FYS-GEO 4500

Lecture Notes #6
Nonlinear Conservation Laws

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Where we are today

	date	Topic	Chapter in LeVeque
1	1.Sep 2011	introduction to conservation laws, Clawpack	1 & 2
2	15.Sep 2011	the Riemann problem, characteristics	3 & 5
3	22.Sep 2011	finite volume methods for linear systems, high resolution	4 & 6
4	29.Sep 2011	boundary conditions, accuracy, variable coeff.	7,8, part 9
5	6.Oct 2011	nonlinear conservation laws, finite volume methods	11 & 12
6	13.Oct 2011	nonlinear equations & systems	13 & 14
7	20.Oct 2011	finite volume methods for nonlinear systems	14 & 15
8	27.Oct 2011	source terms and multidimensions	16,17,18,19
9	3.Nov 2011	multidimensional systems	20 & 21
	10.Nov 2011	no lecture	
10	17.Nov 2011	waves in elastic media	22
11	24.Nov 2011	unfinished business: capacity functions, source terms, project plans	
12	1.Dec 2011	student presentations	
	8.Dec 2011	no lecture	
*13	15.Dec 2011	FINAL REPORTS DUE	

Nonlinear Conservation Laws (Chapter 11 in Leveque)

First we look at **scalar** nonlinear conservation laws

The basic scalar conservation law is $q_t + f(q)_x = 0$ where the flux $f(q)$ is a nonlinear function of q.

Nonlinear equations are interesting because *shocks* and other forms of discontinuities may form.

They are also relevant to a wide variety of physical situations.

A good motivating example, with which most of us are familiar, is *traffic flow.*

A simple nonlinear model for traffic flow

Consider the flow of cars on a one-lane highway. All cars are assumed to be the same length, and we measure the density of cars in units of

<cars per car length>

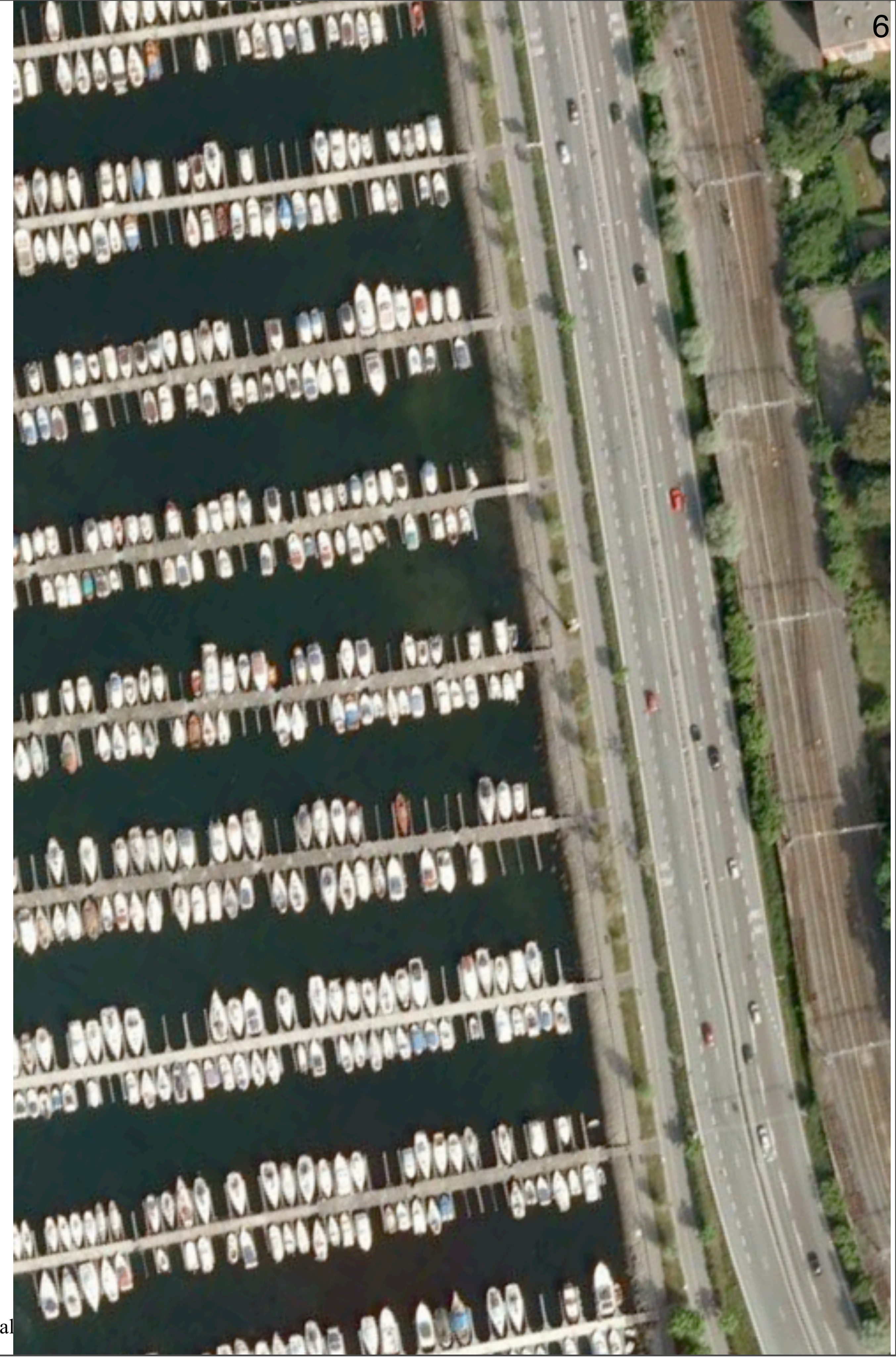
averaged over a reasonably long stretch of road. If $q(x,t)$ is the density of cars at point x and time t, then the number of cars between x_1 and x_2 is

$$\int_{x_1}^{x_2} q(x,t)\,dx.$$

On an empty highway, $q=0$, and in bumper-to-bumper traffic $q=1$. We assume drivers are careful enough so that cars never collide, so that we always have

$$0 \leq q \leq 1$$

Traffic is like a highly compressible gas



Traffic flow is very much like a one-dimensional highly compressible gas of point molecules, so this example provides a good introduction to gas dynamics.

Our equation is $q_t + f(q)_x = 0$

The flux of cars is $f(q) = uq$ where u is the speed of cars measured in units of <car lengths per unit time>.

In very light traffic, the speed u can be constant, and the equation is linear. Leveque treats this case in Section 9.4.2. In heavy traffic, u depends on density q, and the equation is therefore nonlinear.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Traffic speed depends on density

In heavy traffic, drivers will slow down. At this point we can *assume* a form for the dependence of speed on density, such as

$$U(q) = u_{\max}(1-q) \text{ for } 0 \le q \le 1.$$

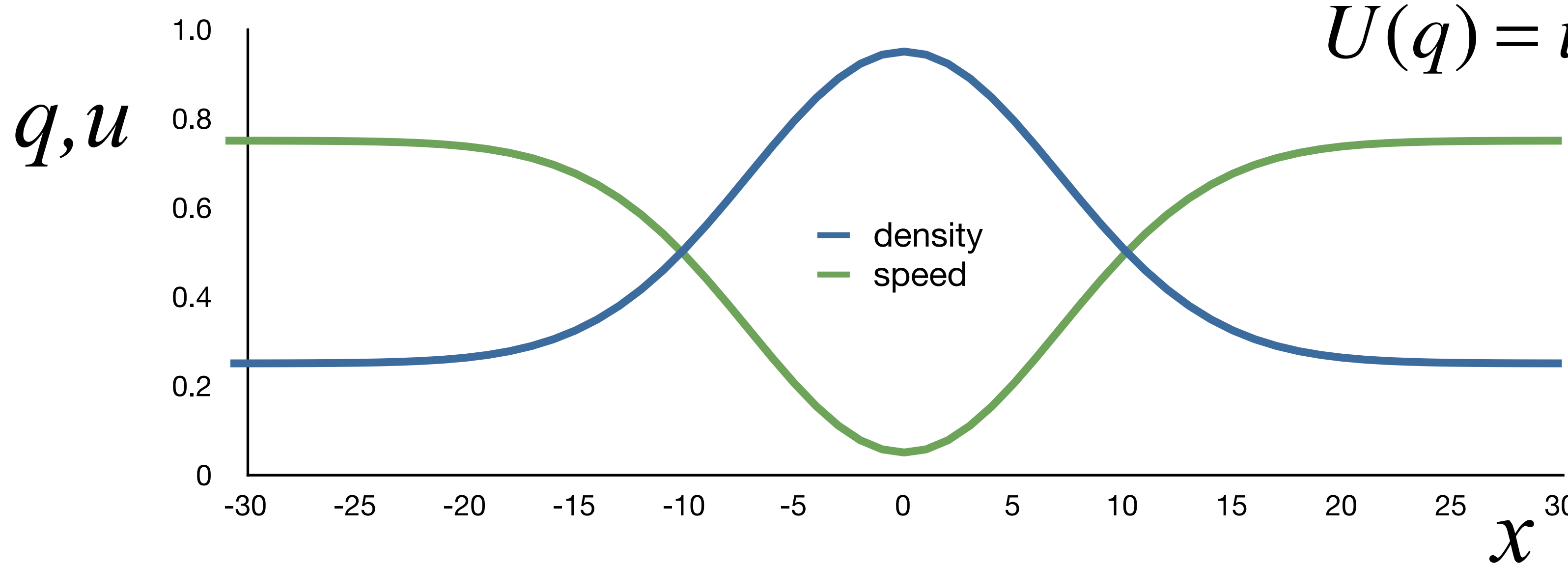
The flux is $f(q) = qU(q)$ and the equation to be solved is

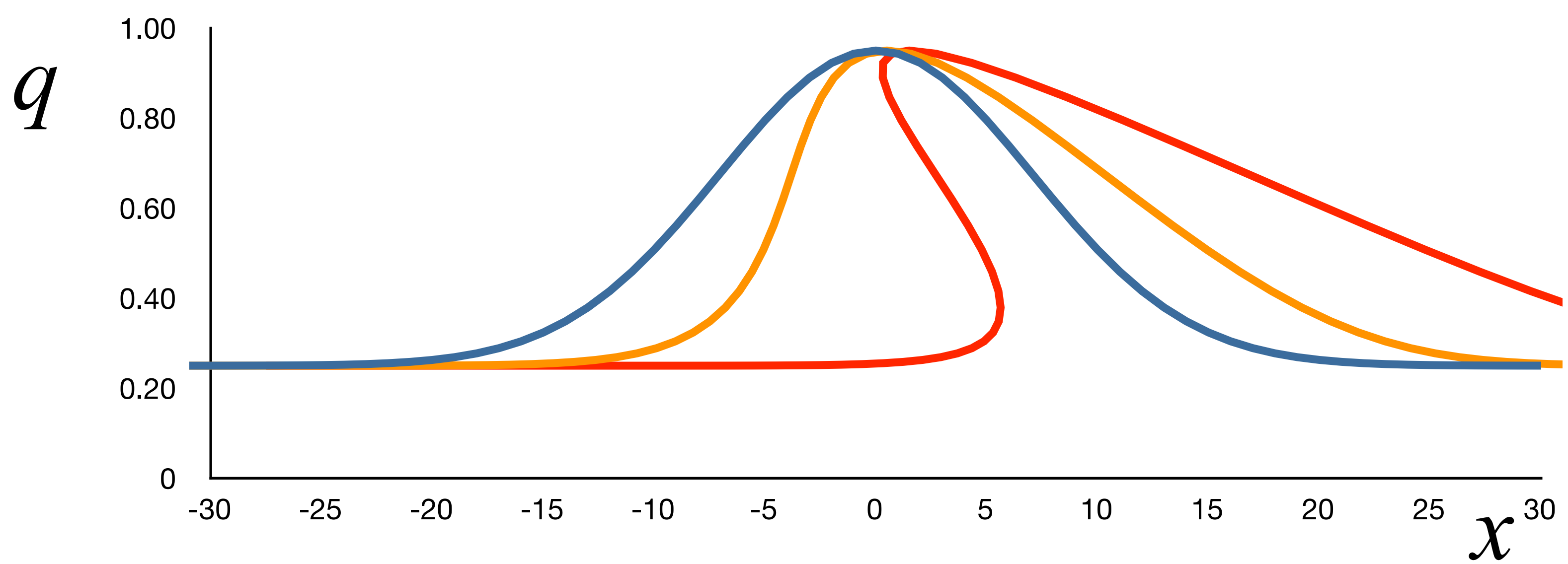
$$\begin{aligned} q_t + f(q)_x &= 0 \\ q_t + f'(q)q_x &= 0 \\ q_t + u_{\max}(1-2q)q_x &= 0 \end{aligned}$$

We will call this the *traffic flow equation*. Simulations for this case demonstrate shocks, rarefaction and compression waves.

If speed depends only on initial density, cars from behind will run into the peak.

$$U(q) = u_{\max}(1-q)$$





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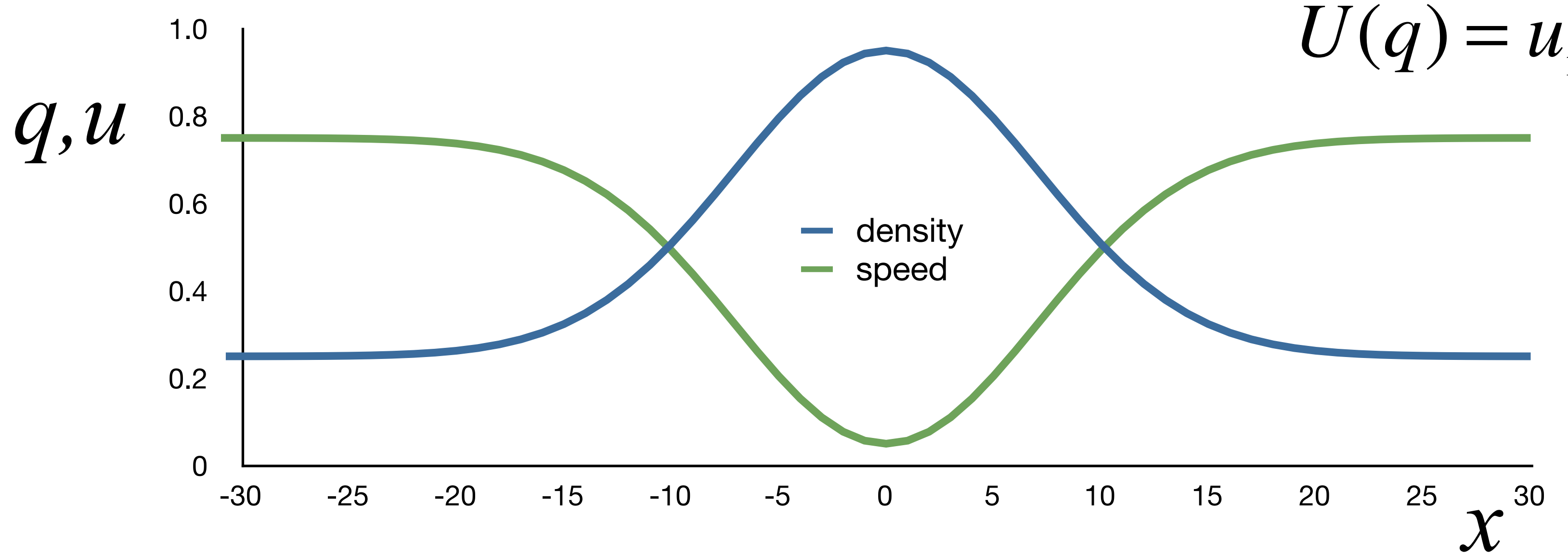

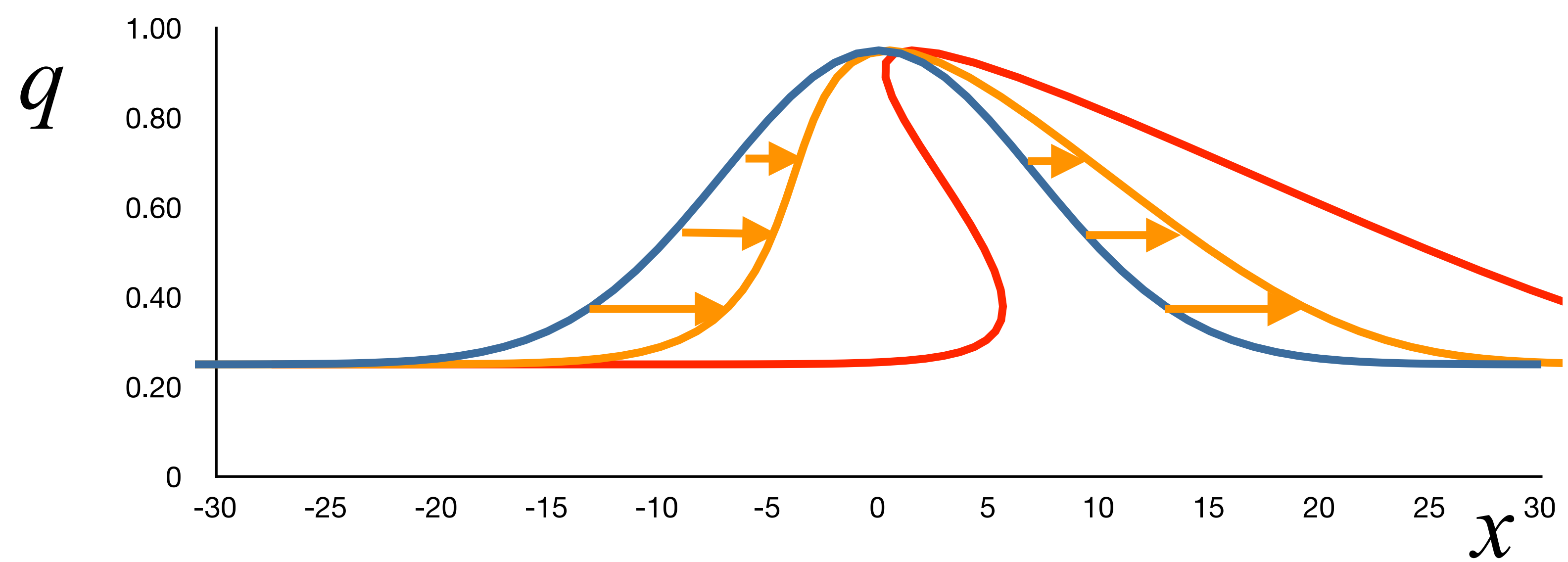

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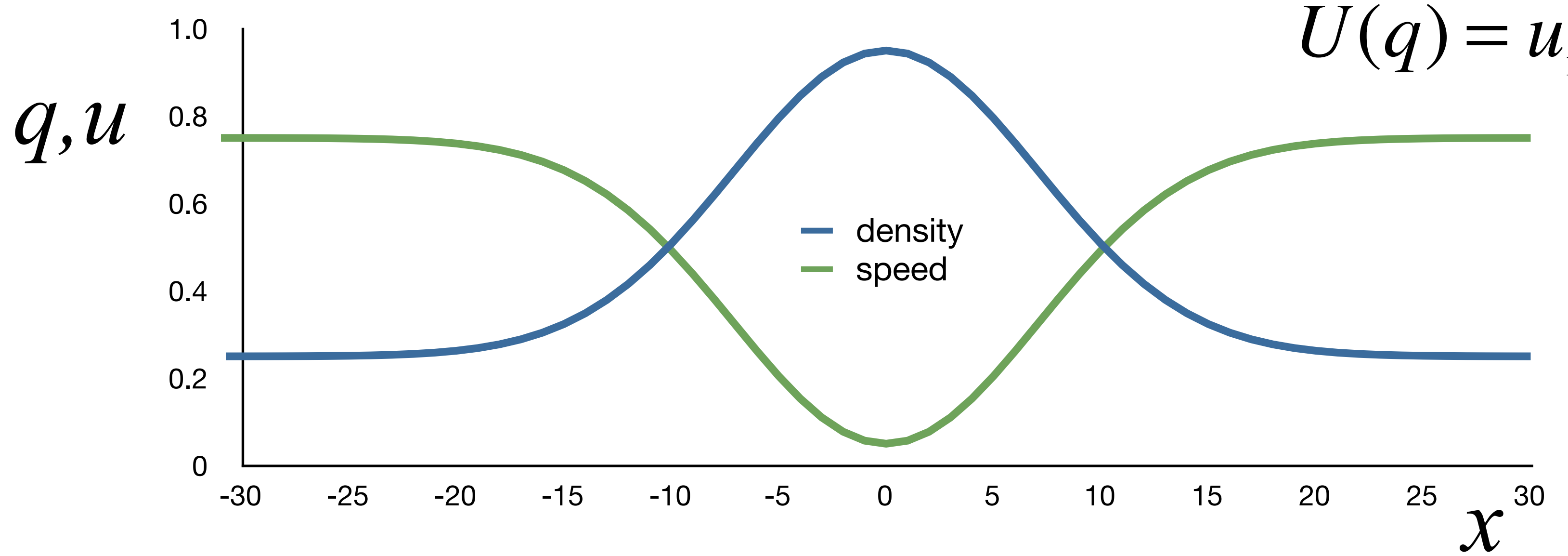

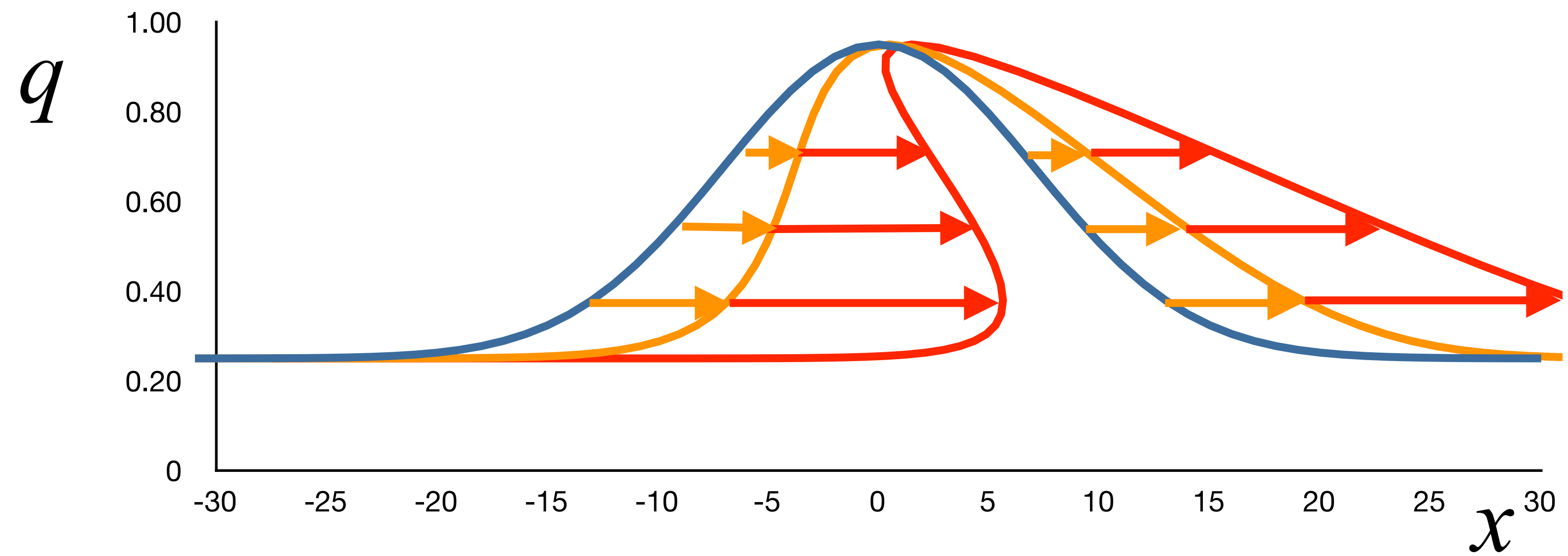

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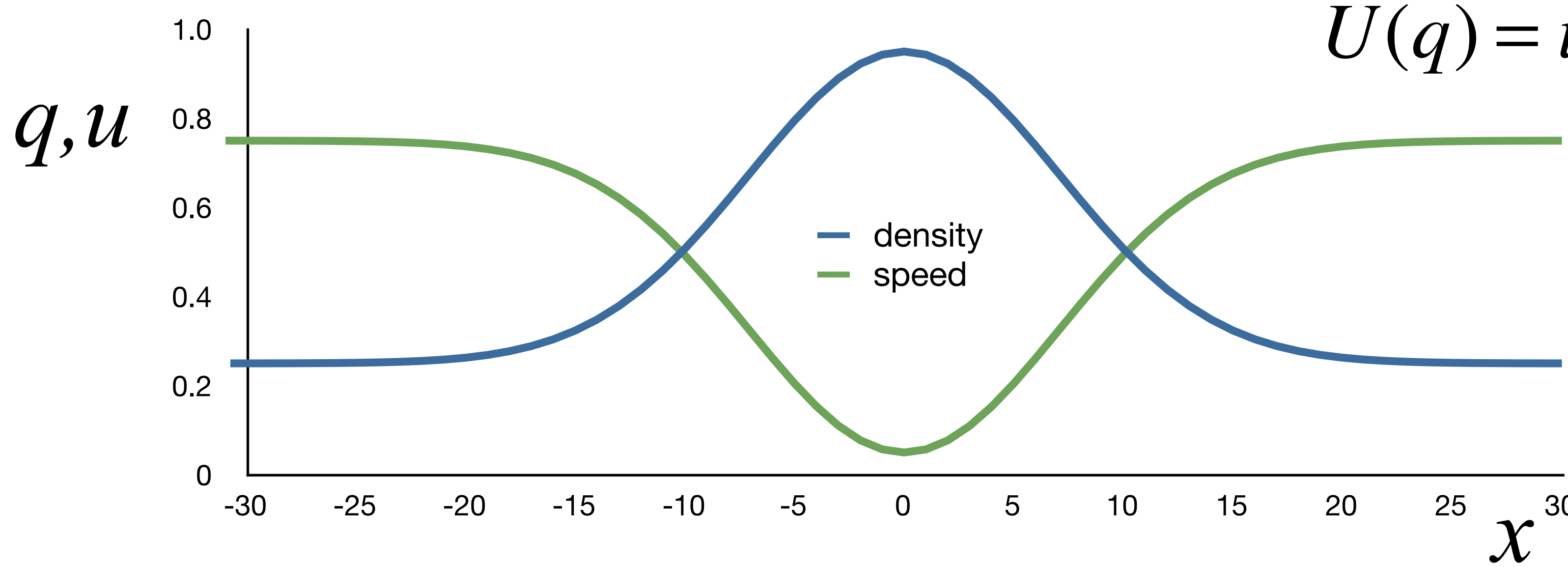
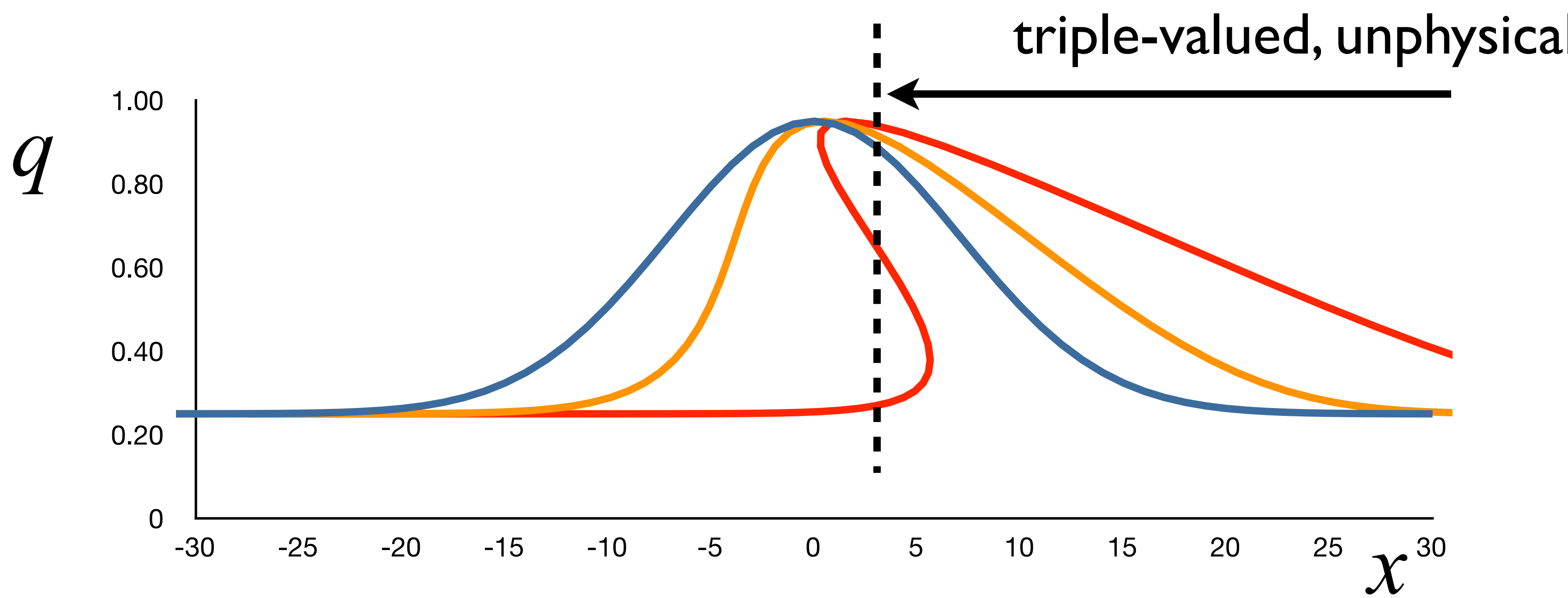
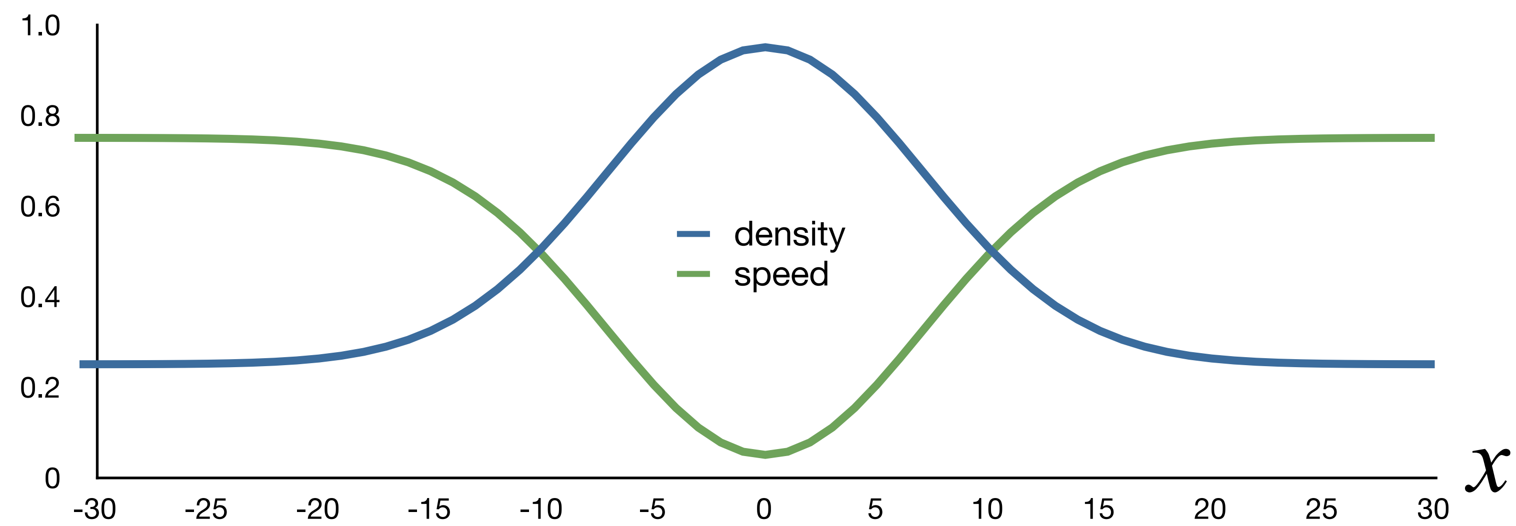
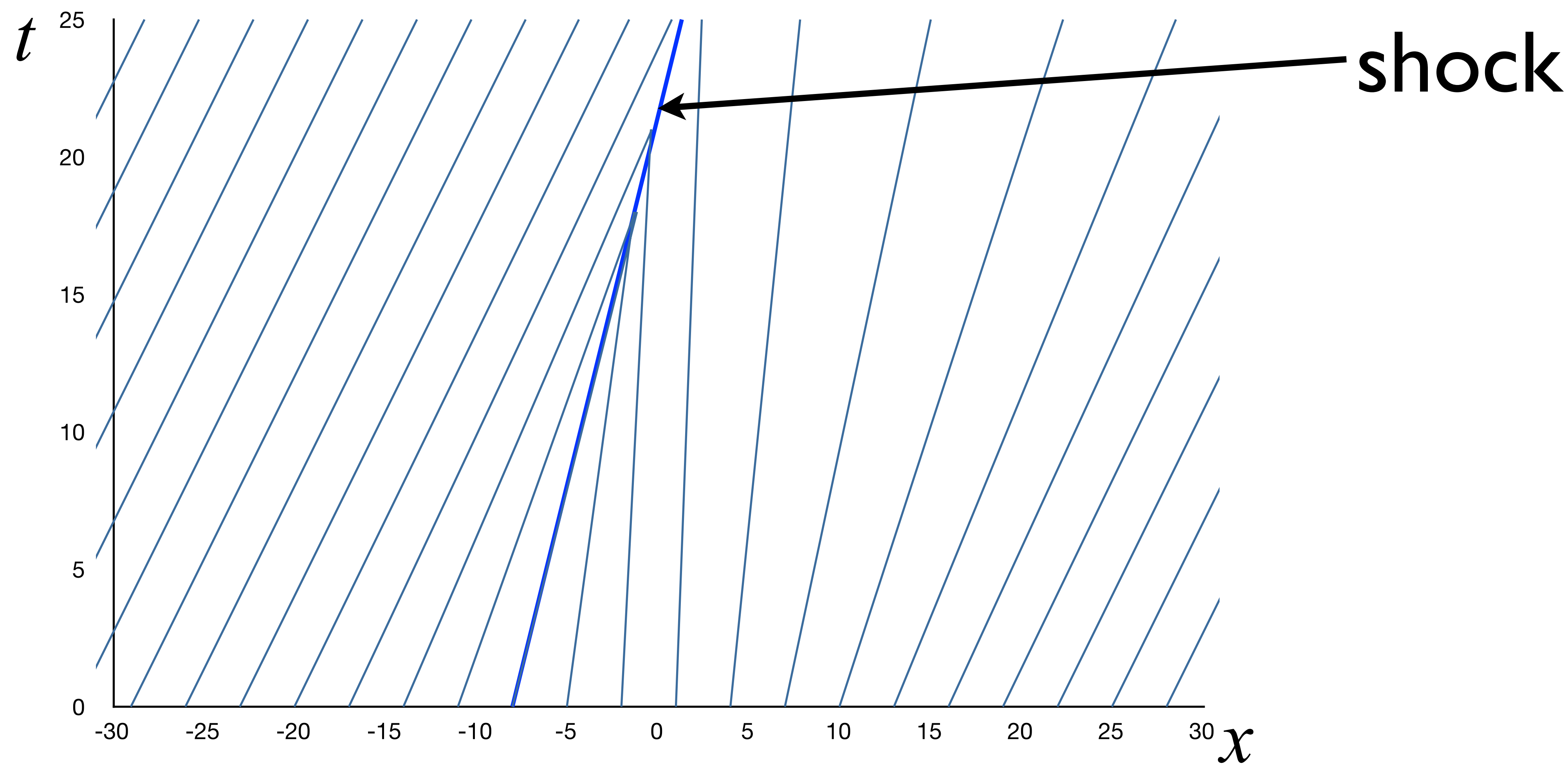




Colliding characteristics make a shock





Shock development in traffic congestion

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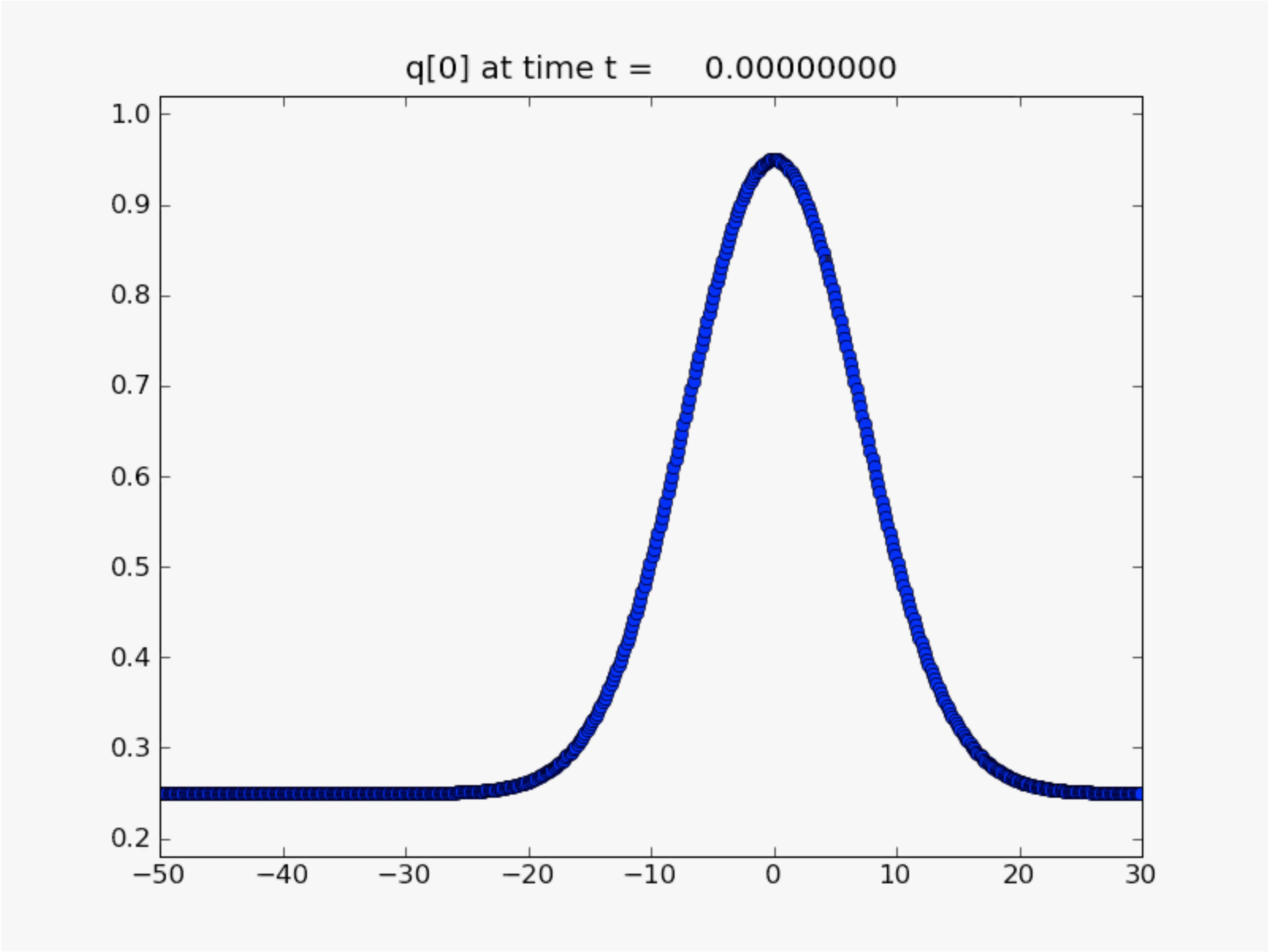
Shock development in traffic congestion



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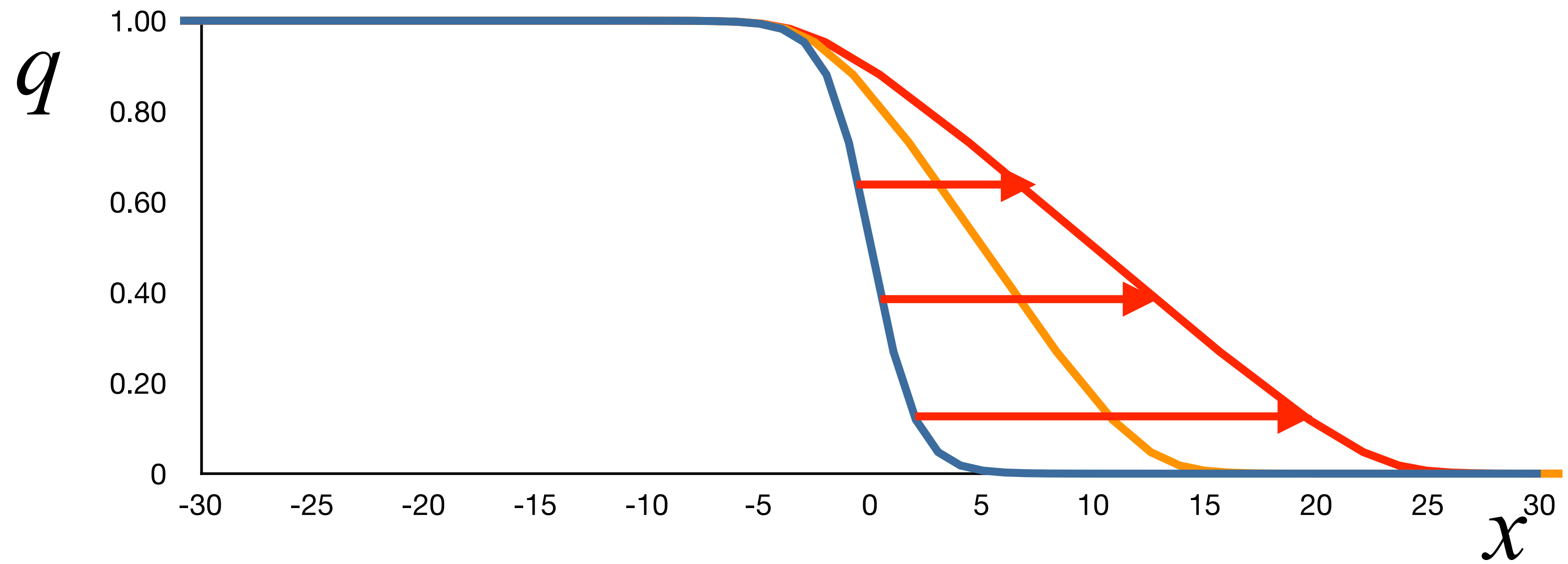
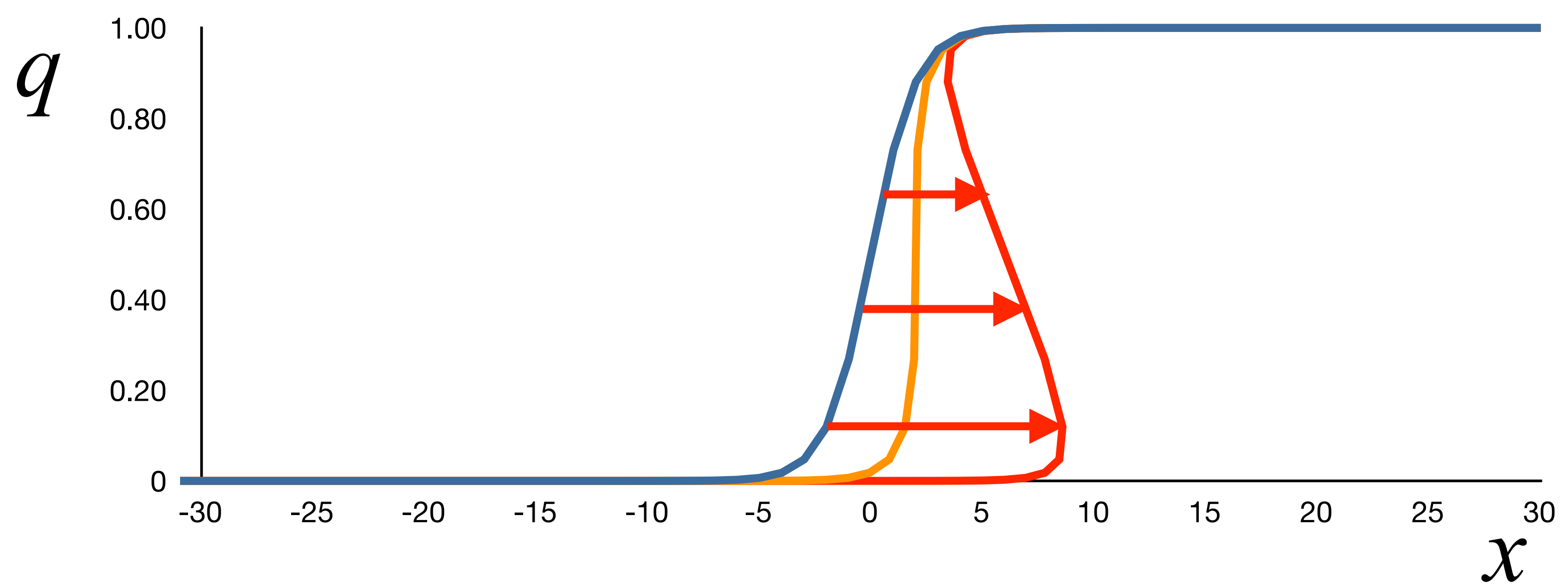
Rarefaction waves and compression waves

In the traffic flow equation, if the initial data q (the density) is a *decreasing* function of x, the cars will spread out in time. This is a *rarefaction wave*.

$$q_t + (1 - 2q)q_x = 0$$



On the other hand, if the initial data q is an *increasing* function of x, the cars pile up. This is a *compression wave*, and will steepen to become a *shock wave* to avoid the nonphysical triple-valued solution.



A green light makes a rarefaction wave propagating downstream

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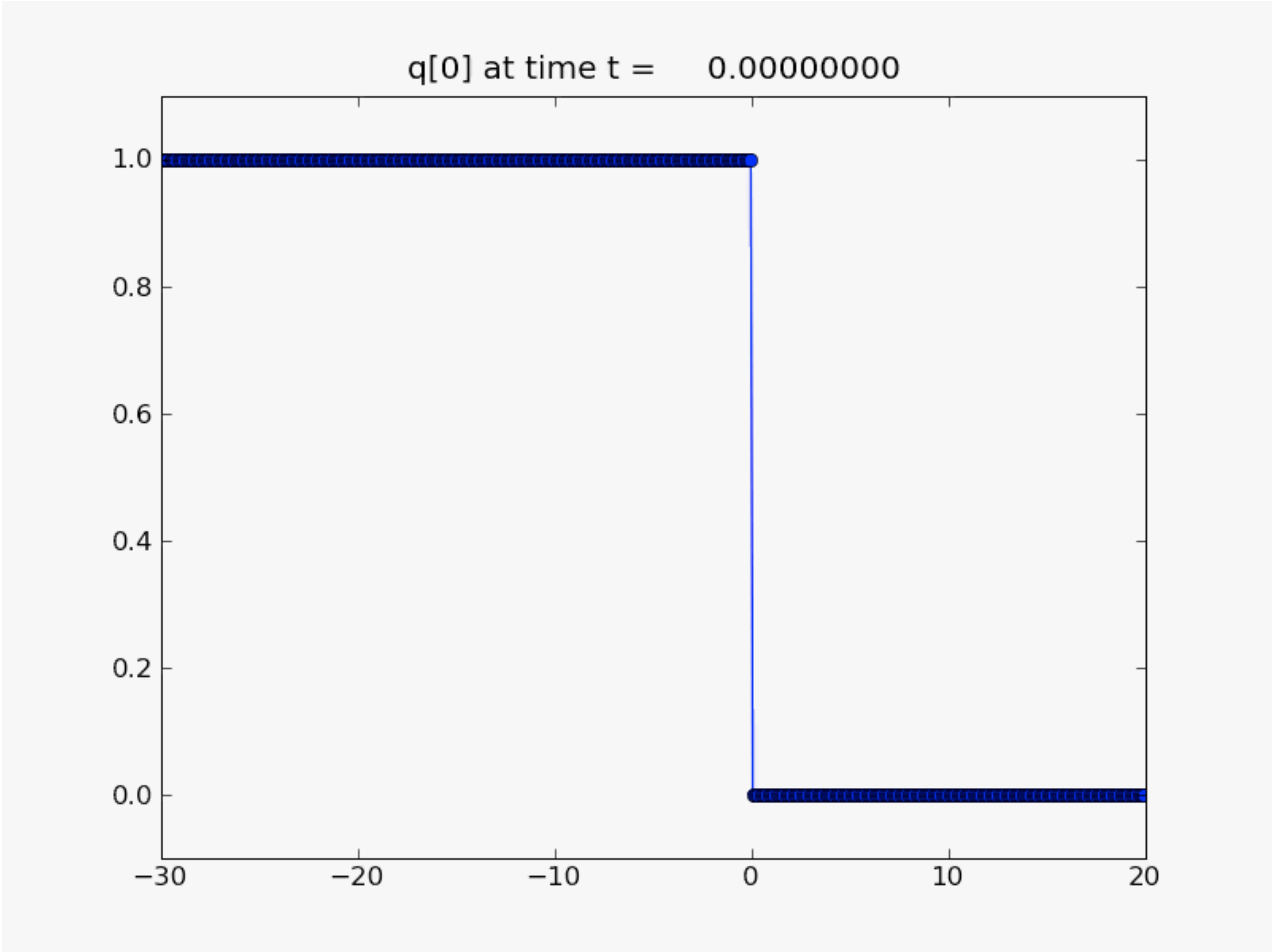
A green light makes a rarefaction wave propagating downstream



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A red light makes a shock propagating upstream

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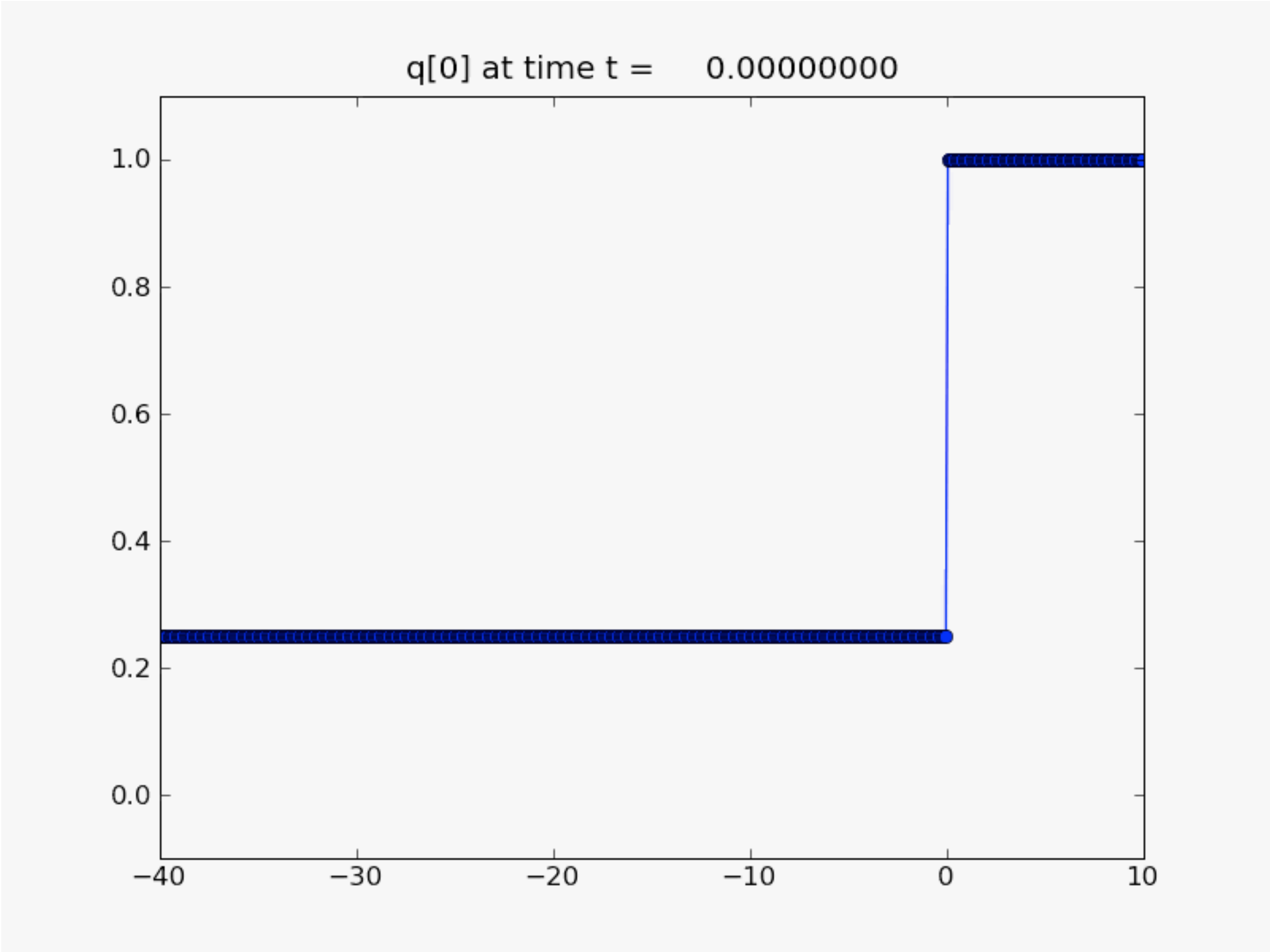
A red light makes a shock propagating upstream



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Burgers' Equation

The traffic flow equation was

$$q_t + u_{\max}(1-2q)q_x = 0$$

An even simpler nonlinear partial differential equation is Burger's equation:

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

$$u_t + uu_x = 0$$

This equation, which has an analytical exact solution, has been extensively studied for use in verification of techniques for solving PDEs.

It is the simplest nonlinear PDE that produces compression waves, rarefaction waves, and shocks.

Burgers' Equation produces "N-waves"

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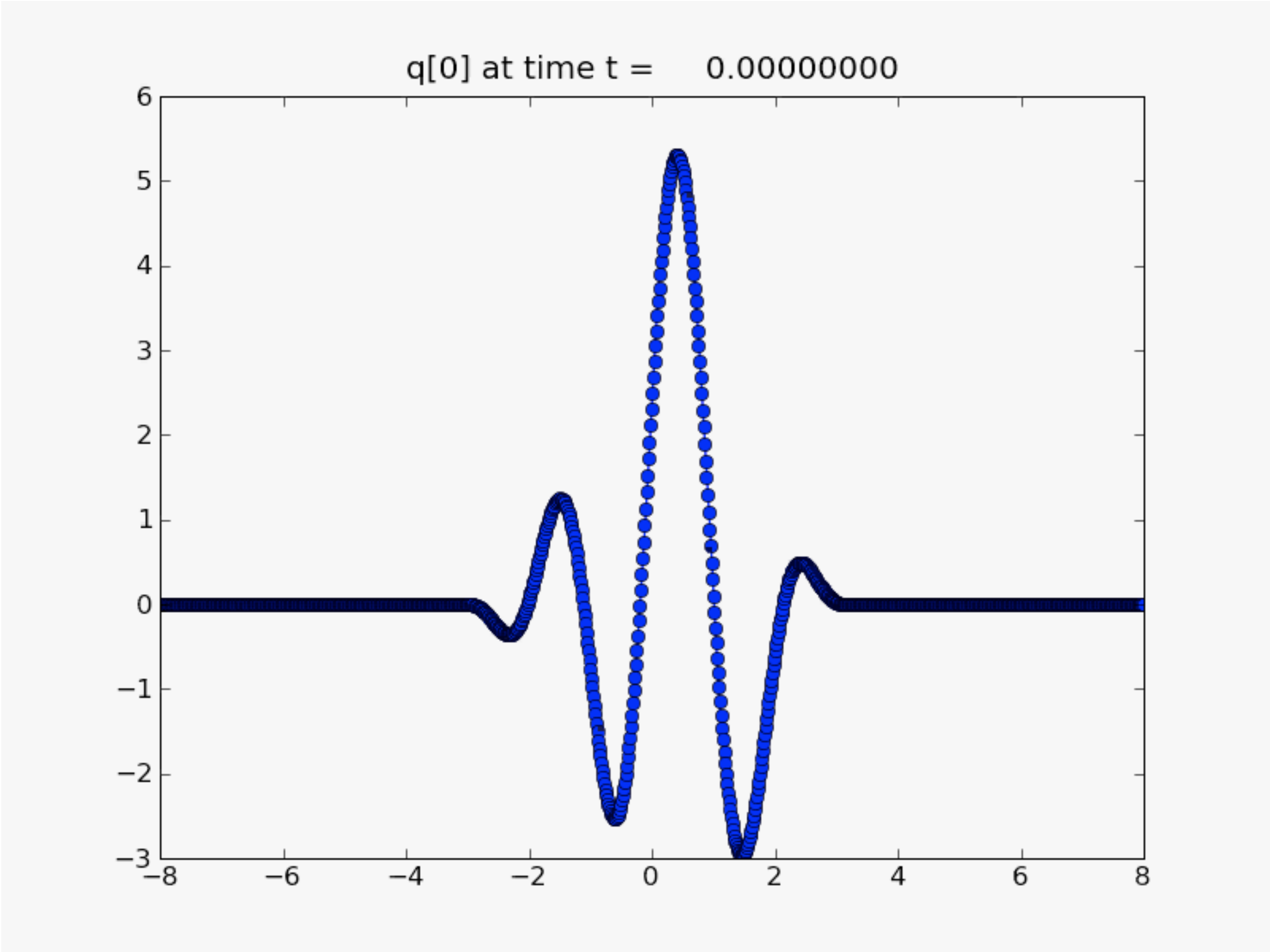

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Remember: The integral form of the conservation law is more fundamental!

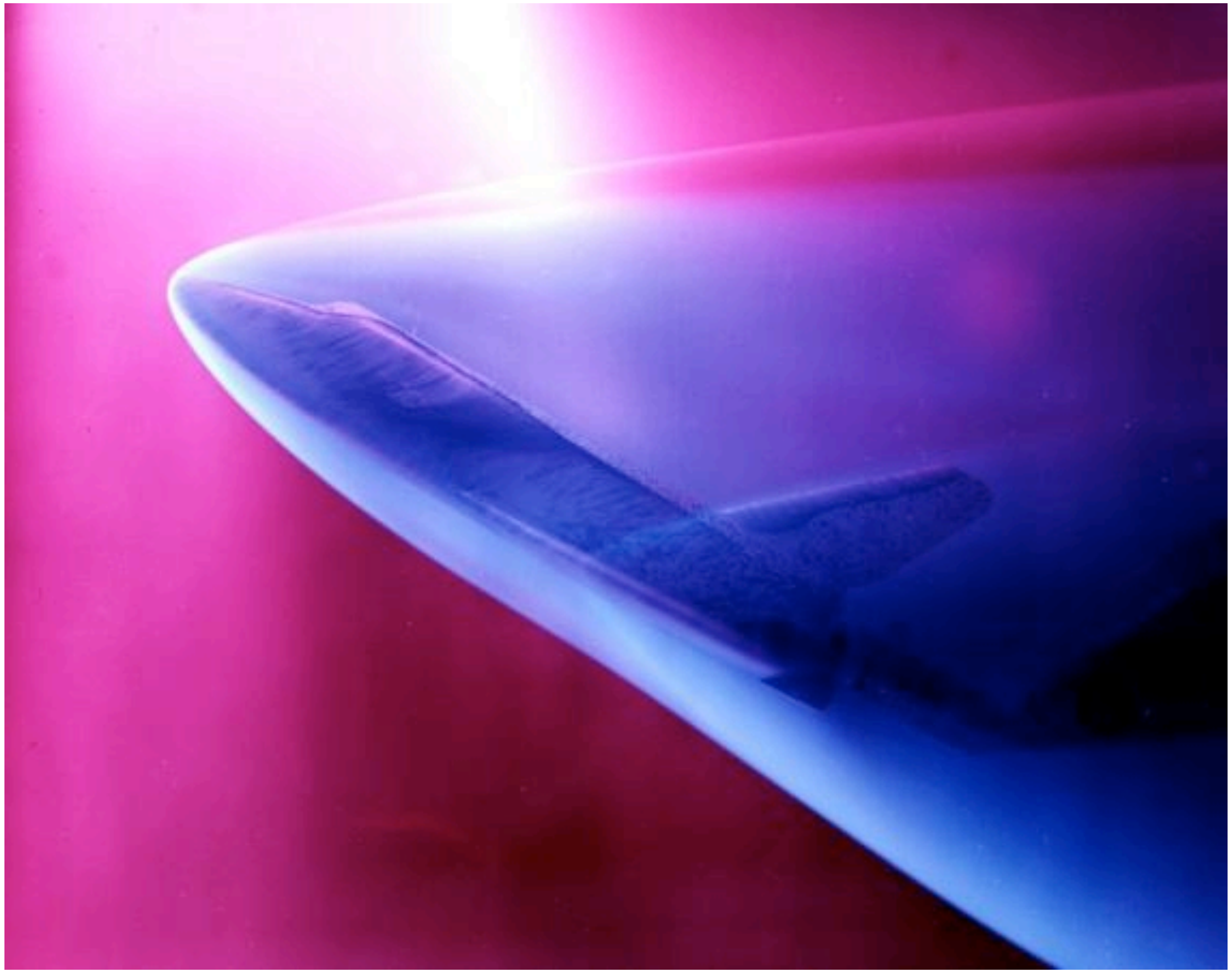
The discontinuities that develop in the traffic flow and Burgers' equations reveal an essential weakness in the differential-equation formulation of conservation laws.

Remember, they were derived in integral form, and converted to differential form under the assumption that the solutions were smooth.

But they are not always smooth.

Still, we continue to write the differential form because it is more compact, but we regard it only as a short-hand for the more fundamental integral form.

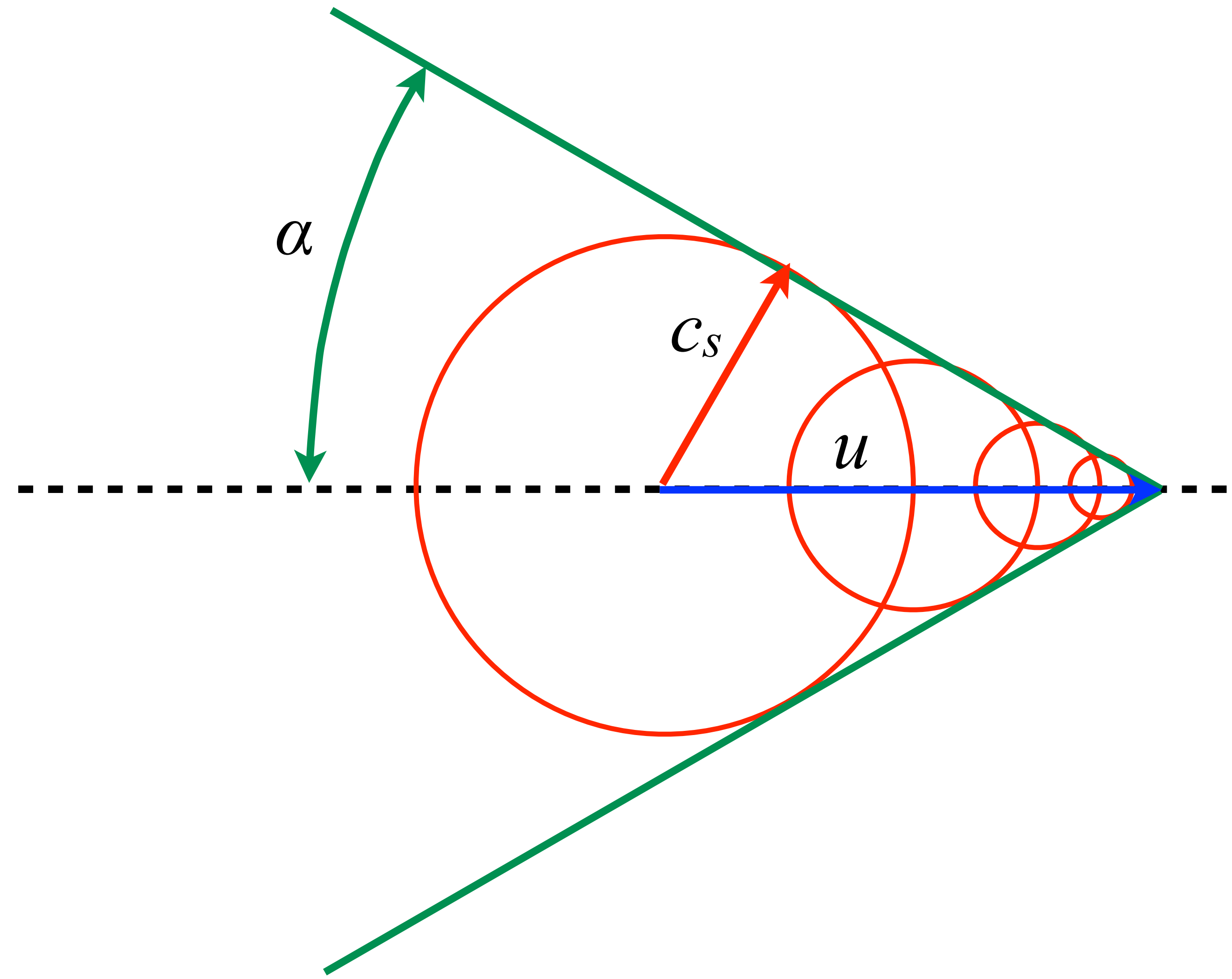
Shock waves usually make you think of military aircraft or rocket ships:



But ducks make them too!

When something tries to move through a medium faster than the speed of characteristic waves in that medium, it makes a “shock” wave.

… rather we should say a “critical” wave …

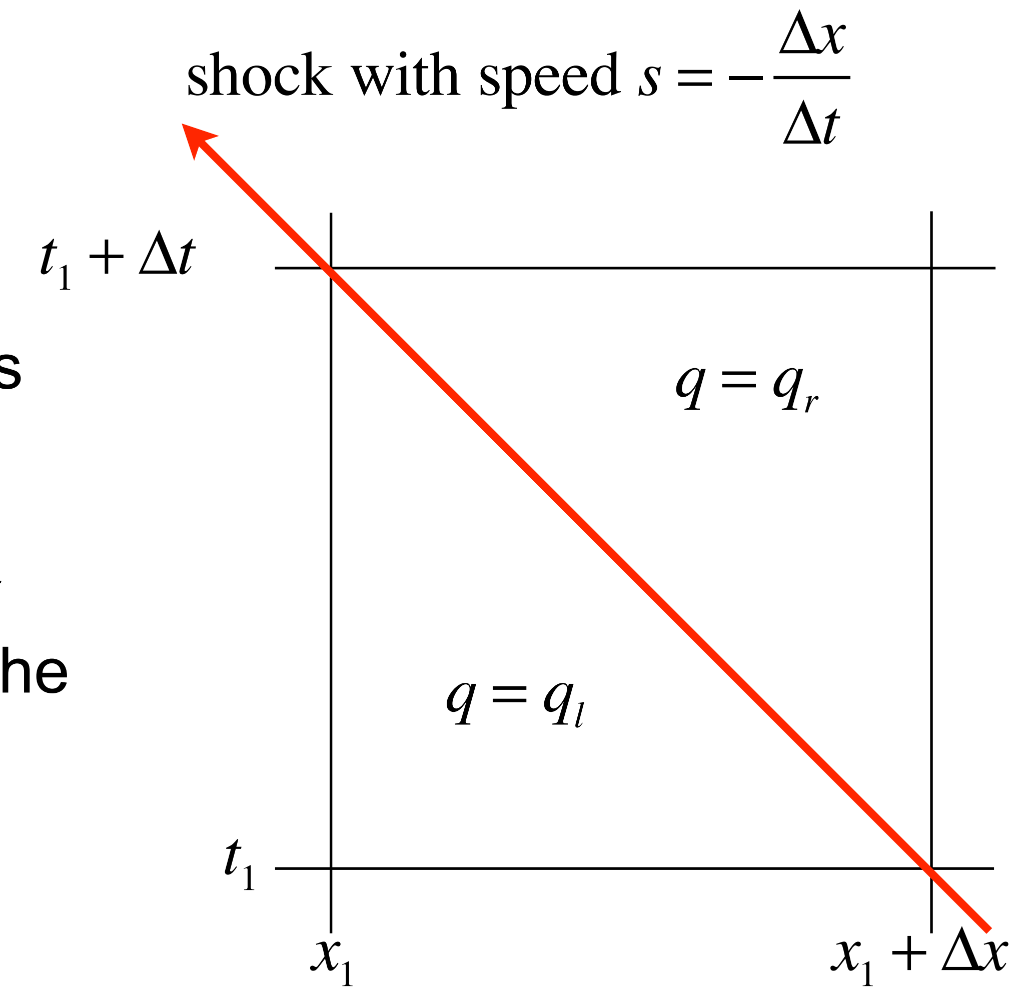


A shock wave can arise from the pile-up of sound waves that are emitted from an object travelling faster than the speed of sound. The opening angle of the *Mach cone* is

$$\alpha = \sin^{-1}\left(\frac{c_s}{u}\right)$$

where u is the speed of the object and c_s is the speed of sound.

Shock speed



The integral form of the conservation law enables us to determine the speed of a shock wave.

The diagram illustrates a small portion of the x–t plane in which the shock speed is constant and the solution is roughly constant on either side of the shock.

The conservation law $$\frac{d}{dt}\int_{x_1}^{x_1+\Delta x} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_1+\Delta x,t))$$

gives us $$\Delta x q_r - \Delta x q_l = \Delta t f(q_l) - \Delta t f(q_r) + \mathcal{O}(\Delta t^2).$$

With shock speed $s = -\frac{\Delta x}{\Delta t}$, in the limit $\Delta t \to 0$,

we get the *Rankine-Hugoniot jump condition*: $$s(q_r - q_l) = f(q_r) - f(q_l)$$

Rankine-Hugoniot Conditions

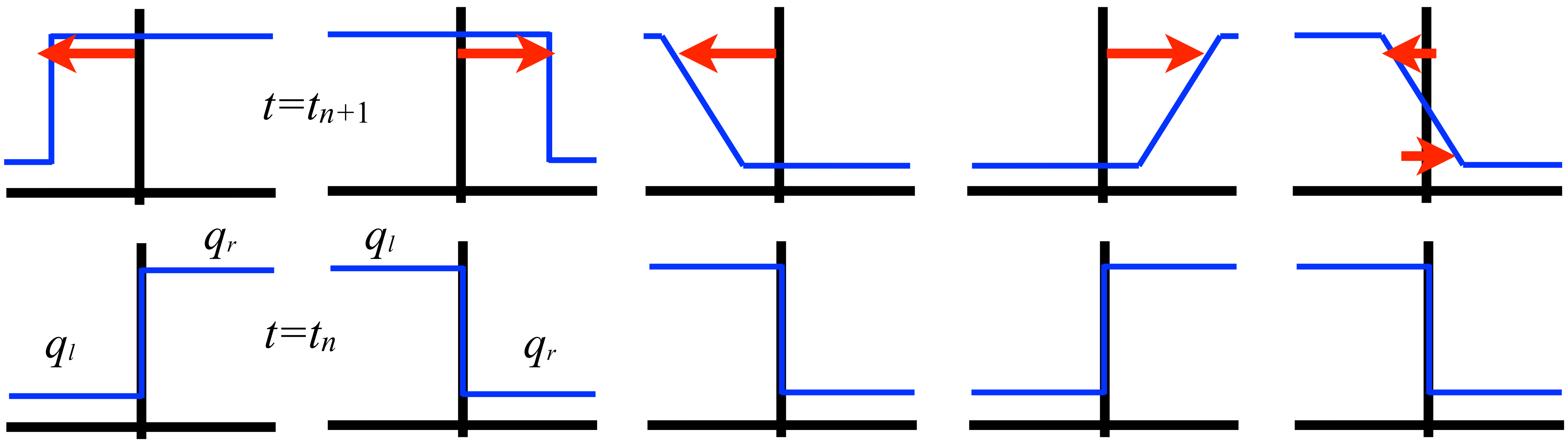
For *systems* of conservations laws, the Rankine-Hugoniot jump conditions also apply.

For *linear* systems, $f(q) = Aq$, and the jump condition becomes:

$$A(q_r - q_l) = s(q_r - q_l)$$

which means that the difference vector $(q_r - q_l)$ must be an *eigenvector* of the system, and the speed s is the corresponding eigenvalue.

An initial discontinuity can lead to a (limited) variety of different states at the next time step:



In most cases, when we do a Riemann problem, we are interested in the future value only at the position of the original discontinuity.

Because the solution to a Riemann problem is a similarity solution, for the scalar equation, the answer is either the right or the left original state, or a simple combination of the two determined by the equation.

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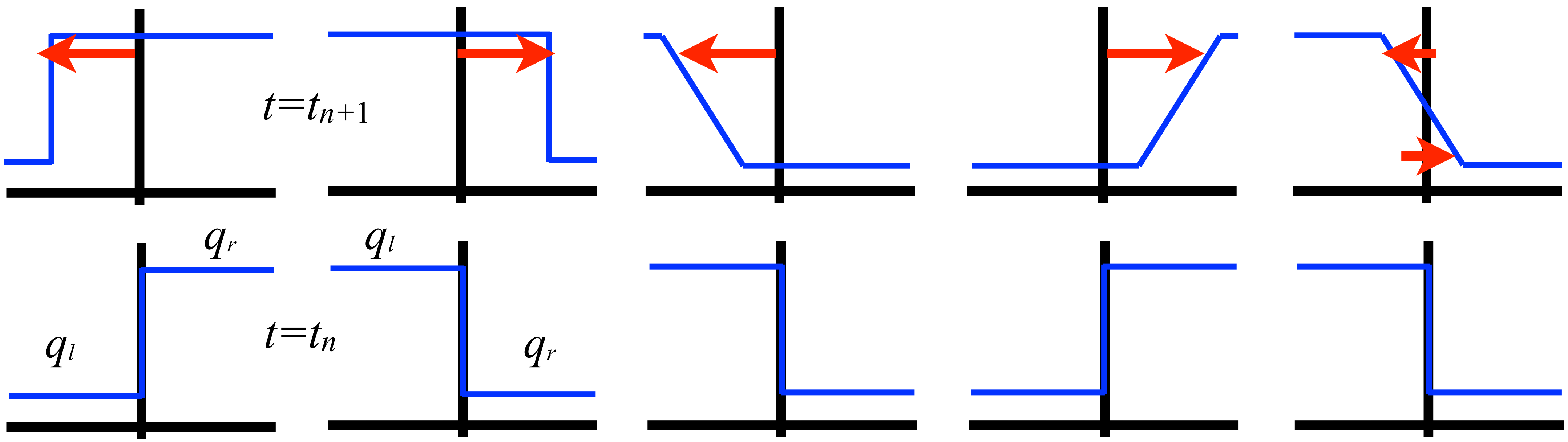

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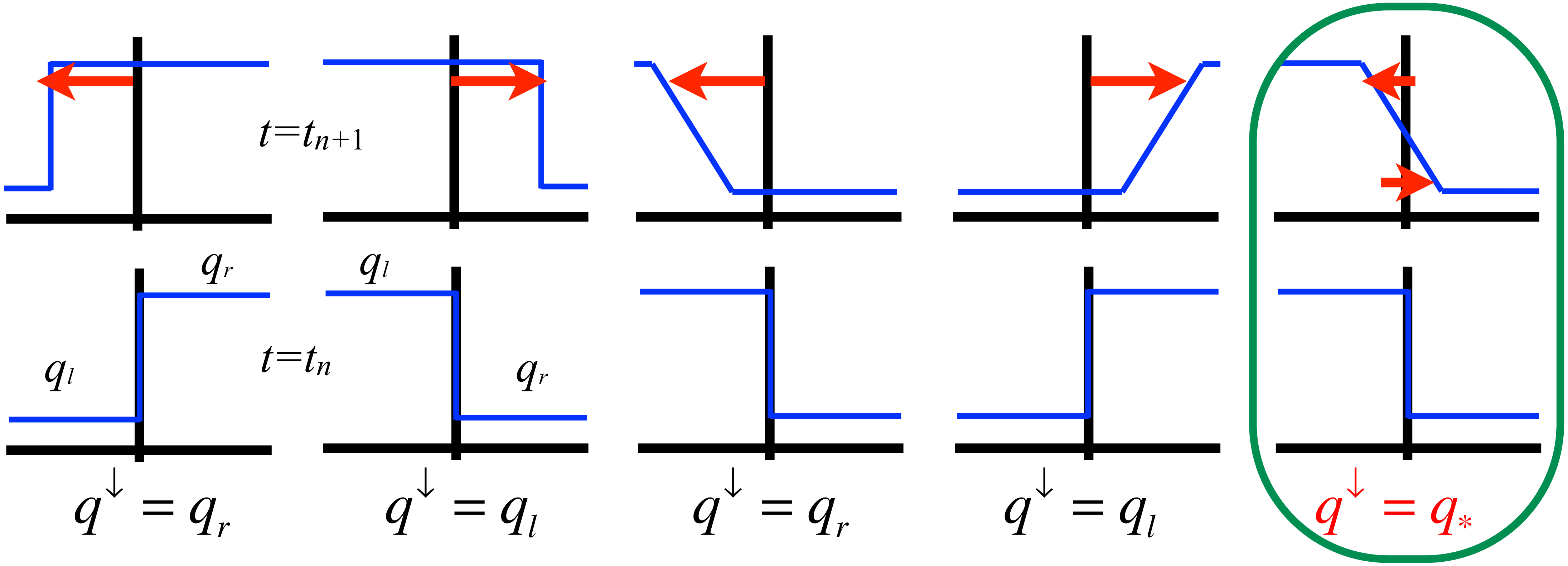

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The centred rarefaction is the only one that requires a separate calculation.

Centred rarefactions

The similarity solution $q(x,t) = \tilde{q}(x/t)$

has derivatives $q_t(x,t) = -\frac{x}{t^2}\tilde{q}'(x/t); \qquad q_x(x,t) = \frac{1}{t}\tilde{q}'(x/t)$

Placing these in the conservation law $q_t + f'(q)q_x = 0$

we get $$f'\left(\tilde{q}(x/t)\right)\tilde{q}'(x/t) = \frac{x}{t}\tilde{q}'(x/t)$$

So either $f'\left(\tilde{q}(x/t)\right) = \frac{x}{t}$ or $\tilde{q}$ is constant.

For the centred rarefaction, the former holds.

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For the centred rarefaction, the former holds.

In the case of the traffic flow equation

$$f'(\tilde{q}(x/t)) = u_{\max}\left[1 - 2\tilde{q}(x/t)\right] = x/t$$

$$\tilde{q}(x/t) = \frac{1}{2}\left(1 - \frac{x}{u_{\max}t}\right)$$

Weak solutions and entropy conditions

A *strong solution* of a differential equation is a solution that is sufficiently smooth that all the derivatives that are needed exist.

A *weak solution* is a solution of the related integral equation, and may have discontinuities so that the derivatives cannot be taken.

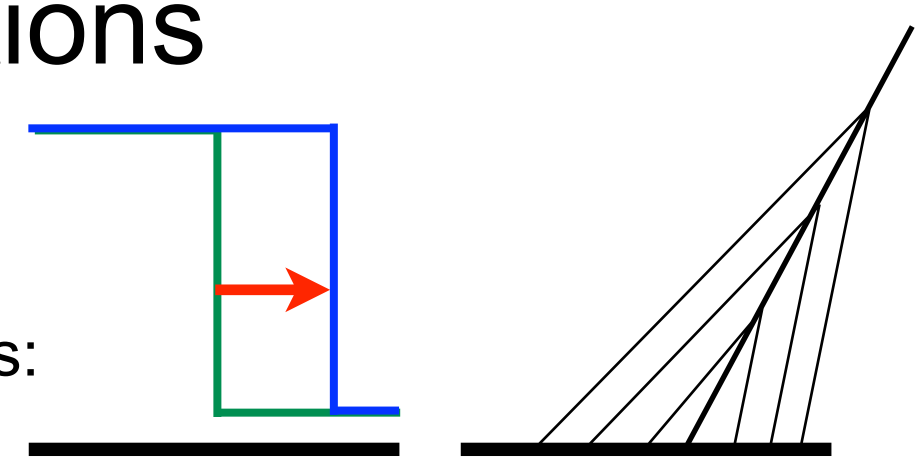
Both weak solutions and strong solutions satisfy the integral equation. Only strong solutions rigorously satisfy the differential equation.

Weak solutions are not unique, unfortunately! (Integrals always introduce arbitrary constants, for example.)

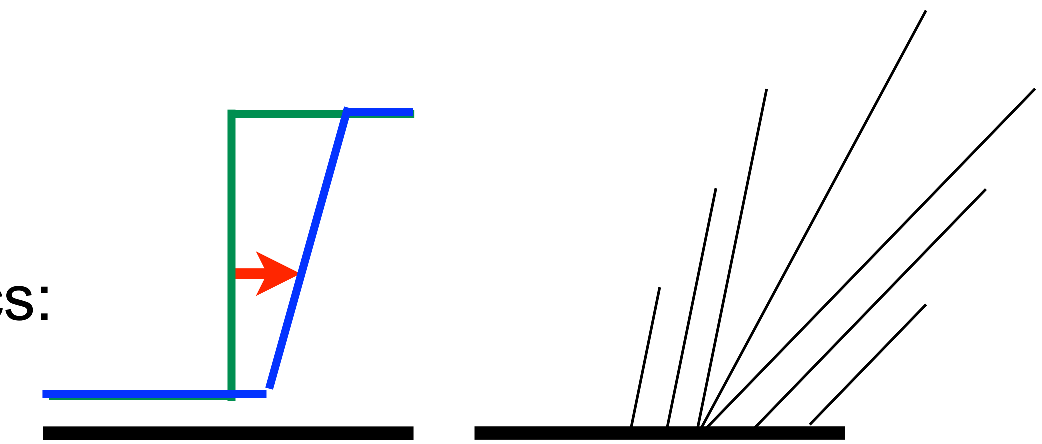
Selecting the appropriate weak solution requires an *entropy condition*.

Eliminating non-unique solutions

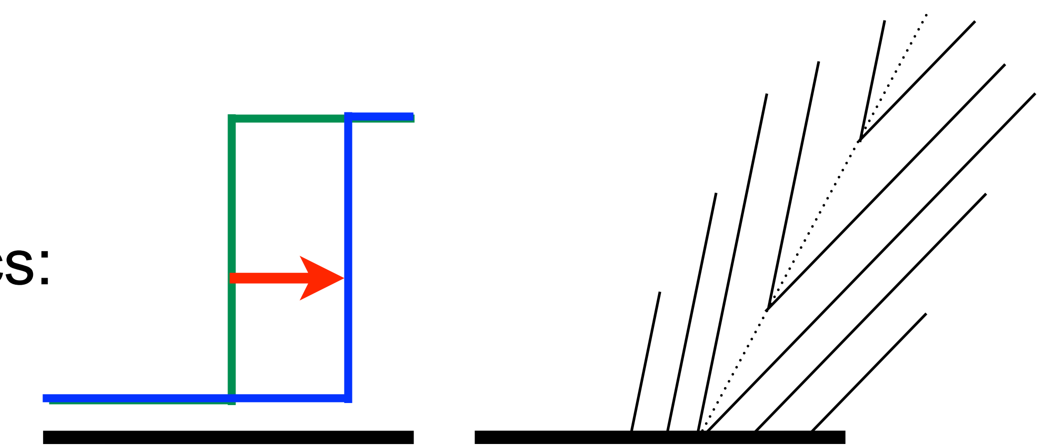
A shock and its characteristics:



A rarefaction wave and its characteristics:



An entropy-violating “shock” and its characteristics:



A discontinuity propagating with speed s must satisfy the entropy condition

$$f'(q_l) < s < f'(q_r)$$

Eliminating non-unique solutions

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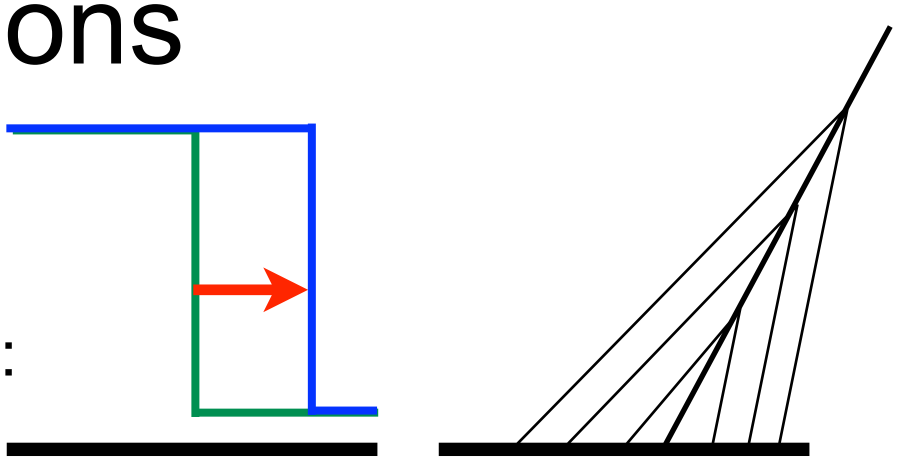

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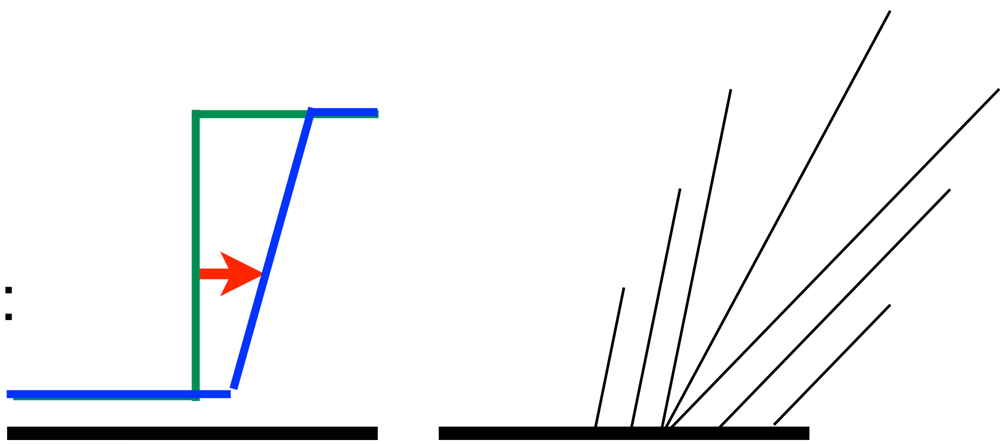

Characteristics can't start in free space!

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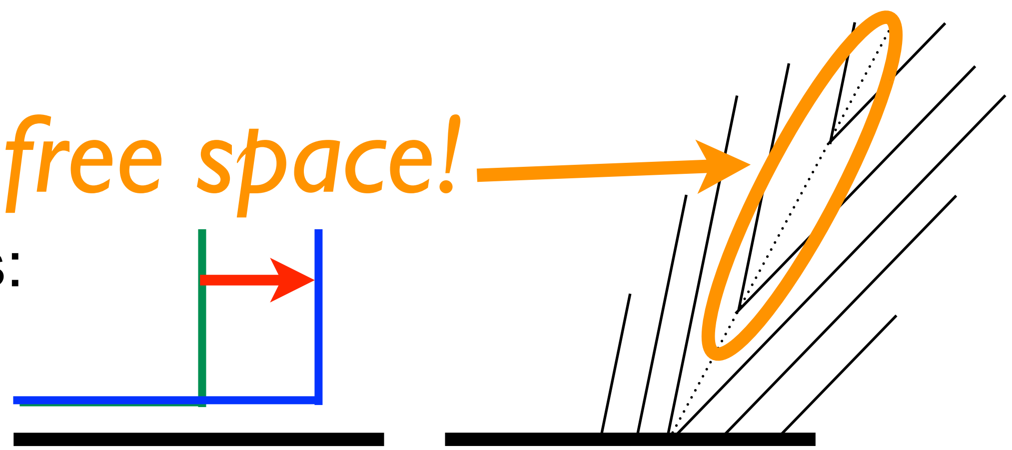


A discontinuity propagating with speed s must satisfy the entropy condition

$$f'(q_l) < s < f'(q_r)$$

Entropy functions

An entropy function is a function that is conserved when the solution is smooth, but changes in magnitude at a discontinuity.

Leveque considers a variety of these from a mathematical/computational point of view. We will usually find a physical or information-theoretic entropy condition to help select the proper weak solution.

The thermodynamic entropy of a gas, for example, increases across a shock but is constant in smooth flow.

We will consider thermodynamic entropy further when we deal with gas dynamics.

Finite Volume Methods for Nonlinear Equations (Chapter 12 in Leveque)

We extend from what we've learned for linear equations

We intend to solve the nonlinear conservation law $q_t + f(q)_x = 0$

using a method that is in conservative form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

and yielding a *weak solution* to this conservation law. To get the *correct* weak solution we must use an appropriate entropy condition.

Things will get a little tricky, so for now we stick to the scalar problem.

Recall Godunov's method:

Given a set of cell quantities Q_i^n at time n:

1. Solve the Riemann problem at $x_{i-1/2}$ to obtain $Q_{i-1/2}^{\downarrow} = q^{\downarrow}(Q_{i-1}^n, Q_i^n)$
2. Define the flux: $F_{i-1/2}^n = f\left(Q_{i-1/2}^{\downarrow}\right)$
3. Apply the flux differencing formula: $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

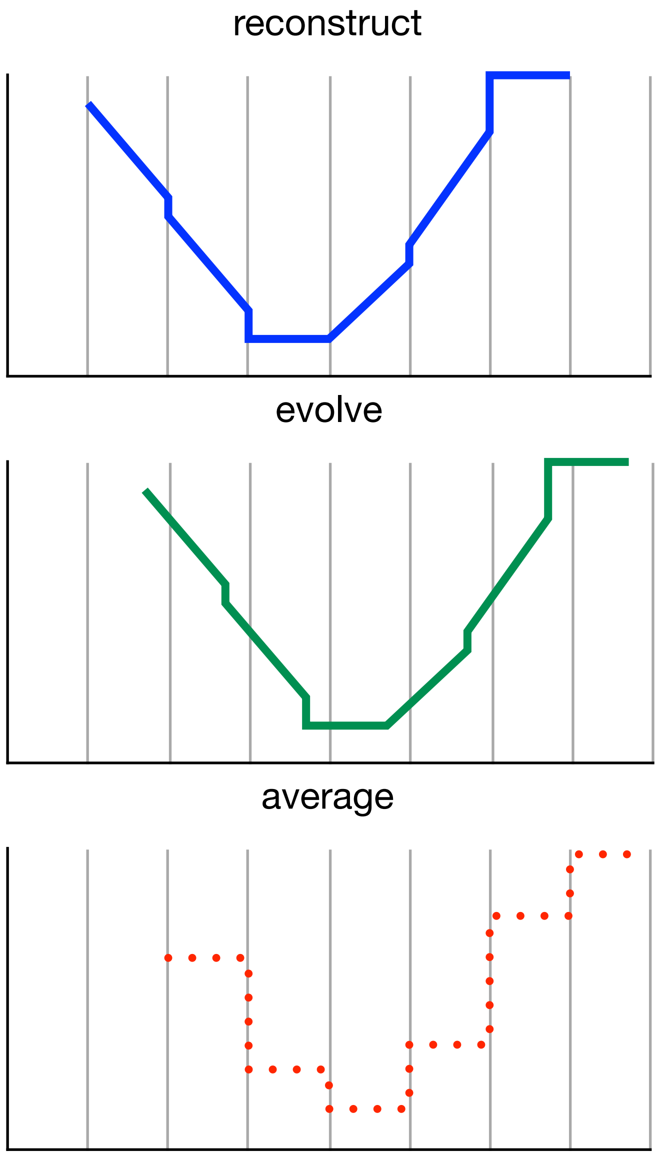
This will work for any general system of conservation laws. Only the formulation of the Riemann problem itself changes with the system.

In terms of the REA scheme we have discussed

1. Reconstruct a piece-wise **linear** function from the cell averages.
$$q^n(x,t_n) = Q_i^n + \sigma_i^n (x - x_i) \text{ for } x \text{ in cell } i$$
with the property that $\mathrm{TV}(q) \le \mathrm{TV}(Q)$

2. Evolve the hyperbolic equation with this function to obtain a later-time function, by solving Riemann problems at the interfaces.
$$\tilde{q}^n(x,t_{n+1})$$

3. Average this function over each grid cell to obtain new cell averages.
$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x,t_{n+1})dx$$



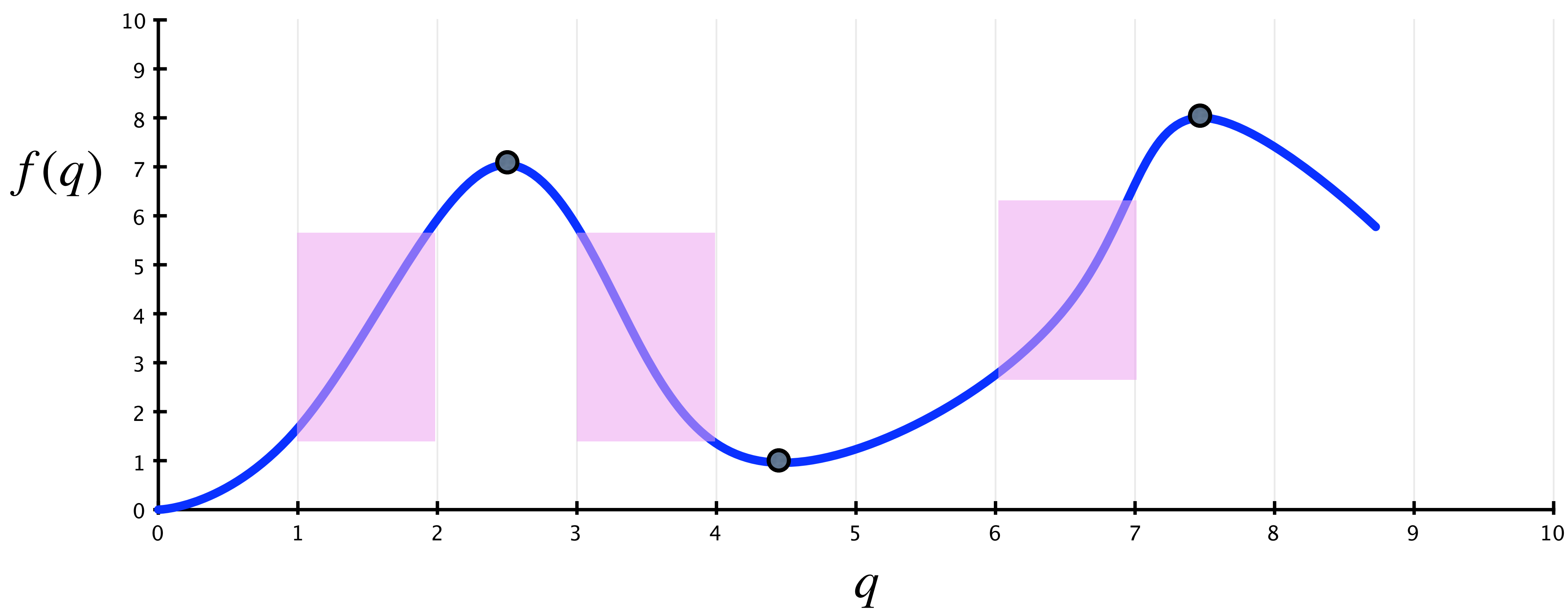
The reconstruction step depends on the slope limiter that is chosen, and should be subject to TVD constraints. The other two steps do not affect TVD.

First assume that the flux function is convex within the interval of interest.

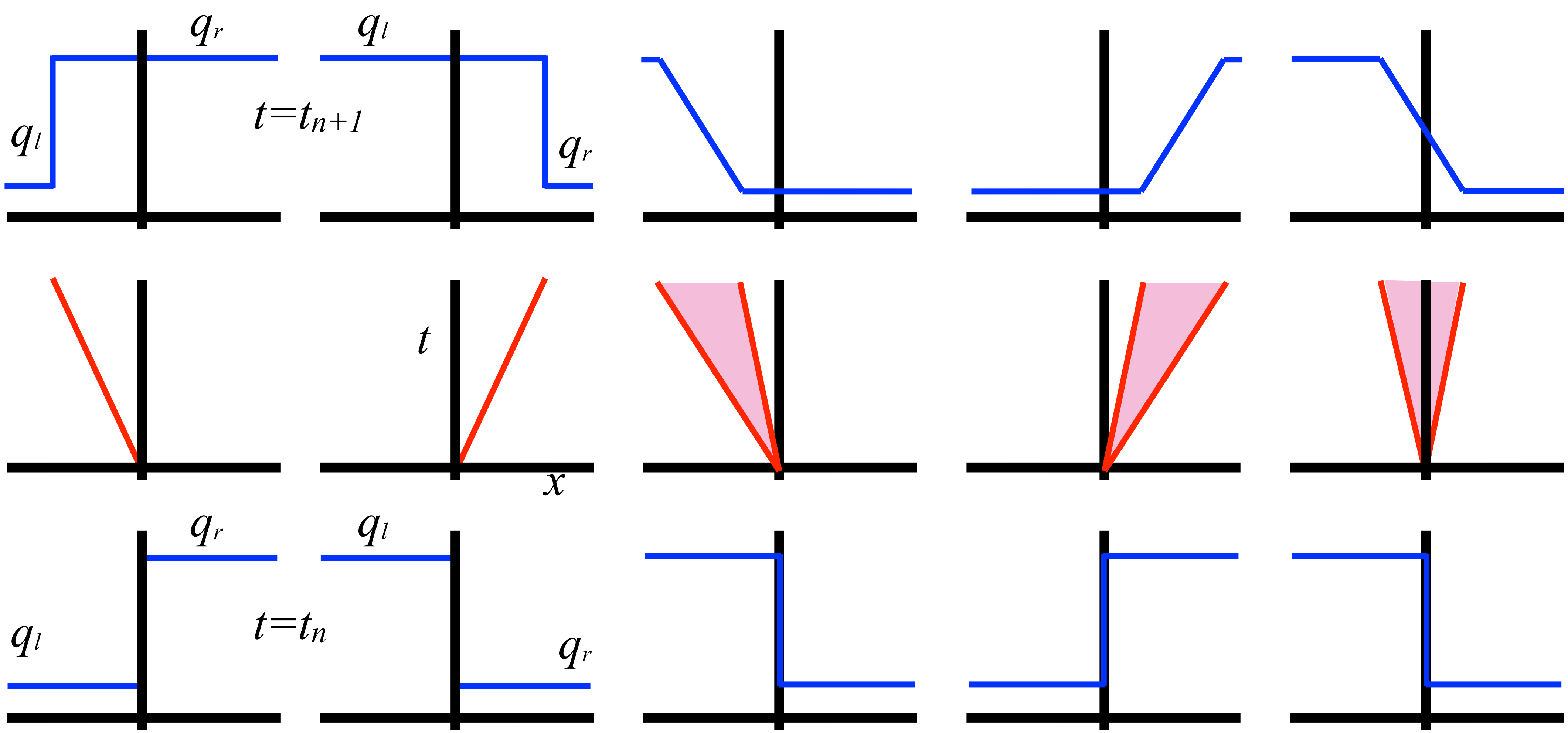
The function $f(q)$ is defined as convex in a given range if its second derivative does not change sign over that range.

In the graph below, the **nonconvex** intervals are shaded and extrema of the flux function are marked with dots. Extrema only occur within convex intervals, but do not occur in all convex intervals.

A fictitious nonlinear flux function



We have one equation, therefore one characteristic — BUT it may be a rarefaction wave!



Only in the case that the rarefaction wave spreads both to left and right, does the Riemann solution give a value different from the right and left values. This is at the stagnation point, or sonic point in a flow calculation.

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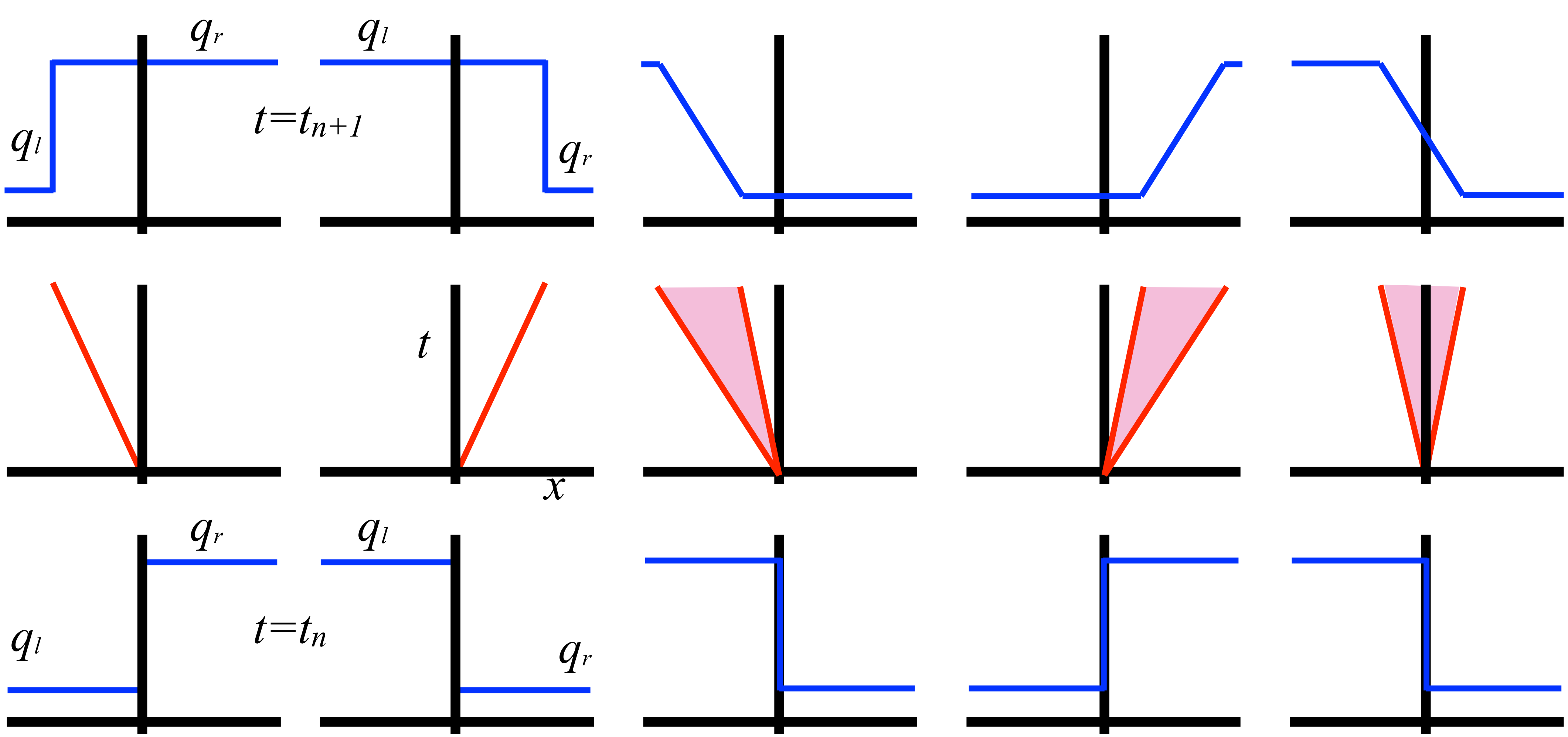

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Using fluctuations

We use our fluctuation notation from before:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right),$$

where we define the fluctuations $\mathcal{A}^{\pm}\Delta Q_{i-1/2}$ as

$$\mathcal{A}^+\Delta Q_{i-1/2} = f\left(Q_i\right) - f\left(Q_{i-1/2}^{\downarrow}\right)$$

$$\mathcal{A}^-\Delta Q_{i-1/2} = f\left(Q_{i-1/2}^{\downarrow}\right) - f\left(Q_{i-1}\right).$$

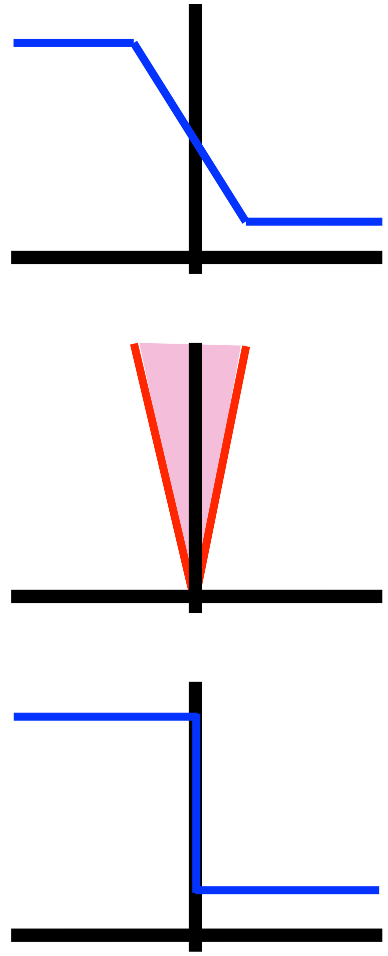
The wave *strength* is simply, as before for linear systems,

$$Q_i - Q_{i-1} = \mathcal{W}_{i-1/2},$$

but the wave *speed* is given by the Hugoniot jump condition:

$$s_{i-1/2} = \frac{f(q_i) - f(q_{i-1})}{(q_i - q_{i-1})}.$$

Entropy fix



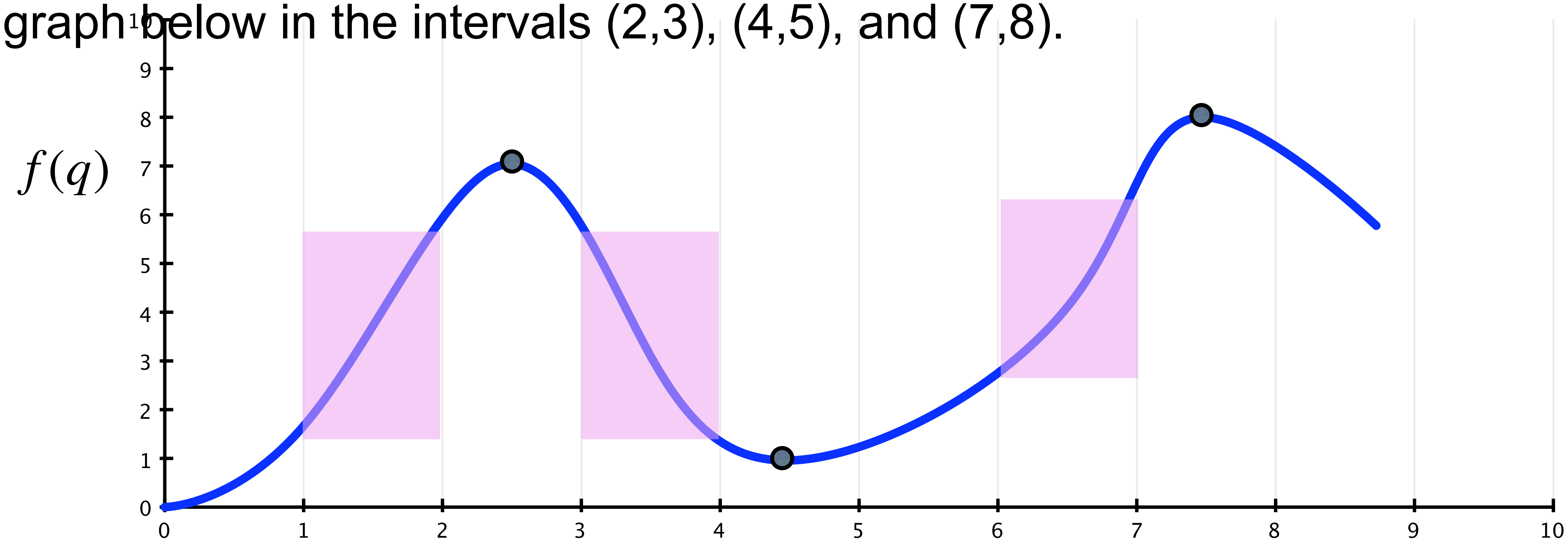
Unless the wave is a *transonic rarefaction*, we can use

$$\mathcal{A}^+\Delta Q_{i-1/2} = s^+_{i-1/2}\mathcal{W}_{i-1/2}$$
$$\mathcal{A}^-\Delta Q_{i-1/2} = s^-_{i-1/2}\mathcal{W}_{i-1/2}$$

If it is a transonic rarefaction (as at right) then we must use

$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - f(q_s)$$
$$\mathcal{A}^-\Delta Q_{i-1/2} = f(q_s) - f(Q_{i-1})$$

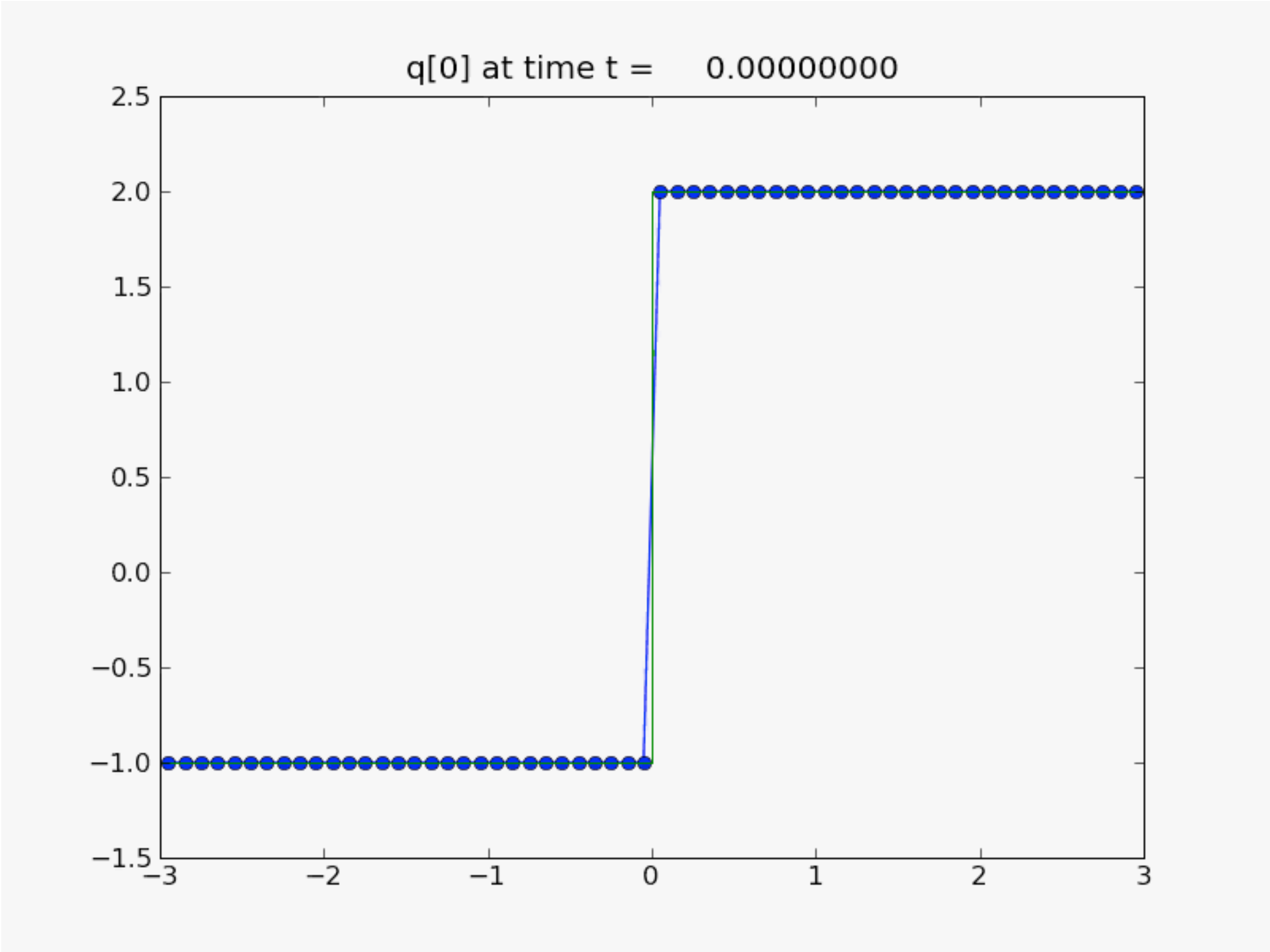
where q_s is the value of q for which $f'(q) = 0$. If the flux function is convex within the interval, this value is unique within the interval, as marked in the graph below in the intervals (2,3), (4,5), and (7,8).



Rarefaction wave without entropy fix

in $CLAW/book/chap12/efix, with efix = .false. in rp1.f

Rarefaction wave without entropy fix

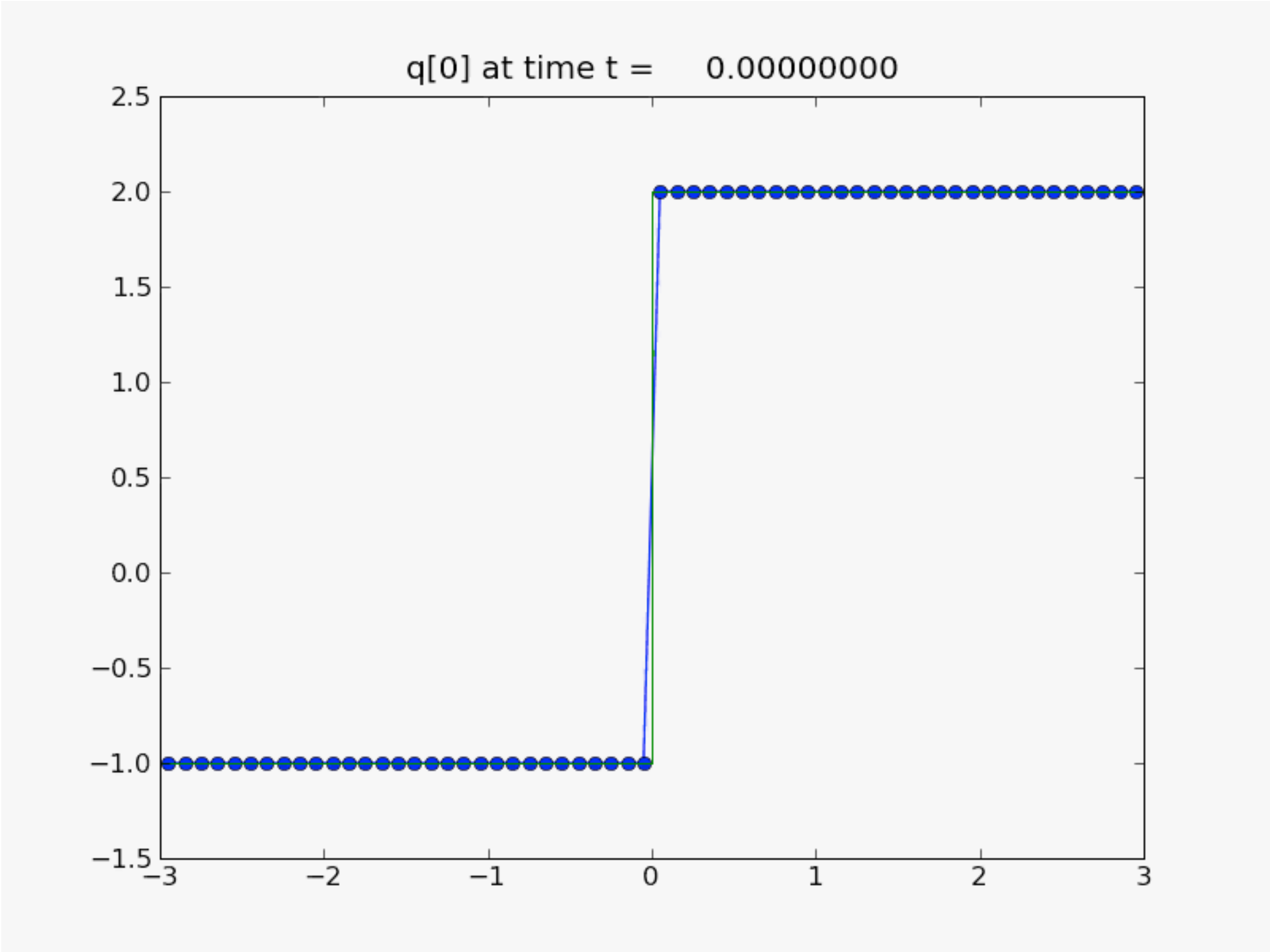


in $CLAW/book/chap12/efix, with efix = .false. in rp1.f

Rarefaction wave with entropy fix

in $CLAW/book/chap12/efix, with efix = .true. in rp1.f

Rarefaction wave with entropy fix



in $CLAW/book/chap12/efix, with efix = .true. in rp1.f rp1.f

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File Path ▾ : ~/ClawpackAll/clawpac....0rc2/myclaw/efix/rp1.f

rp1.f (no symbol selected)

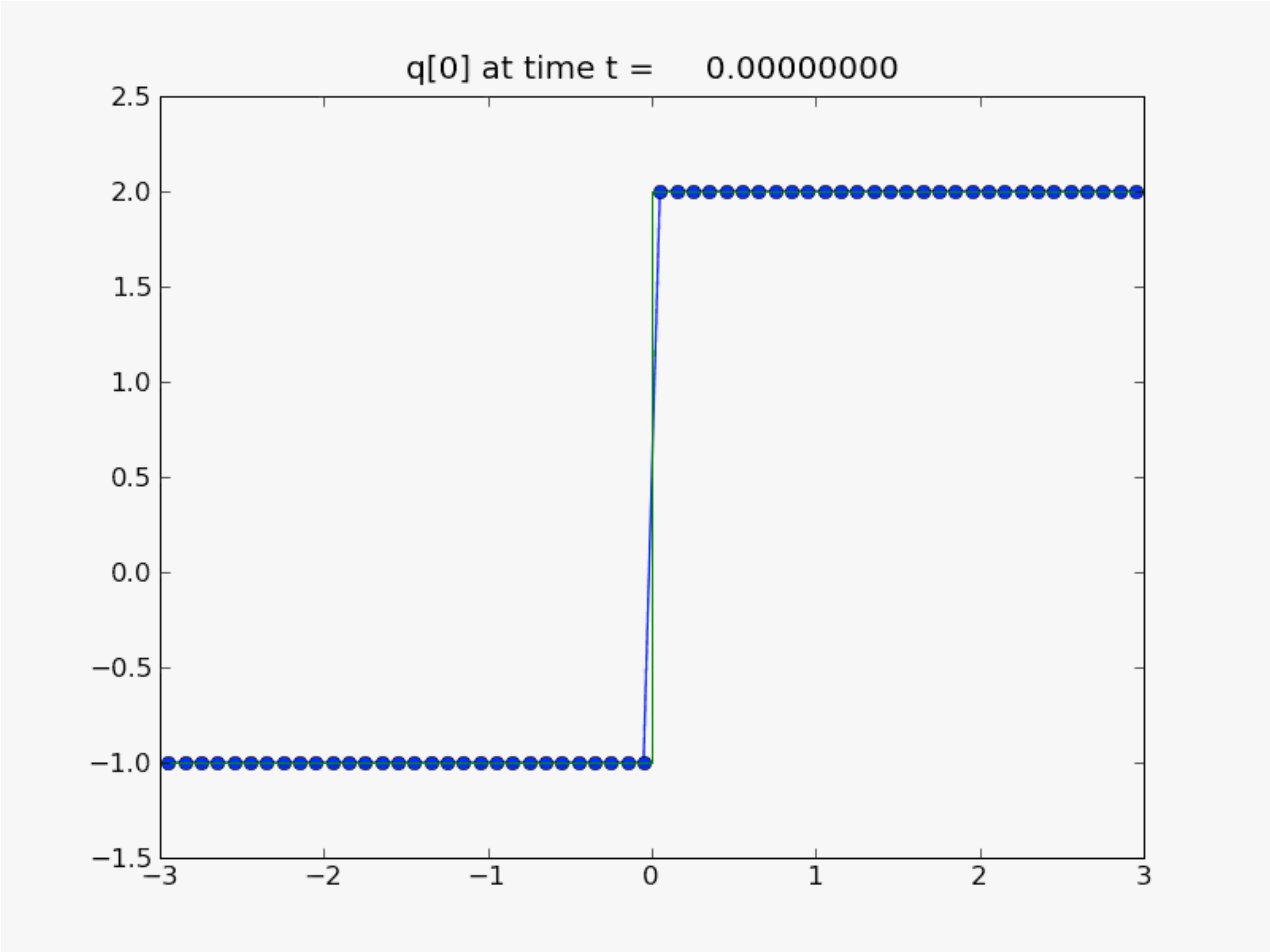
```fortran
c
      efix = .true.    !# Compute correct flux for transonic rarefactions
c
      do 30 i=2-mbc,mx+mbc
c
c        # Compute the wave and speed
c
         wave(i,1,1) = ql(i,1) - qr(i-1,1)
         s(i,1) = 0.5d0 * (qr(i-1,1) + ql(i,1))
c
c
c        # compute left-going and right-going flux differences:
c        ------------------------------------------------------
c
      amdq(i,1) = dmin1(s(i,1), 0.d0) * wave(i,1,1)
      apdq(i,1) = dmax1(s(i,1), 0.d0) * wave(i,1,1)
c
      if (efix) then
c           # entropy fix for transonic rarefactions:
            if (qr(i-1,1).lt.0.d0 .and. ql(i,1).gt.0.d0) then
          amdq(i,1) = - 0.5d0 * qr(i-1,1)**2
          apdq(i,1) =   0.5d0 * ql(i,1)**2
          endif
        endif
   30   continue
c
```

58 | 1 | Fortran | Western (Mac OS Roman) | Unix (LF) | 1 966 / 282 / 58

Rarefaction wave with entropy fix and limiters

in $CLAW/book/chap12/efix, efix = .true. and 2nd order

Rarefaction wave with entropy fix and limiters



in $CLAW/book/chap12/efix, efix = .true. and 2nd order

Vanishing viscosity / numerical viscosity

One way to introduce an entropy fix is to add a little numerical viscosity. We consider the flux function in the Lax-Friedrichs method:

$$F_{i-1/2}^{n} = \tfrac{1}{2}\left[f(Q_{i-1}^{n}) + f(Q_{i}^{n}) \right] - \frac{\Delta x}{2\Delta t}\left(Q_{i}^{n} - Q_{i-1}^{n} \right)$$

This can be viewed as having a small numerical viscosity $a = \frac{\Delta x}{\Delta t}$ throughout the computational domain.

An improvement can be made if we make this viscosity dependent on the local derivative of the flux:

$$F_{i-1/2}^{n} = \tfrac{1}{2}\left[f(Q_{i-1}^{n}) + f(Q_{i}^{n}) - a_{i-1/2}\left(Q_{i}^{n} - Q_{i-1}^{n} \right) \right]$$

where $a_{i-1/2} = \max\left(\left| f'(q) \right| \right)$ over the interval. Note that $\left| f'(q) \right| \leq \frac{\Delta x}{\Delta t}$ if the CFL condition is satisfied.

High-resolution methods, nonlinear case

We can extend the high resolution methods we developed earlier to the nonlinear conservation law by writing

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}\right),$$

where

$$\tilde{F}^n_{i-1/2} = \tfrac{1}{2}\left|s^p_{i-1/2}\right|\left(1 - \frac{\Delta t}{\Delta x}\left|s^p_{i-1/2}\right|\right)\tilde{\mathcal{W}}^p_{i-1/2},$$

and $\tilde{\mathcal{W}}^p_{i-1/2} = \alpha^p_{i-1/2}\phi(\theta^p_{i-1/2})r^p$ is the wave strength limited by the chosen slope limiter (e.g. minmod, superbee, MC, or vanLeer).

Importance of Conservation Form

Solutions that have shocks are inconsistent with the differential equation, but obey the Rankine-Hugoniot conditions, which are derived from the integral equation.

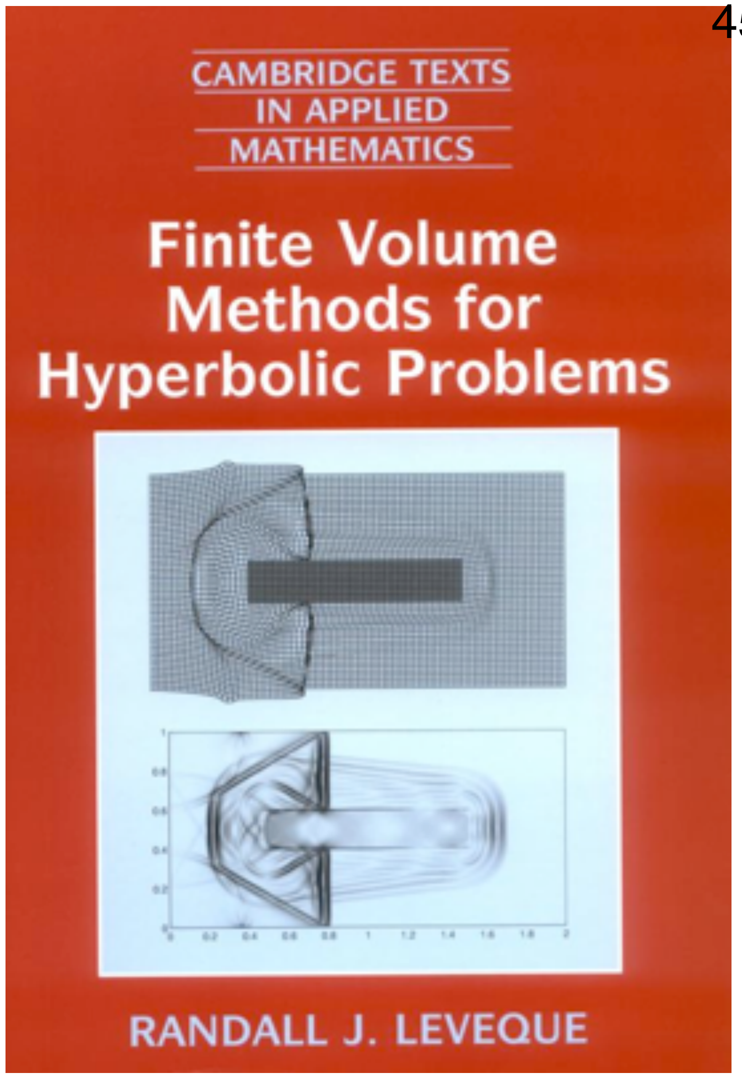
The differential equation can be manipulated in a variety of ways, but these involve the assumption of smoothness. It is important to keep the equation in a form that conserves the quantity that is actually physically conserved.

A conservative finite volume method based on an integral conservation law, *if that method converges*, will converge to a solution to the conservation law.

This is the Lax-Wendroff theorem.

Additional work must be done to establish convergence, mainly stability of the method, and that an entropy condition is satisfied so that the weak solution is in fact the correct solution.

Assignment for next time



Read Chapter 11 and Chapter 12.

Work problem 11.1. Also work problem 11.8 analytically and with Clawpack, and compare the results. Reproduce all 4 panels of Fig 12.2 using $CLAW/book/chap12/efix, and try other techniques of doing the entropy fix.

Next: Nonlinear Systems of Conservation Laws (Chapter 13)