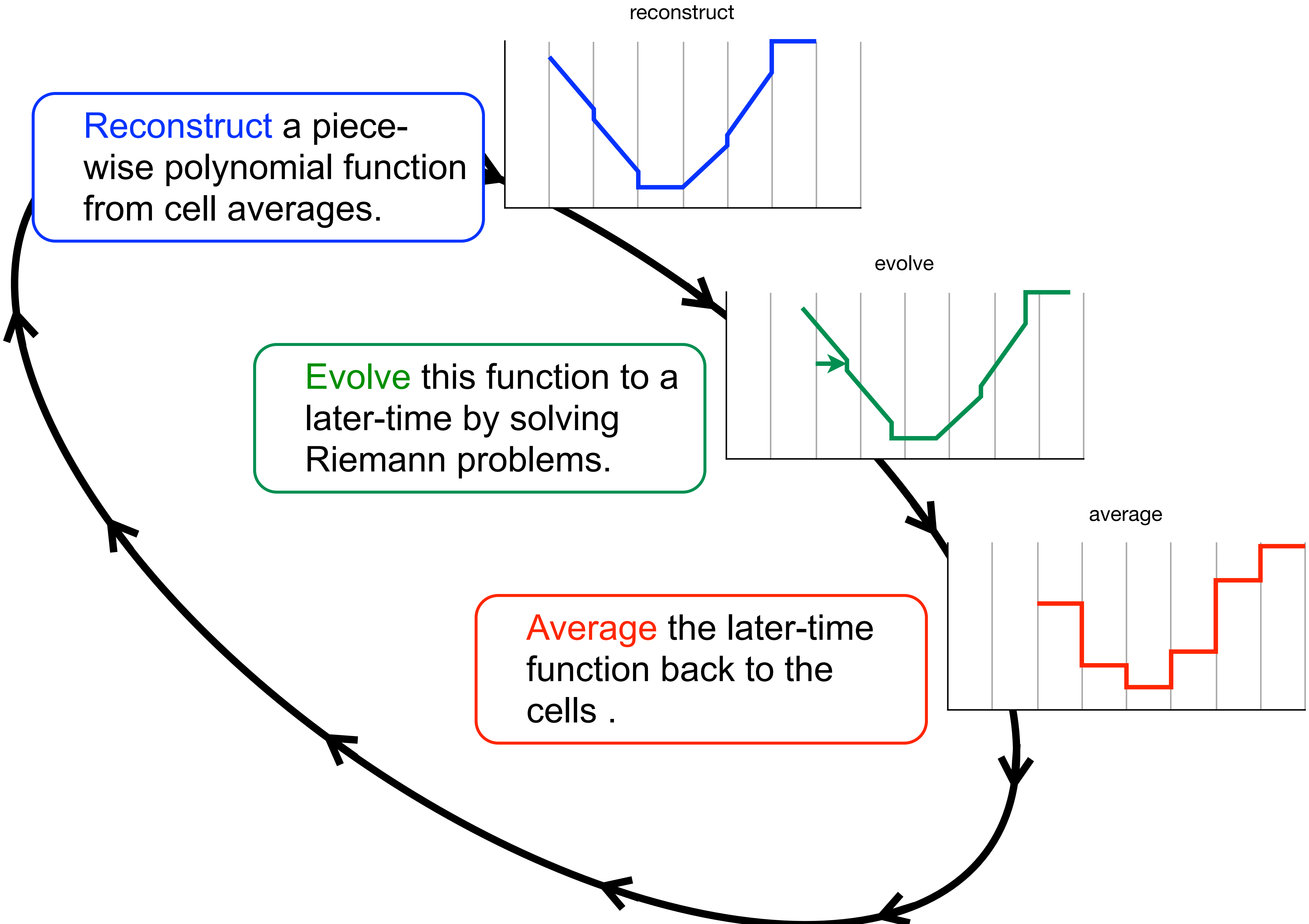
If you don't know *a priori* which direction the flux is, you can use:

$$F_{i-1/2}^n = u^- Q_i^n + u^+ Q_{i-1}^n$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)\right)$$

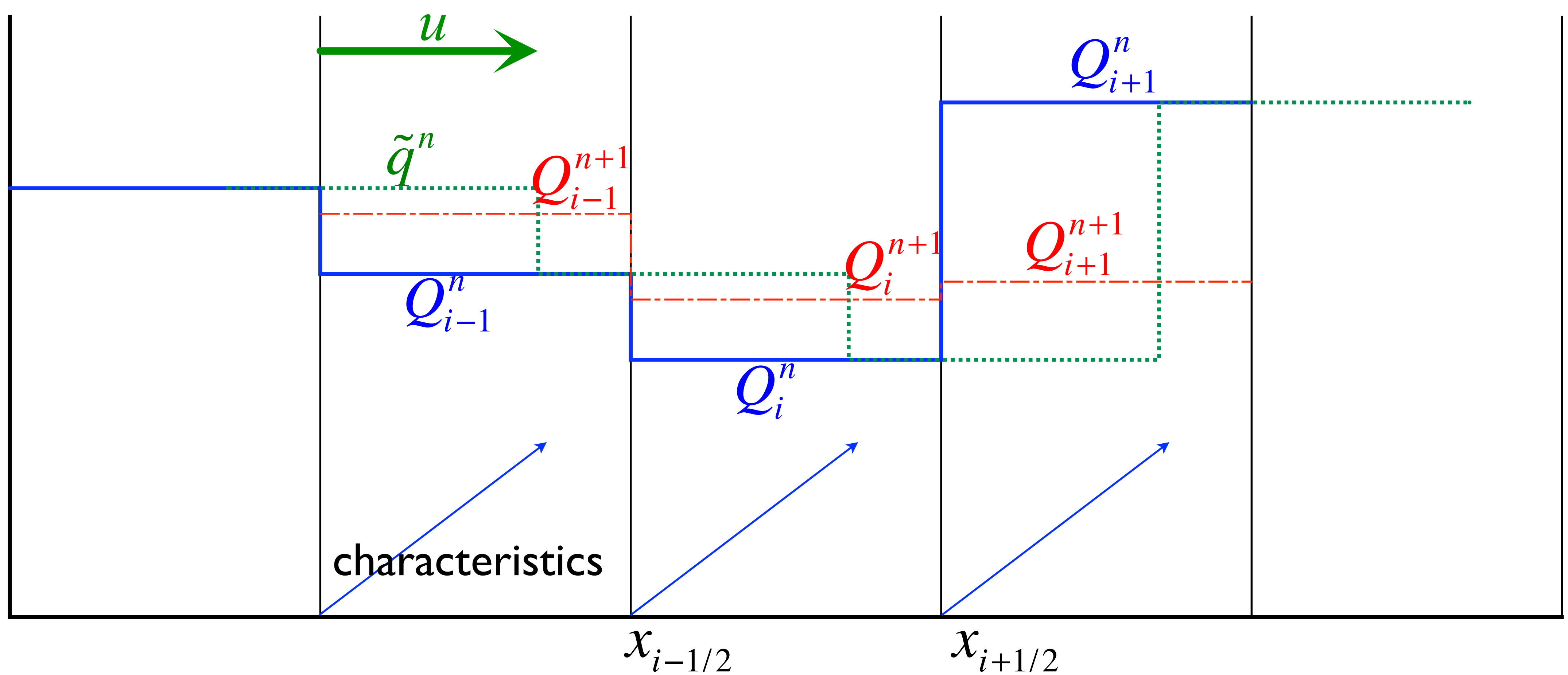
where

$$u^+ = \max(u,0), \quad u^- = \min(u,0).$$



We have just now simply prototyped the Godunov REA method.

How does this work?



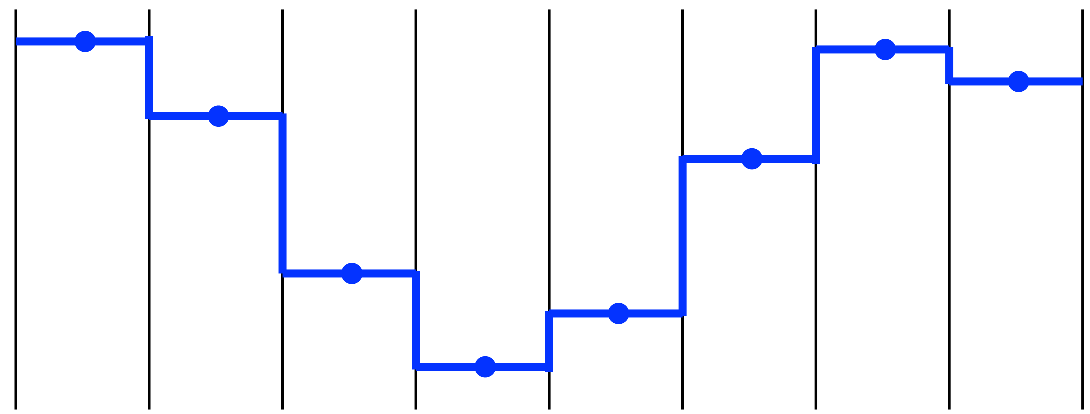
An illustration of the upwind method for CFL number $\nu \equiv u \dfrac{\Delta t}{\Delta x} \simeq 0.7$.

The cell averages Q_i^n are advected by the velocity u from time n to $n+1$,to produce an intermediate value $\tilde{q}^n$. Because the shift is less than a full cell, new cell averages must be computed to obtain the new quantities Q_i^{n+1}.

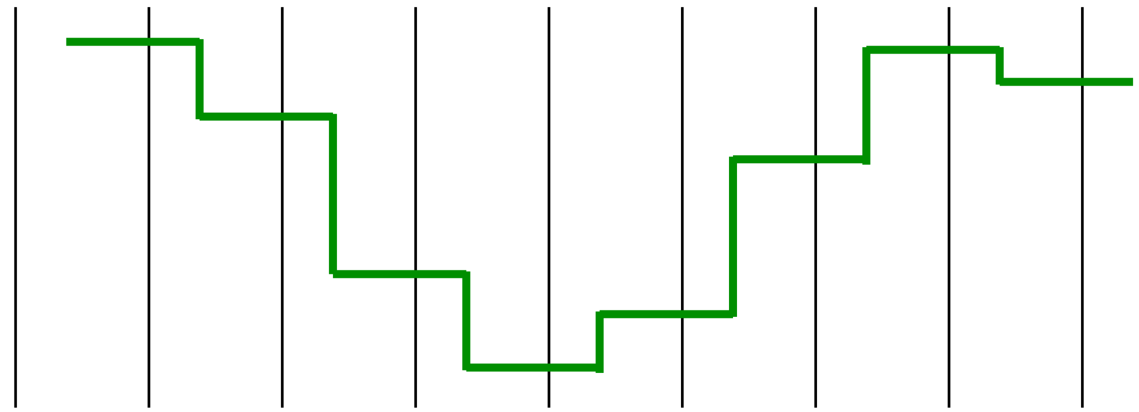
Each cell edge has a discontinuity: we can solve for the new cell value either directly (as we have done), or with the help of the Riemann technique.

First-order upwind for advection problem

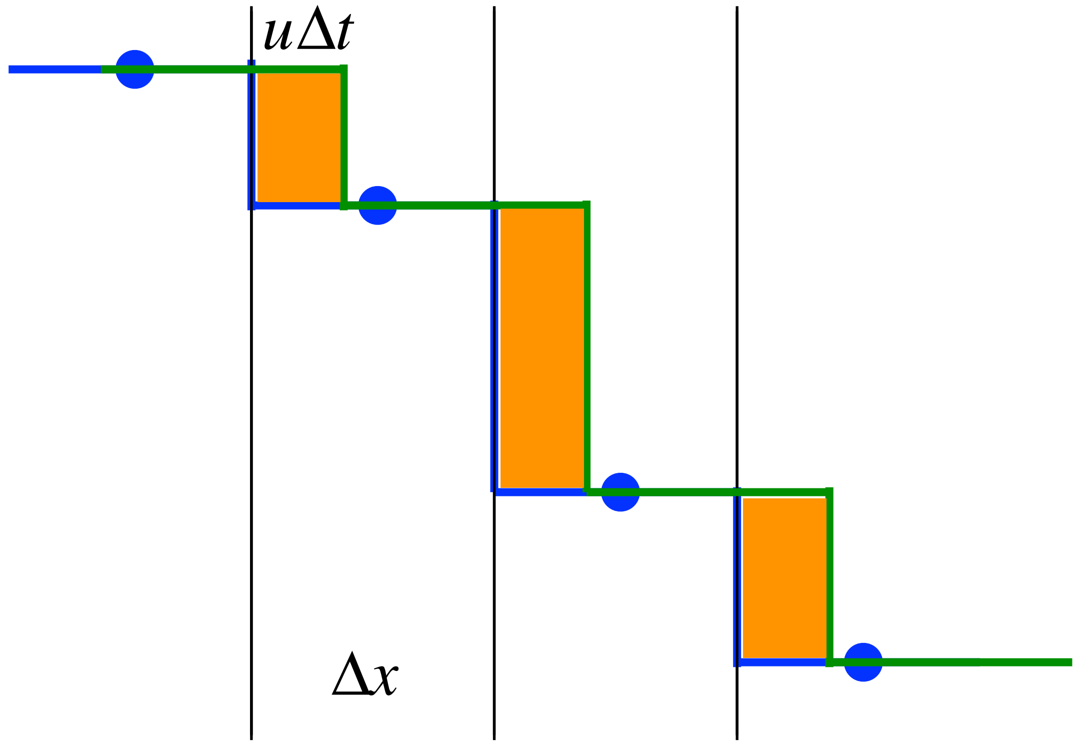
Reconstruct a function from the cell averages: piecewise constant in this case



Evolve the solution: advect it with the characteristic speed



Then compute the new cell averages



The cell average is changed by $$+u\frac{\Delta t}{\Delta x}\left(Q_{i-1}^{n} - Q_{i}^{n}\right)$$

So the upwind method is, as before, simply

$$Q_i^{n+1} = Q_i^n - u\frac{\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right)$$

To generalise, let's write it in wave-propagation form

We write the change in the cell average as

$$u\frac{\Delta t}{\Delta x}\left(Q_{i-1}^{n}-Q_{i}^{n}\right)=-s\frac{\Delta t}{\Delta x}\mathcal{W}_{i-1/2}$$

Where $\mathcal{W}_{i-1/2}=\left(Q_{i}^{n}-Q_{i-1}^{n}\right)$ is the wave strength and s is the wave speed.

At this point, this is only a change in notation, to prepare for the use of the method with *systems* of equations. But this is the same $\mathcal{W}$ we have already encountered in the Riemann problem.

In the advection equation there is (of course) only one upwind direction.

In a system of equations, waves may travel in any direction. We have to handle this now.

That's where the Riemann solver comes in.

Generalising the upwind method to systems

The general upwind method for s of either sign for a single wave is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(s^+ \mathcal{W}_{i-1/2} + s^- \mathcal{W}_{i+1/2} \right)$$

and as before, we define

$$s^+ = \max(s,0), \quad s^- = \min(s,0).$$

Now recall the Riemann solution for a many-wave problem:

$$q(x,t) = q_l + \sum_{p=1}^{m} H(x-\lambda^p t)\mathcal{W}^p; \;\; H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

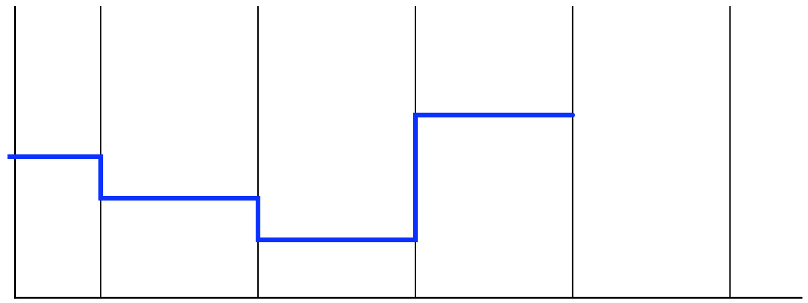
We just put these together.

Godunov's method for linear systems

The upwind scheme is representative of REA algorithms, first invented by S.K. Godunov in 1959. REA stands for:

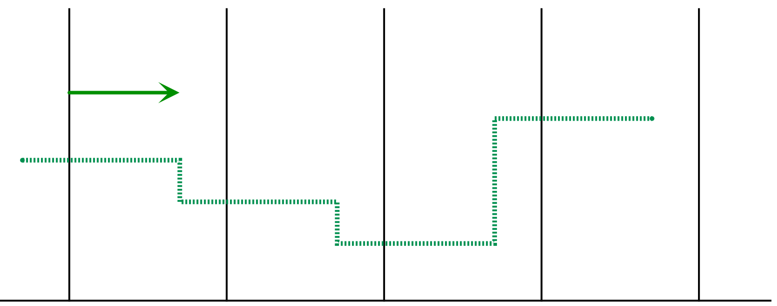
1. Reconstruct a piece-wise constant function from the cell averages.

$$q^n(x,t_n) = Q_i^n \text{ for } x \text{ in cell } i$$



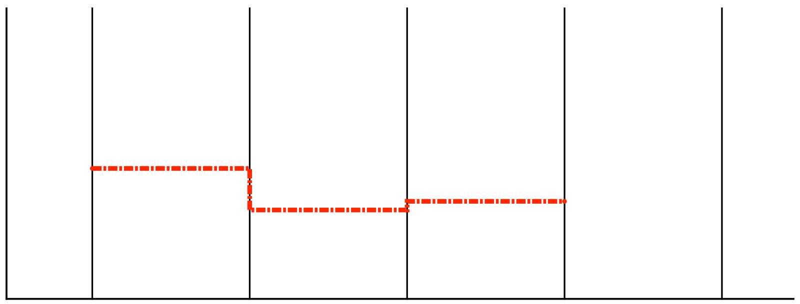
2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces.

$$\tilde{q}^n(x,t_{n+1})$$



3. Average this function over each grid cell to obtain new cell averages.

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



This is done at each time step. The method can be improved by using other interpolation functions, polynomials for example, to improve the accuracy.

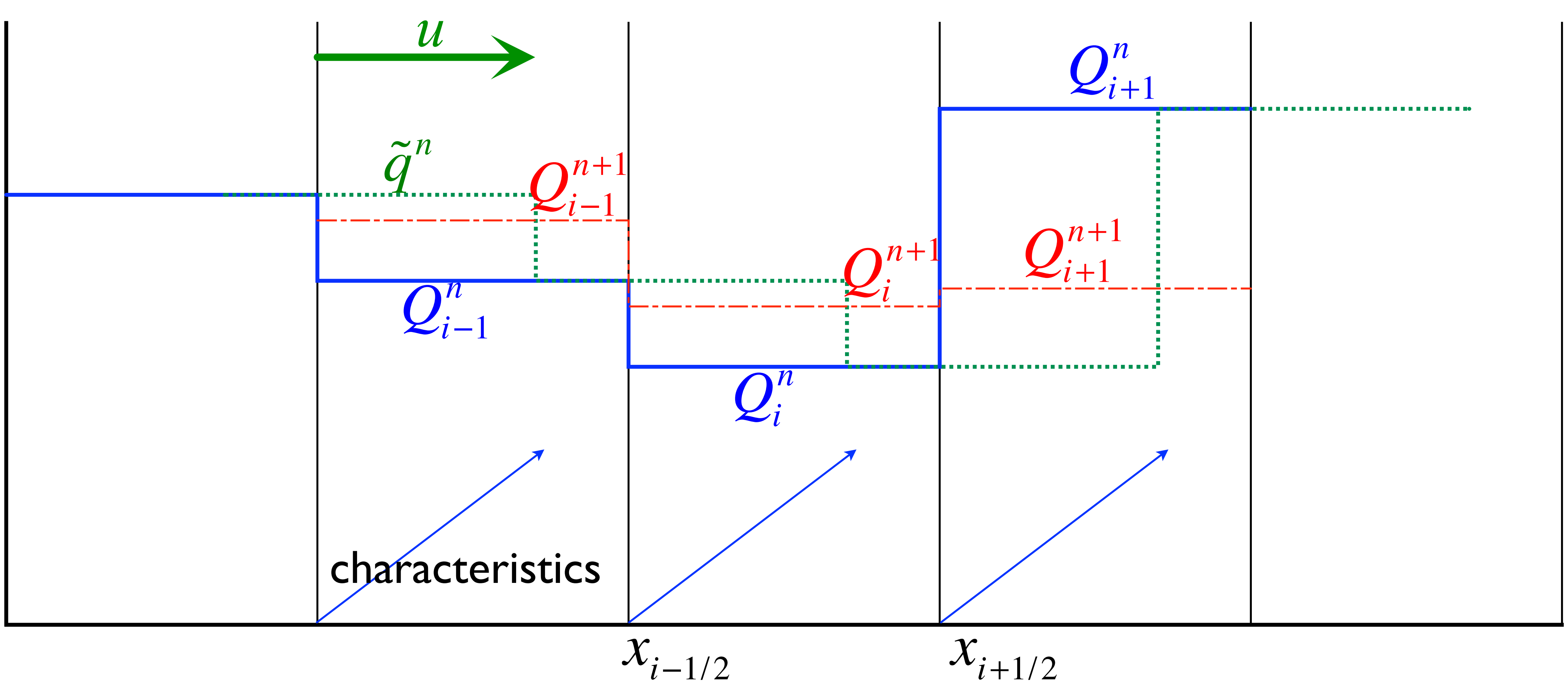
Physics is needed in the second step (evolution stage), as all the characteristics must be known and used in the solution. The first and third steps (projection stages) are entirely numerical (and problem independent).

Reconstruct - Evolve - Average

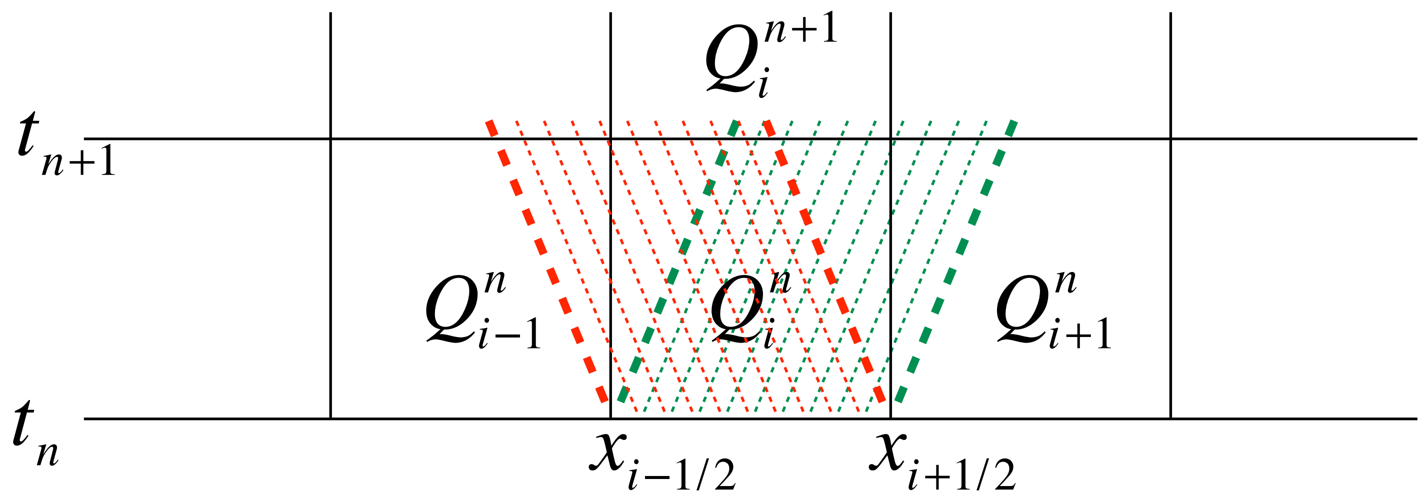
$q^n(x,t_n) = Q_i^n$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

$$Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



Care must be taken with interacting characteristics



In problems where the characteristics travel in both directions, solving the Riemann problem *independently* at each interface requires that the characteristics from neighbouring cell boundaries do not intersect.

This apparently gives a considerably stricter CFL limit: $\nu \equiv u\frac{\Delta t}{\Delta x} < \frac{1}{2}$.

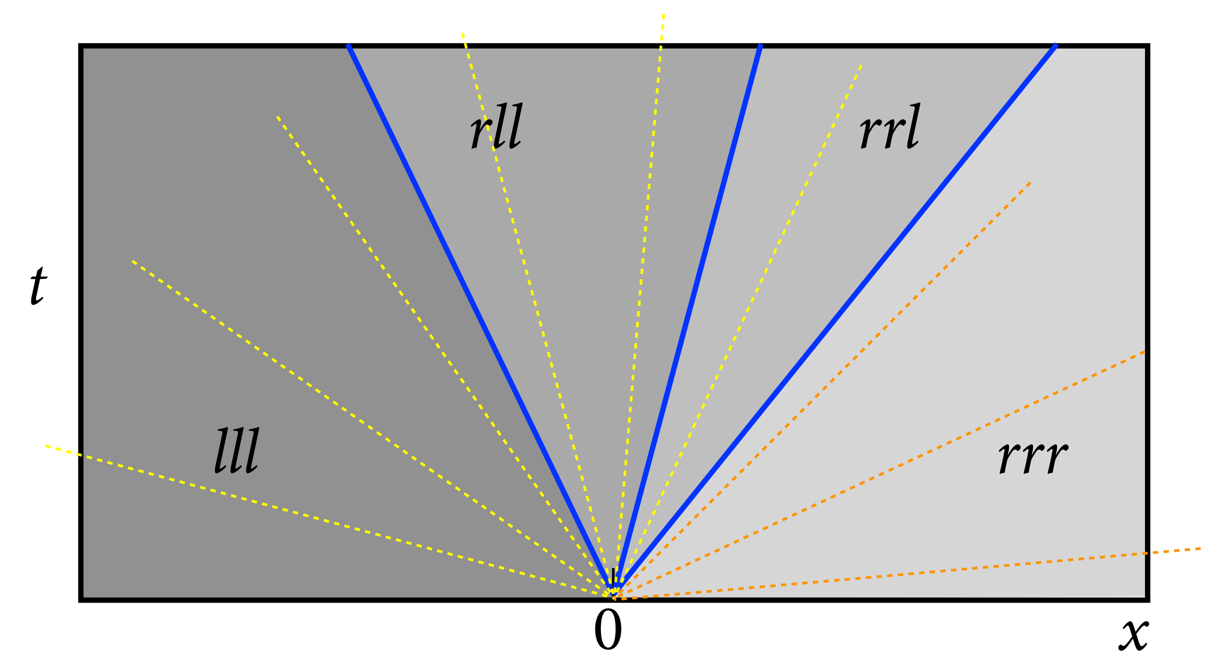
But in fact there are ways of solving the Riemann problem (cooperatively among adjacent cells) that relax this limit.

Numerical flux function in Godunov's method

Recall the formula for the numerical flux: $F_{i+1/2}^{n} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))\,dt$

The numerical flux should be the average of the true flux over the time step, but we don't know how the true flux varies.

But if we replace $q^n(x,t)$ by $\tilde{q}^n(x,t)$ we have a tremendous advantage, since the solution to the Riemann problem is a similarity solution, constant along rays from the interface (yellow, orange dashed lines).



Leveque defines a special symbol for $\tilde{q}^n(x_{i-1/2},t)$, namely $q^{\downarrow}(Q_{i-1}^n, Q_i^n)$ and then the flux function is simply

$$F_{i-1/2}^{n} = f\left(q^{\downarrow}(Q_{i-1}^n, Q_i^n)\right)$$

Godunov's method for a general system

Given a set of cell quantities Q_i^n at time n:

1. Solve the Riemann problem at $x_{i-1/2}$ to obtain $q^{\downarrow}(Q_{i-1}^n, Q_i^n)$
2. Define the flux: $F_{i-1/2}^n = f\left(q^{\downarrow}(Q_{i-1}^n, Q_i^n)\right)$
3. Apply the flux differencing formula: $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

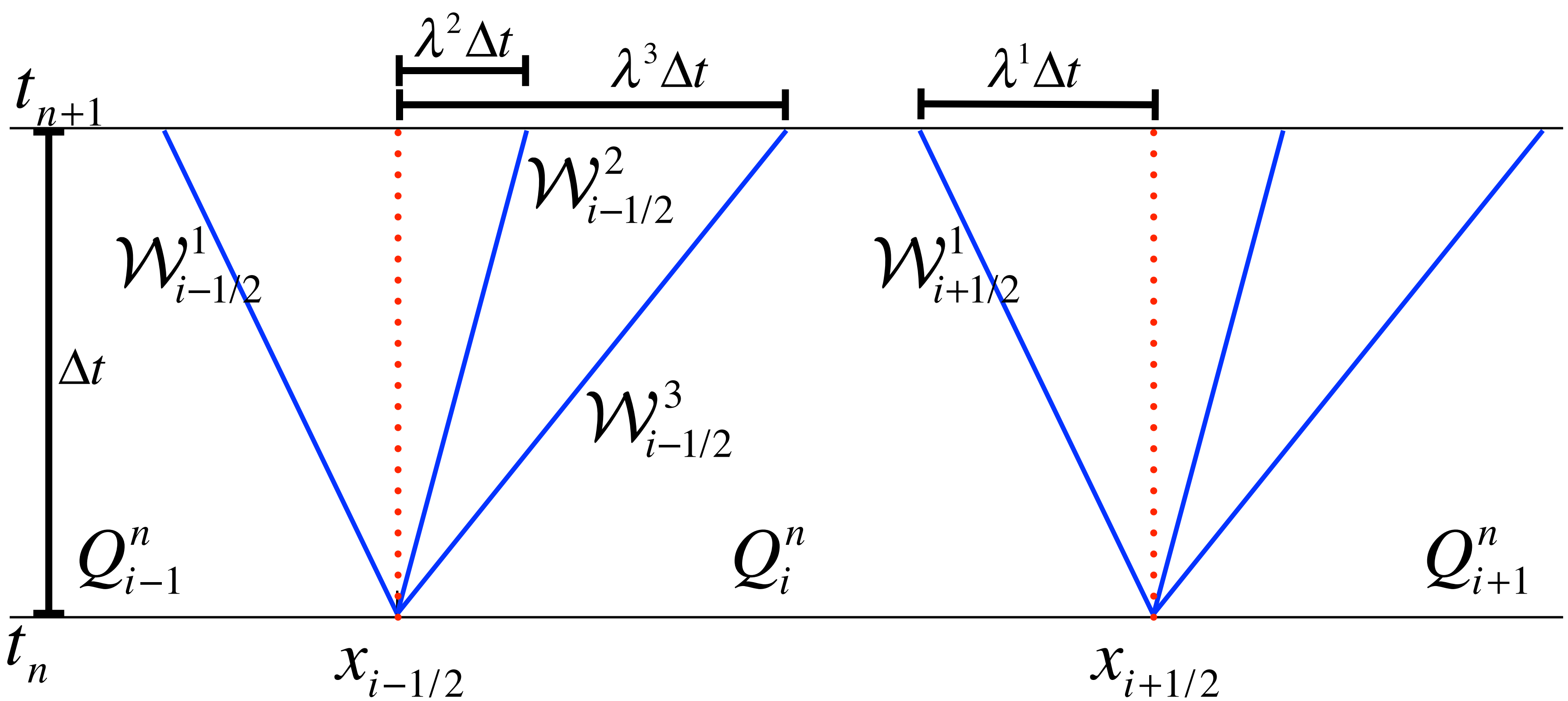
This will work for any general system of conservation laws. Only the formulation of the Riemann problem itself changes with the system.

The wave propagation implementation of Godunov's method

For a linear $m \times m$ system $q_t + Aq_x = 0$, the Riemann problem consists of m waves $\mathcal{W}^p$ propagating with constant speed λ^p.

Then

$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha_{i-1/2}^p r^p = \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p$$

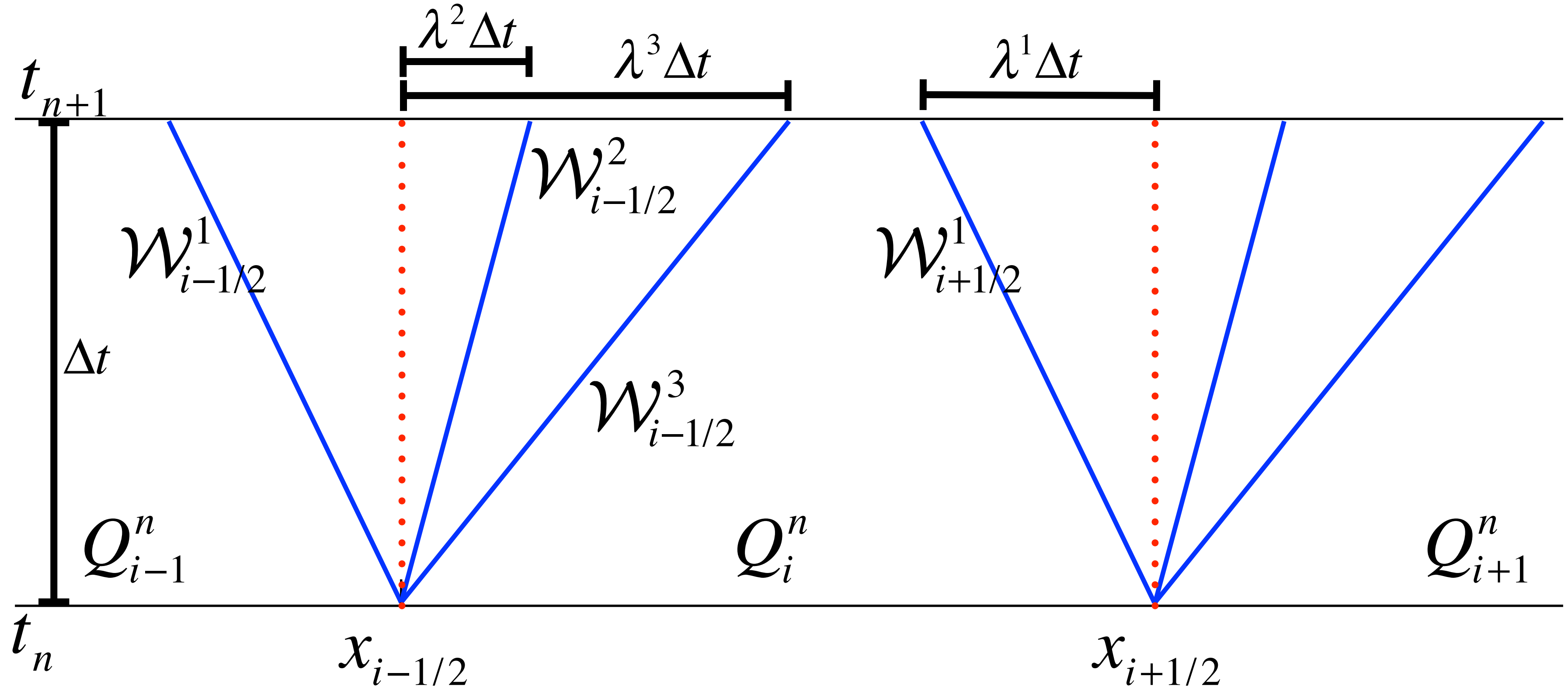


The wave propagation implementation of Godunov's method

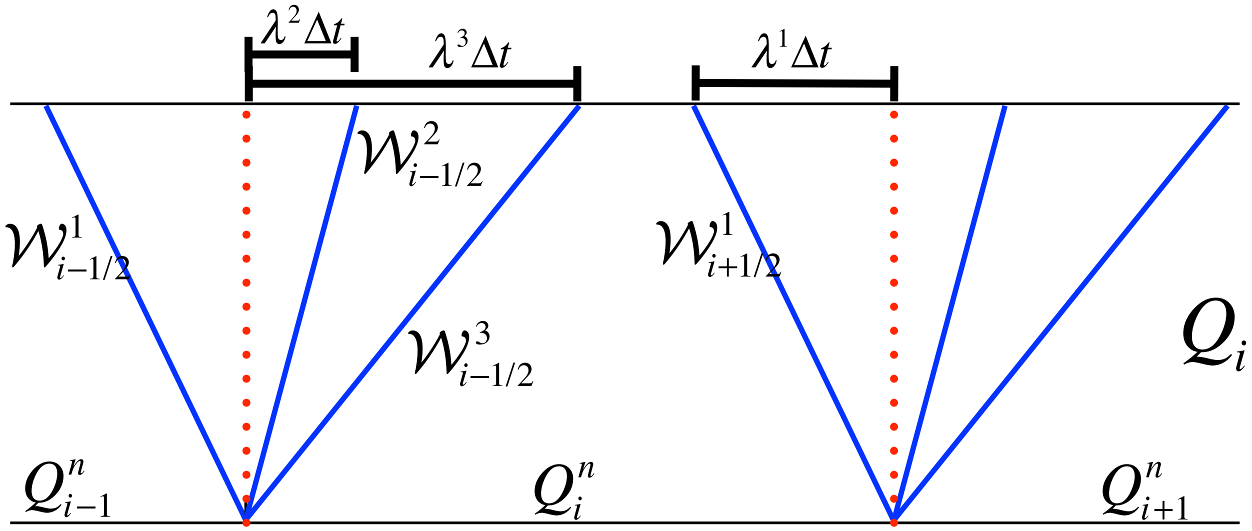
This is analogous to the basic upwind scheme.

A three-equation system has three characteristics. At timestep n, there is a discontinuity at the cell edge between Q_i^n and Q_{i+1}^n. As we evolve the Riemann solution forward to form $\tilde{q}^n(x, t_{n+1})$, this discontinuity splits into three pieces.

We use our knowledge of the splitting to compute the new cell averages.



The waves split the discontinuity



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

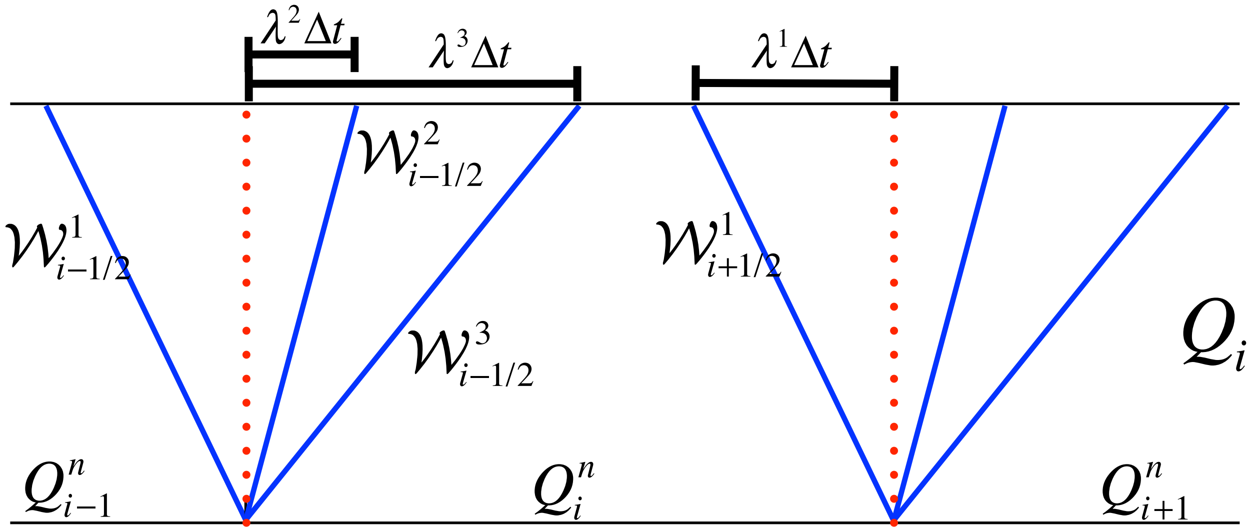
The wave $\mathcal{W}^2_{i-1/2}$ changes the cell average by $-\frac{\lambda^2 \Delta t}{\Delta x}\mathcal{W}^2_{i-1/2}$. Taking all three waves, keeping track of which direction the information is coming from, we have:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right).$$

Defining $\lambda^+ = \max(\lambda, 0), \quad \lambda^- = \min(\lambda, 0)$ (as we did for the upwind advection case), we generalise to the $m \times m$ case:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}^p_{i-1/2} + \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}^p_{i+1/2}\right]$$

The waves split the discontinuity



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Introduce the notion of fluctuations

If $\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}^p$ is the effect of all right-going waves, and

$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i-1/2}^p$ is the effect of all left-going waves from $x_{i-1/2}$,

then we can write the update as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

Notice that we take the right-going waves from the left interface and the left-going waves from the right interface!

The symbols $\mathcal{A}^\pm\Delta Q_{i\pm1/2}$ will be referred to as *fluctuations*.

This notation will be useful for nonlinear systems.

What are these fluctuations?

The symbols $\mathcal{A}^{\pm}\Delta Q_{i\pm 1/2}$ are the *fluctuations*, and we will use these heavily when we get to nonlinear systems.

But for *linear systems,* these are easily resolved into $A^{\pm}(Q_i^n - Q_{i-1}^n)$ etc.

Here's how…

To resolve the fluctuations in a linear system:

For the **linear** $m \times m$ system $q_t + Aq_x = 0$, remember we had

$$R^{-1}AR = \Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix}$$

Now we separate this into matrices of positive and negative eigenvalues:

$$\Lambda^+ = \begin{bmatrix} (\lambda^1)^+ & & & \\ & (\lambda^2)^+ & & \\ & & \ddots & \\ & & & (\lambda^m)^+ \end{bmatrix} \qquad \Lambda^- = \begin{bmatrix} (\lambda^1)^- & & & \\ & (\lambda^2)^- & & \\ & & \ddots & \\ & & & (\lambda^m)^- \end{bmatrix}$$

and we define $A^+ = R\Lambda^+R^{-1},\ A^- = R\Lambda^-R^{-1}$ so $\Lambda^+ + \Lambda^- = \Lambda,\ A^+ + A^- = A$.

Then $$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right]$$

The fluctuations for a linear system

Recall the solution in terms of waves for the $m \times m$ case

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i+1/2}^p\right]$$

and remember that by our definition of the waves for a linear system:

$$A\left(Q_i - Q_{i-1}\right) = \sum_{p=1}^{m}\lambda^p\alpha_{i-1/2}^p r^p = \sum_{p=1}^{m}\lambda^p\mathcal{W}_{i-1/2}^p$$

so, keeping careful track of where the left-going and right-going waves come from, we have

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right]$$

in analogy with

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

Flux-difference splitting

For the linear system, $\mathcal{W}^p_{i-1/2} = \alpha^p_{i-1/2} r^p$ and since $A^{\pm} = R\Lambda^{\pm}R^{-1}$

then $A^{\pm}\alpha^p_{i-1/2} r^p = (\lambda^p)^{\pm}\alpha^p_{i-1/2} r^p$. From this we get

$$\mathcal{A}^{\pm}\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^{\pm}\mathcal{W}^p_{i-1/2} = A^{\pm}(Q^n_i - Q^n_{i-1})$$

and then the update is

$$Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left(A^+(Q^n_i - Q^n_{i-1}) + A^-(Q^n_{i+1} - Q^n_i)\right)$$

or, written in terms of the flux, $Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right)$

with $F^n_{i-1/2} = A^+Q_{i-1} + A^-Q_i$

Flux-difference splitting

For the more general conservation law, $q_t + f(q)_x = 0$ we define

$$F^n_{i-1/2} = f(Q_{i-1}) + \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}^p_{i-1/2} \equiv f(Q_{i-1}) + \mathcal{A}^- \Delta Q_{i-1/2}$$

$$F^n_{i-1/2} = f(Q_i) - \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}^p_{i-1/2} \equiv f(Q_i) - \mathcal{A}^+ \Delta Q_{i-1/2}$$

These two are equivalent, the same flux through the same cell border, representing either a left-going flux that updates Q_{i-1} or a right-going fluctuation that updates Q_i .

If we subtract one from the other, we have

$$f(Q_i) - f(Q_{i-1}) = \mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2}$$

directly showing the difference in fluxes split into right- and left-going fluctuations.

Generalisation to nonlinear problems

For the nonlinear Riemann problem, the solution is still a *similarity solution*:

$$q(x,t) = q^{*}(x/t)$$

A system of m equations consists of m_W waves propagating at constant speed.

Often $m_w = m$ but not always.

Some waves may be *rarefaction waves* instead of discontinuities (as in the shock tube problem).

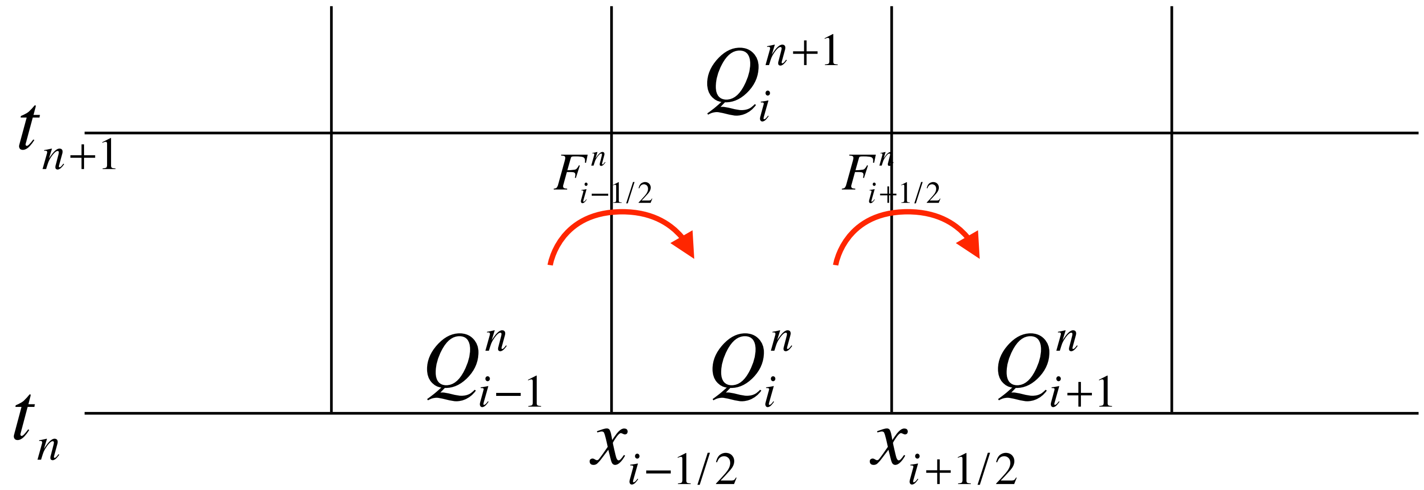
The numerical method is based on an *approximate* Riemann solution with the decomposition

$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^{p}$$

where $\mathcal{W}_{i-1/2}^{p}$ is a wave propagating at some speed $s_{i-1/2}^{p}$.

We'll get much more of this later ...

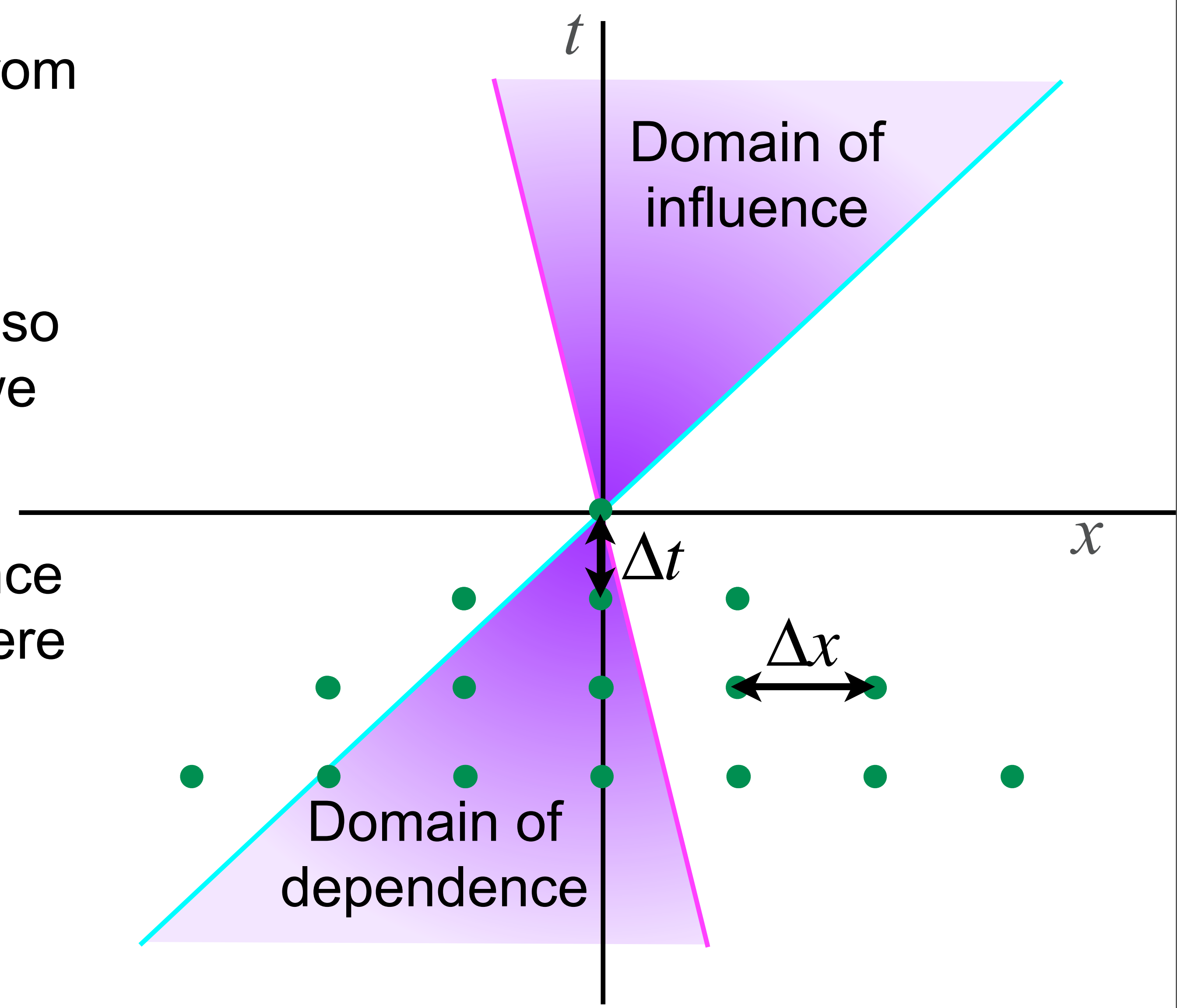
Review of Finite Volume Methods



We need a scheme for obtaining the fluxes from one cell to the next in terms of the available solution values at the present time step.

The conservation law tells us how to do this, so we must ensure that the difference formula we produce is in conservation form.

In hyperbolic problems, the domain of influence is limited; we use this limitation to decide where to take information from.



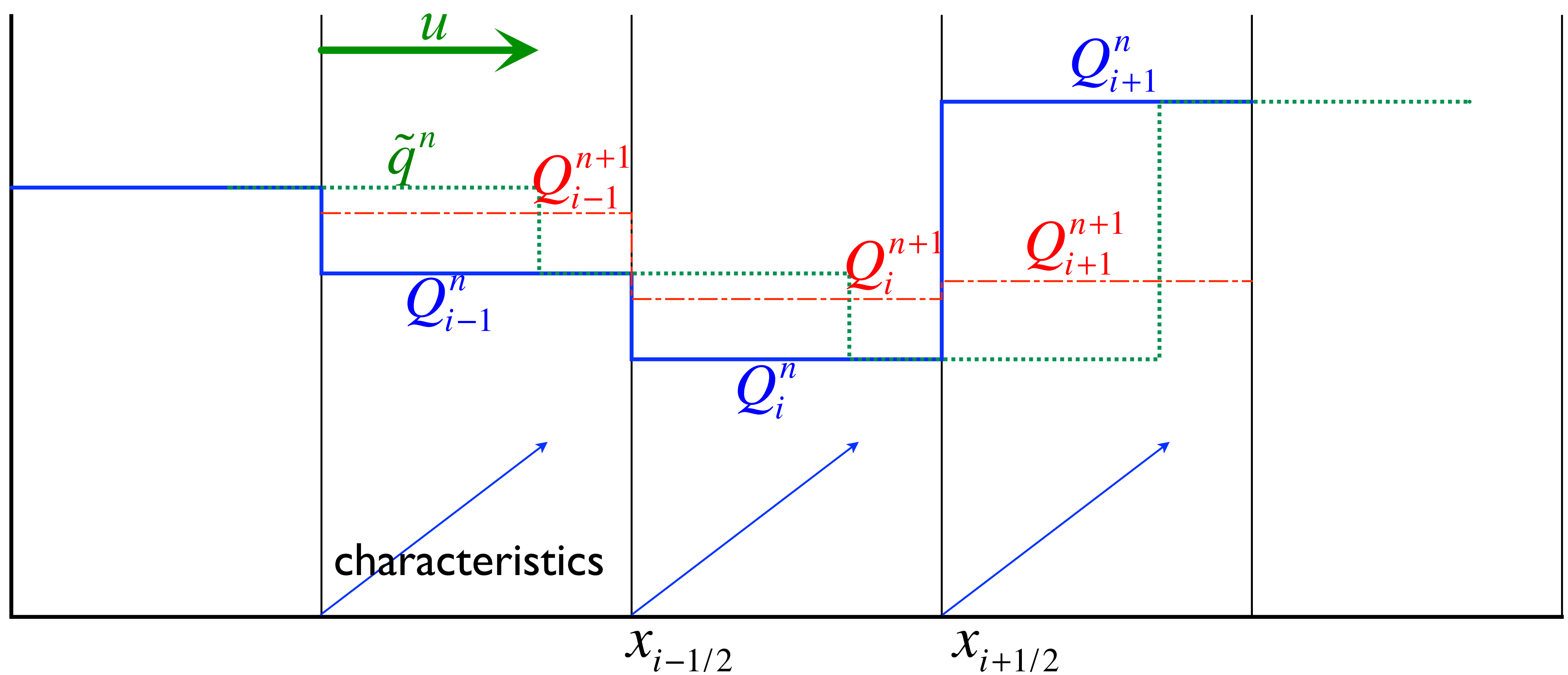
Review of the upwind method

Reconstruct - Evolve - Average

$q^n(x,t_n) = Q_i^n$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

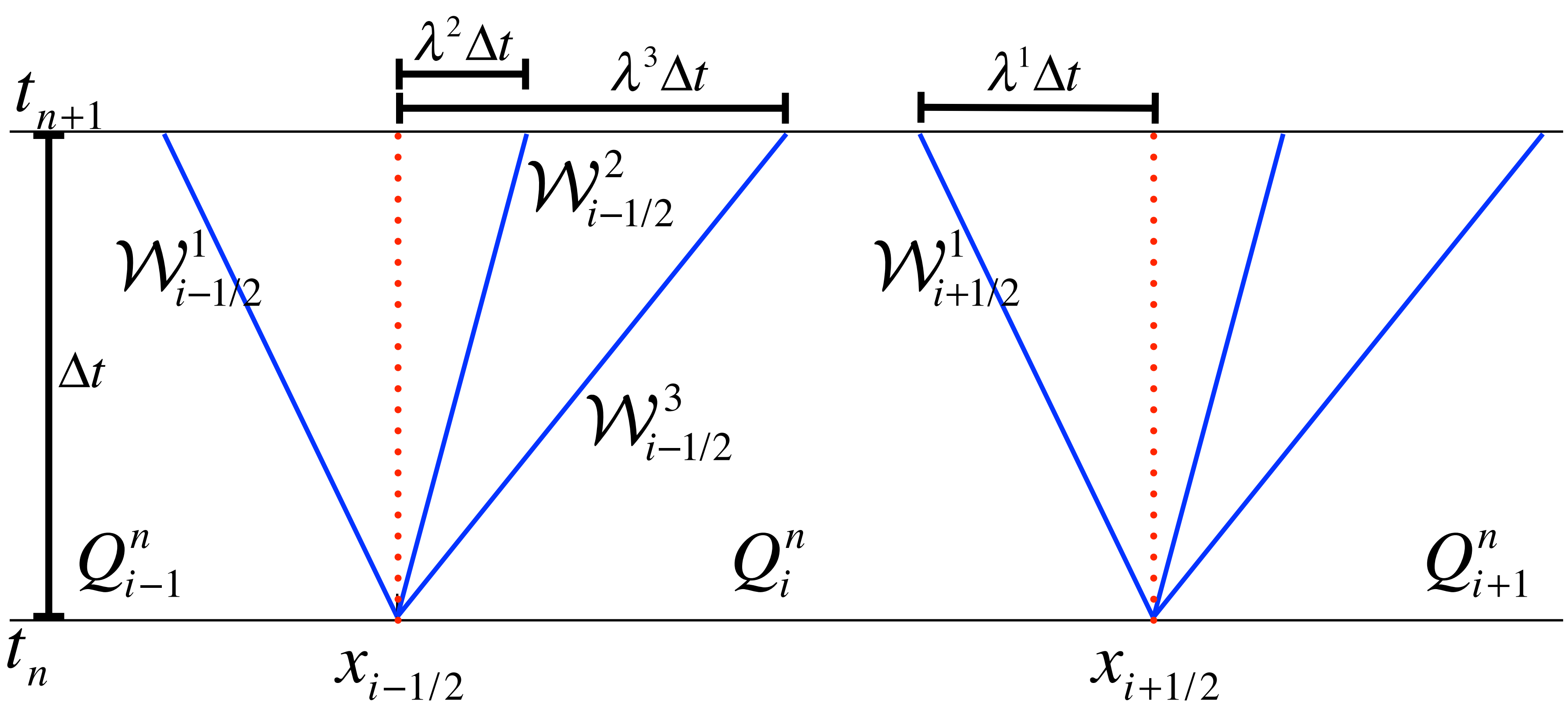
$$Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



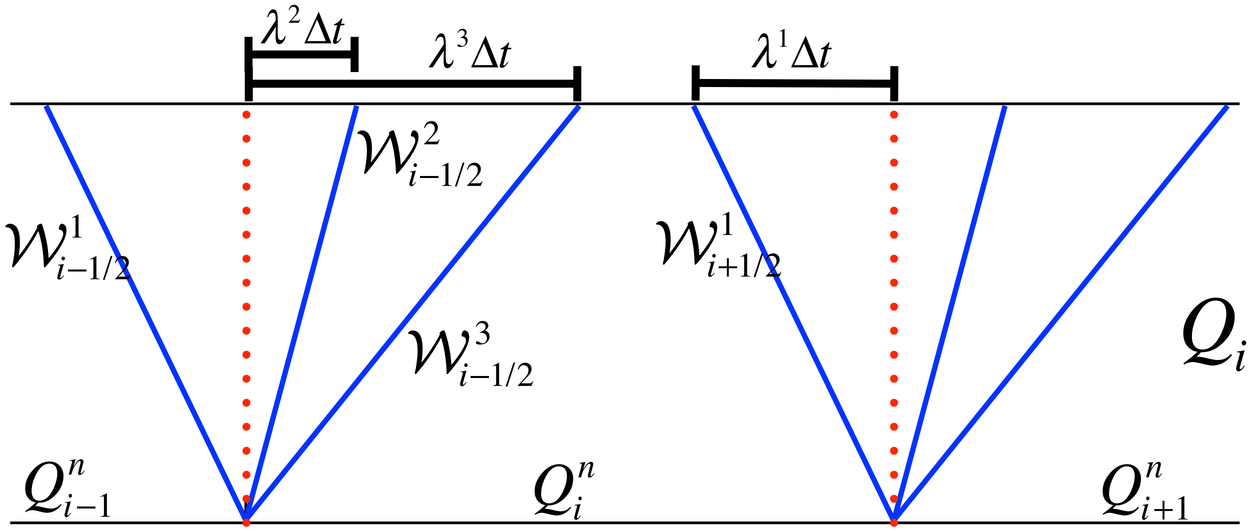
Review: The wave propagation implementation of Godunov's method

A three-equation system has three characteristics. At timestep n, there is a discontinuity at the cell edge between Q_i^n and Q_{i+1}^n. As we evolve the Riemann solution forward to form $\tilde{q}^n(x, t_{n+1})$, this discontinuity splits into three pieces.

We use our knowledge of the splitting to compute the new cell averages.



Review: splitting the discontinuity



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

The wave $\mathcal{W}^2_{i-1/2}$ changes the cell average by $-\frac{\lambda^2 \Delta t}{\Delta x}\mathcal{W}^2_{i-1/2}$. The three waves together give us:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\lambda^2\mathcal{W}^2_{i-1/2} + \lambda^3\mathcal{W}^3_{i-1/2} + \lambda^1\mathcal{W}^1_{i+1/2}\right)$$

.

Defining $\lambda^+ = \max(\lambda,0), \quad \lambda^- = \min(\lambda,0)$, we generalise to m waves:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p_{i-1/2} + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p_{i+1/2}\right]$$

Review: Fluctuations

If $\mathcal{A}^{+}\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^{p})^{+} \mathcal{W}_{i-1/2}^{p}$ is the effect of all right-going waves, and

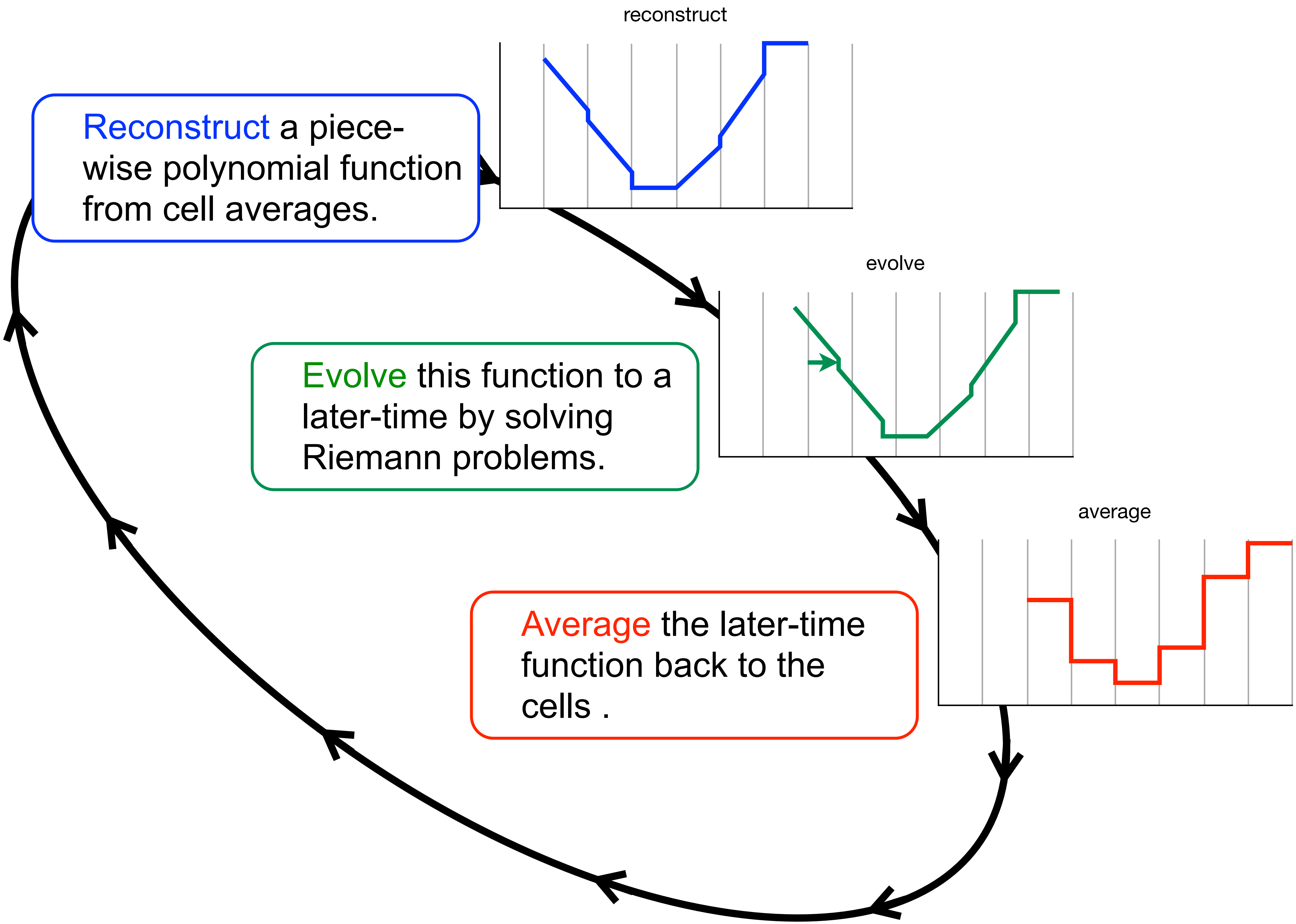
$\mathcal{A}^{-}\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^{p})^{-} \mathcal{W}_{i-1/2}^{p}$ is the effect of all left-going waves from $x_{i-1/2}$,

then we can write the update as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^{+}\Delta Q_{i-1/2} + \mathcal{A}^{-}\Delta Q_{i+1/2}\right)$$

We take the right-going waves from the left interface and the left-going waves from the right interface.

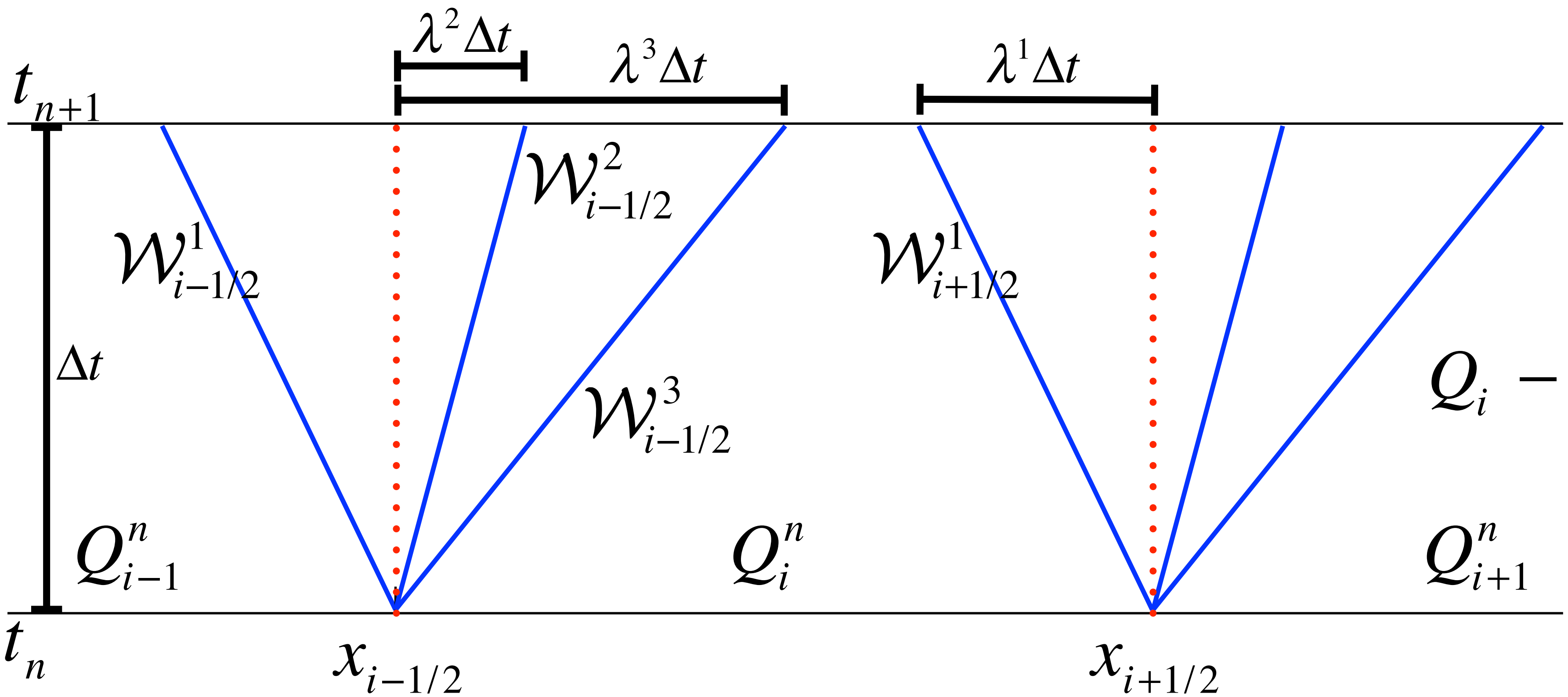
The symbols $\mathcal{A}^{\pm}\Delta Q_{i\pm 1/2}$ are the *fluctuations*.



Next we examine high-resolution implementations of the Godunov REA method.

Next: High Resolution Methods (Ch 6)

Extending Godunov's method to high-resolution



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

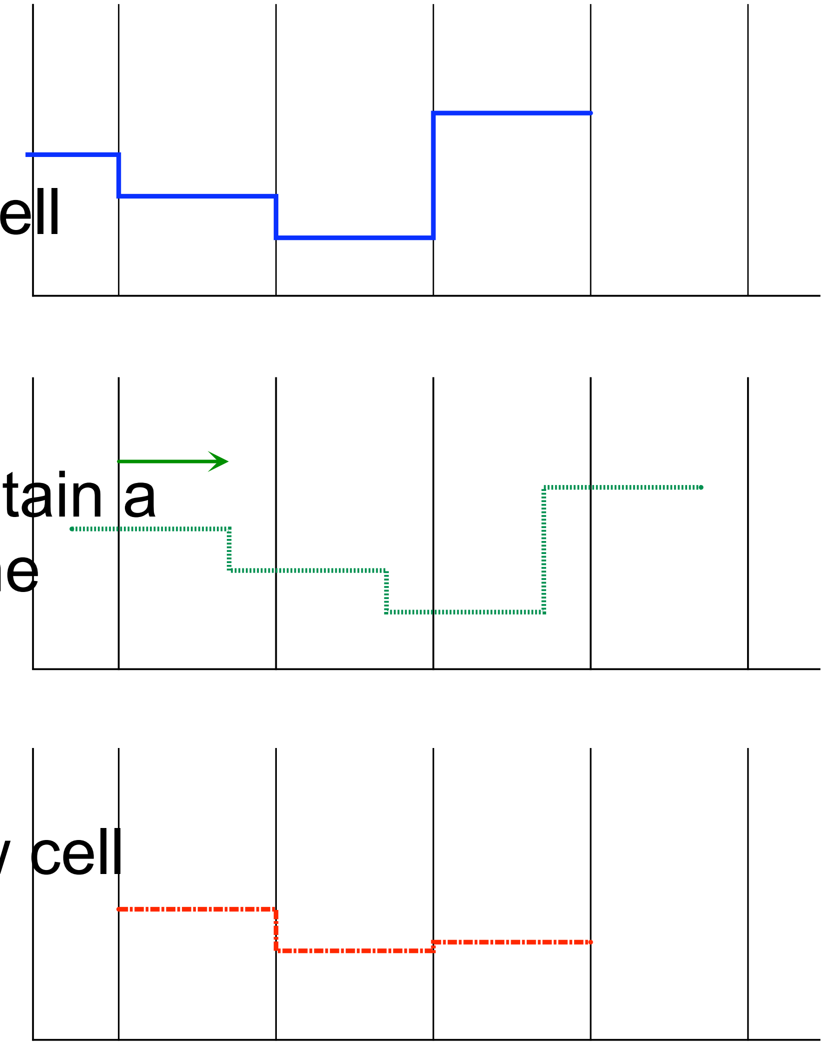
Q_i^n defines a piece-wise *constant* function. The discontinuities at the cell interfaces give rise to Riemann problems

$$F^n_{i-1/2} = f\left(q^{\downarrow}(Q^n_{i-1}, Q^n_i)\right),$$

and the solution at the next time step is obtained from

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right).$$

Godunov's method with piece-wise constant reconstruction is only first order



1. Reconstruct a piece-wise *constant* function from the cell averages.

$$q^n(x,t_n) = Q_i^n \text{ for } x \text{ in cell } i$$

2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces.

$$\tilde{q}^n(x,t_{n+1})$$

3. Average this function over each grid cell to obtain new cell averages.

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

This is done at each time step. The method can be improved by using other interpolation functions, polynomials for example, to improve the accuracy.

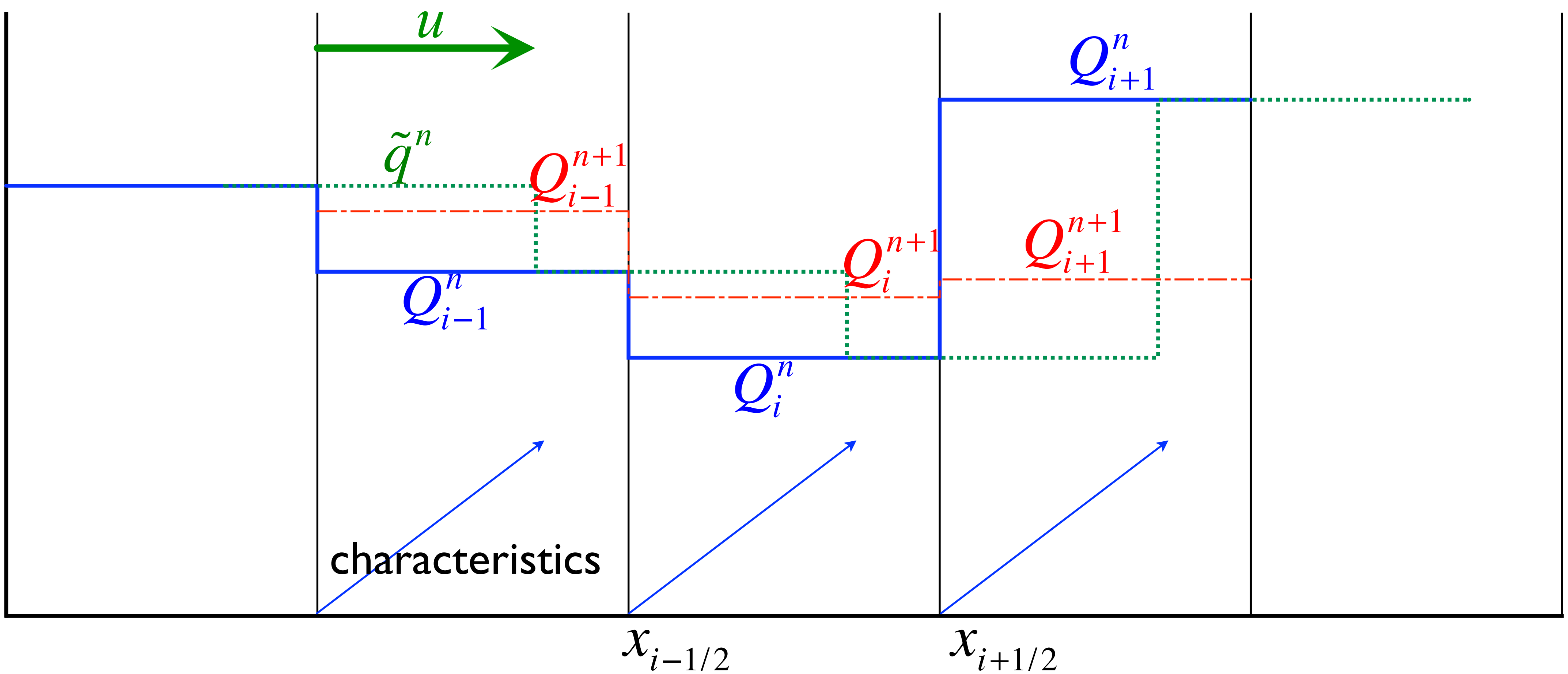
Physics is needed in the second step (evolution stage), as all the characteristics must be known and used in the solution. The first and third steps (projection stages) are entirely numerical (and problem independent).

Reconstruct - Evolve - Average

$q^n(x,t_n) = Q_i^n$ for x in cell i

$\tilde{q}^n(x,t_{n+1})$

$$Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



Improvements on the simple system

Recall the update formula developed in chapter 4 that uses the notion of fluctuations:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

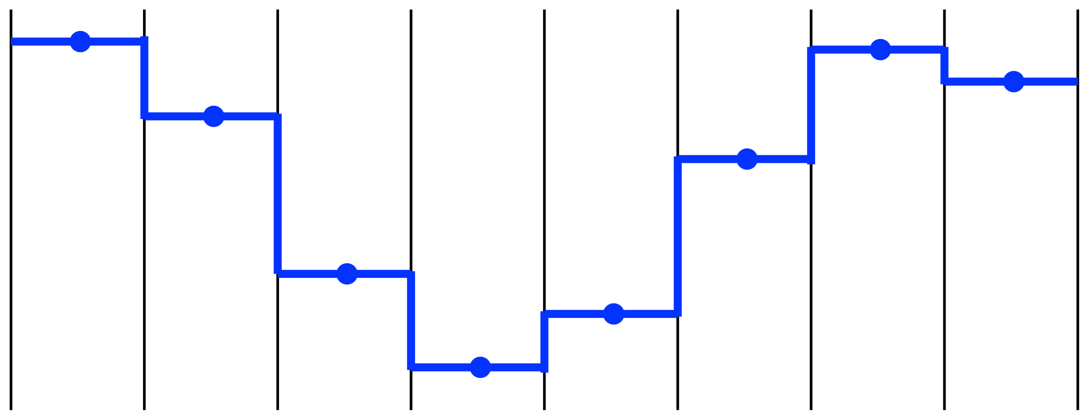
This gives a method that is only first-order accurate. We can improve it by introducing corrections, and writing:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}\right)$$

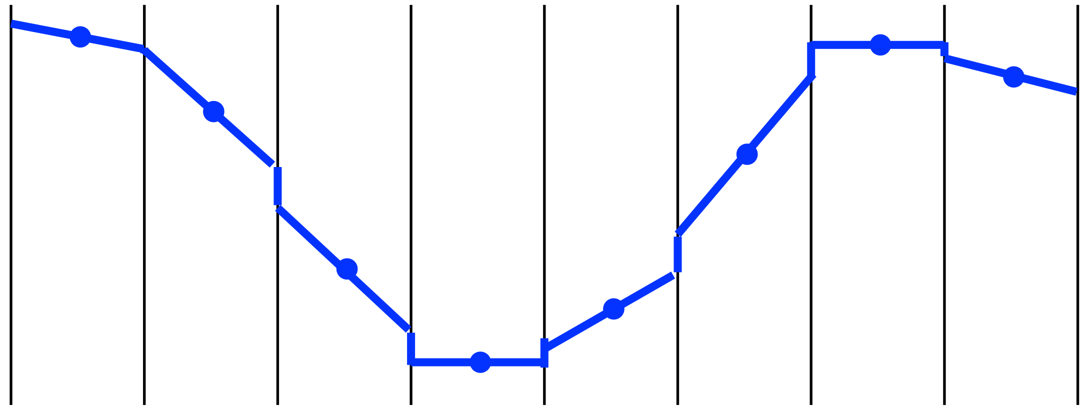
There are several possible techniques, and we illustrate a few here and show how well or how poorly they do.

Piece-wise Linear Reconstruction

Instead of using piece-wise constant reconstruction as in the simple REA update,



We could use a piece-wise linear reconstruction:



We can choose how to do this, subject to the constraint that the cell averages are conserved, and that the slopes somehow reflect the local function behaviour. This is how second-order and high-resolution methods are done.

Second-order methods:

Start with the linear system $q_t + Aq_x = 0$

Write the Taylor series expansion about the present time for the solution q at the advanced time:

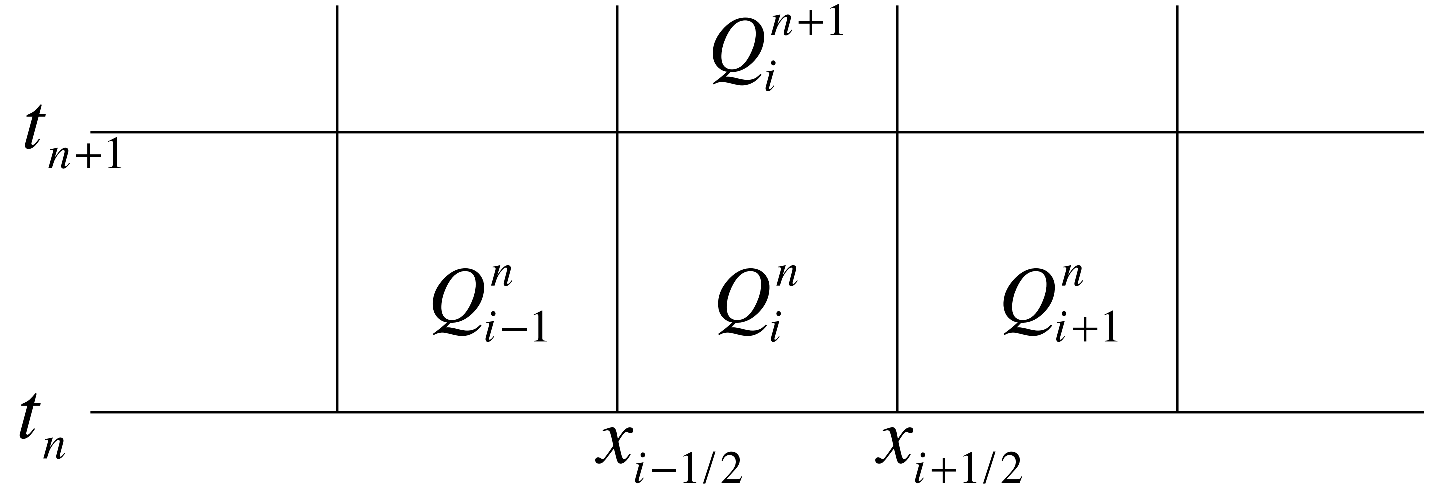
$$q(x,t_{n+1}) = q(x,t_n) + \Delta t q_t(x,t_n) + \frac{1}{2}(\Delta t)^2 q_{tt}(x,t_n) + \ldots$$

The differential equation gives us $q_t = -Aq_x$ and therefore $q_{tt} = A^2 q_{xx}$

so that:

$$q(x,t_{n+1}) = q(x,t_n) - \Delta t A q_x(x,t_n) + \frac{1}{2}(\Delta t)^2 A^2 q_{xx}(x,t_n) + \ldots$$

Lax-Wendroff:



From the first three terms of the Taylor expansion

$$q(x,t_{n+1}) \approx q(x,t_n) - \Delta t A q_x(x,t_n) + \frac{1}{2}(\Delta t)^2 A^2 q_{xx}(x,t_n)$$

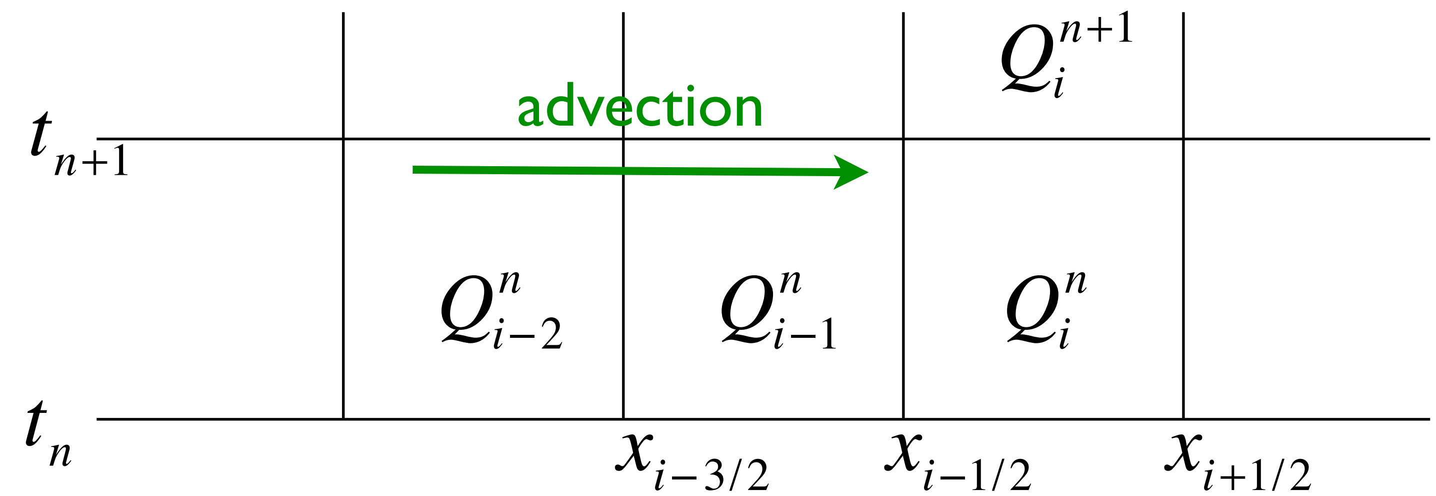
using centred differences:

$$\begin{cases} q_x(x,t_n) \approx \dfrac{1}{2\Delta x}\left(Q_{i+1}^n - Q_{i-1}^n\right) \\ q_{xx}(x,t_n) \approx \left(\dfrac{1}{\Delta x}\right)^2 \left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right) \end{cases}$$

we come to the Lax-Wendroff (1960) formula:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2 \left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

Beam-Warming:



From the first three terms of the Taylor expansion

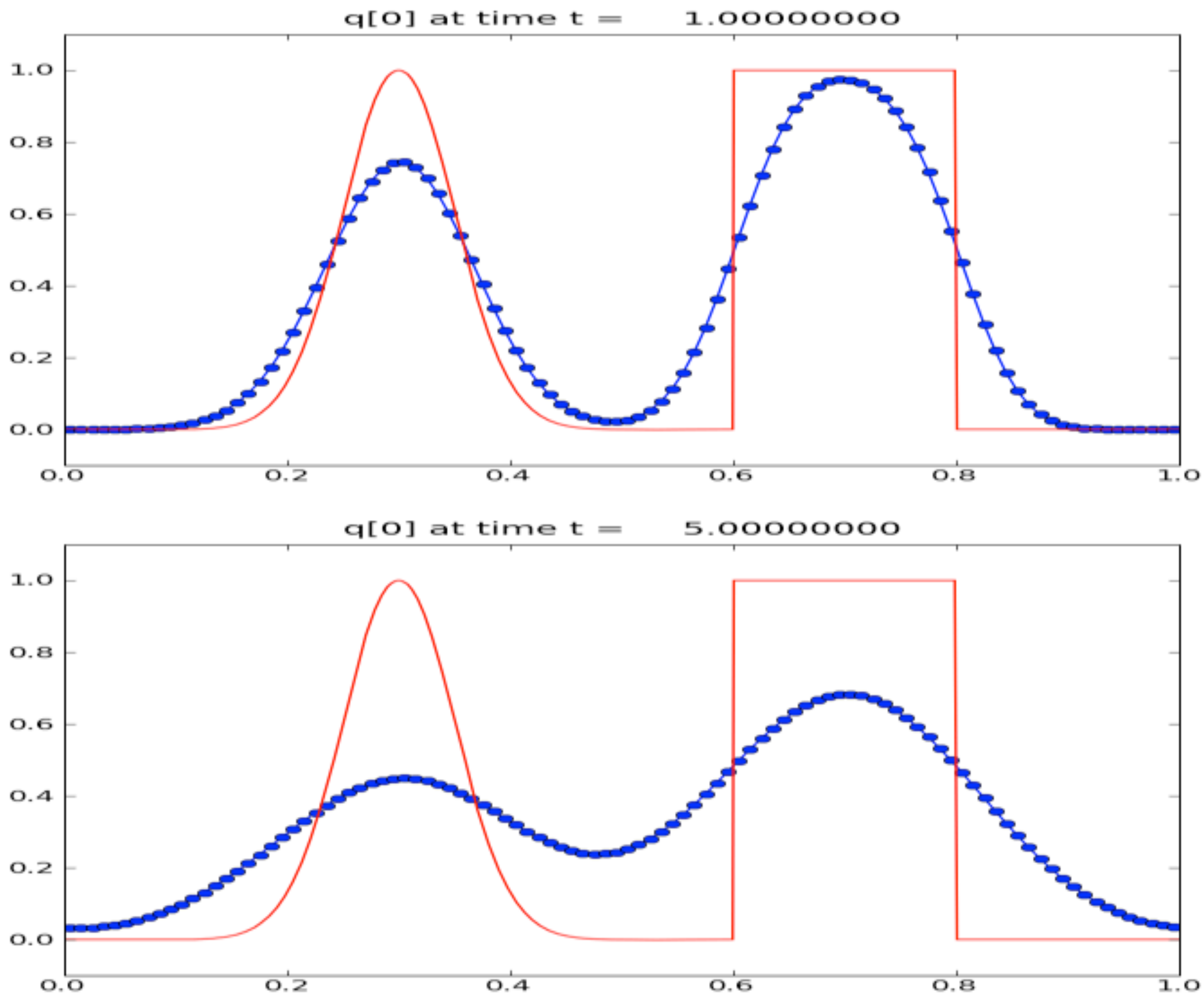
$$q(x,t_{n+1}) \approx q(x,t_n) - \Delta t A q_x(x,t_n) + \frac{1}{2}(\Delta t)^2 A^2 q_{xx}(x,t_n)$$

Using upwind differences:
$$\begin{cases} q_x(x,t_n) \approx \dfrac{1}{2\Delta x}\left(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n\right) \\ q_{xx}(x,t_n) \approx \left(\dfrac{1}{\Delta x}\right)^2 \left(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n\right) \end{cases}$$

leads to the Beam-Warming (1976) formula for one-sided flows:

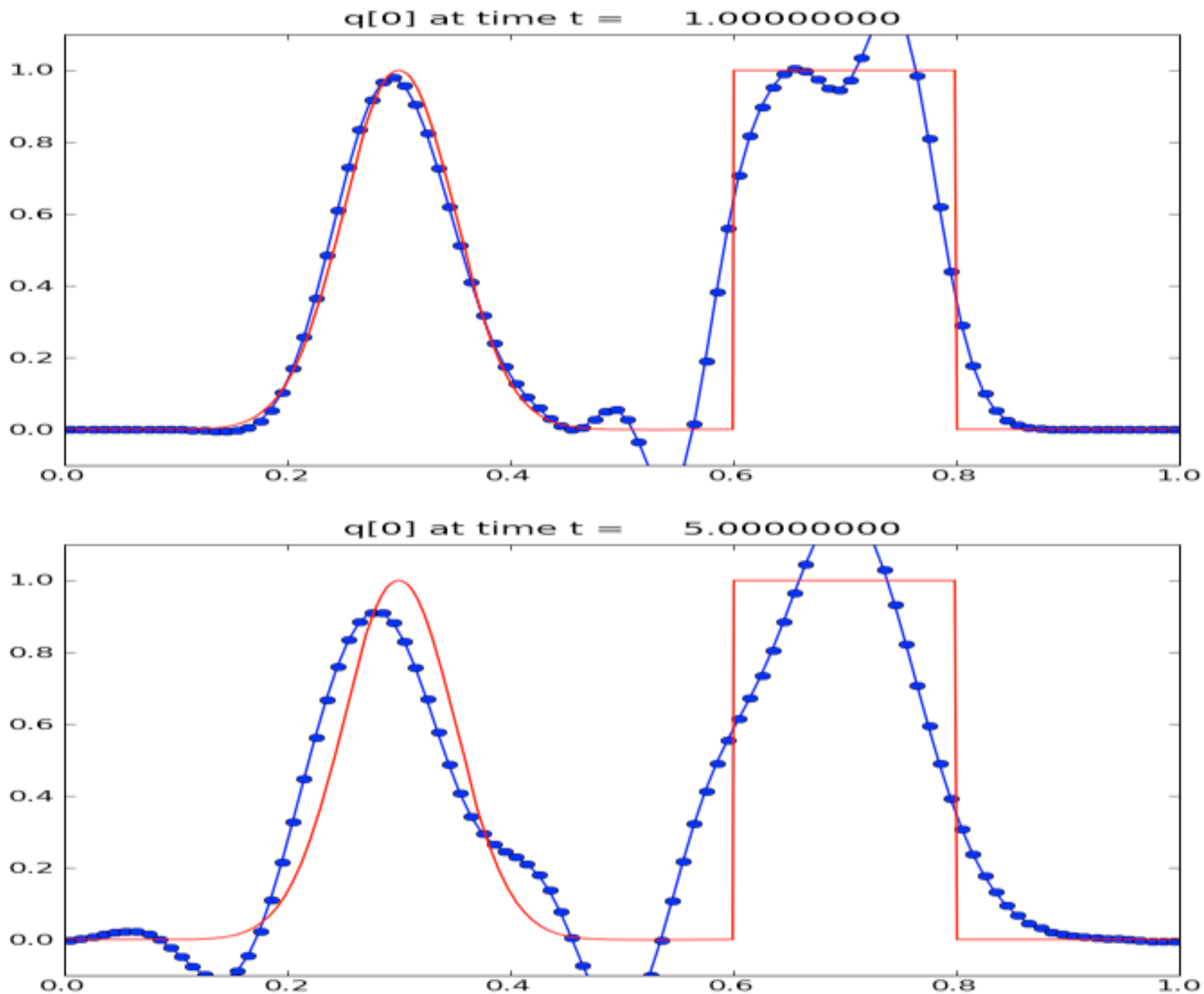
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\left(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n\right)$$

Demonstration of simple methods



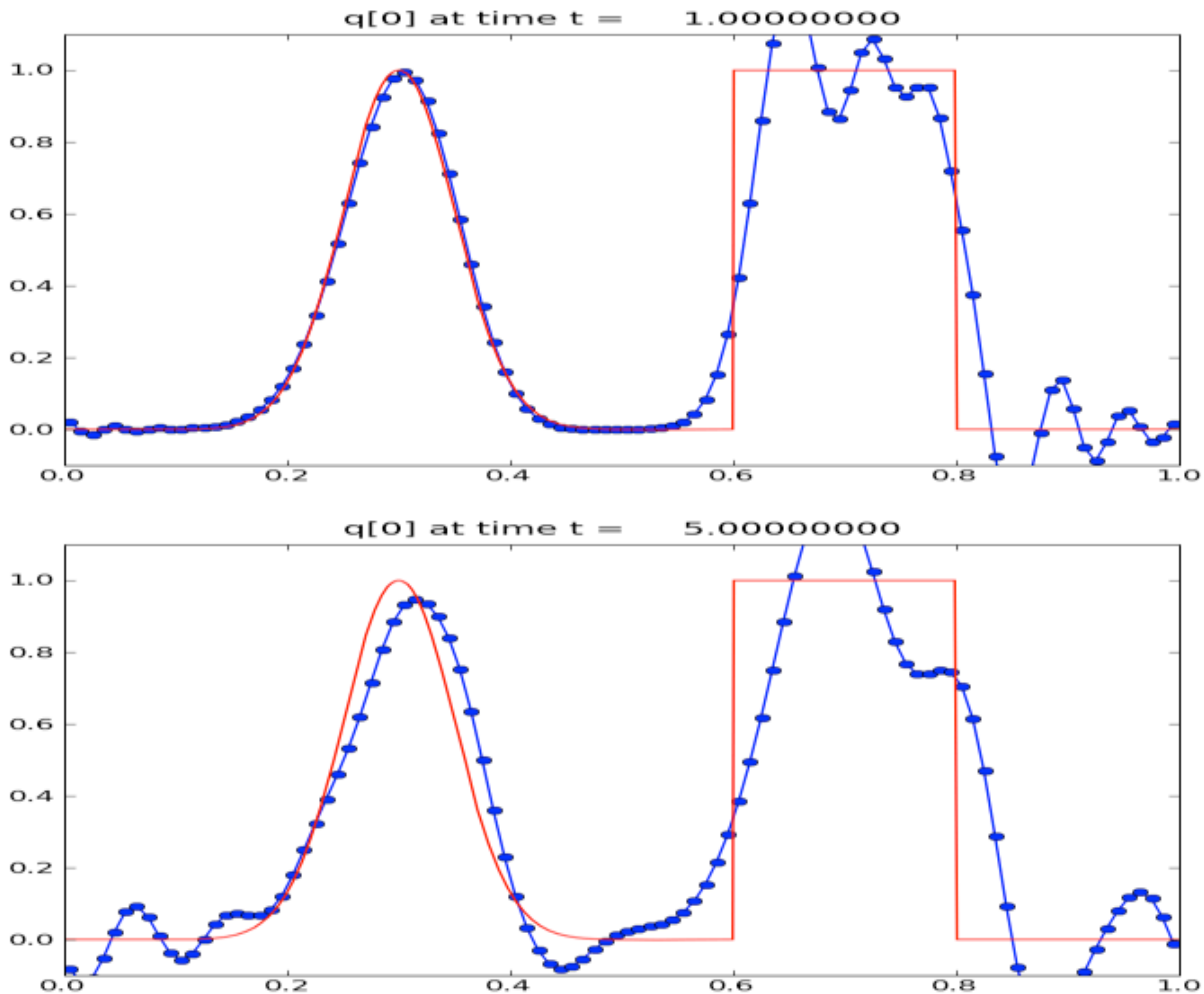
1st Order Godunov (Lax-Friedrichs)

Demonstration of simple methods



2nd Order Lax-Wendroff

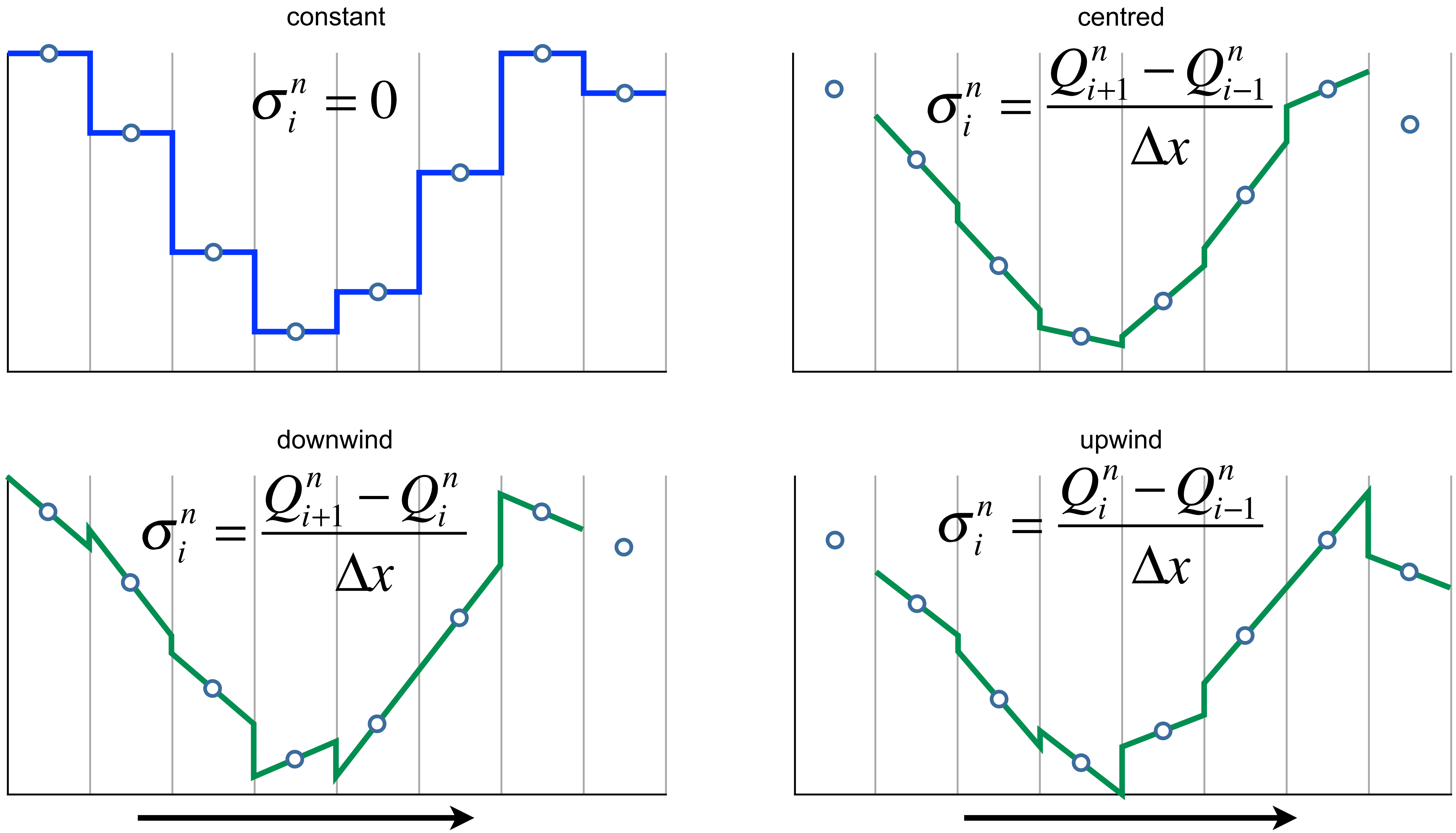
Demonstration of simple methods



2nd Order Beam-Warming

Why is Lax-Wendroff Oscillatory?

We can choose a variety of slopes for a piecewise linear reconstruction.



The aim is to approximate the derivative over the i^{th} cell, for second-order accuracy. The overshoots in these methods cause oscillatory behaviour near discontinuities.

Why is Lax-Wendroff Oscillatory?

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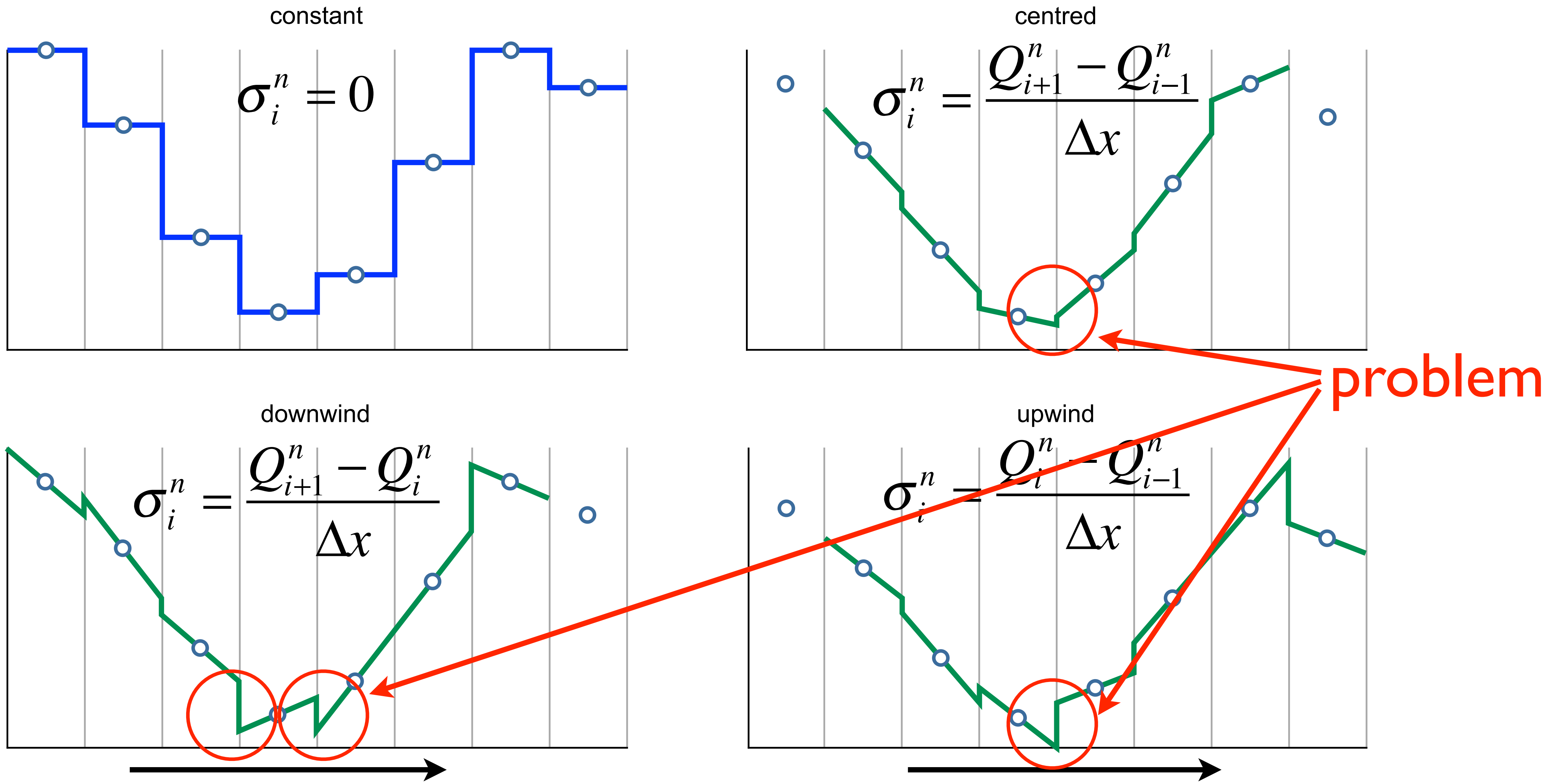


The aim is to approximate the derivative over the i^{th} cell, for second-order accuracy. The overshoots in these methods cause oscillatory behaviour near discontinuities.

Need for limiters

Second-order methods give good results when the solutions are smooth but generate oscillations where discontinuities occur.

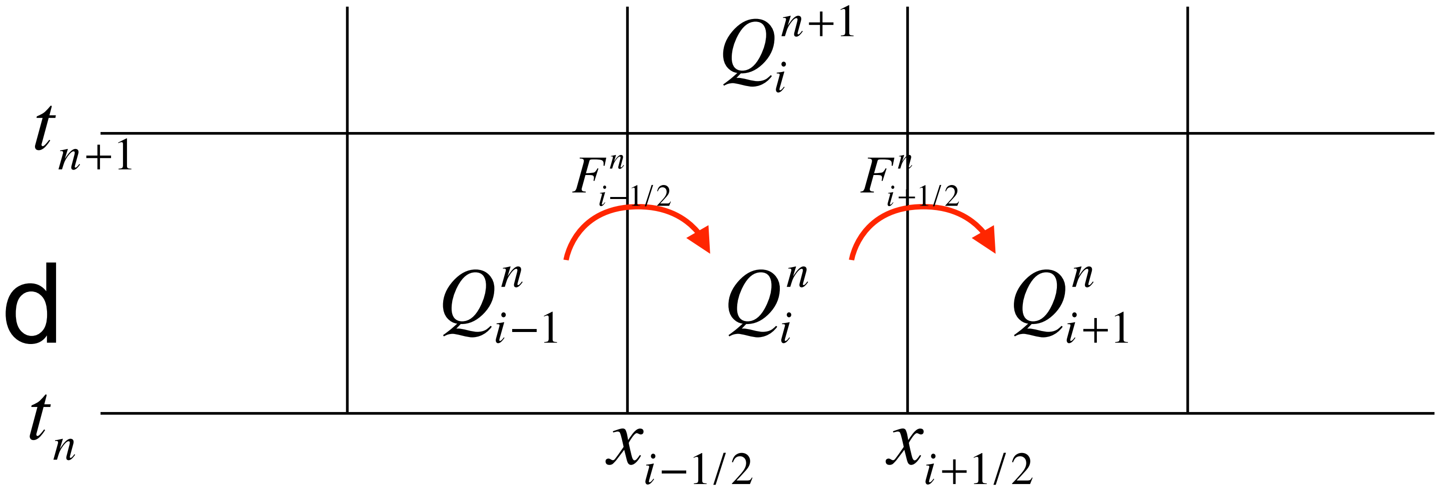
First-order methods give poorer results, but do not generate oscillations near discontinuities. That is, they keep the solution varying *monotonically*.

The idea behind high-resolution methods is to get second-order accuracy when possible, but to keep the solution *monotonic* where the solution is not smooth.

Limiters are introduced to manage this.

The breakthrough work in this area was made by Bram van Leer in a series of papers culminating in 1979.

Lax-Wendroff as a finite-volume method



The basic finite-volume update formula is $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

We can put Lax-Wendroff in this form if we write:

$$F_{i+1/2}^n = \frac{1}{2}A\left(Q_{i+1}^n + Q_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2\left(Q_{i+1}^n - Q_i^n\right)$$

$$F_{i-1/2}^n = \frac{1}{2}A\left(Q_i^n + Q_{i-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2\left(Q_i^n - Q_{i-1}^n\right)$$

then:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}A\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

First we rewrite the Lax-Wendroff flux

$$F_{i-1/2}^{n} = \frac{1}{2} A\left(Q_i^n + Q_{i-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A^2\left(Q_i^n - Q_{i-1}^n\right)$$

using

$$\begin{cases} A^{\pm} = R\Lambda^{\pm}R^{-1} \\ A = A^{+} + A^{-} \\ \left|A\right| = A^{+} - A^{-} \end{cases} \qquad \Lambda^{\pm} = \begin{bmatrix} (\lambda^1)^{\pm} & & & \\ & (\lambda^2)^{\pm} & & \\ & & \ddots & \\ & & & (\lambda^m)^{\pm} \end{bmatrix}$$

we get

$$F_{i-1/2}^{n} = \left(A^{-}Q_i^n + A^{+}Q_{i-1}^n\right) + \frac{1}{2}\left|A\right|\left(I - \frac{\Delta t}{\Delta x}\left|A\right|\right)\left(Q_i^n - Q_{i-1}^n\right)$$

which is like the upwind flux with an added (antidiffusive) correction term. We can improve this by limiting the amount of correction actually applied, based on the solution behaviour.

We illustrate how this is done with the scalar advection equation.

How do we choose a slope limiter?

We want to use the slope when the function is smooth to achieve second-order accuracy.

But when the function is not smooth, using the slope results in overshoots, causing oscillatory behaviour.

So we *limit the slope*, based on the local behaviour of the solution.

We write the slope as $\sigma_i^n = \left(\dfrac{Q_{i+1}^n - Q_i^n}{\Delta x} \right) \phi_i^n$, where ϕ is the *flux limiter function*, and

$\phi = 1$ in the Lax-Wendroff scheme,

$\phi = 0$ in the piecewise-constant upwind scheme.

The REA algorithm suggests ...

that we update the advection equation by

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

where the slope is given by

$$\sigma_i^n = \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\phi_i^n$$

and ϕ is the *flux limiter function*.

How do we choose a slope limiter?

Monotonicity preserving methods:

If a grid function that is initially monotone, i.e. $Q_i^n \geq Q_{i-1}^n$ for all i at step n

remains monotone at the next time: $Q_i^{n+1} \geq Q_{i-1}^{n+1}$ for all i at step $n+1$

then the method is monotonicity preserving.

Total Variation Diminishing (TVD) methods:

Define the total variation of a grid function Q as: $\mathrm{TV}(Q) = \sum_{\text{grid}} \left| Q_i - Q_{i-1} \right|$

A method is Total Variation Diminishing if $\mathrm{TV}(Q^{n+1}) \leq \mathrm{TV}(Q^n)$

TVD methods are monotonicity preserving. We chose slope limiters that ensure the method is TVD.

The minmod slope limiter

Define the minmod function: $$\text{minmod}(a,b) \equiv \begin{cases} a \text{ if } |a| < |b| \text{ and } ab > 0 \\ b \text{ if } |b| < |a| \text{ and } ab > 0 \\ 0 \text{ if } ab \leq 0 \end{cases}$$

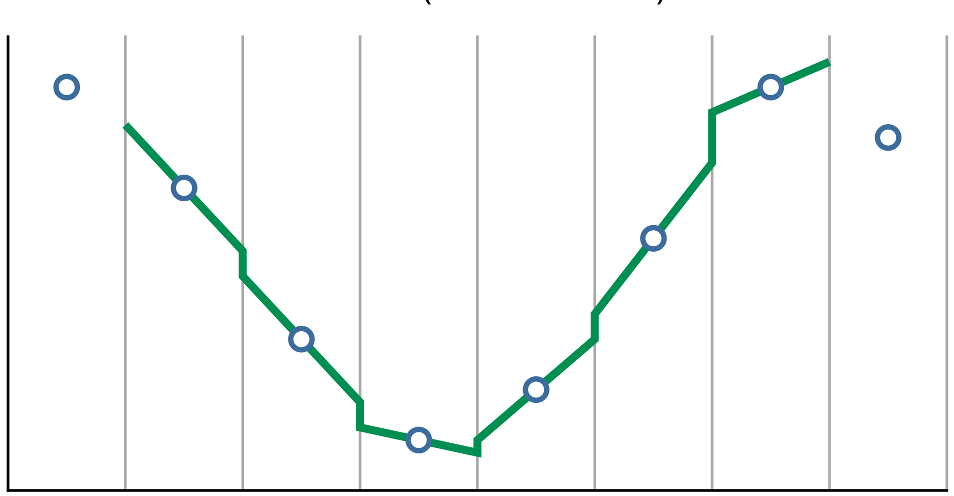
Then choose the slope to use by:

$$\sigma_i^n = \text{minmod}\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}, \frac{Q_{i+1}^n - Q_i^n}{\Delta x} \right)$$

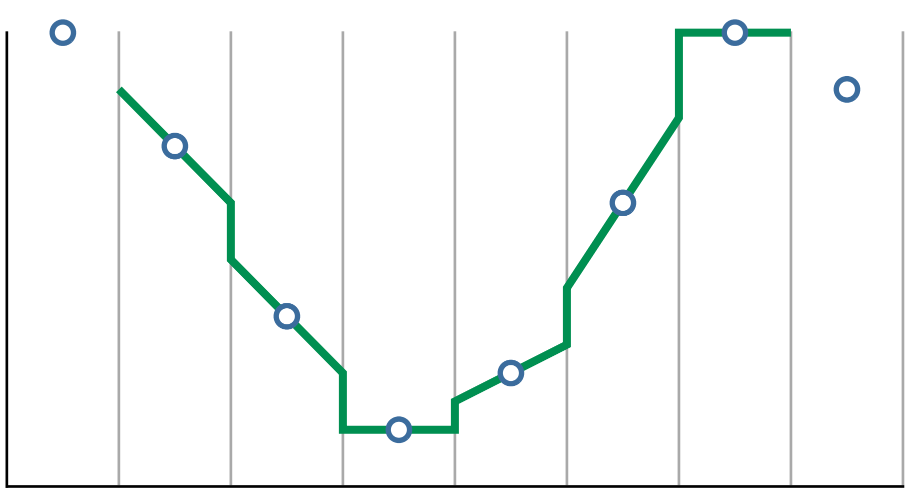
If the slopes have the same sign, the one with the smaller absolute value is chosen; if they have opposite signs, the slope is 0.

Extended to more arguments, the minmod function returns 0 unless all the arguments are the same sign, otherwise it returns the argument with the smallest absolute value.

centred (Lax-Wendroff)



minmod



For generality, we write the slope in terms of the flux-limiter function ϕ

For minmod:

$$\sigma_i^n = \text{minmod}\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}, \frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)$$

$$= \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\phi(\theta_i^n)$$

where $\phi(\theta) = \text{minmod}(\theta, 1)$

and $\theta_i^n = \dfrac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n}$

θ measures the local smoothness of the data. At extrema, θ is negative; if the data are smooth, $\theta \approx 1$ and at discontinuities, θ can be very large.

Widely used flux limiters are:

Linear methods

upwind: $\phi(\theta) = 0$

Lax-Wendroff: $\phi(\theta) = 1$

Beam-Warming: $\phi(\theta) = \theta$

Fromm: $\phi(\theta) = \frac{1}{2}(1+\theta)$

High-resolution methods

minmod: $\phi(\theta) = \text{minmod}(1,\theta)$

superbee: $\phi(\theta) = \max(0,\min(1,2\theta),\min(2,\theta))$

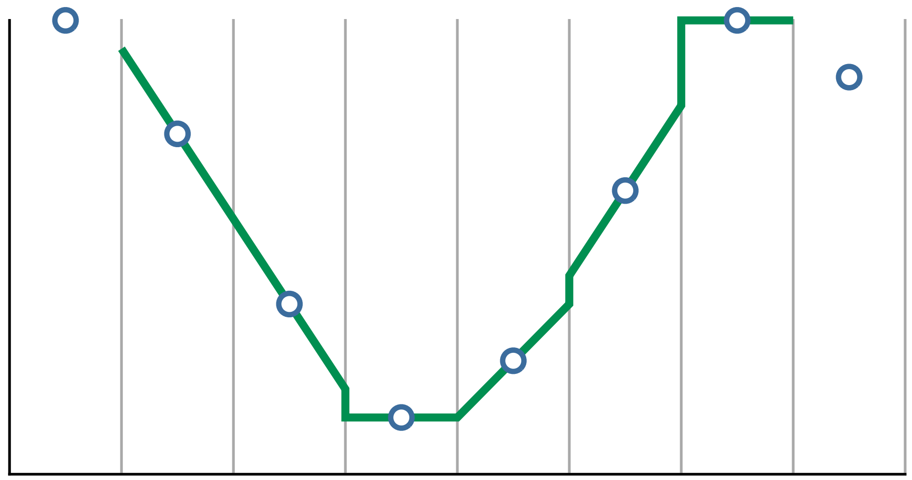
MC: $\phi(\theta) = \max(0,\min((1+\theta)/2,2,2\theta))$

vanLeer: $\phi(\theta) = \frac{(\theta+|\theta|)}{(1+|\theta|)}$

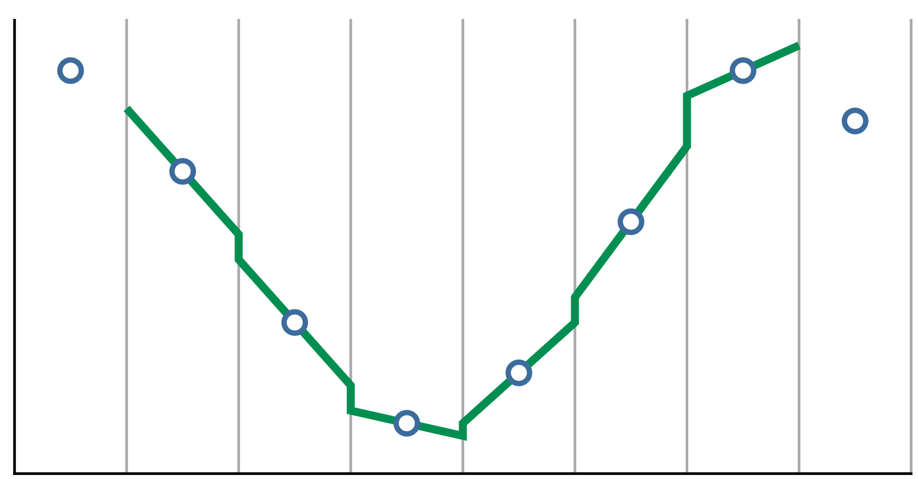
Comparing minmod, superbee, MC and vanLeer limiters

cell	data	left slope	right slope	centred slope	theta	minmod	superbee	MC	vanLeer
0.5	8		-2						
1.5	6	-2	-3	-2.5	0.666667	-2	-3	-2.5	-2.4
2.5	3	-3	-2	-2.5	1.5	-3	-3	-2.5	-2.4
3.5	1	-2	1	-0.5	-2	0	0	0	0
4.5	2	1	3	2	0.333333	1	2	2	1.5
5.5	5	3	3	3	1	3	3	3	3
6.5	8	3	-1	1	-3	3	0	0	0
7.5	7	-1							

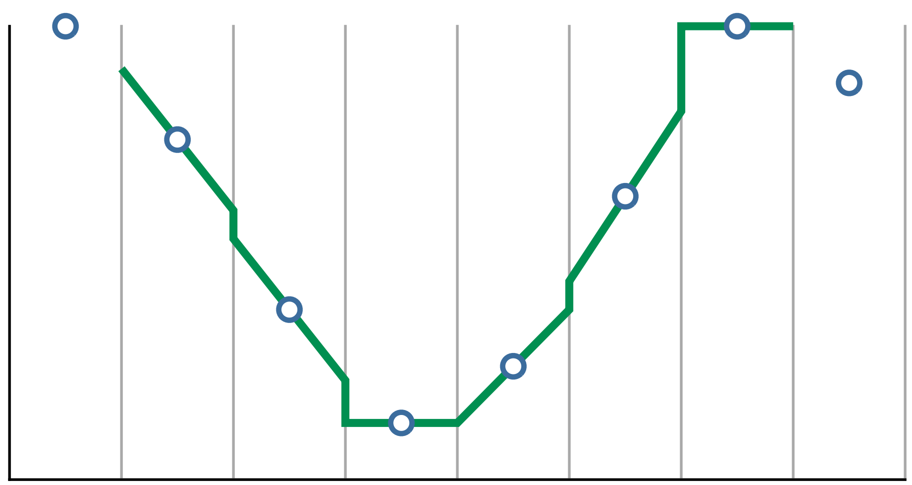
superbee



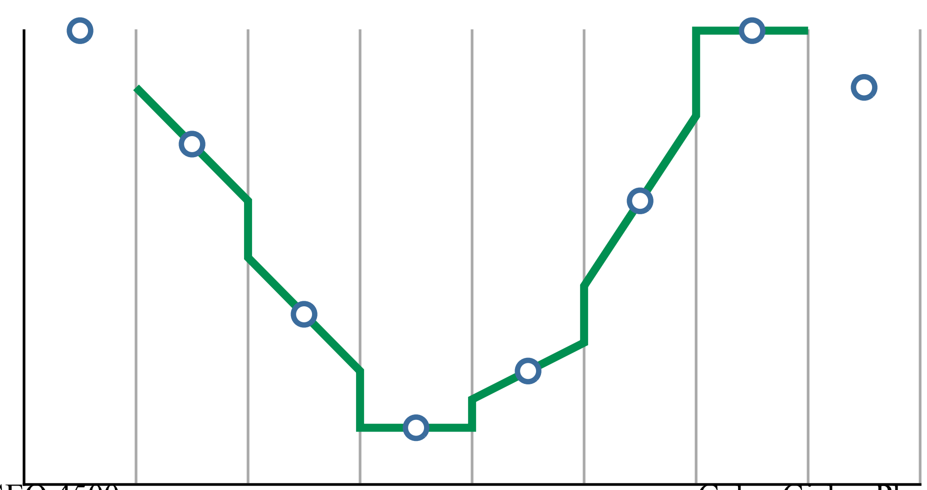
centred



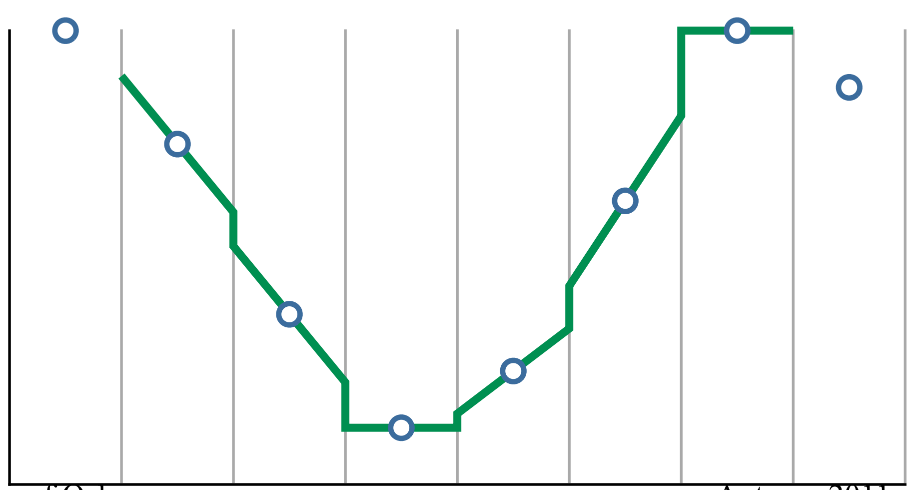
MC



minmod



vanLeer



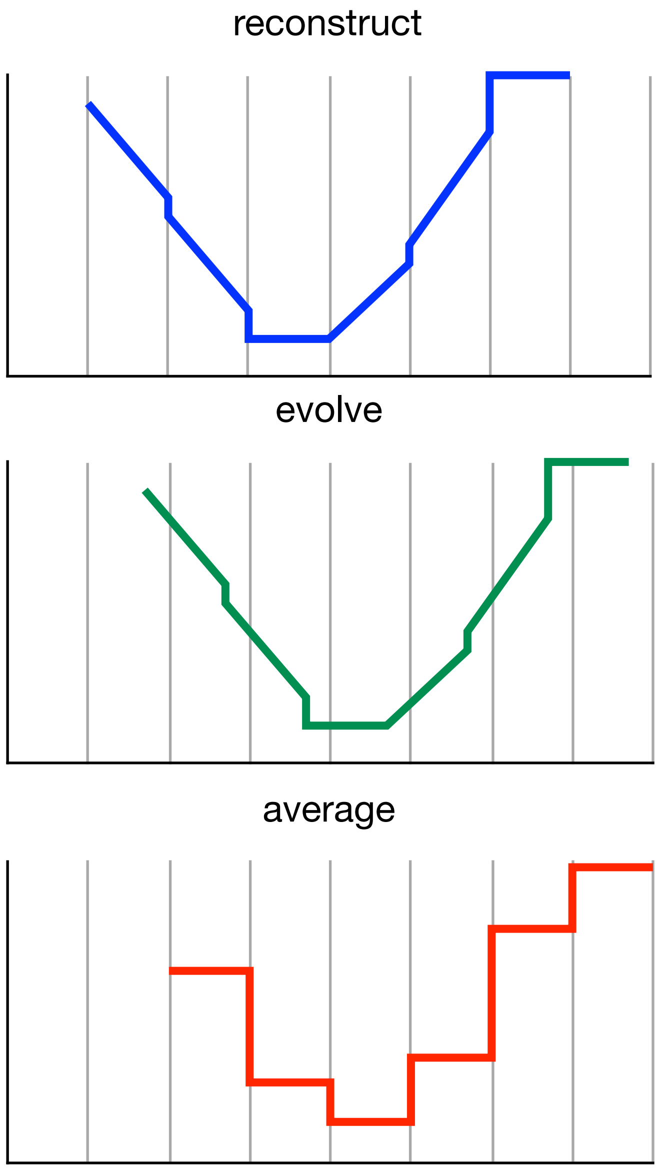
Now we have an REA second order scheme

The steps are identical to the first order REA scheme, except for reconstruction:

1. Reconstruct a piece-wise **linear** function from the cell averages.
$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$
with the property that $\mathrm{TV}(q) \le \mathrm{TV}(Q)$

2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces.
$$\tilde{q}^n(x,t_{n+1})$$

3. Average this function over each grid cell to obtain new cell averages.
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



The reconstruction step depends on the slope limiter that is chosen, and should be subject to TVD constraints. The other two steps do not affect TVD.

Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n(x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

Reconstruct - Evolve - Average

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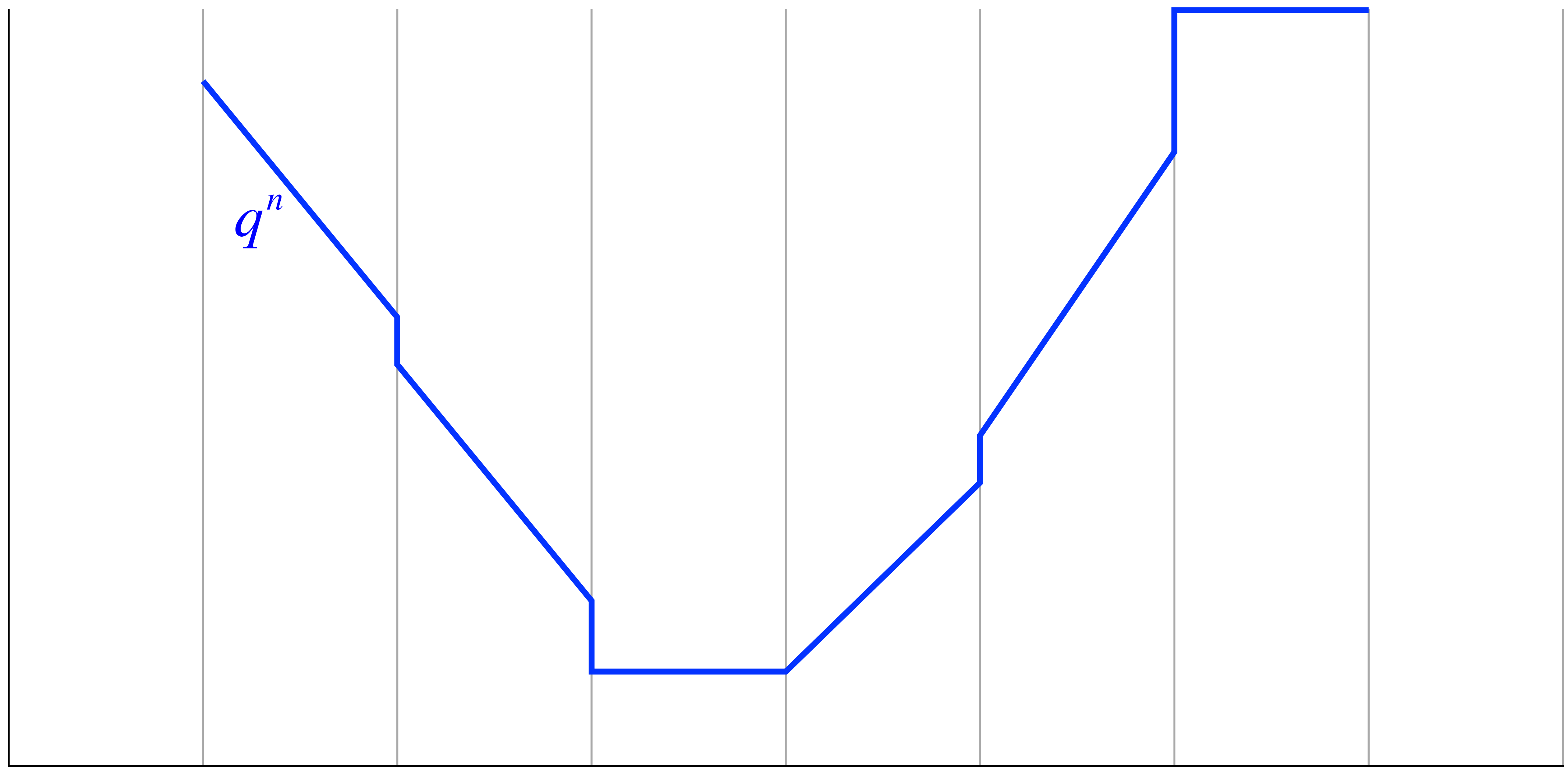
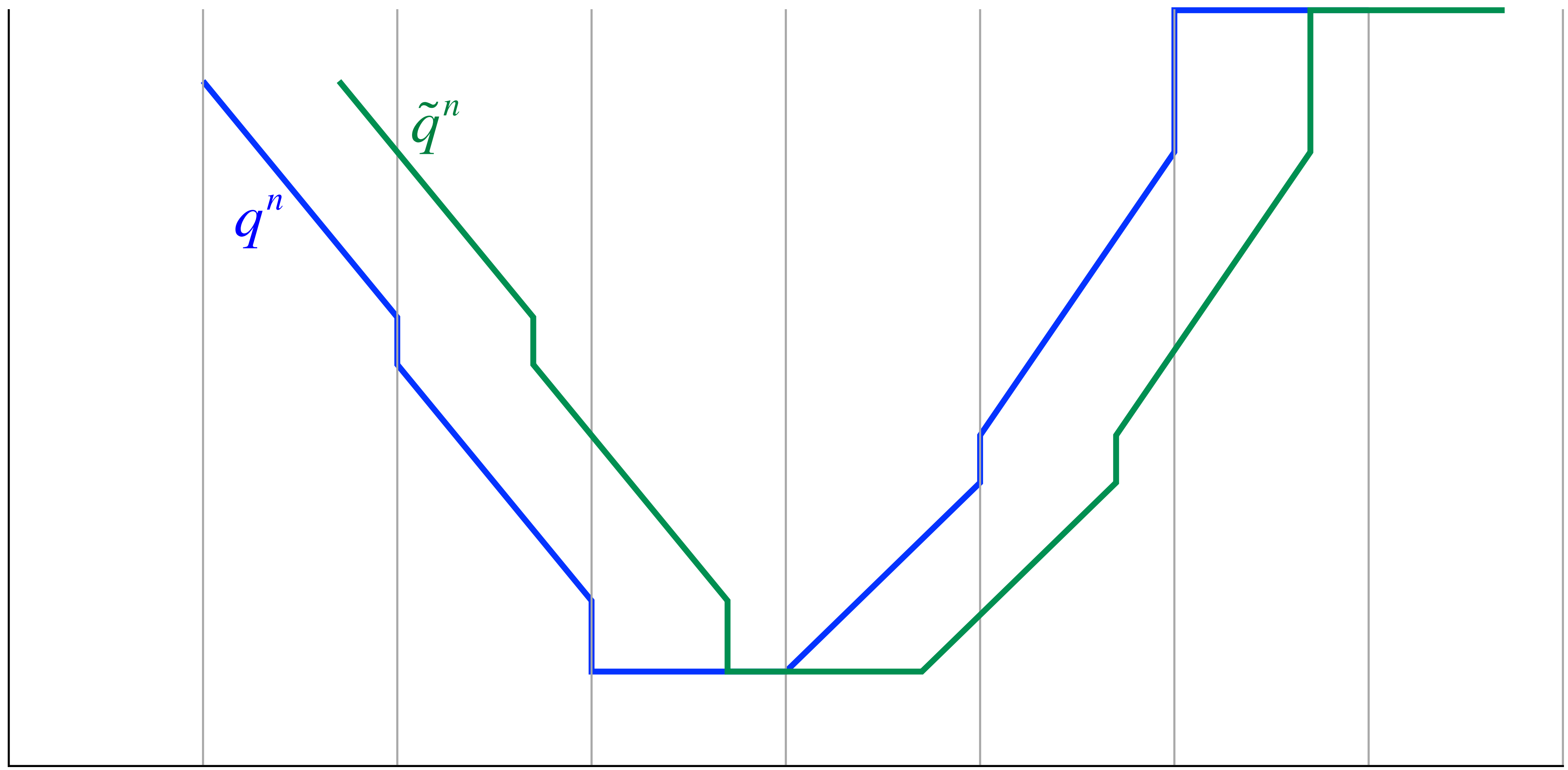


Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1}) dx$$

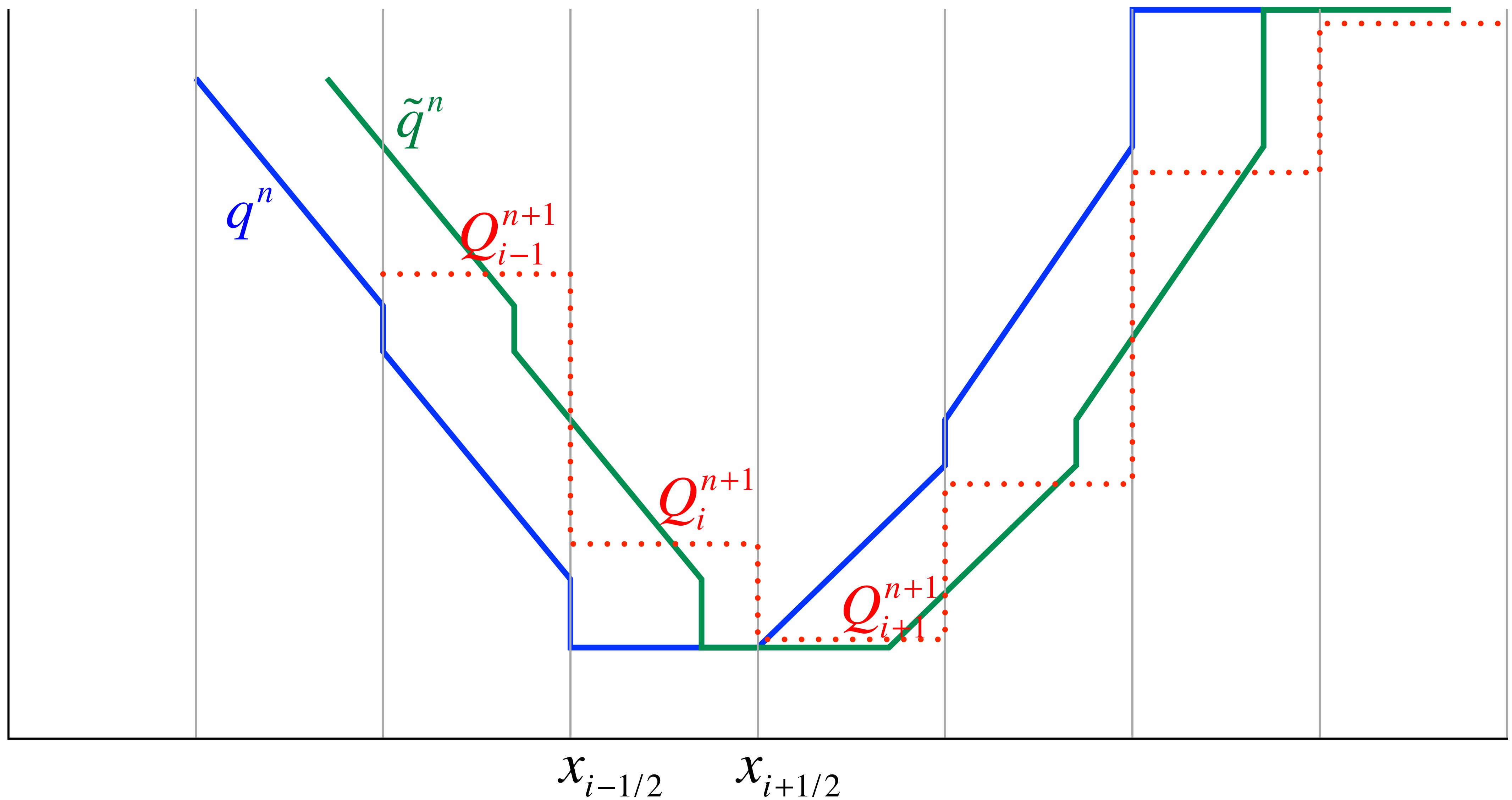


Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$

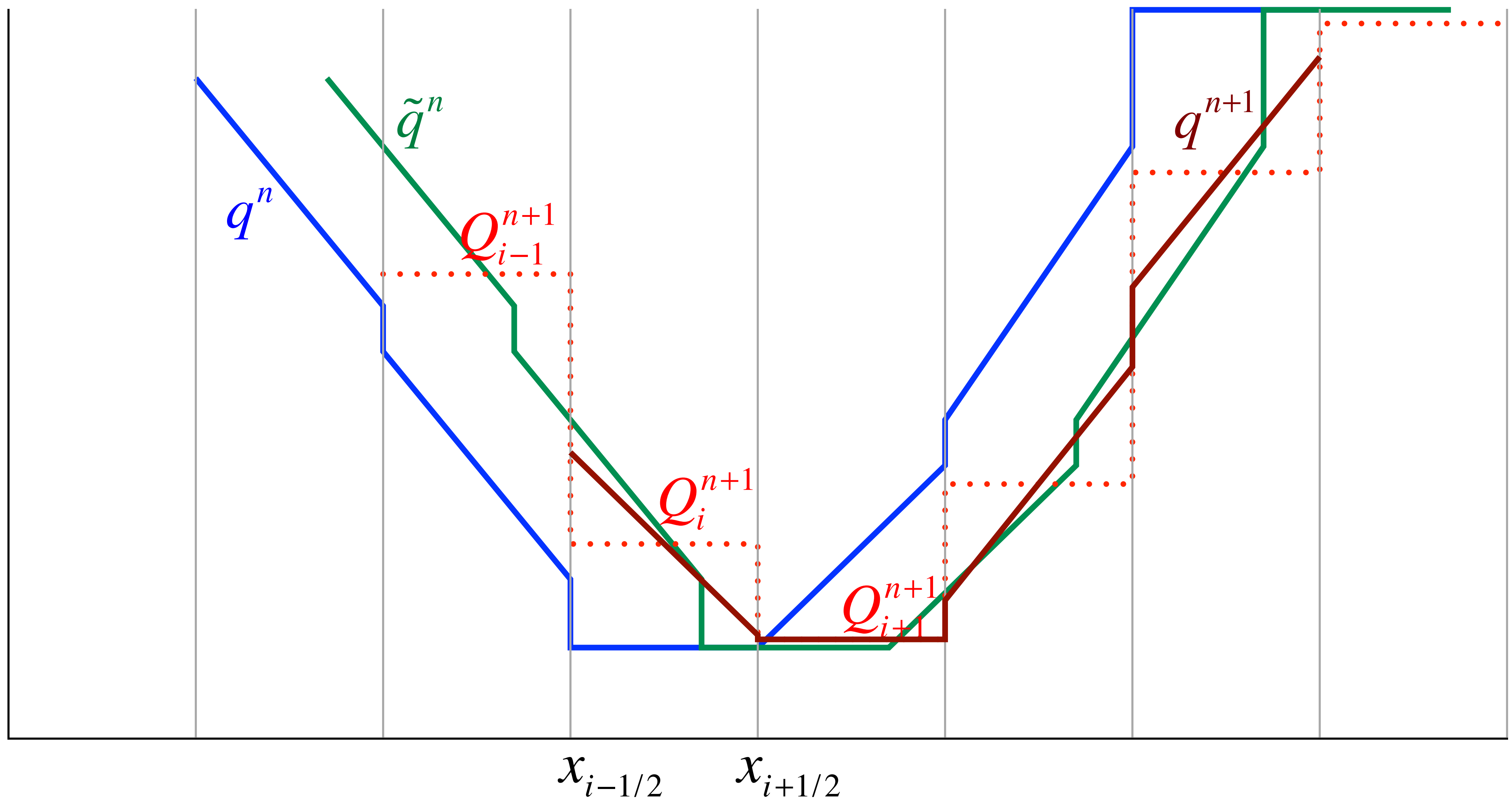


Reconstruct - Evolve - Average

$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$

$$\tilde{q}^n(x,t_{n+1})$$

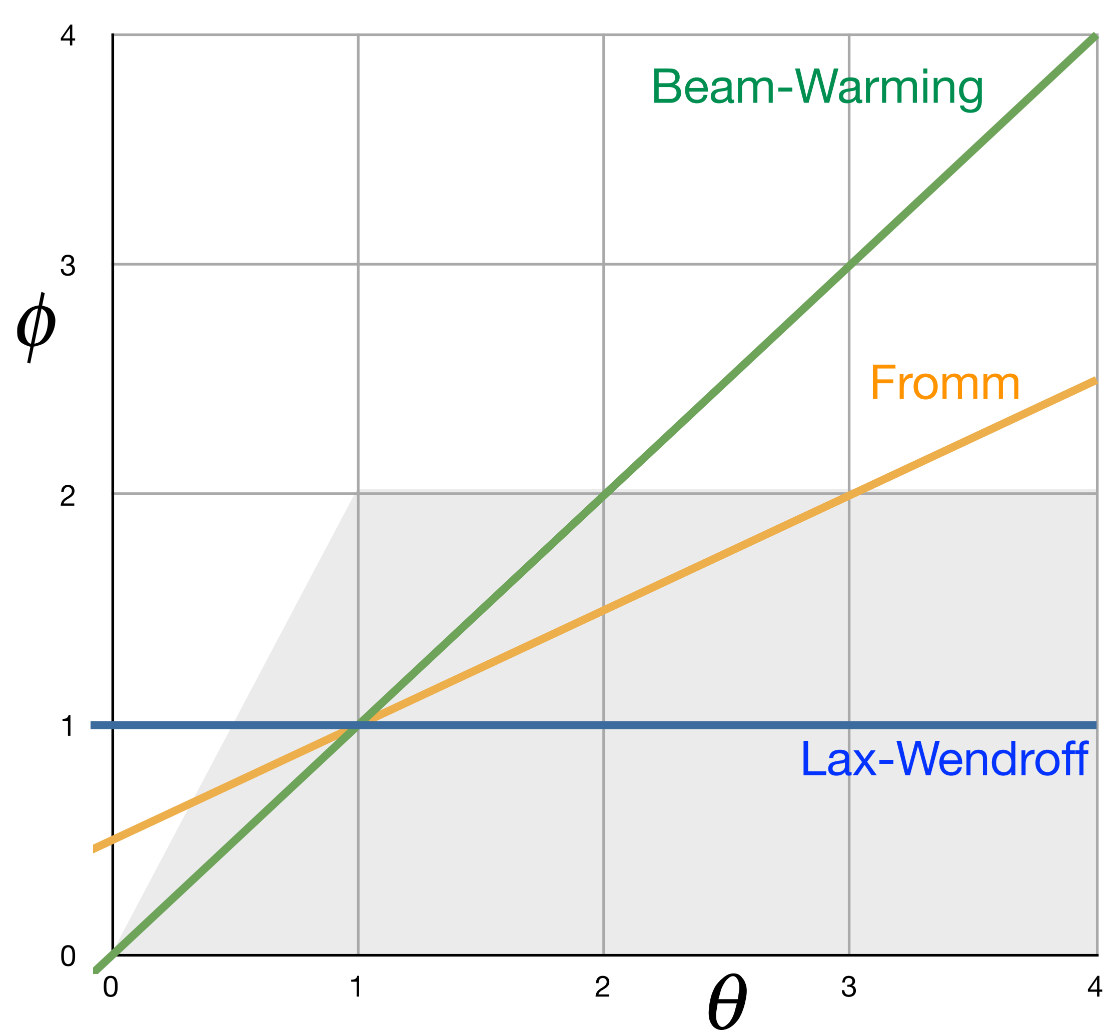
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



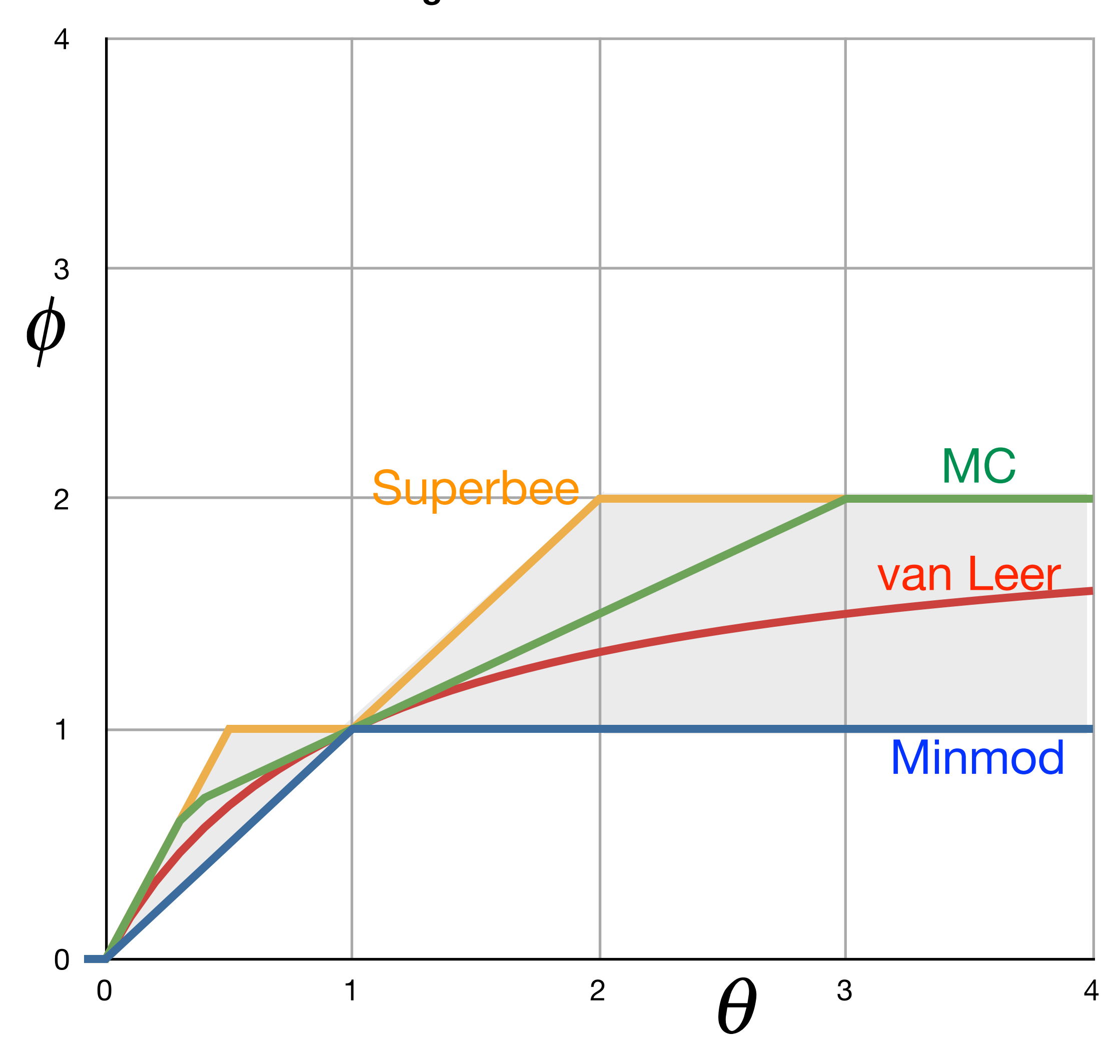
… and then a new piecewise linear reconstruction is done …

How do we make sure we satisfy the Total Variation Diminishing Constraint?

Compare the limiter functions $\phi(\theta)$ where $\theta = \dfrac{\Delta Q_{upwind}}{\Delta Q_{downwind}}$.

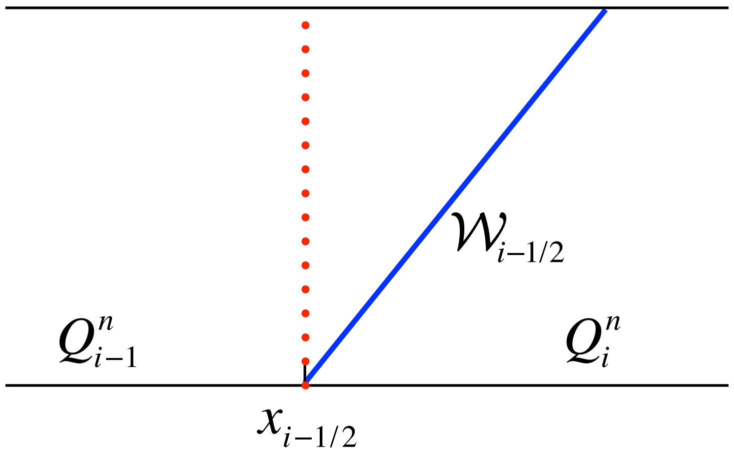


TVD is satisfied when $0 \le \phi(\theta) \le \mathrm{minmod}(2, 2\theta)$



Sweby's region where TVD is satisfied is shaded. Any second-order accurate method must have $\phi(1) = 1$.

Wave limiters



We can think of slope limiters as limiters on the wave strengths. Let $\mathcal{W}_{i-1/2} = Q_i - Q_{i-1}$.

Then the upwind method for the scalar advection equation is

$$Q_i^{n+1} = Q_i^n - u\frac{\Delta t}{\Delta x}\mathcal{W}_{i-1/2}.$$

The Lax-Wendroff method is:

$$Q_i^{n+1} = Q_i^n - u\frac{\Delta t}{\Delta x}\mathcal{W}_{i-1/2} - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right),$$

where

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|u\frac{\Delta t}{\Delta x}\right|\right)|u|\mathcal{W}_{i-1/2}.$$

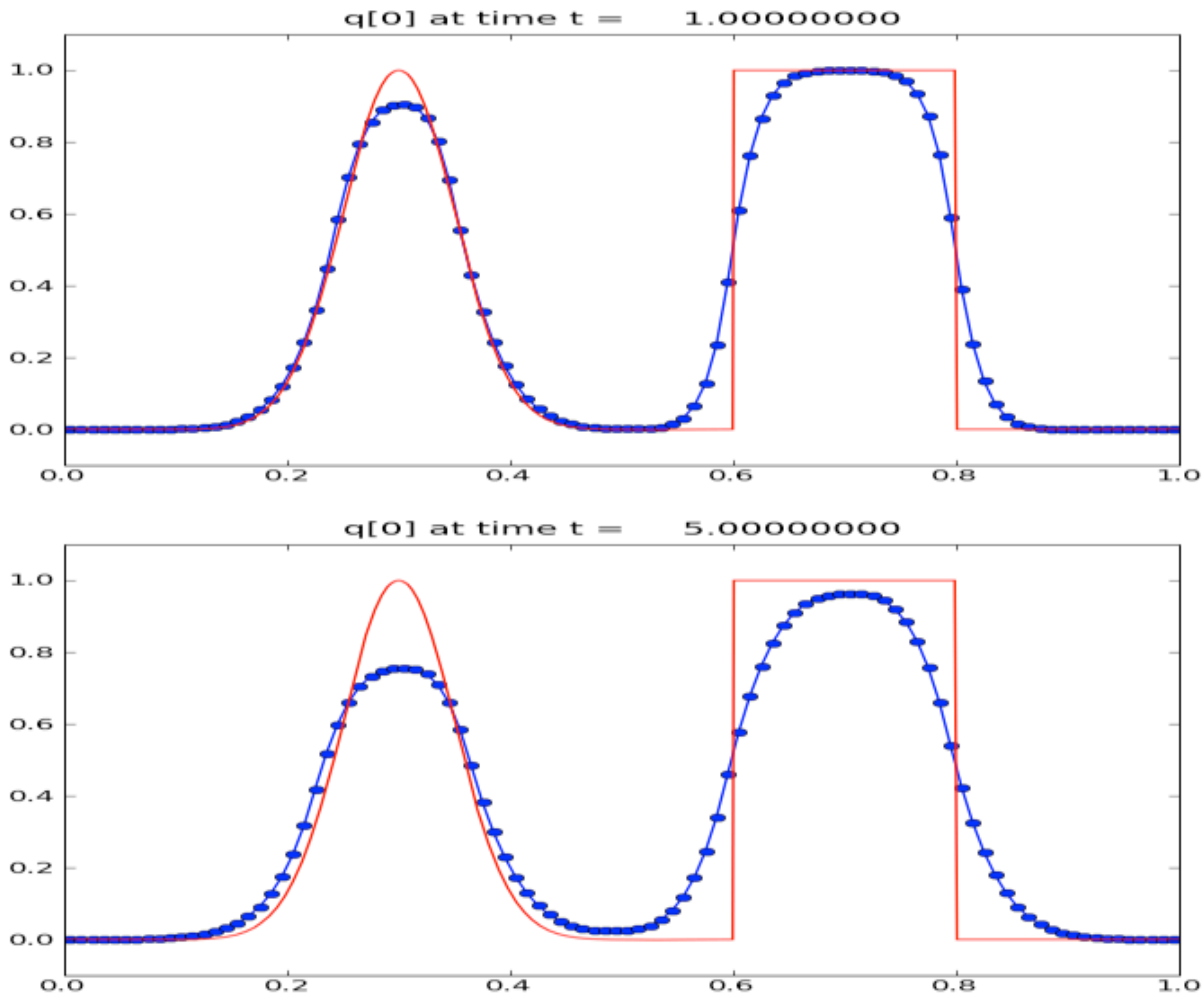
For a high-resolution we use

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|u\frac{\Delta t}{\Delta x}\right|\right)|u|\widetilde{\mathcal{W}}_{i-1/2},$$

where

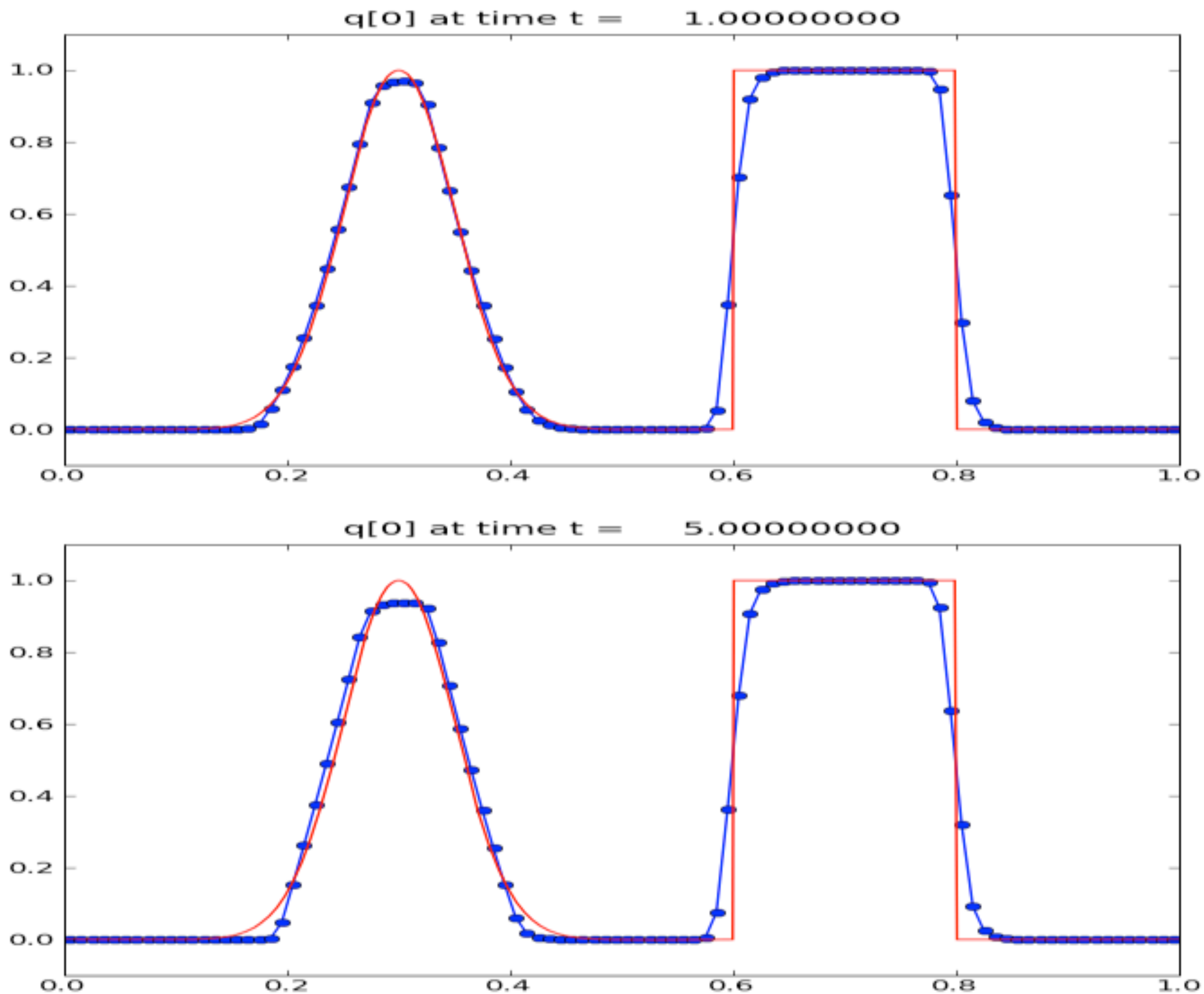
$$\widetilde{\mathcal{W}}_{i-1/2} = \phi_{i-1/2}\mathcal{W}_{i-1/2}.$$

Demonstration of methods with limiters



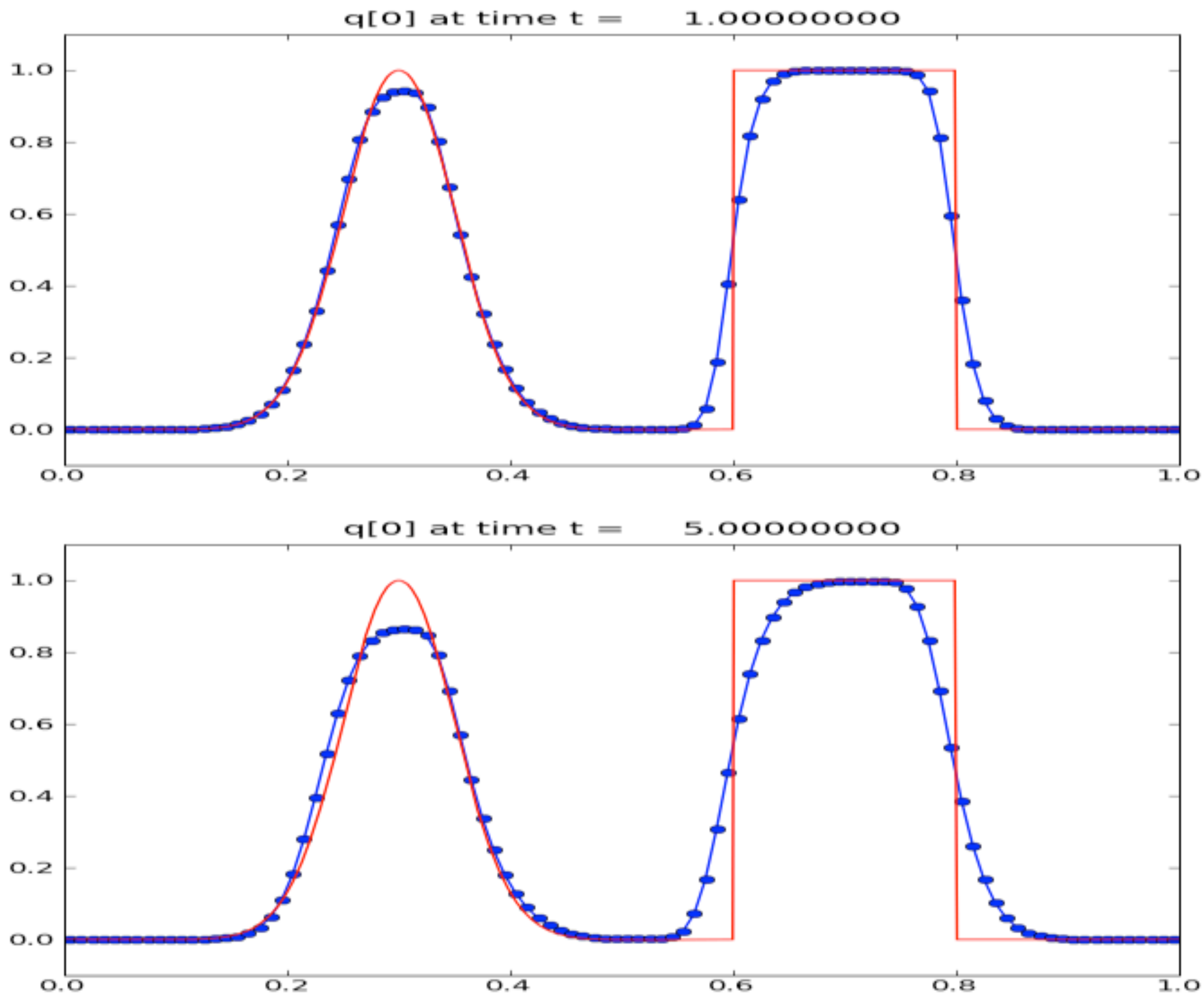
Minmod

Demonstration of methods with limiters



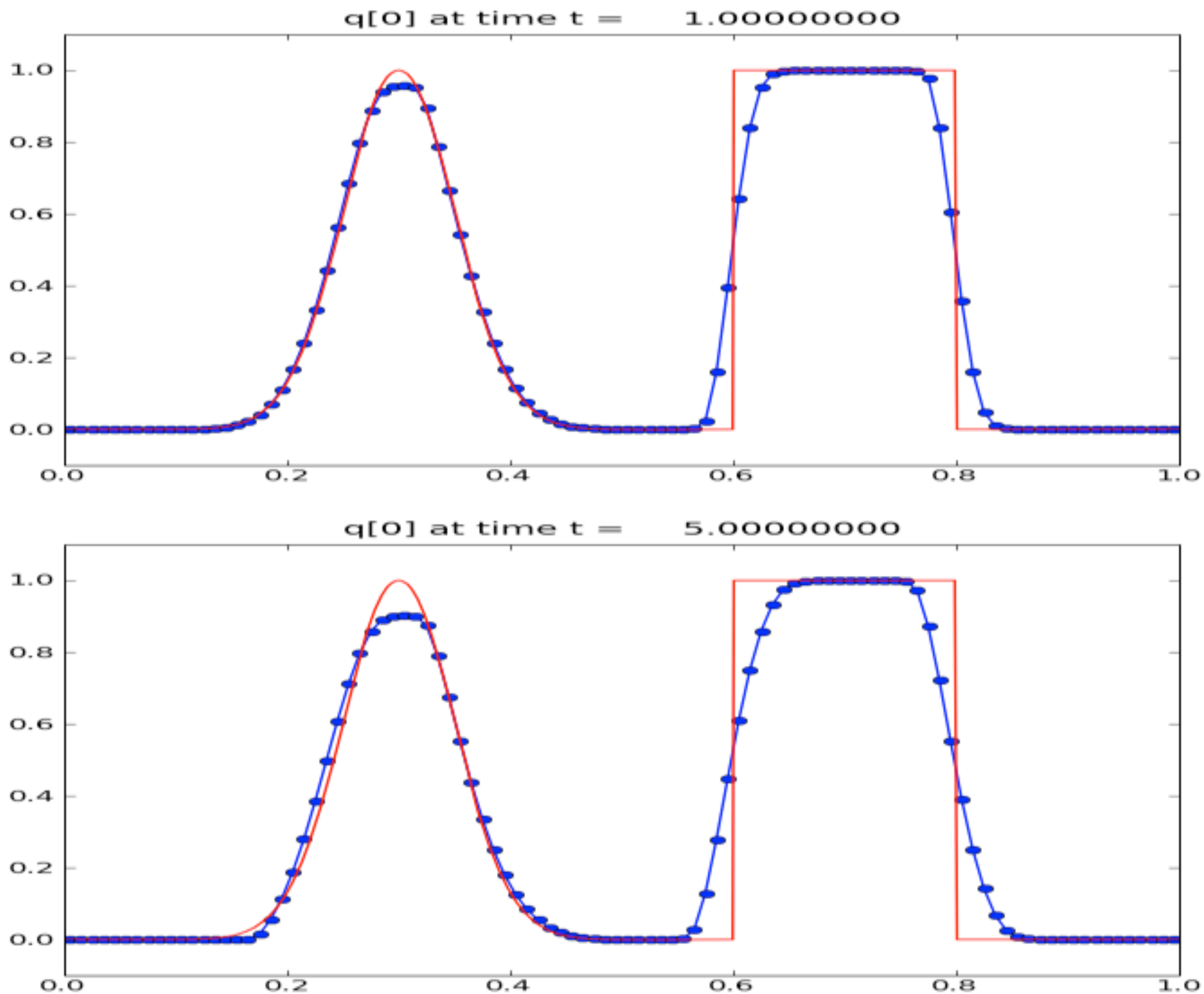
Superbee

Demonstration of methods with limiters



van Leer

Demonstration of methods with limiters



Monotonised Centred

Set order and limiters in setrun.py

setrun.py

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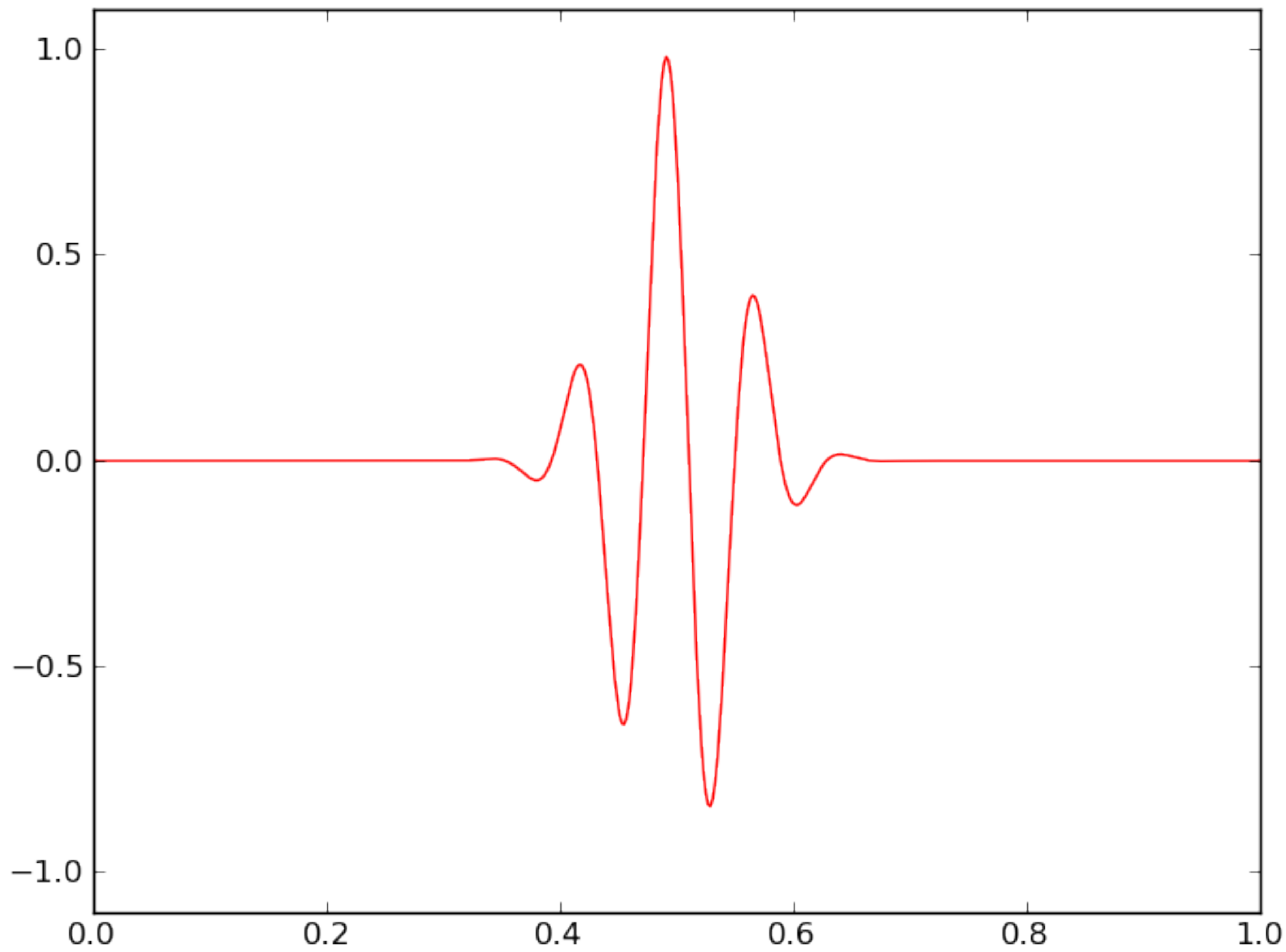
File Path ▾ : ~/ClawpackAll/clawpac.../compareadv/setrun.py

setrun.py | setrun

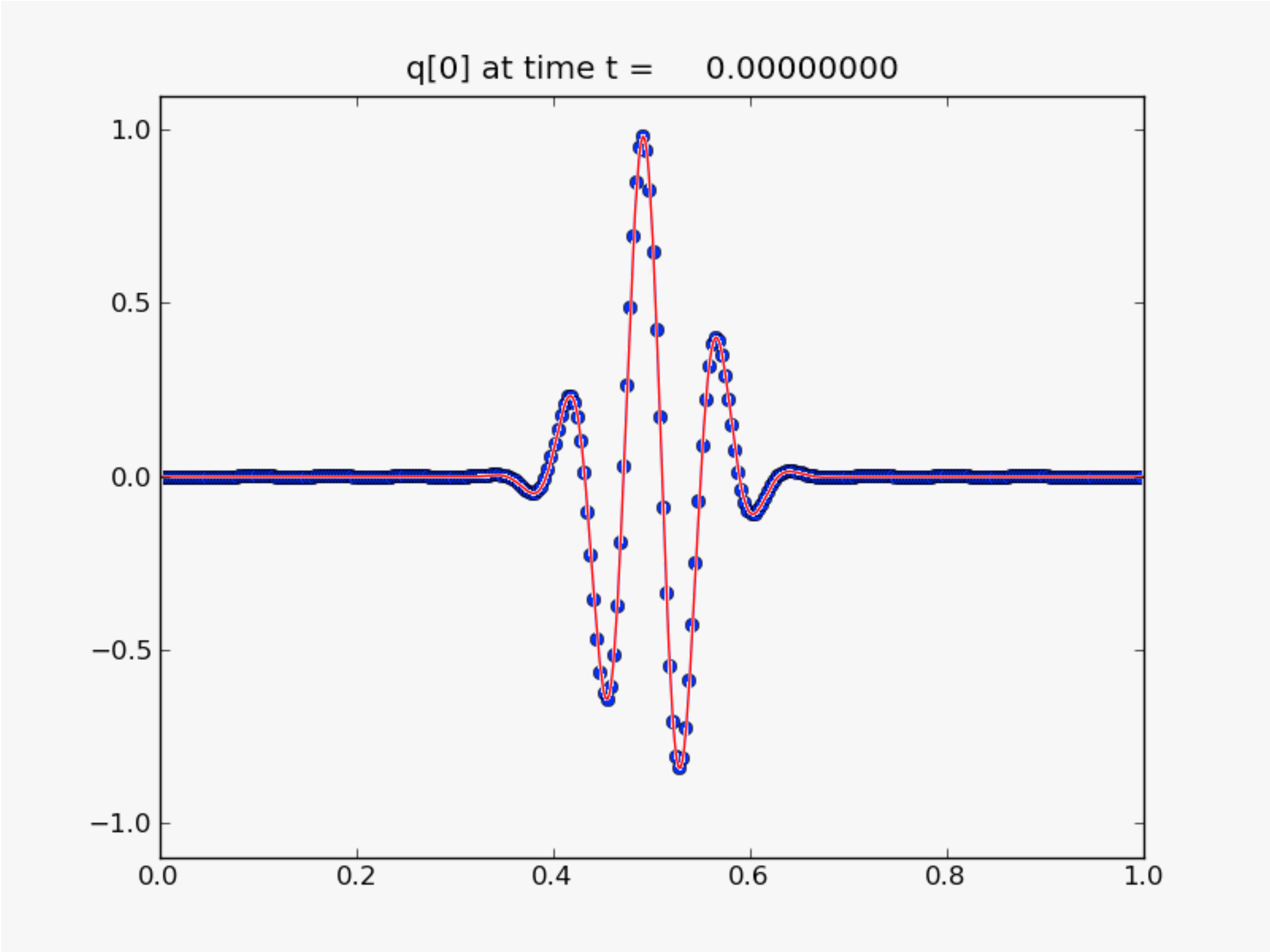
```
    # ------------------
    # Method to be used:
    # ------------------

    # Order of accuracy:  1 => Godunov,  2 => Lax-Wendroff plus limiters
    clawdata.order = 2

    # Transverse order for 2d or 3d (not used in 1d):
    clawdata.order_trans = 0

    # Number of waves in the Riemann solution:
    clawdata.mwaves = 1

    # List of limiters to use for each wave family:
    # Required:  len(mthlim) == mwaves
    # 1 => minmod, 2 => superbee, 3 => van Leer, 4 => monotonised centred
    clawdata.mthlim = [3]

    # Source terms splitting:
    #   src_split == 0  => no source term (src routine never called)
    #   src_split == 1  => Godunov (1st order) splitting used,
    #   src_split == 2  => Strang (2nd order) splitting used,  not recommended.
    clawdata.src_split = 0

```

165 | 25 | Python | Western (Mac OS Roman) | Unix (LF) | 5 719 / 649 / 205

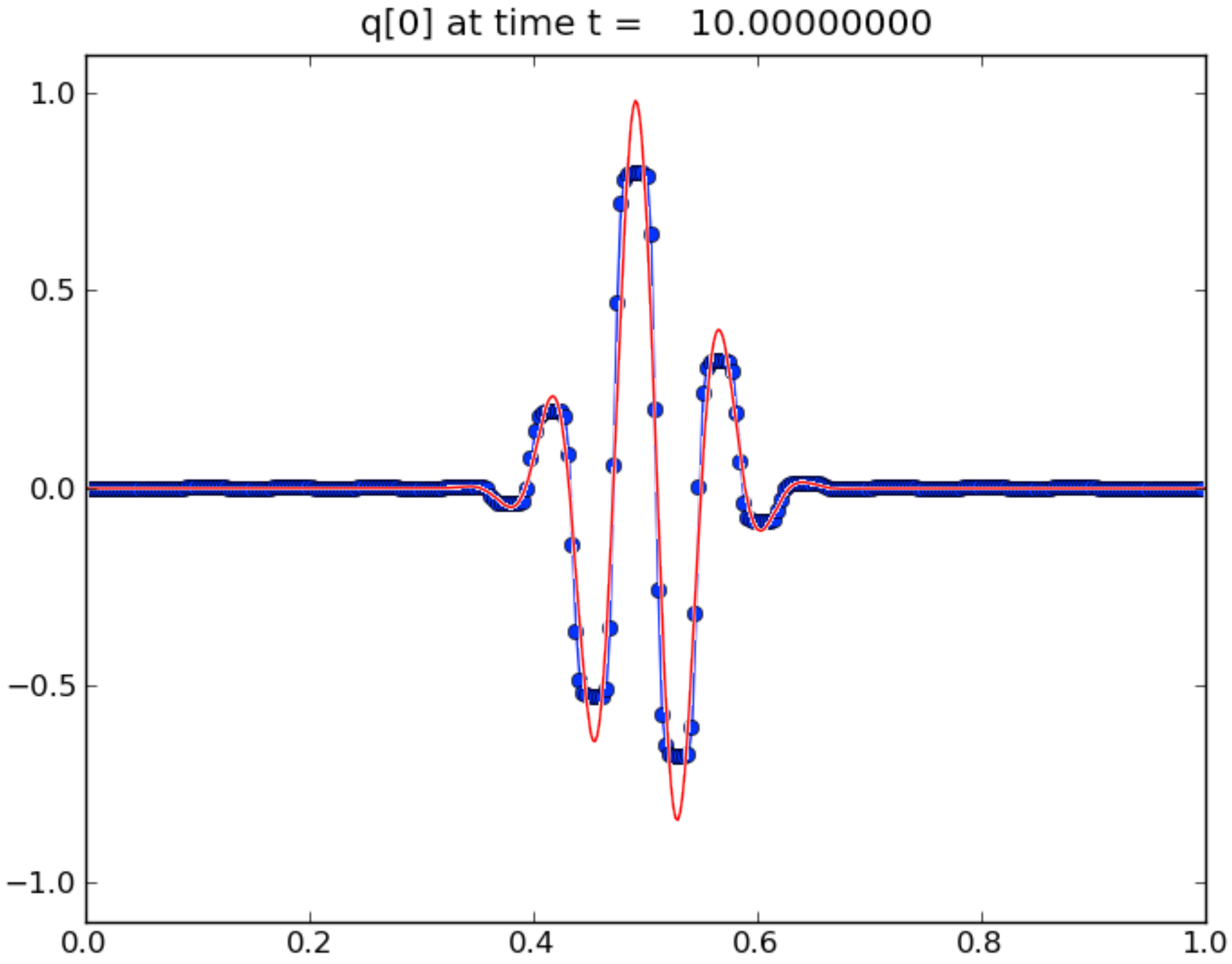
Wavepacket advection with superbee



Wavepacket advection with superbee

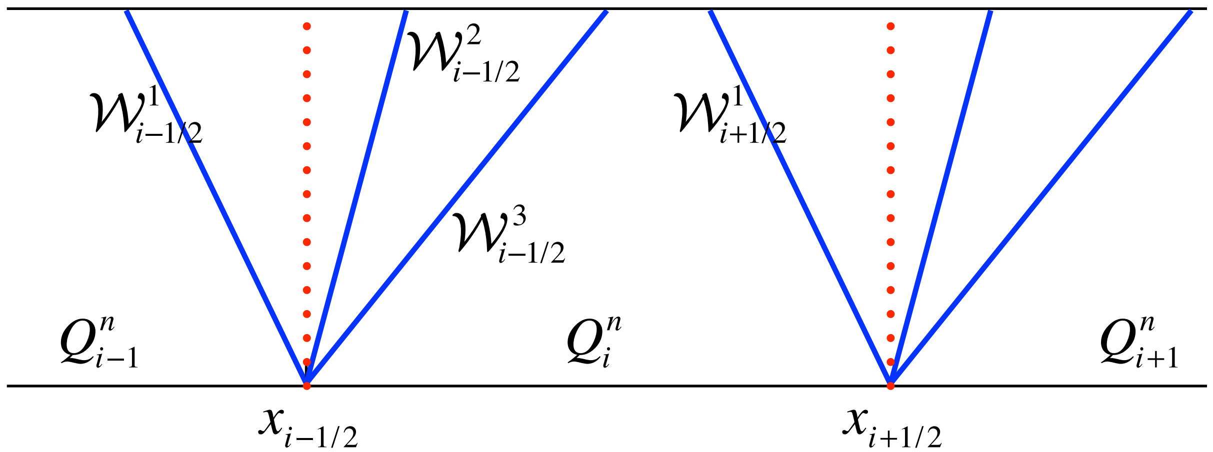


Wavepacket advection with superbee



Note that the extrema are *clipped*. This is a limitation of the methods with the limiters we've discussed.

Extension to linear systems



Approach 1:

Diagonalise the system to $q_t + \Lambda q_x = 0$

Apply the scalar algorithm to each component separately.

Approach 2:

Solve the linear Riemann problem to decompose $Q_i^n - Q_{i-1}^n$ into a number of waves.

Apply a wave limiter to each wave.

These approaches are equivalent, but we'll use the wave propagation method. Note that it is important to apply the limiters to the *waves* rather than to the original variables.

High-resolution methods for systems

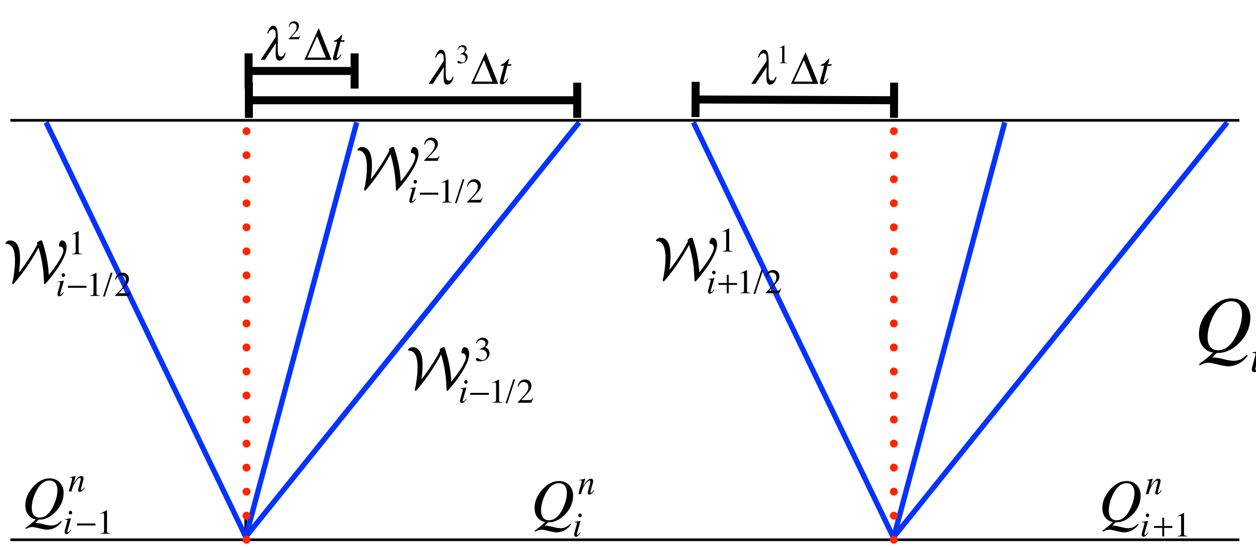
The Lax-Wendroff method in flux difference form had the flux written as:

$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q^n_i - Q^n_{i-1}\right)$$

We need to separate the eigenvectors in order to apply flux limiters, so we rewrite the correction term, using the Godunov-Riemann splititng:

$$\frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q^n_i - Q^n_{i-1}\right) = \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\sum_{p=1}^{m} \alpha^p_{i-1/2} r^p$$

Recall from before that the discontinuity between cells i and i+1 is split into m pieces by the Riemann characteristics:



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

High-resolution methods for systems

Now we apply the limiter to the coefficients of the eigenvectors:

$$\tilde{\alpha}^p_{i-1/2} = \alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})$$

$$\theta^p_{i-1/2} = \frac{\alpha^p_{l-1/2}}{\alpha^p_{i-1/2}};\; l = \begin{cases} i-1 \text{ if } \lambda^p > 0 \\ i+1 \text{ if } \lambda^p < 0 \end{cases}$$

Then the flux function is

$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}\sum_{p=1}^{m}\left|\lambda^p\right|\left(1 - \frac{\Delta t}{\Delta x}\left|\lambda^p\right|\right)\alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})r^p.$$

If we write $\widetilde{\mathcal{W}}^p_{i-1/2} = \alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})r^p$ as a limited version of the wave strength, and $s^p_{i-1/2} = \lambda^p$ for a generalised wave speed, we have:

$$F^n_{i-1/2} = \left(A^- Q^n_i + A^+ Q^n_{i-1}\right) + \frac{1}{2}\sum_{p=1}^{m}\left|s^p_{i-1/2}\right|\left(1 - \frac{\Delta t}{\Delta x}\left|s^p_{i-1/2}\right|\right)\widetilde{\mathcal{W}}^p_{i-1/2}$$

Generalisation for Nonlinear Systems

For linear systems, we can rearrange the update into the form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

with

$$\tilde{F}_{i-1/2}^n = \frac{1}{2}\sum_{p=1}^{m}\left|s_{i-1/2}^p\right|\left(1 - \frac{\Delta t}{\Delta x}\left|s_{i-1/2}^p\right|\right)\widetilde{\mathcal{W}}_{i-1/2}^p$$

Generalising to nonlinear systems we can write the update as:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

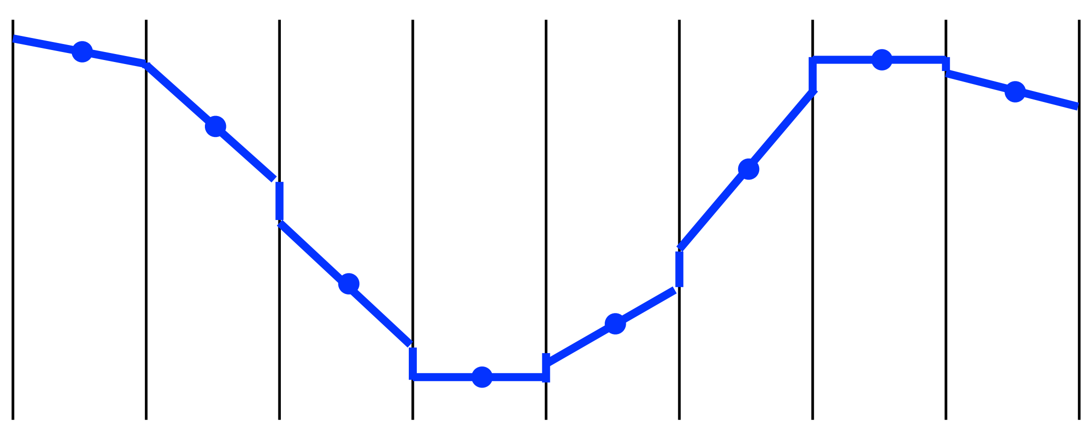
with the fluctuations suitably defined. There are some subtleties we'll get into later, associated with rarefaction waves and entropy conditions.

Review of High-Resolution Methods

We improve the first-order upwind method by introducing corrections, and writing:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}\right)$$

We derive the corrections by considering piece-wise linear (instead of piece-wise constant) reconstructions.



Review of High-Resolution Methods

Taking the basic Lax-Wendroff formula:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

we re-write it in the flux form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

with

$$F_{i-1/2}^n = \frac{1}{2} A\left(Q_i^n + Q_{i-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A^2\left(Q_i^n - Q_{i-1}^n\right)$$

Then making use of the divided matrices $A^{\pm}$ we can write this as

$$F_{i-1/2}^n = \left(A^- Q_i^n + A^+ Q_{i-1}^n\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q_i^n - Q_{i-1}^n\right)$$

Review of High-Resolution Methods

$$F_{i-1/2}^{n} = \left(A^{-}Q_{i}^{n} + A^{+}Q_{i-1}^{n}\right) + \frac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(Q_{i}^{n} - Q_{i-1}^{n}\right)$$

This version of the Lax-Wendroff formula has a correction term that can be limited, if we choose, to avoid oscillations around extrema.

For a one-equation system (the advection equation), we can apply a simple functional limiter to the slope:

$$\sigma_{i}^{n} = \left(\frac{Q_{i+1}^{n} - Q_{i}^{n}}{\Delta x}\right)\phi_{i}^{n}$$

Examples of limiters:

Lax-Wendroff: $\phi(\theta) = 1$

minmod: $\phi(\theta) = \text{minmod}(1,\theta)$

superbee: $\phi(\theta) = \max(0,\min(1,2\theta),\min(2,\theta))$

MC: $\phi(\theta) = \max(0,\min((1+\theta)/2,2,2\theta))$

vanLeer: $\phi(\theta) = \frac{(\theta + |\theta|)}{(1+|\theta|)}$

Review of High-Resolution Methods

For a system of equations, we use limiters on the waves. The wave-propagation form for a high-resolution version of Lax-Wendroff is:

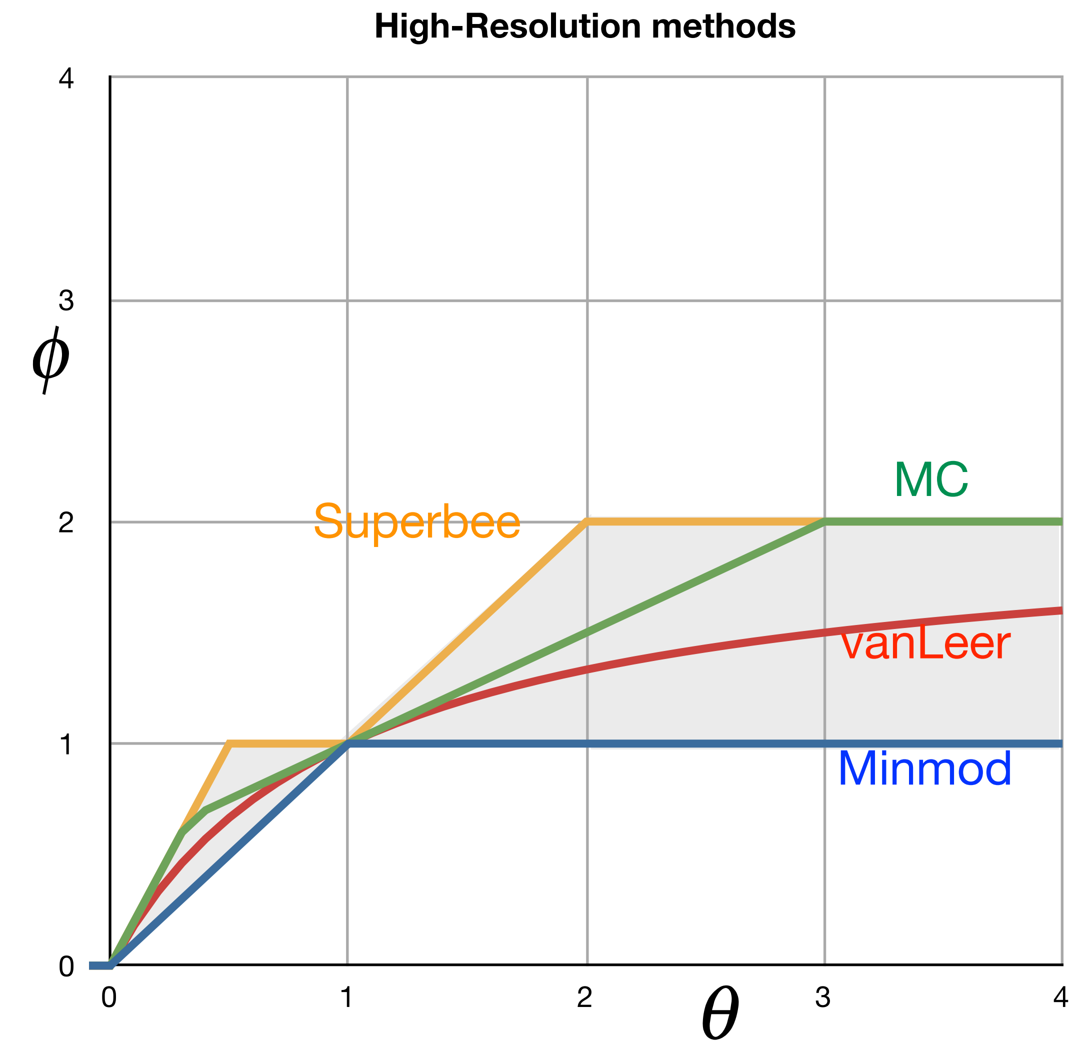
$$F_{i-1/2}^{n} = \left(A^{-}Q_{i}^{n} + A^{+}Q_{i-1}^{n}\right) + \frac{1}{2}\sum_{p=1}^{m}\left|s_{i-1/2}^{p}\right|\left(1 - \frac{\Delta t}{\Delta x}\left|s_{i-1/2}^{p}\right|\right)\widetilde{\mathcal{W}}_{i-1/2}^{p}$$

with the limited version of the waves defined as :

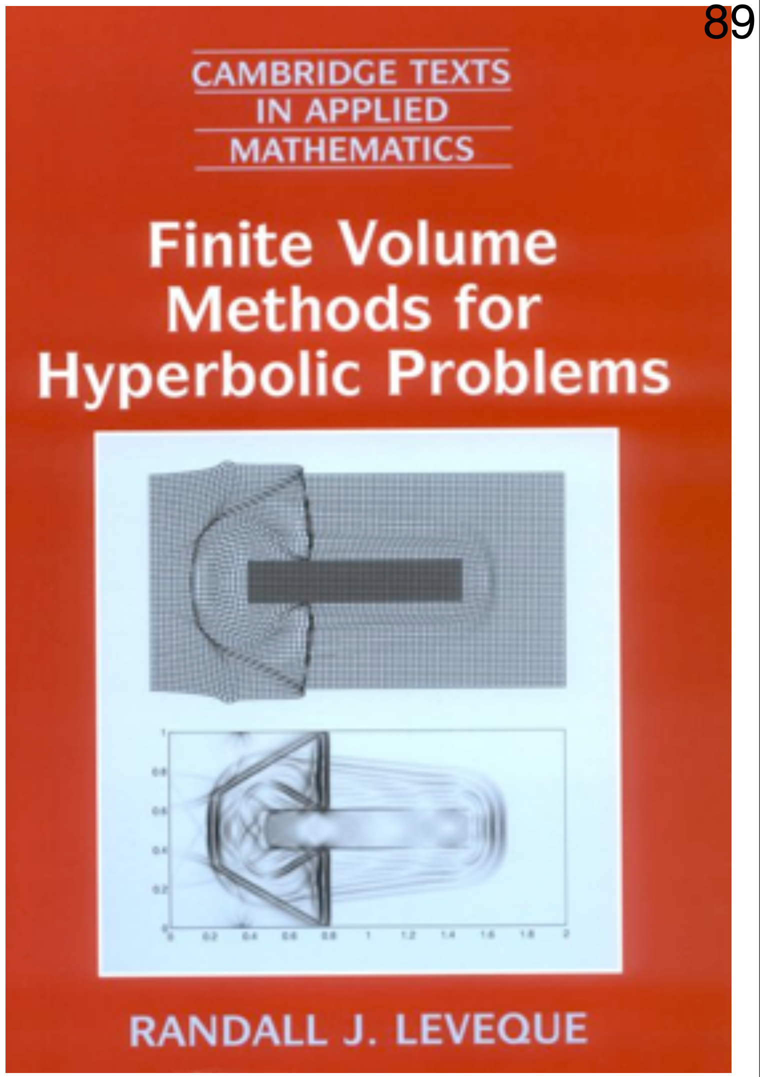
$$\widetilde{\mathcal{W}}_{i-1/2}^{p} = \alpha_{i-1/2}^{p}\phi(\theta_{i-1/2}^{p})r^{p}$$

and a generalised wave speed

$$s_{i-1/2}^{p} = \lambda^{p}$$



Assignment for next time



Read all of Chapter 4.

Work problems 4.1 and 4.2.

Read Chapter 6 at least through 6.15.

Run tests of different methods in claw/book/chap6/compareadv and claw/book/chap6/wavepacket. Reproduce Fig 6.1, 6.2, and 6.3. Add the other limiters discussed in the book. Try increasing the number of gridpoints. Does this help? Try advection with a triangular pulse and discuss which limiter does best in that case. Hand in resulting output plots together with your conclusions.

Work problems 6.1, 6.5, and 6.10.