FYS-GEO 4500

Lecture Notes #6
Nonlinear Systems and Gas Dynamics

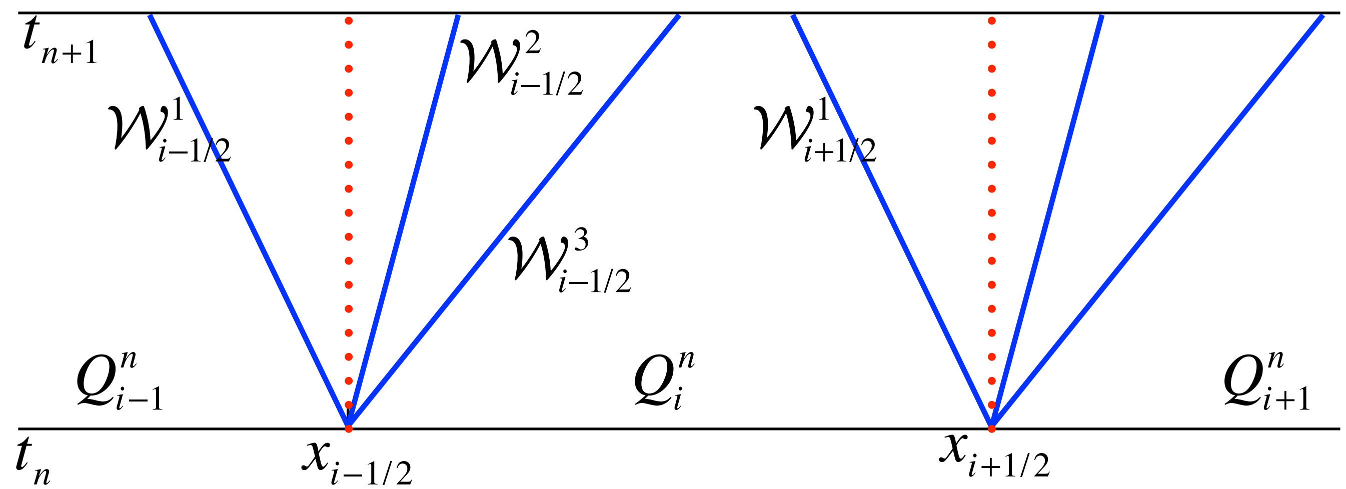
Where we are today

	date	Topic	Chapter in LeVeque
1	1.Sep 2011	introduction to conservation laws, Clawpack	1 & 2
2	15.Sep 2011	the Riemann problem, characteristics	3 & 5
3	22.Sep 2011	finite volume methods for linear systems, high resolution	4 & 6
4	29.Sep 2011	boundary conditions, accuracy, variable coeff.	7,8, part 9
5	6.Oct 2011	nonlinear conservation laws, finite volume methods	11 & 12
6	13.Oct 2011	nonlinear equations & systems	13 & 14
7	20.Oct 2011	**no lecture**	
8	27.Oct 2011	finite volume methods for nonlinear systems	15,16,17
9	3.Nov 2011	multidimensional systems and source terms, etc.	18, 19, 20, 21
	10.Nov 2011	no lecture	
10	17.Nov 2011	waves in elastic media	22
11	24.Nov 2011	unfinished business: capacity functions, source terms, project plans	
12	1.Dec 2011	student presentations	
	8.Dec 2011	no lecture	
*13	15.Dec 2011	FINAL REPORTS DUE	

Nonlinear Systems (Chapter 13 in Leveque)

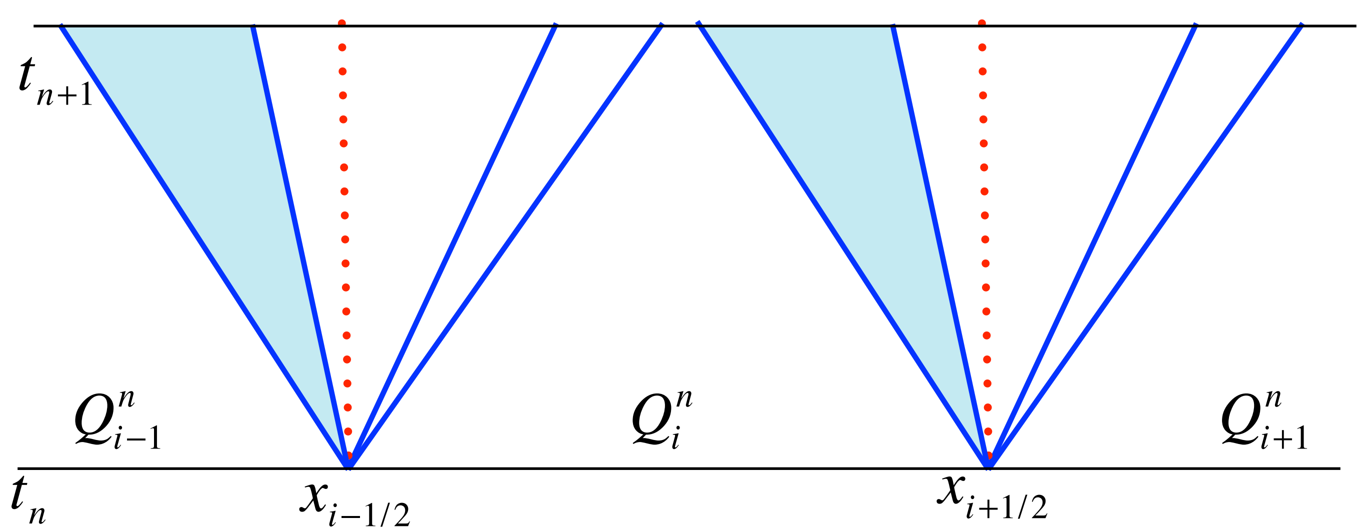
Nonlinear systems

For linear systems we learned how to apply high-resolution methods based on the Godunov technique, resolving cell interfaces into a series of Riemann-problem waves.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}$$

For nonlinear systems the procedure will be similar, except that some of the waves may be shock waves and others may be rarefaction fans.



Real Waves

At last we get to the meat of this course: the treatment of real waves.

Up to now we've been using toy problems with limited applicability.

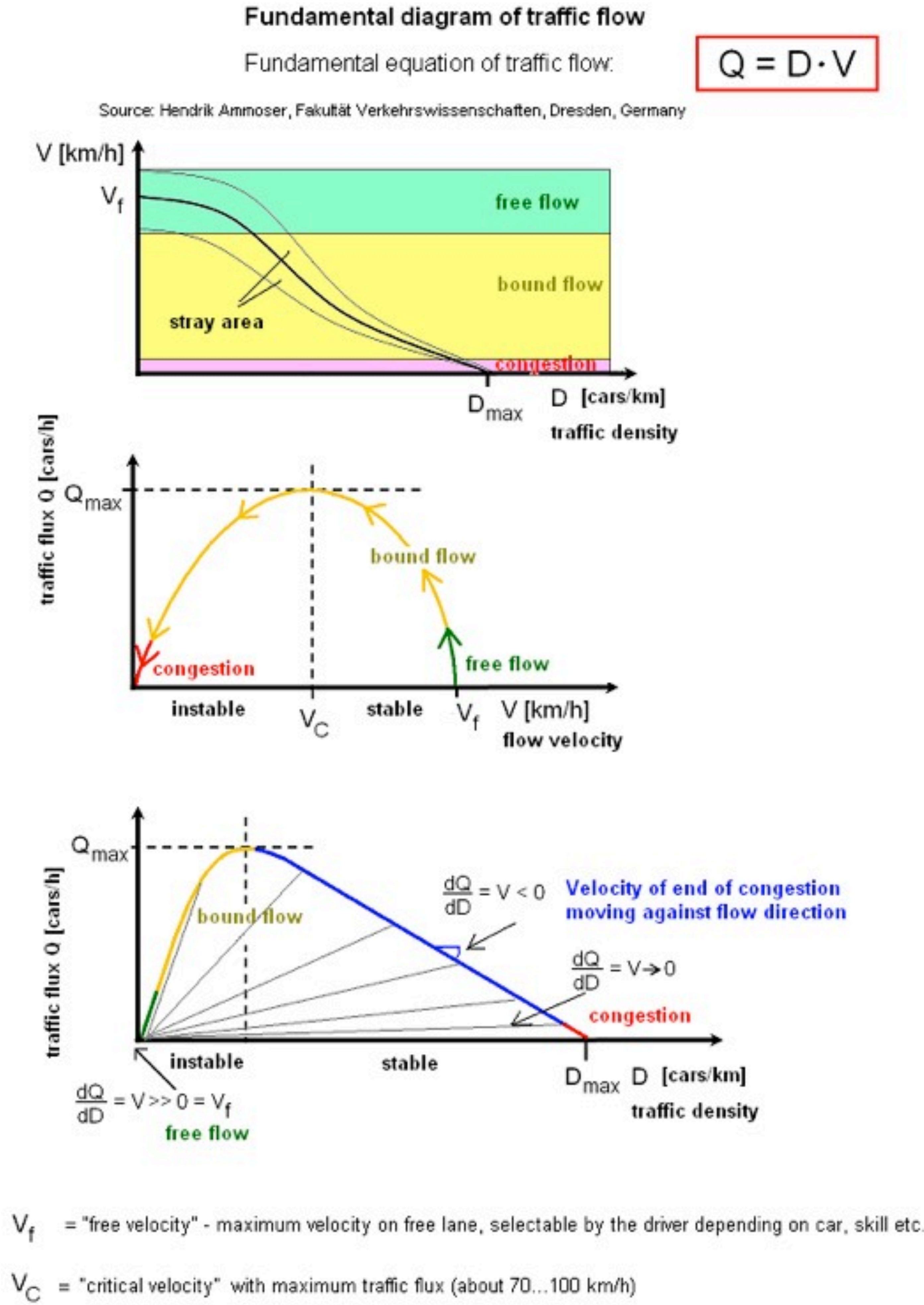
Even the traffic-flow problem we discussed last time oversimplified by assuming a velocity law dependent only on density, with no concept of reaction or acceleration, in order to reduce the problem to a single equation.

This has been necessary to help us develop the relatively complex framework that is necessary to calculate wave-like phenomena in the real world. But we are now becoming familiar with *Riemann solutions*, how they work, what their limitations are, and how they can be improved.

We will soon be able to write our own Riemann solvers and insert them into Clawpack to solve real problems.

From now on, we can start thinking of projects that we want to develop with what we've learned.

Traffic, anyone?



Found on Wikipedia:

Considerable contemporary research, especially in Germany, on traffic flow as a fluid problem. The "fundamental diagram" at right is criticised by proponents of the "three-phase traffic theory".

B.S. Kerner, *The Physics of Traffic*, Springer 2004, ISBN 3-540-20716-3.

Examples of potential projects using Clawpack

Euler equations:
- Explosive volcanic eruptions
- High-energy meteor impacts

Shallow-water equations:
- Tsunami in a fjord system or in a basin of varied bathymetry

Dusty gas equations:
- Fluidisation and hydrothermal venting
- Geysers
- Volcanic jets
- Pyroclastic flows
- DeLaval nozzle in a dusty gas

Airy-wave equations:
- Normal (deep or intermediate) water waves
- Pockmarks
- Atmospheric dispersal of contaminants
- Climate patterns

Elastic equations:
- Seismic waves and deformations

Project recipe:

Write down all the equations of the problem in conservative form, including closure relations (equations of state, for example). You should also prepare an entropy equation that will be calculable should transonic conditions (or centred rarefactions) arise.

Find the Jacobian of the corresponding quasilinear system, and calculate its eigenvalues and eigenvectors.

For an arbitrary pair of right and left states, solve the Riemann problem: write down general formulas for the waves, $\mathcal{W}^p_{i-1/2}$, the wave speeds $s^{\pm}$, the fluctuations $\mathcal{A}^{\pm}\Delta Q_{i-1/2}$ and the entropy fix for the transonic case.

Prepare a new directory for the routines you must write. Write the Riemann solver (**rp1.f**) in Fortran in this directory. Figure out what special work space you need, what boundary conditions, source terms, and other things that you want, and what special variables you need to input or initialise, and write the appropriate routines (**driver**, **setprob**, setaux, qinit, bcN, b4stepN, srcN) in the same directory. Write a **Makefile** that points to these files, and construct **setrun.py** and **seplot.py** to fill the data files and make the plots.

Then compile, run, and check your results.

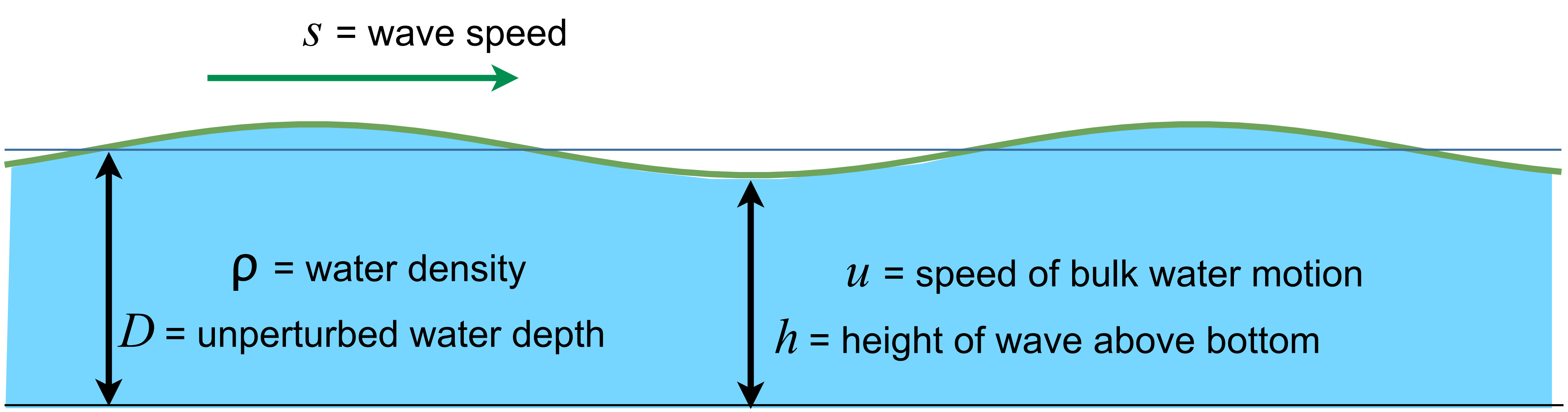
Riemann solver for Burgers' equation

```
c =====================================================================
      subroutine rp1(maxmx,meqn,mwaves,mbc,mx,ql,qr,auxl,auxr,
     &                  wave,s,amdq,apdq)
c =====================================================================
c
c     # solve Riemann problems for the 1D Burgers' equation.
c     # On input, ql contains the state vector at the left edge of each cell
c     #           qr contains the state vector at the right edge of each cell
c     # On output, wave contains the waves,
c     #            s the speeds,
c     #            amdq the  left-going flux difference  A^- \Delta q
c     #            apdq the right-going flux difference  A^+ \Delta q
c
c     # Note that the i'th Riemann problem has left state qr(i-1,:)
c     #                                     and right state ql(i,:)
c     # From the basic clawpack routine step1, rp is called with ql = qr = q.
c
c
      implicit double precision (a-h,o-z)
      dimension   ql(1-mbc:maxmx+mbc, meqn)
      dimension   qr(1-mbc:maxmx+mbc, meqn)
      dimension    s(1-mbc:maxmx+mbc, mwaves)
      dimension wave(1-mbc:maxmx+mbc, meqn, mwaves)
      dimension amdq(1-mbc:maxmx+mbc, meqn)
      dimension apdq(1-mbc:maxmx+mbc, meqn)
      logical efix
c
c
      efix = .true.   !# Compute correct flux for transonic rarefactions
c
      do 30 i=2-mbc,mx+mbc
c
c        # Compute the wave and speed
c
         wave(i,1,1) = ql(i,1) - qr(i-1,1)
         s(i,1) = 0.5d0 * (qr(i-1,1) + ql(i,1))
c
c
c        # compute left-going and right-going flux differences:
c        ------------------------------------------------------
c
         amdq(i,1) = dmin1(s(i,1), 0.d0) * wave(i,1,1)
         apdq(i,1) = dmax1(s(i,1), 0.d0) * wave(i,1,1)
c
         if (efix) then
c           # entropy fix for transonic rarefactions:
            if (qr(i-1,1).lt.0.d0 .and. ql(i,1).gt.0.d0) then
               amdq(i,1) = - 0.5d0 * qr(i-1,1)**2
               apdq(i,1) =   0.5d0 * ql(i,1)**2
               endif
            endif
   30   continue
c
      return
      end
```

The shallow-water equations

We start this investigation with a system of 2 equations, the equations of wave propagation in shallow water. And we stay in one horizontal dimension.

The shallow-water equations are used, among other things, to calculate the propagation of tsunamis in mid ocean. Because tsunami wavelengths are often 20 times greater than typical ocean depths, their propagation closely follows the shallow-water equations. The water is assumed incompressible.



Conservation of mass: $$\rho h_t + \rho(uh)_x = 0$$

Conservation of momentum: $$\rho(hu)_t + \rho(hu^2)_x + p_x = 0$$

Hydrostatic equation for pressure: $$p = \frac{1}{2}\rho g h^2$$

The shallow-water equations

Substituting the pressure equation into the momentum conservation equation and eliminating the constant density from both equations, we obtain the system of one-dimensional shallow-water equations:

$$q_t + f(q)_x = \begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0.$$

Where the solution is smooth, we can linearise in the form

$$q_t + f'(q)q_x \text{ with } q(x,t) = \begin{bmatrix} h \\ hu \end{bmatrix},$$

and obtain the Jacobian matrix

$$f'(q) = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix}.$$

Eigenvalues and eigenvectors of the shallow-water equation

The Jacobian of the shallow-water system is

$$f'(q) = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix},$$

whose eigenvalues are

$$\lambda^1 = u - \sqrt{gh},\ \lambda^2 = u + \sqrt{gh},$$

and eigenvectors

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix},\ r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}.$$

Small-amplitude shallow-water waves

If the water has constant unperturbed depth D and is moving with constant velocity (or is stationary), and if $y = h - D \ll D$, then the Jacobian is

$$f'(q) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + gD & 2u_0 \end{bmatrix},$$

with eigenvalues $\lambda^1 = u_0 - \sqrt{gD}$, $\lambda^2 = u_0 + \sqrt{gD}$.

Small-amplitude shallow-water waves thus move at speed $\sqrt{gD}$ relative to the motion of the water. In the deep ocean, D ~ 5000 m, and the speed is therefore > 200 m/s, as fast as a jet plane. Tsunamis in mid ocean have typical amplitudes of centimetres and wavelengths of 10s of kilometres, so this small-amplitude shallow-water approximation applies for their mid-ocean propagation.

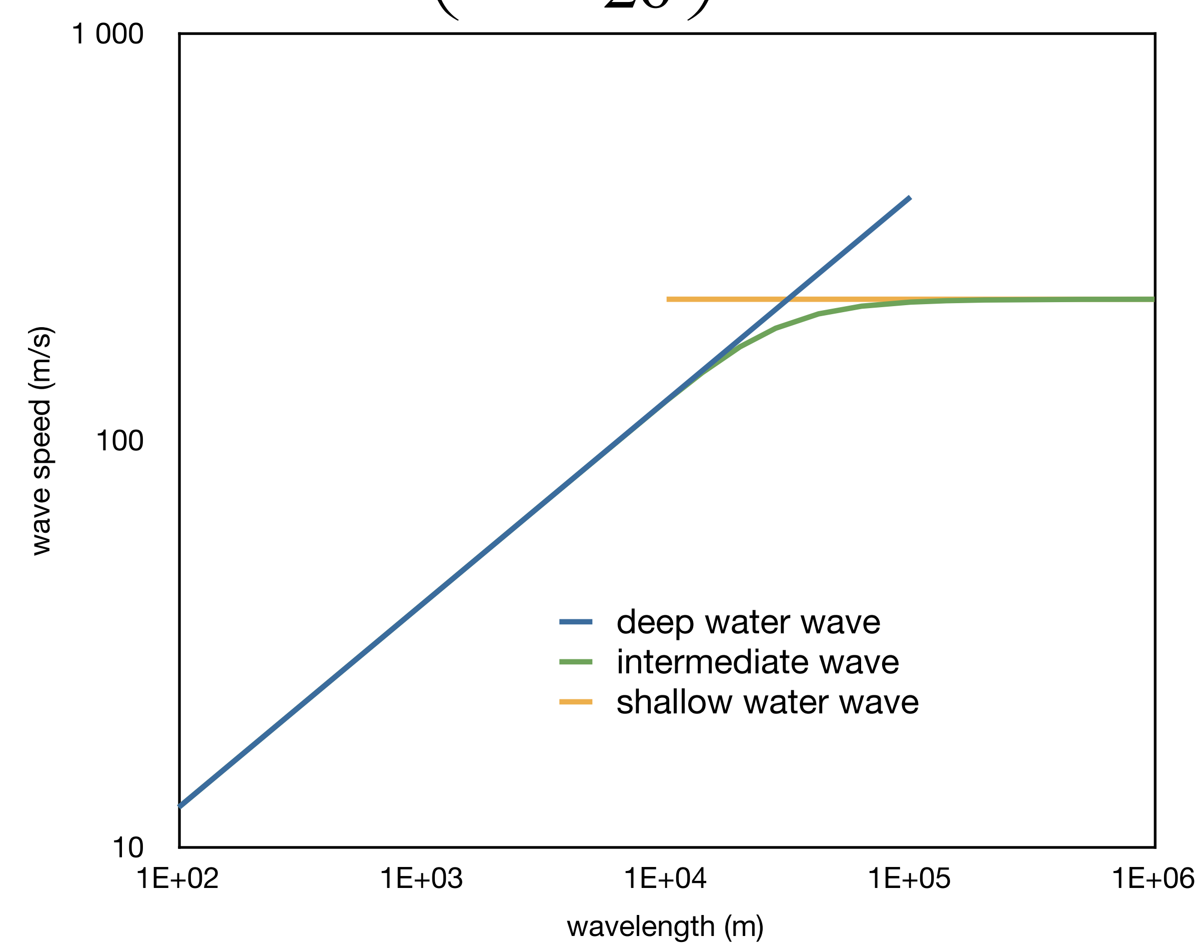
As these waves approach a shore, the water depth decreases, the speed decreases, and the wave piles up; eventually the waves become nonlinear, steepen, and break.

Other types of waves on water have different speeds (from Airy wave theory)

$$c_p = \begin{cases} \sqrt{\dfrac{g\lambda}{2\pi}} & \text{for deep-water waves} \quad \left(D > \dfrac{\lambda}{2}\right) \\ \sqrt{\dfrac{g\lambda}{2\pi}\tanh\left(\dfrac{2\pi D}{\lambda}\right)} & \text{for intermediate waves} \quad \left(\dfrac{\lambda}{20} > D > \dfrac{\lambda}{2}\right) \\ \sqrt{gD} & \text{for shallow-water waves} \quad \left(D < \dfrac{\lambda}{20}\right) \end{cases}$$

For deep-water and intermediate waves, the wave speed also depends on the wavelength λ.

The diagram is for waves in a deep (D=5000 m) ocean basin. The transition to shallow-water behaviour occurs at wavelengths greater than $2\pi D$.



Shallow-water waves: initial hump

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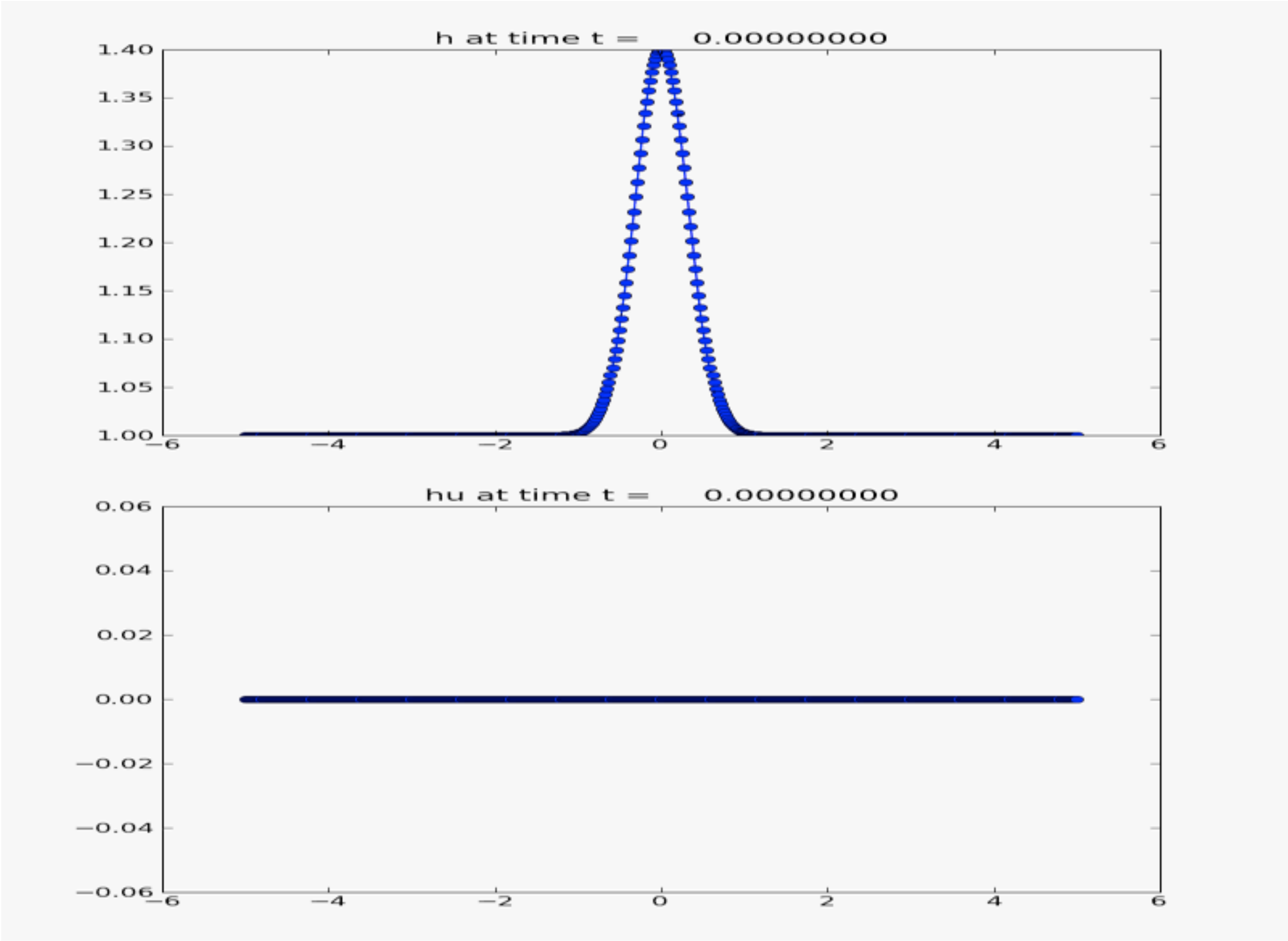
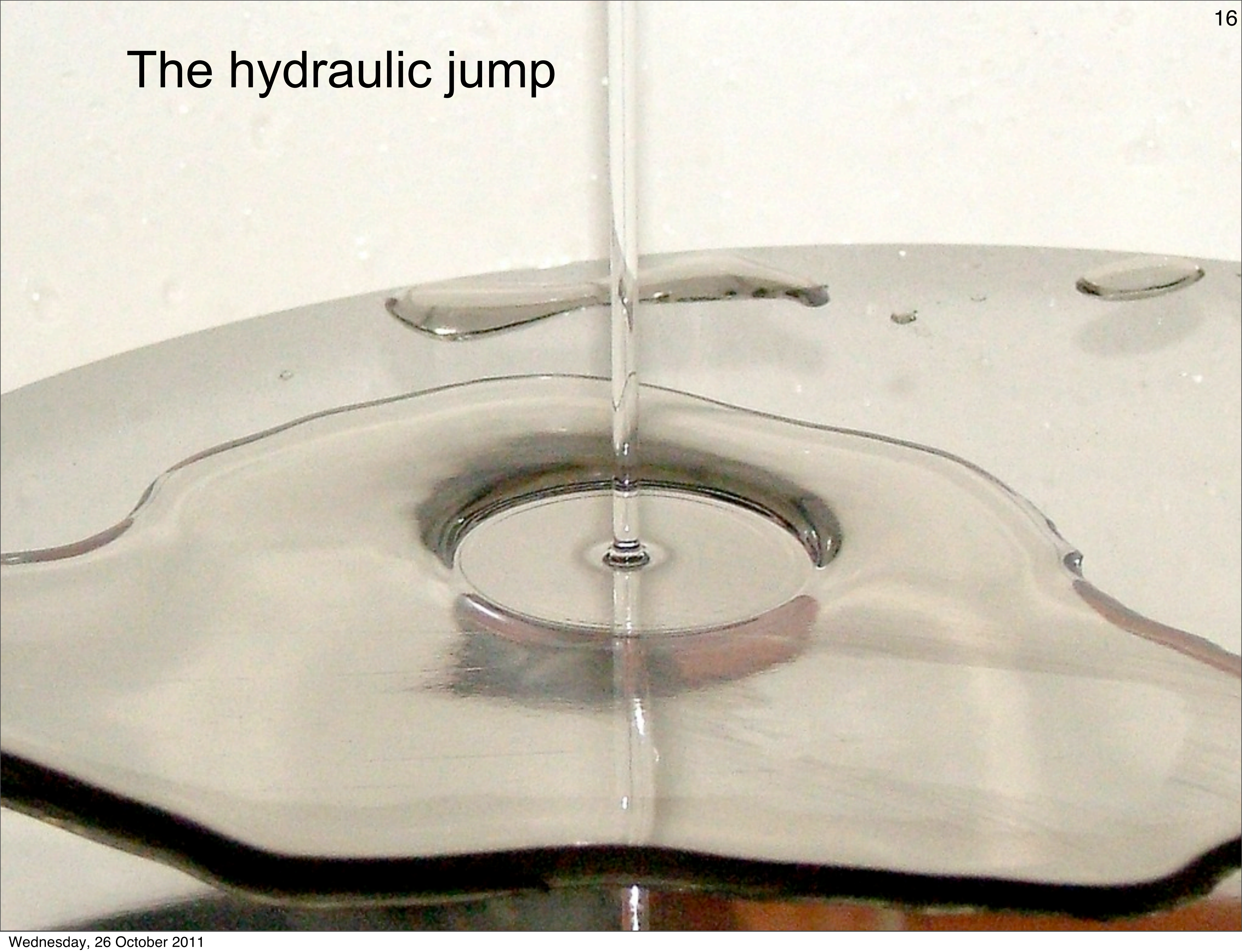


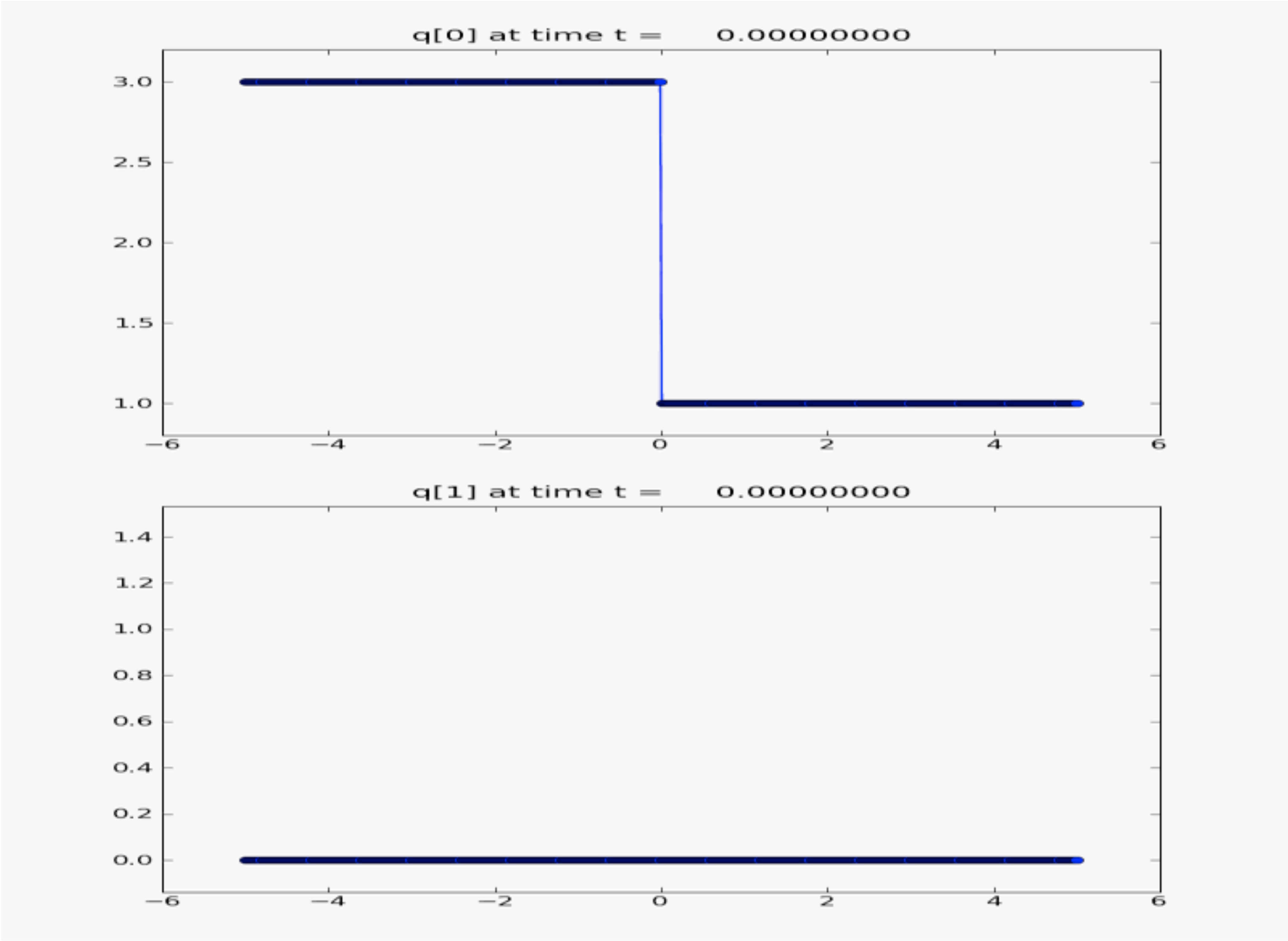
The hydraulic jump



Shallow-water waves: the dam break

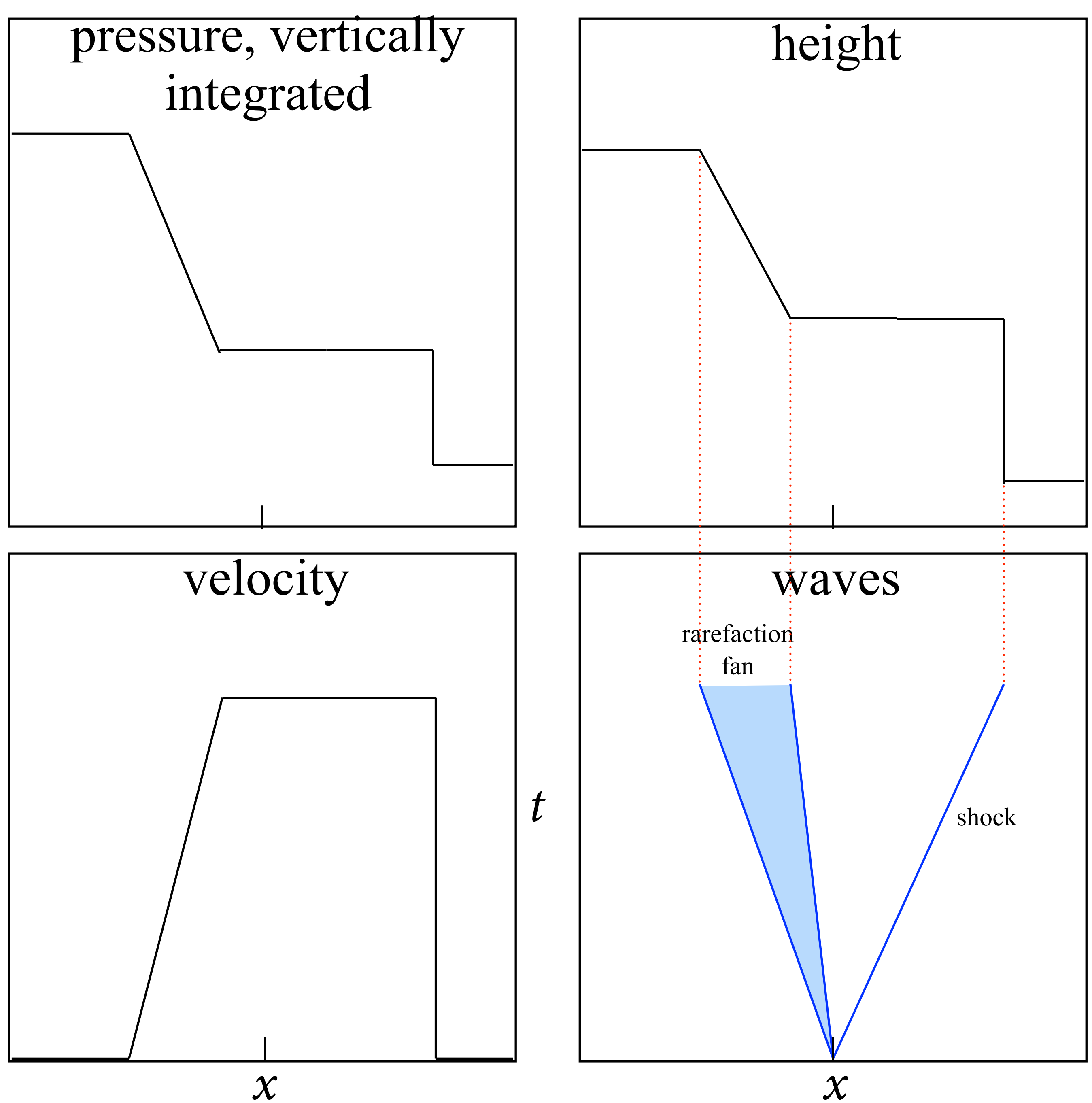
with both a shock and a rarefaction

Shallow-water waves: the dam break



with both a shock and a rarefaction

The dam-break problem is a macroscopic Riemann problem



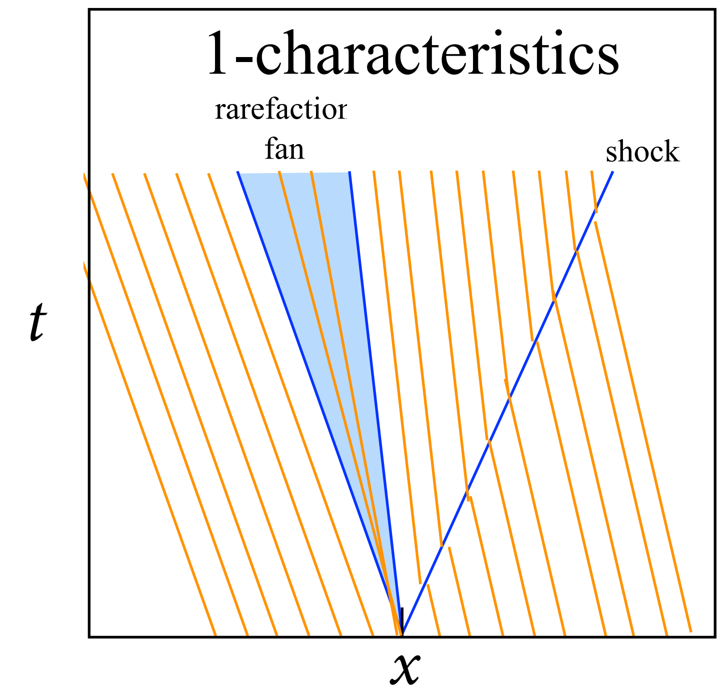
It resembles the shock tube from gas dynamics, but has only two waves, the shock and the rarefaction wave.

(*Pressure and height are not independent*)

Characteristics in the dam-break problem

Since there are two equations, there are two sets of characteristics. The 1-characteristics satisfy

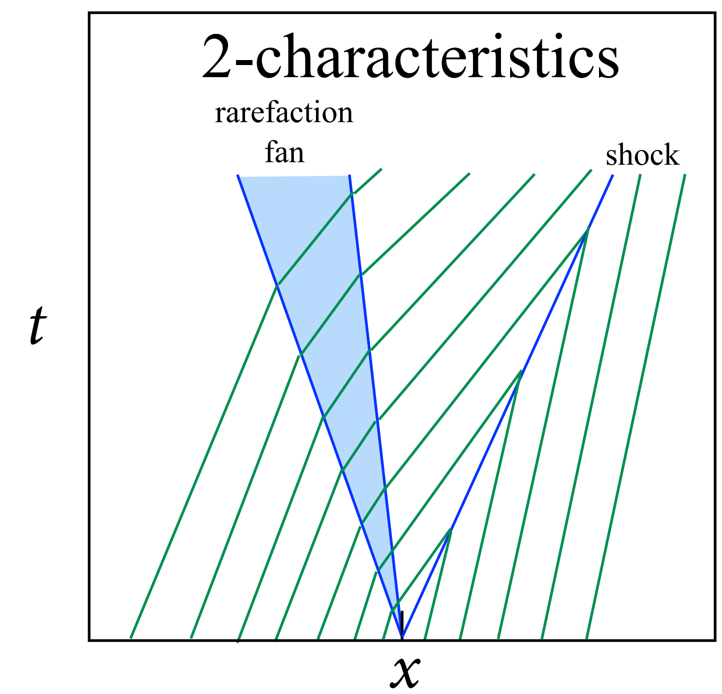
$$\frac{dX}{dt} = \lambda^1 = u - \sqrt{gh},$$



and the 2-characteristics satisfy

$$\frac{dX}{dt} = \lambda^2 = u + \sqrt{gh}.$$

The 1-characteristics spread out in the rarefaction 1-wave, and they cross the 2-shock, bending because of the discontinuity in h.



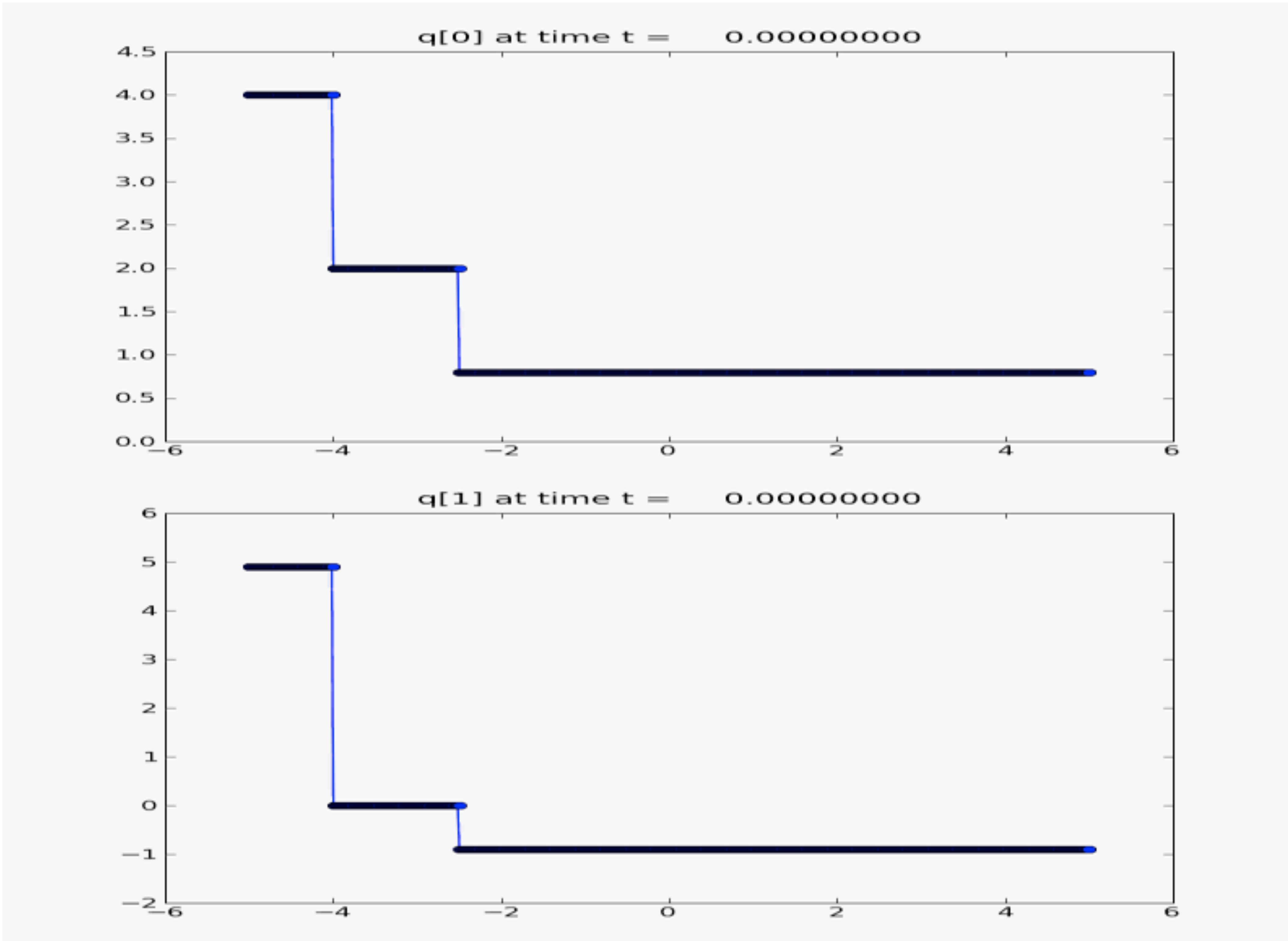
The 2-characteristics *meet* each other at the 2-shock and cross the rarefaction 1-wave, curving slightly as they go through it.

The characteristics are straight elsewhere.

What happens when one shock overtakes another?

In $CLAW/book/chap13/collide

What happens when one shock overtakes another?



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Strategy for the Riemann problem in a nonlinear system

Given arbitrary right and left states q_l, and q_r representing two adjoining cells:

Determine which wave(s) are shocks and which are rarefactions.

Determine the intermediate states between the waves.

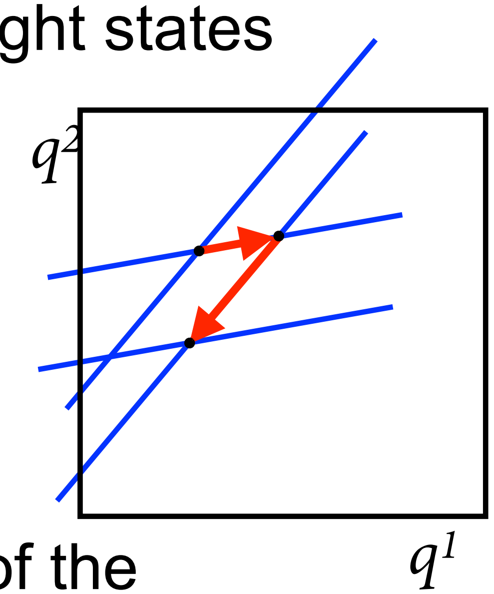
Determine the solution structure through the rarefaction waves - this is the only tricky part.

Usually an *approximate* Riemann solution will be used in practical finite volume methods.

Such an approximate solver (the *Roe* solver) is found in $CLAW/book/chap13/collide/rp1sw.f

An isolated shock

Across an isolated shock, propagating with speed s with left and right states q_l, and q_r the Rankine- Hugoniot condition must be satisfied:

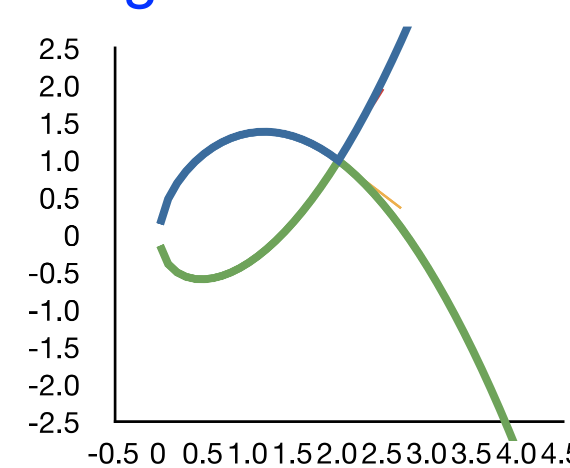


$$f(q_r) - f(q_l) = s(q_r - q_l)$$

This can only hold for certain pairs q_l and q_r.

For example, for a *linear system*, $q_r - q_l$ must be an eigenvector of the system. For any given left state q_l, the only possible right states q_r are those that lie (in state space) on straight lines in an eigen-direction from q_l.

For a *nonlinear system,* there will be an equivalent requirement, but instead of straight lines, the allowable right states q_r lie on curves, called Hugoniot loci, through q_l.



The Hugoniot loci for shallow water equations

Fix $q_* = (h_*, u_*)$. We may think of this as either q_r or q_l in a Riemann problem, in which we need to find q_m, the middle state.

We must ask: Which states q can be connected to q_* by an isolated shock?

The Rankine-Hugoniot condition $s(q_* - q) = f(q_*) - f(q)$ gives:

$$s(h_* - h) = h_* u_* - hu$$

$$s(h_* u_* - hu) = h_* u_*^2 - hu^2 + \tfrac{1}{2} g(h_*^2 - h^2)$$

These are two equations with 3 unknowns (h, u, s). We expect 1-parameter families of solutions, i.e. curves in the (u, uh) state space.

We'll get a quadratic equation, so in fact we will have two curves, or families of solutions. These are the Hugoniot loci, and on these curves lie *candidates* for states connected by 1-shocks or 2-shocks.

The Hugoniot loci for shallow water equations

$$s(h_* - h) = h_* u_* - hu$$

$$s(h_* u_* - hu) = h_* u_*^2 - hu^2 + \tfrac{1}{2} g(h_*^2 - h^2)$$

We want to examine curves in the state space (u, uh) so we eliminate s from the system:

$$s = \frac{h_* u_* - hu}{h_* - h}$$

Substituting into the second equation, we get the quadratic equation

$$u^2 - 2u_* u + \left[u_*^2 - \frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*} \right)(h_* - h) \right] = 0$$

which has the two solutions

$$u(h) = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*} \right)(h_* - h)}$$

The Hugoniot loci for shallow water equations

The graph at right shows the two solutions for the Hugoniot loci for a particular case.

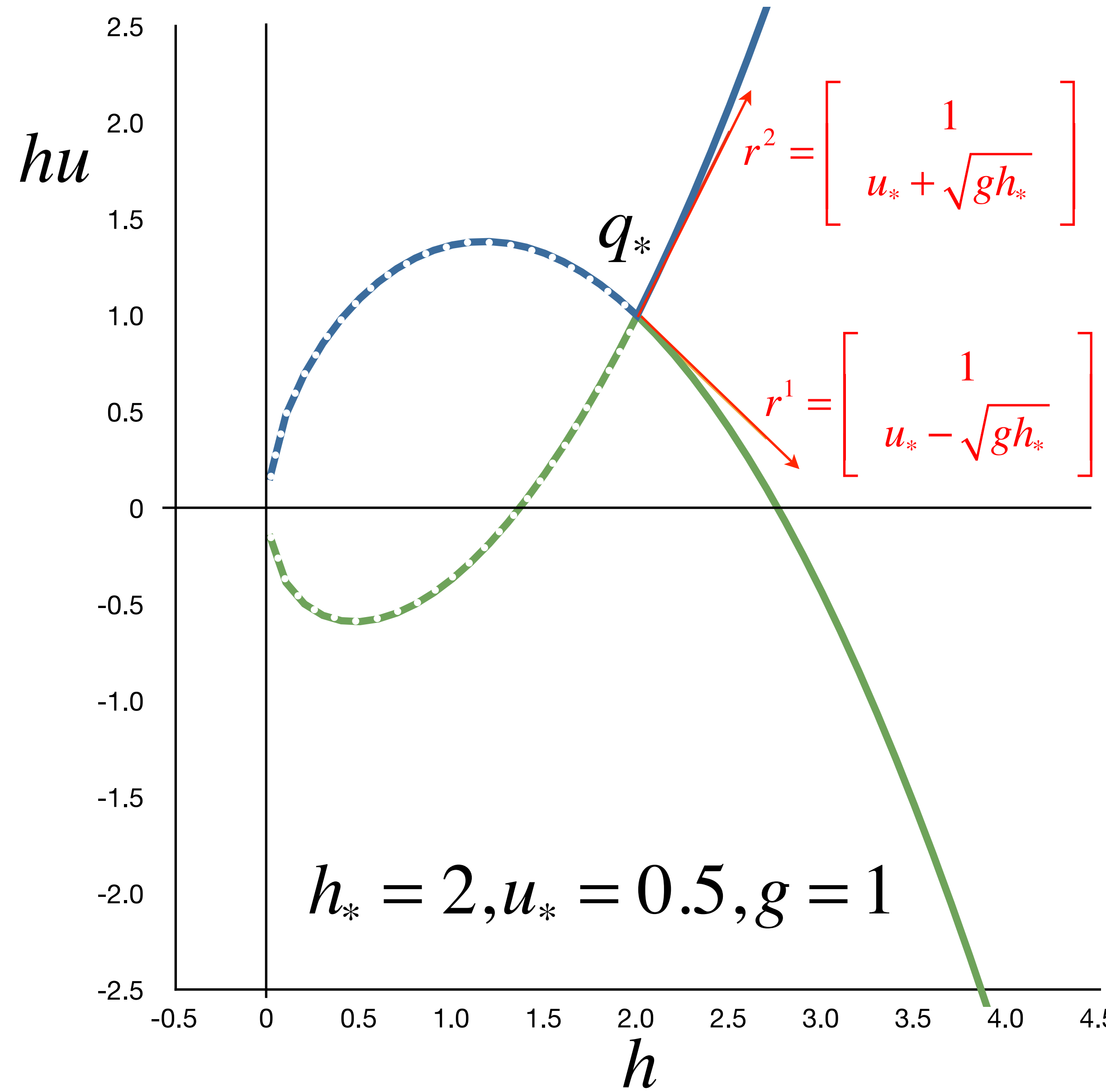
The blue curve is the solution with the + sign, the green one with the – sign. The system eigenvectors at q_* are shown as red arrows, tangent to the curves. Observe that the families switch curves at q_*.

Consider that q_* could be either q_r or q_l.

The states accessible from q_r are on the green curve to the left of q_* but on the blue curve to the right of q_*, following the 2-eigenvector. Similarly the states accessible from q_l lie on the other curve.

Bot only the states to the right satisfy the entropy condition.

$$u(h) = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h_* - h)}$$



The entropy condition for the shallow-water equations

The Lax Entropy condition is:

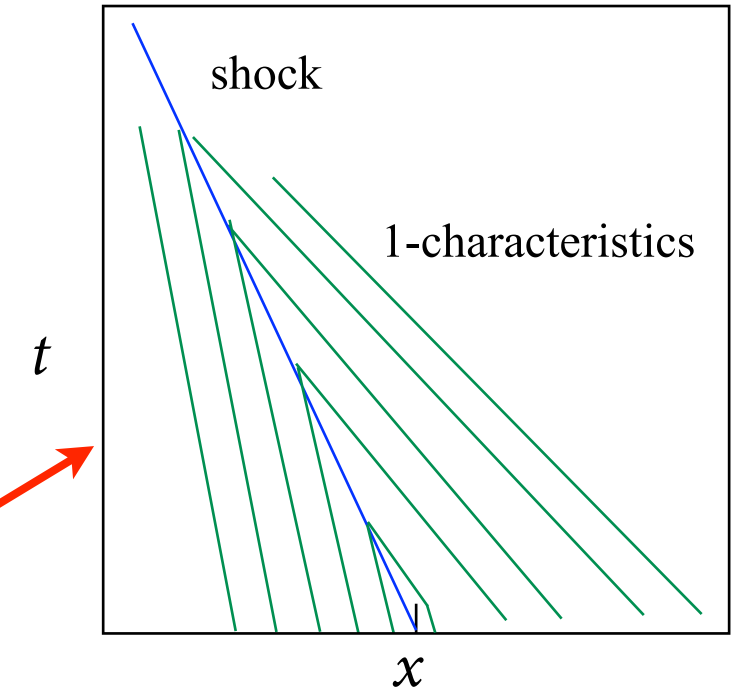
$$\lambda^p(q_l) > s > \lambda^p(q_r)$$

The characteristic velocities at $q_* = \begin{bmatrix} u_* \\ u_* h_* \end{bmatrix}$ are:

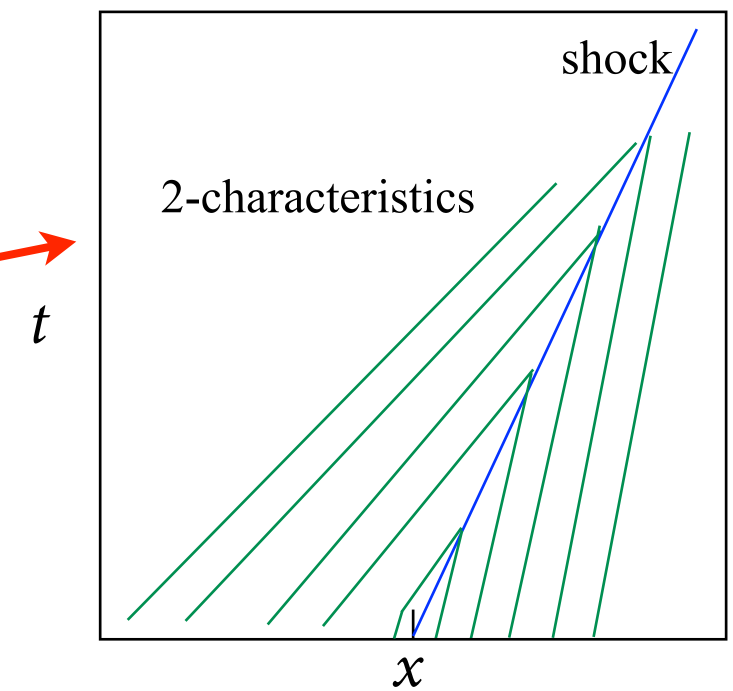
$$\lambda^1 = u_* - \sqrt{g h_*}$$

$$\lambda^2 = u_* + \sqrt{g h_*}$$

Across a 1-shock connecting q_l to q_m, the characteristic velocity must decrease, implying h must *increase*.



Across a 2-shock connecting q_m to q_r, the characteristic velocity must decrease, but in this case h must *decrease*.



The entropy condition for the shallow-water equations

$$\lambda^1 = u_* - \sqrt{gh_*}$$

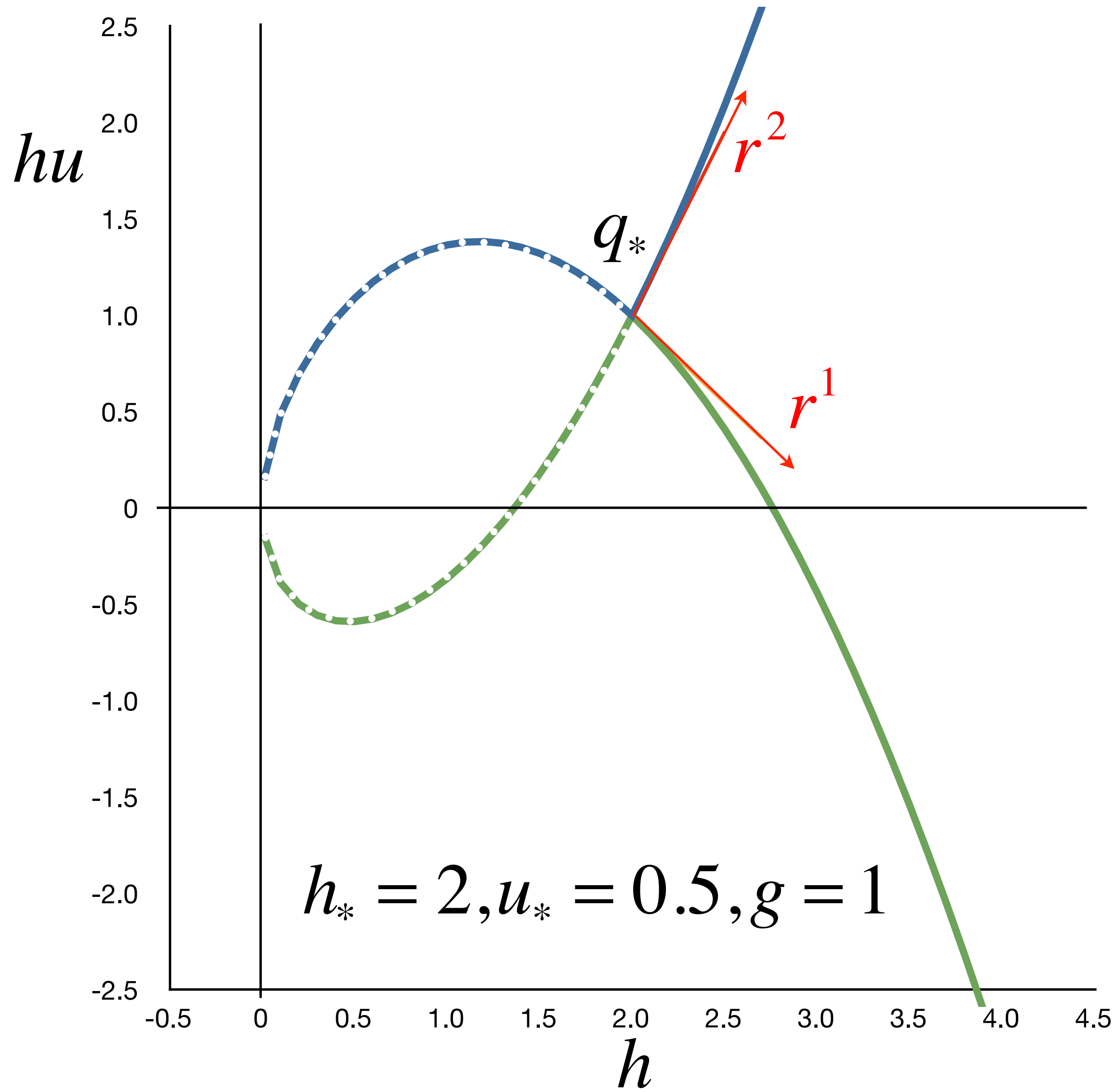
$$\lambda^2 = u_* + \sqrt{gh_*}$$

Across a 1-shock connecting q_l to q_m, the characteristic velocity must decrease, implying h must *increase*.

Across a 2-shock connecting q_m to q_r, the characteristic velocity must decrease, but in this case h must *decrease.*

Put q_l at q_*, then q_m must lie on the green curve below and to the right.

Put q_r at q_*, then q_m must lie on the blue curve above and to the right.



The Lax entropy condition

More fully, the Lax entropy condition is that the speed s of a shock between states q_l and q_r, is such that

$$\lambda^j(q_l) < s \quad \text{and} \quad \lambda^j(q_r) < s \quad \text{for } j < p$$

$$\lambda^p(q_l) > s > \lambda^p(q_r)$$

$$\lambda^j(q_l) > s \quad \text{and} \quad \lambda^j(q_r) > s \quad \text{for } j > p$$

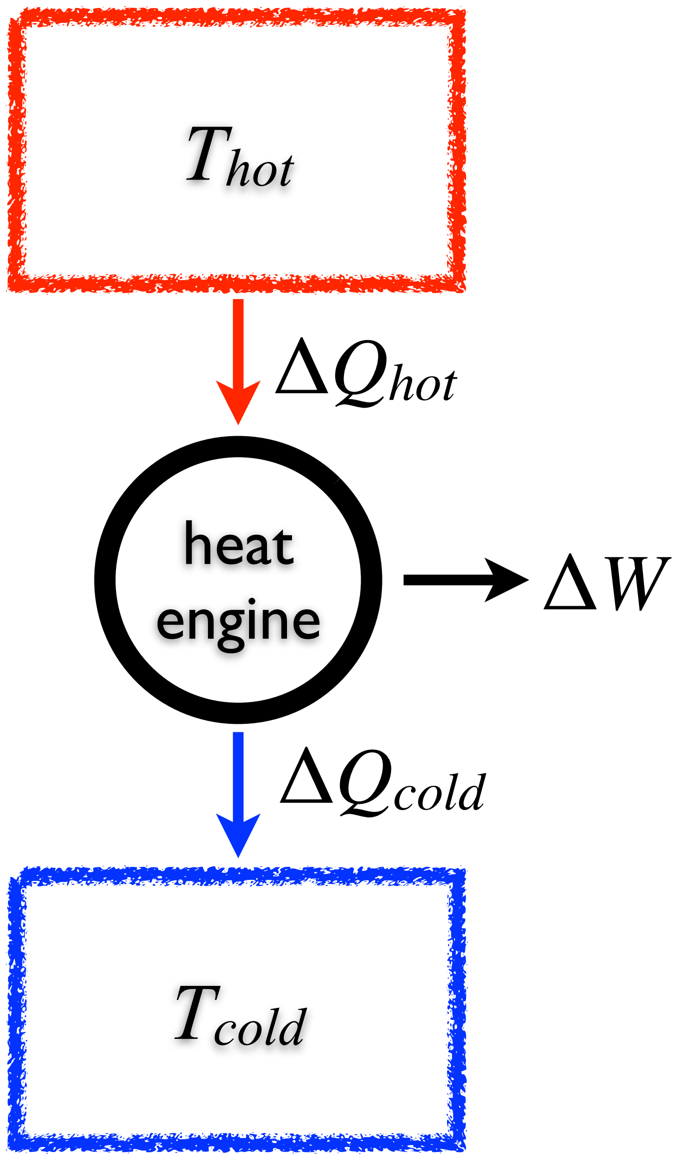
for some index p which defines the shock.

That is, the p-characteristics impinge on the p-shock, while the other characteristics simply cross the shock.

The eigenvalues and characteristics are assumed to be ordered:

$$\lambda^1 < \lambda^2 < \cdots < \lambda^m$$

Entropy: what is it anyway?



A little thermodynamics: for now, $Q, S, T,$ and W represent heat, entropy, temperature, and work.

A body giving off heat to its environment experiences a change in entropy defined by

$$\Delta S \equiv \frac{\Delta Q}{T}.$$

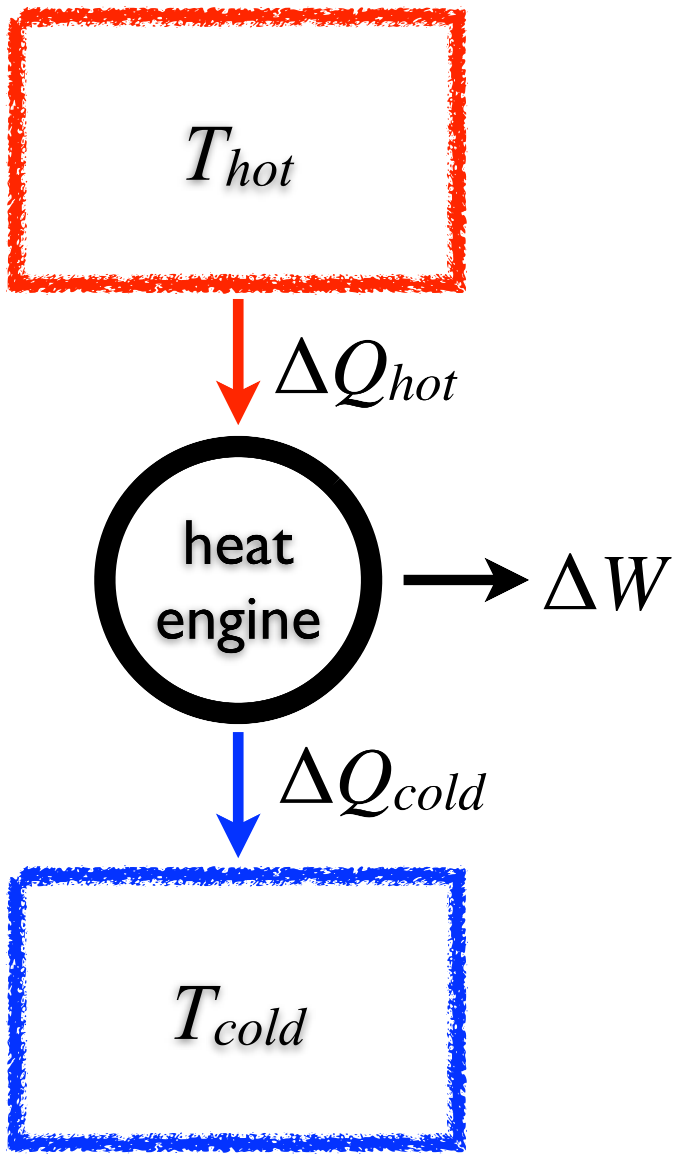
The second law of thermodynamics says that a cool body cannot spontaneously transfer heat to a warmer one. This implies that, for any system,

$$\sum_{\text{system}} \Delta S = \sum_{\text{system}} \frac{\Delta Q}{T} \geq 0$$

The first law of thermodynamics says that a cyclical heat engine (perfectly reversible) operating between two reservoirs produces work equal to the amount of heat exchanged:

$$\Delta W_{\text{reversible}} = \Delta Q_{\text{hot}} - \Delta Q_{\text{cold}}.$$

Entropy represents a *degradation* of energy



$$\Delta W_{\text{reversible}} = \Delta Q_{\text{hot}} - \Delta Q_{\text{cold}}.$$

The condition for reversibility is that the entropy change be zero:

$$\Delta S = \frac{\Delta Q_{hot}}{T_{hot}} - \frac{\Delta Q_{cold}}{T_{cold}} = 0$$

so the reversible work is:

$$\Delta W_{\text{reversible}} = \left(1 - \frac{T_{\text{cold}}}{T_{\text{hot}}}\right)\Delta Q_{\text{hot}}.$$

But if $\Delta S > 0$, the work produced is reduced:

$$\Delta W_{\text{irreversible}} = \Delta W_{\text{reversible}} - T_{\text{cold}}\Delta S.$$

The entropy increase thus degrades the energy into a form less available for producing work. This is the **only** completely objective and rigorous definition of entropy.

What does entropy mean for us?

In gas dynamics, a shock converts kinetic energy (upstream) into thermal energy (downstream), which is less useful for work.

In the shallow-water equations, the hydraulic jump converts velocity into height. There is no energy equation, so height isn't immediately useful.

Usually in dynamic systems, it is bulk velocity or kinetic energy that suffers going across a shock. A collection of particles that enters a shock with high speed, leaves it at lower bulk speed but higher temperature.

A fluid element that enters a shock with high speed leaves it at lower speed. This is indeed the essence of the Lax entropy condition:

$$\lambda^p(q_l) > s > \lambda^p(q_r).$$

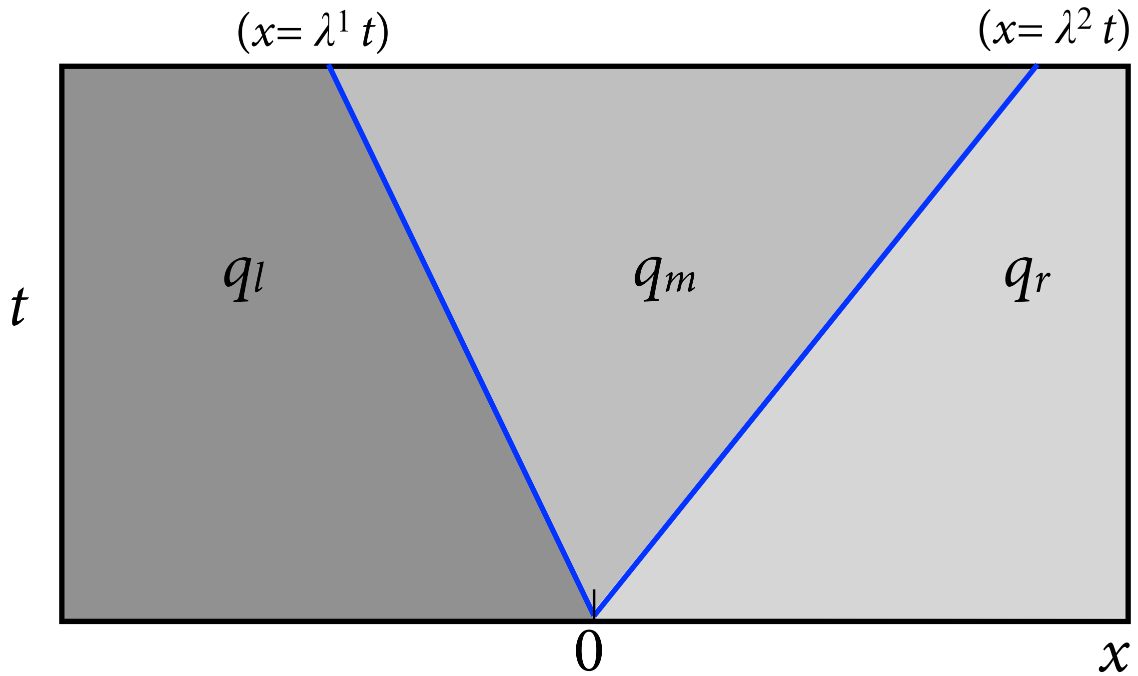
There are other notions of entropy that can be useful in other systems: for example, lack of information, larger number of microstates for a given macrostate, etc. The widely used notion of “disorder” is subjective and difficult to quantify.

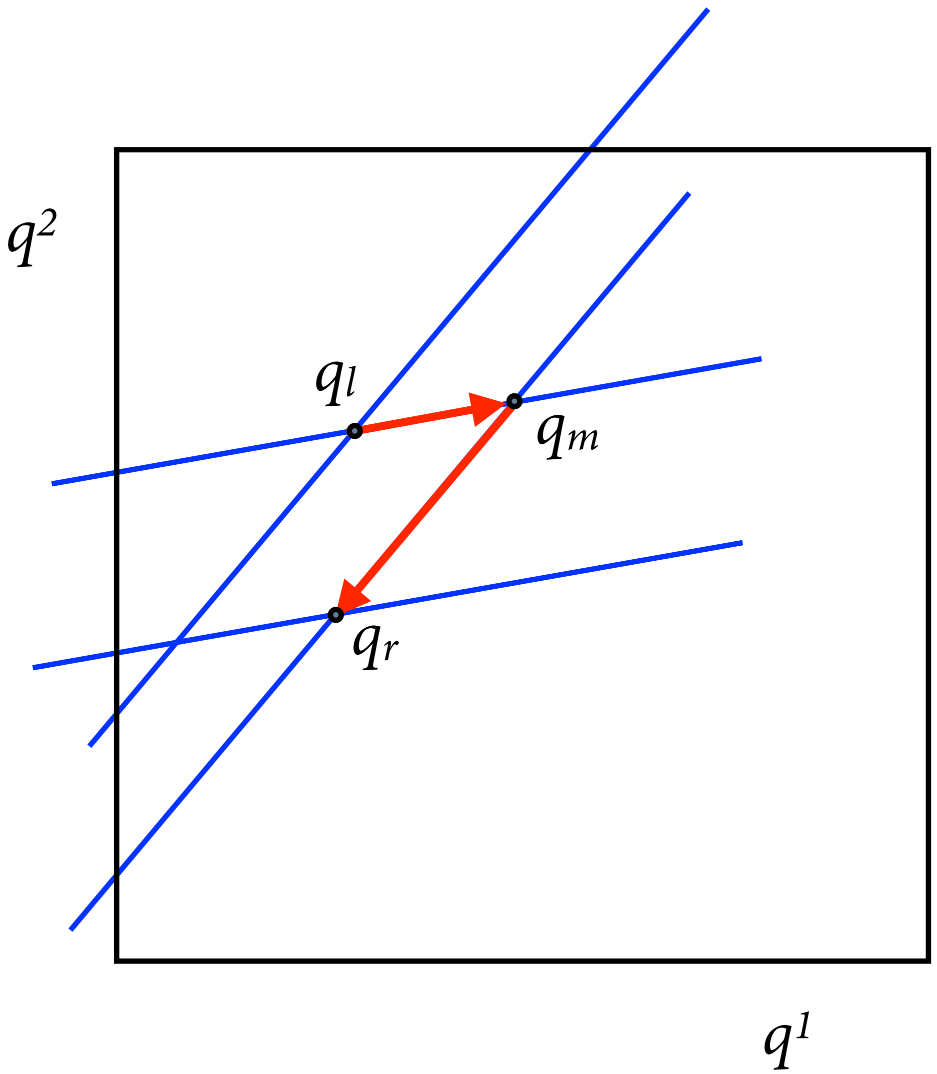
What if there are two shocks?

Since only certain pairs q_l and q_r can be connected by a single shock, what do we do in the general case when we have an arbitrary pair q_l and q_r and we know there are two shocks?

Answer: we find two shocks to connect them. Specifically, we find an intermediate state q_m that is connected to both q_l and q_r by opposing shocks.

This is just as we did for the linear system, recall:





The two-shock solution - collision!

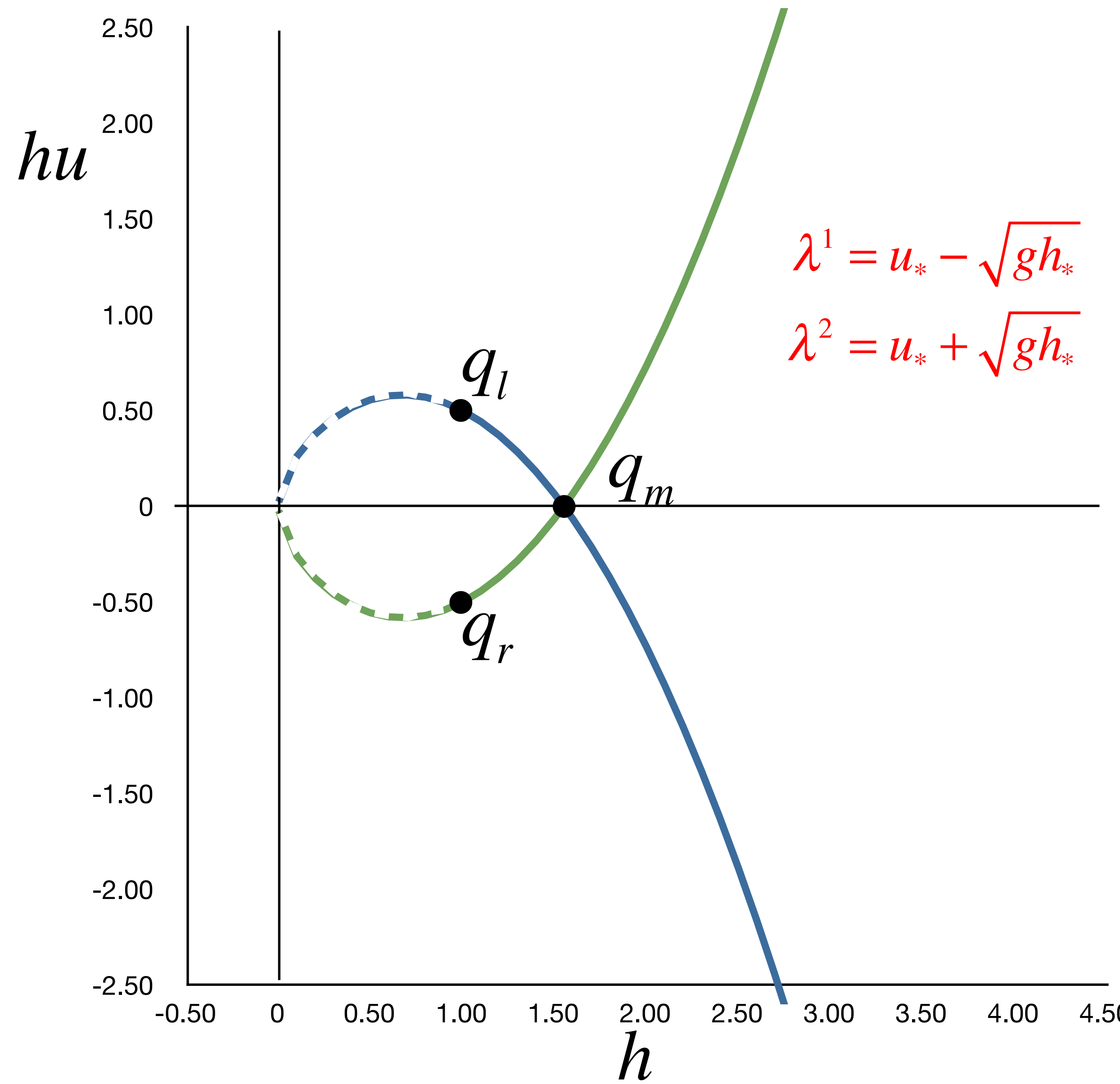
$$h_l = h_r = 1, u_l = 0.5, u_r = -0.5, g = 1$$

But now we have a nonlinear system, and we must solve

$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

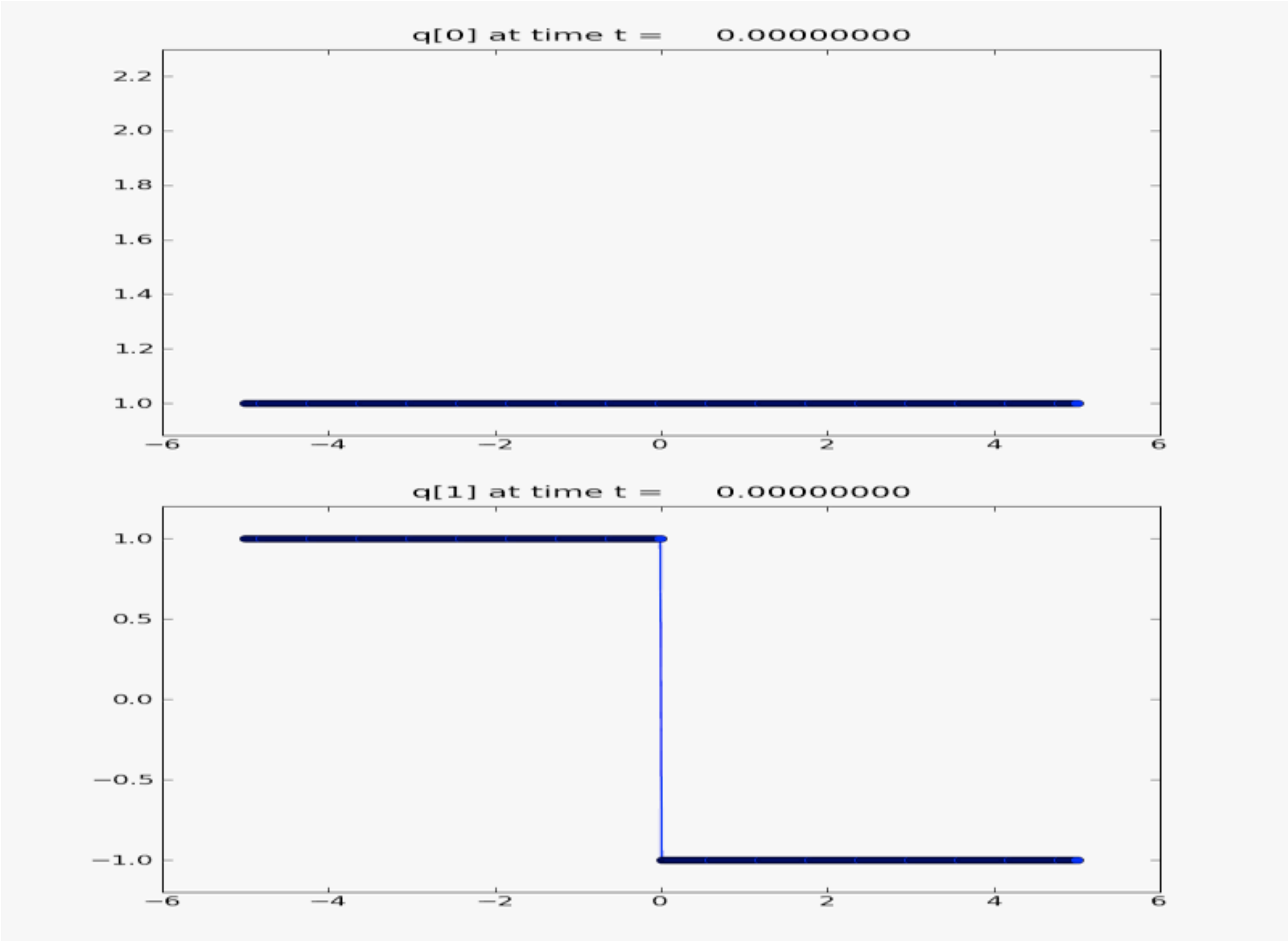
$$u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)};$$

for the intermediate state. This can be done graphically, but in a code it would be done by Newton's method.

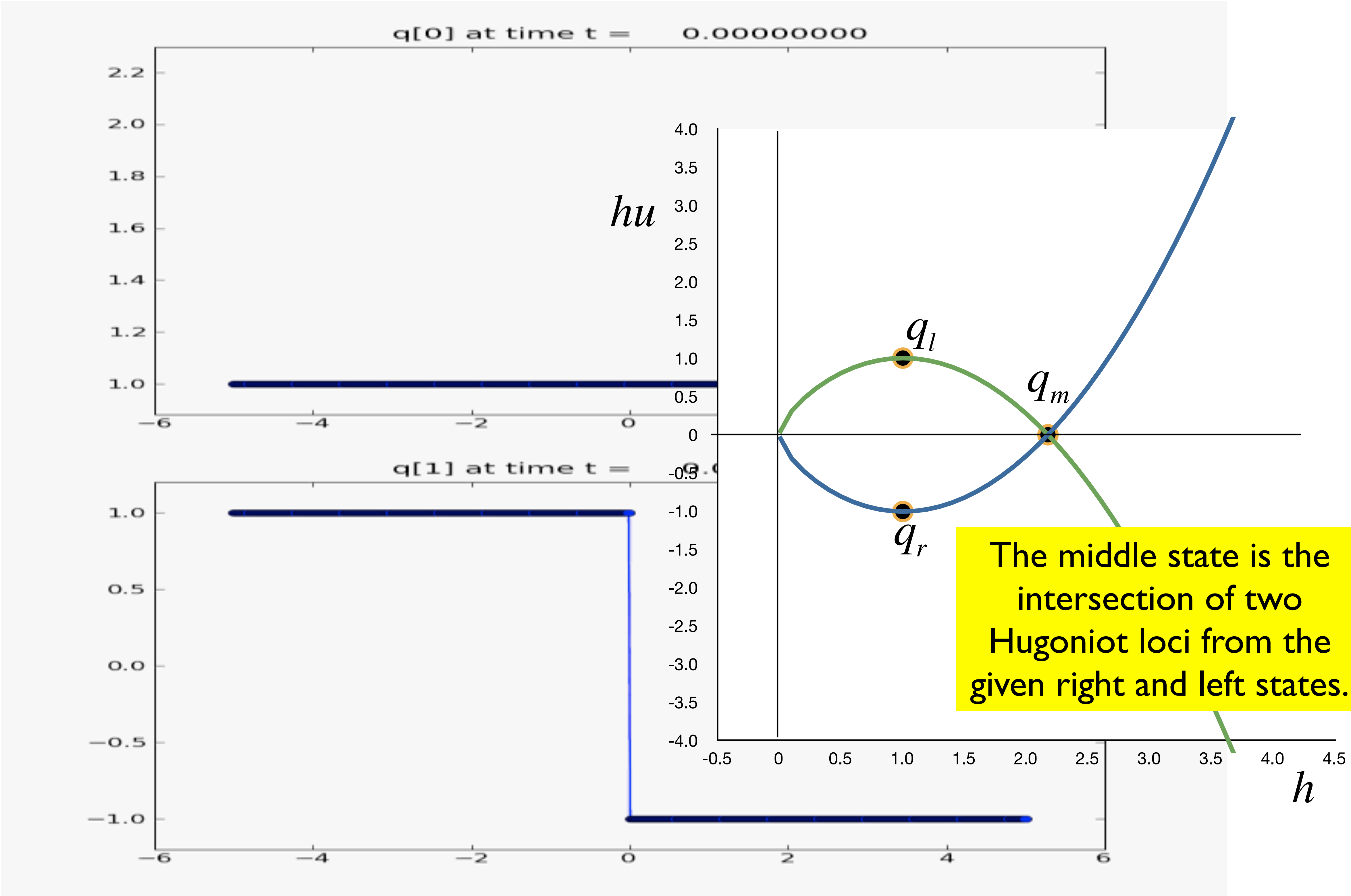


The two-shock problem

The two-shock problem



The two-shock problem



We can show that the dam-break has only one shock

A trial intermediate state is found by solving the same two equations:

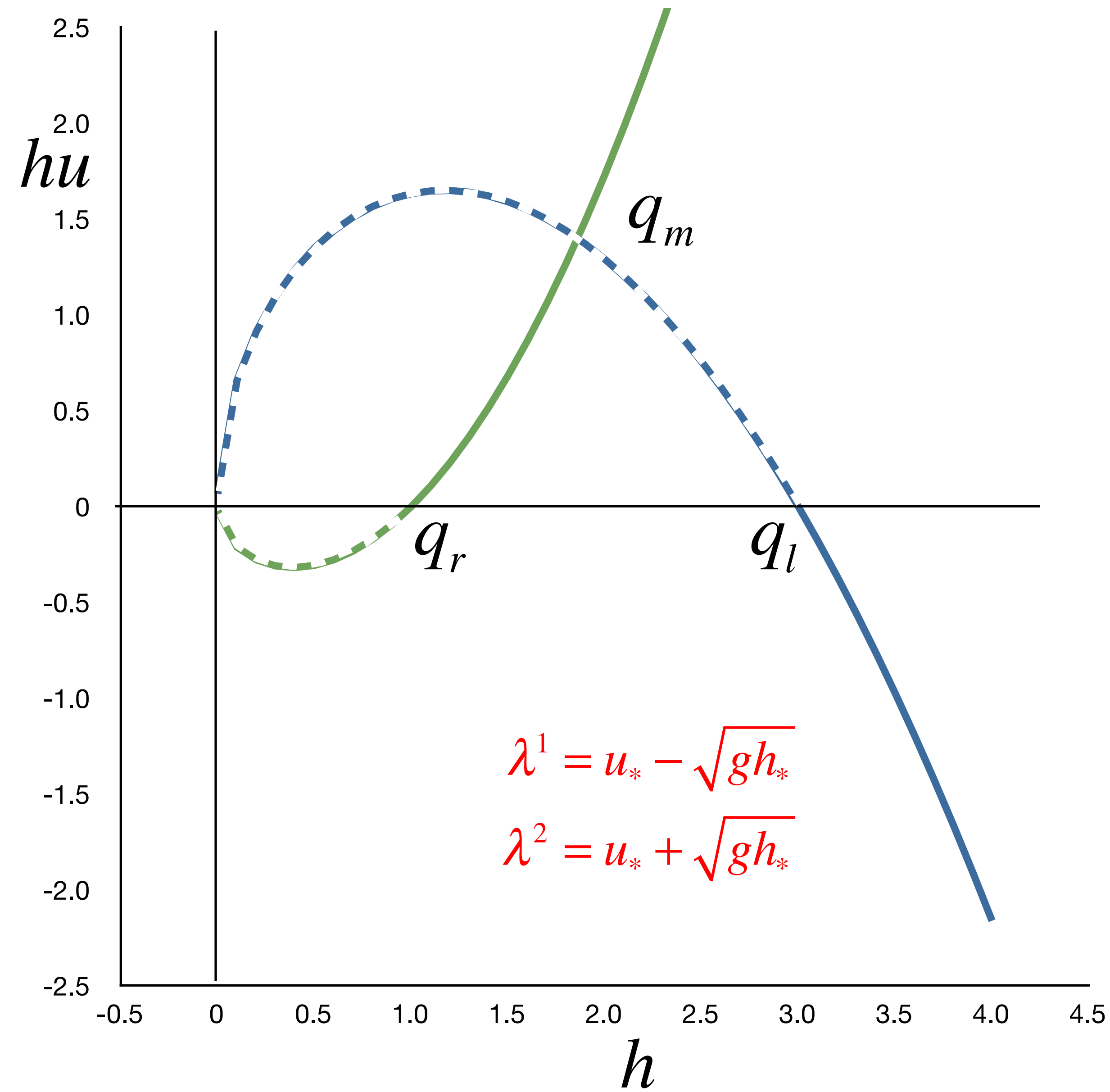
$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

$$u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)};$$

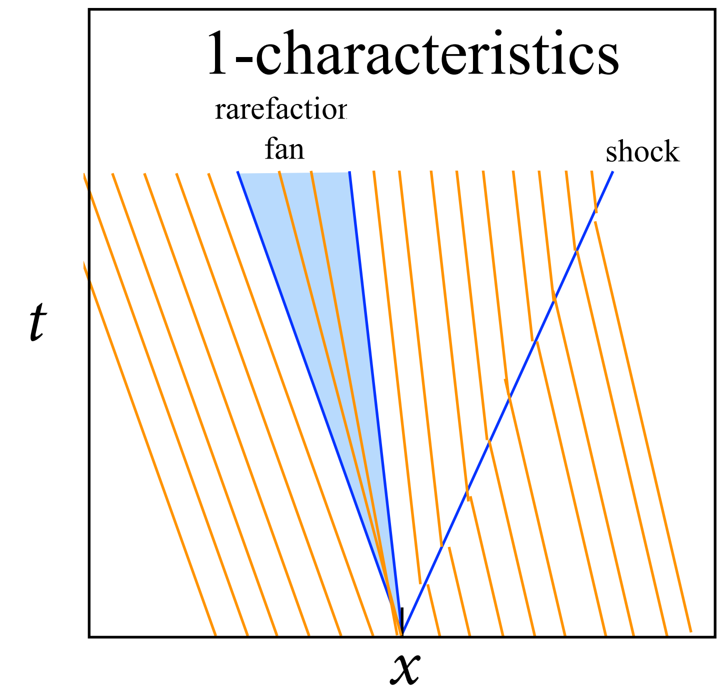
but we see that the Hugoniot locus between q_l and q_m is not an allowable shock, since the 1-characteristic increases along that path.

Instead those states must be connected by a rarefaction wave. We have to replace the Hugoniot locus with the *integral curve* of r^1.

$$h_l = 3, h_r = 1, u_l = u_r = 0, g = 1$$



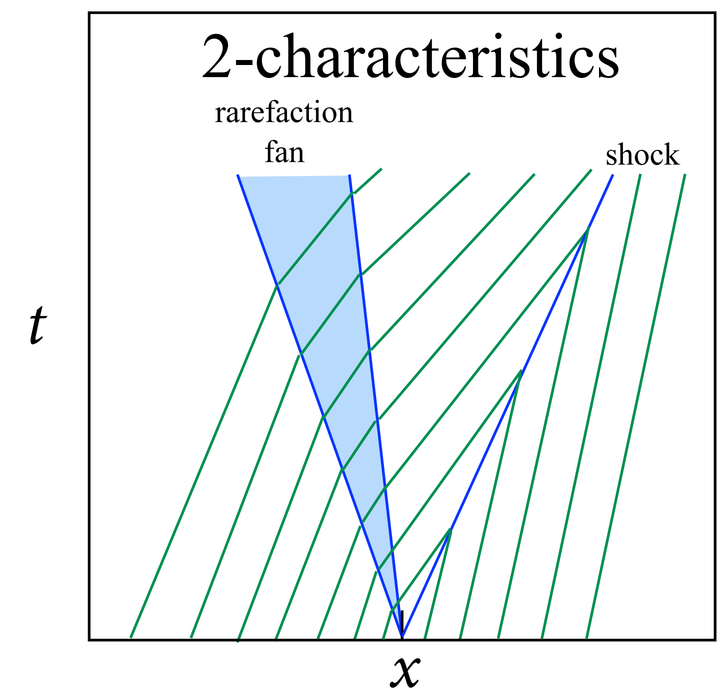
Integral curves for the dam-break problem



Since a shock is not an allowable connection between q_l and q_m,

we need to connect these states via the rarefaction wave, within which the 1-characteristics vary smoothly varying.

So we construct *integral curves* of the appropriate eigenvector, r^1 in this case.



These are curves in the state plane that are everywhere tangent to r^1. We obtain the integral curves by solving

$$\tilde{q}'(\xi) = r^1(\tilde{q}(\xi)) = \begin{bmatrix} 1 \\ \dfrac{\tilde{q}^2}{\tilde{q}^1} - \sqrt{g\tilde{q}^1} \end{bmatrix}$$

The integral curves and Riemann invariants

The integral curves for r^1, shown at right, have the form

$$u = u_* + 2\left(\sqrt{gh_*} - \sqrt{gh}\right)$$

for the curve passing through $\left(u_*, h_* u_*\right)$

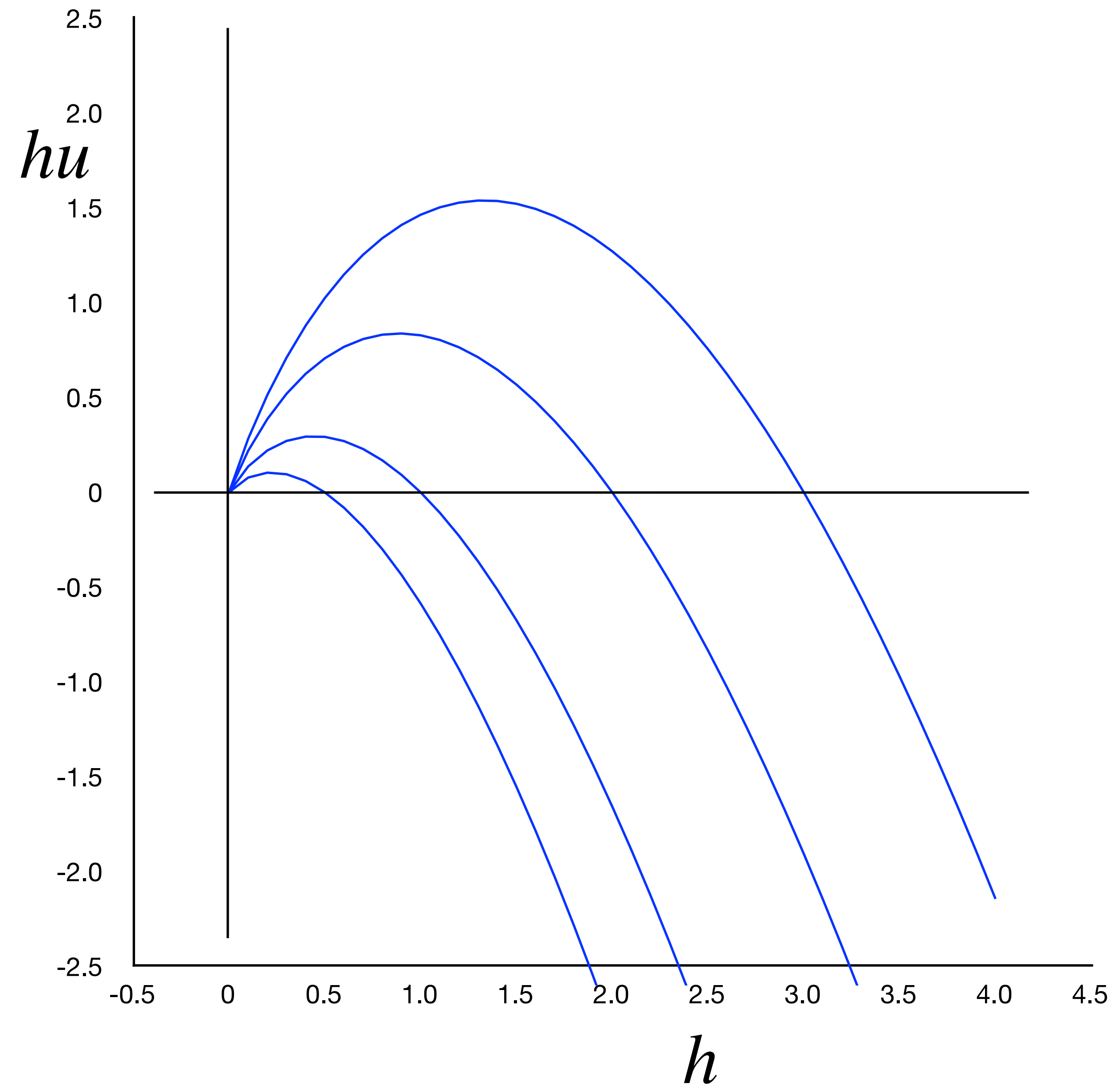
and therefore the function

$$w^1(q) = u + 2\sqrt{gh}$$

is invariant along each curve. These curves are therefore a contour diagram for this function, called the *1-Riemann invariant*.

Compare the integral curves for r^1 to the equation for the 1-Hugoniot locus:

$$u = u_* - (h - h_*)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_*}\right)}$$



The integral curves and Riemann invariants

Similarly, r^2 has integral curves given by

$$u = u_* - 2\left(\sqrt{gh_*} - \sqrt{gh}\right)$$

for the curve passing through $\left(u_*, h_* u_*\right)$

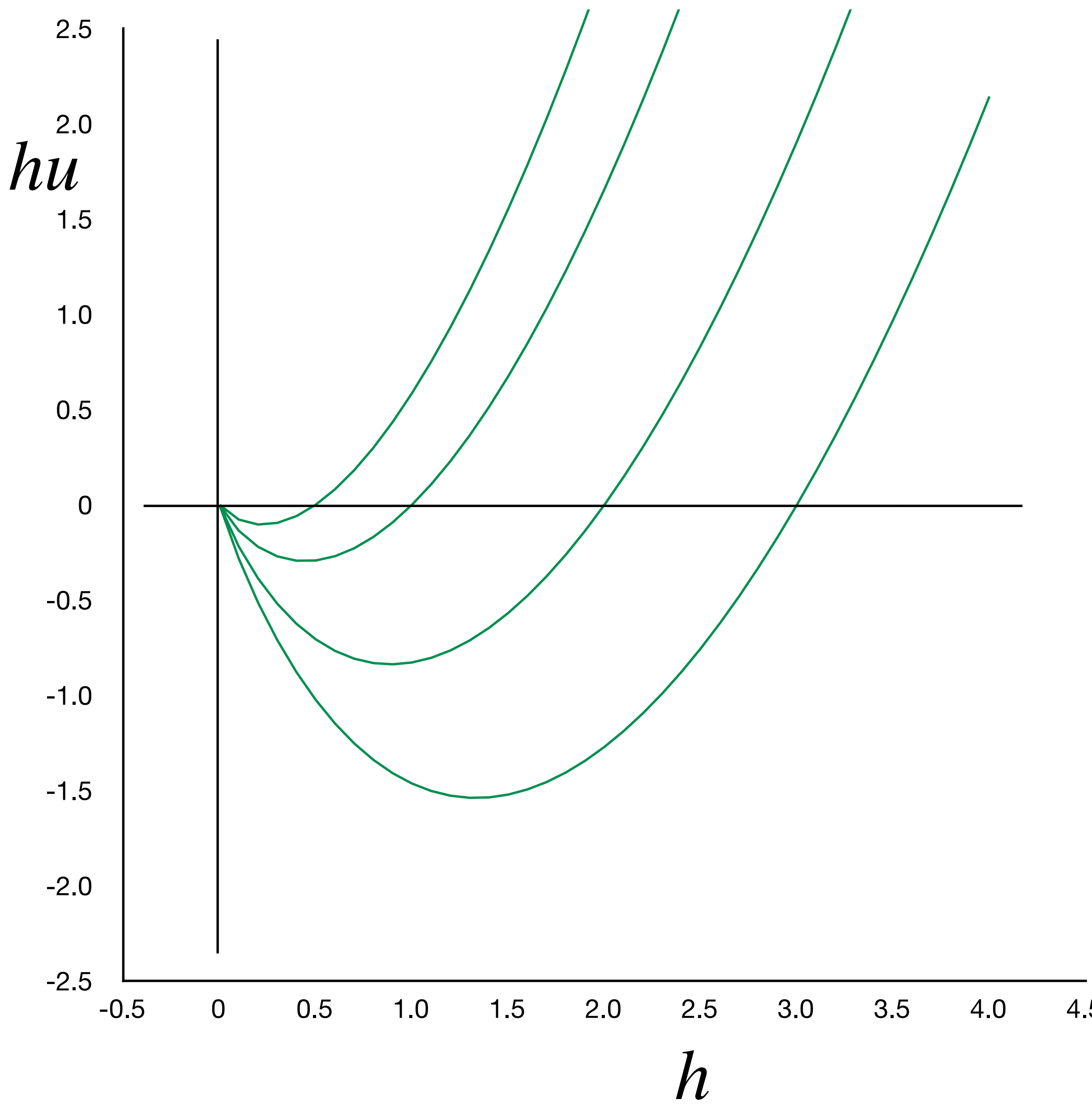
and therefore the function

$$w^2(q) = u - 2\sqrt{gh}$$

is the *2-Riemann invariant*.

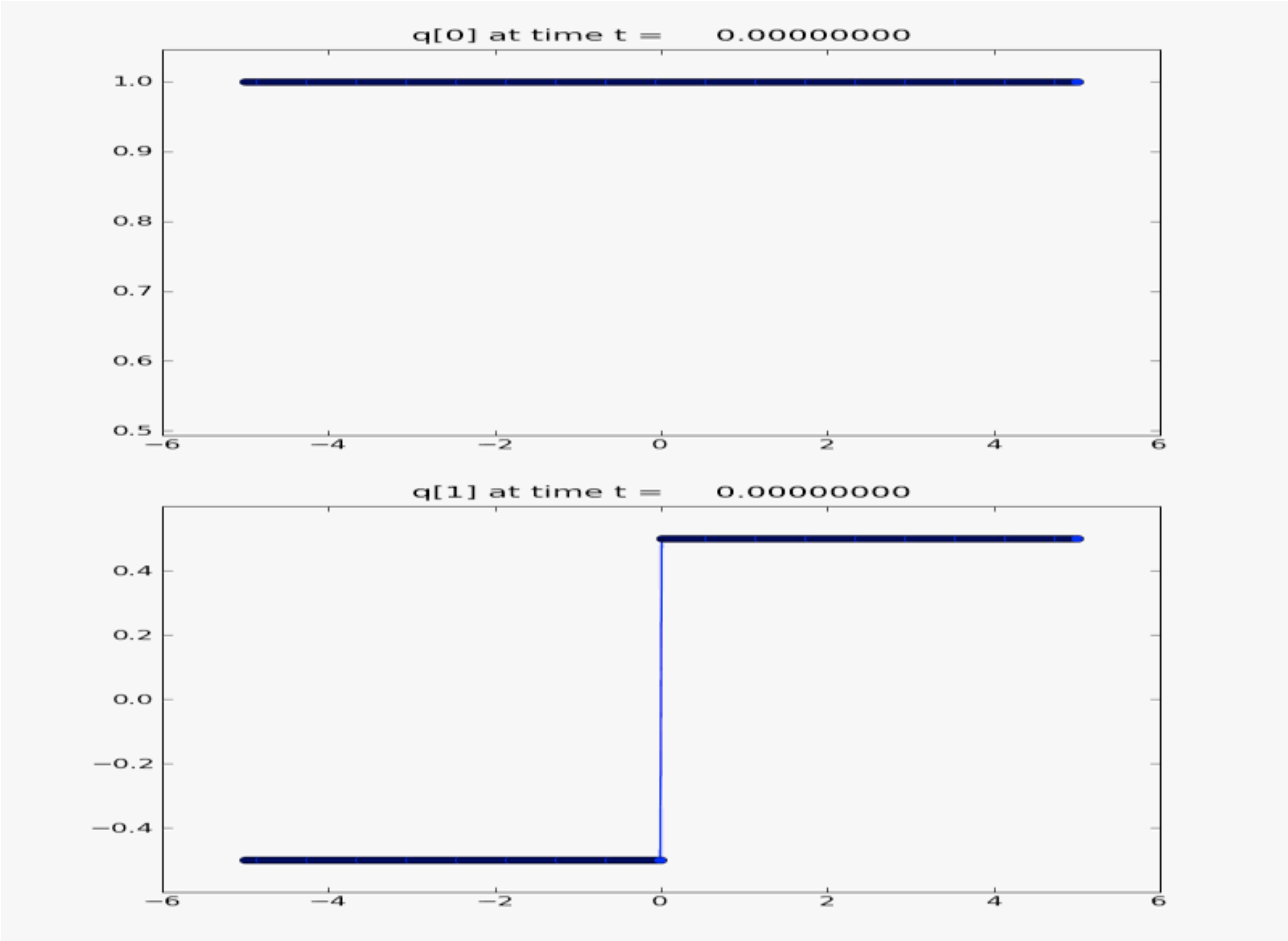
For comparison, the equation for the 2-Hugoniot locus is:

$$u = u_* + (h - h_*)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_*}\right)}$$

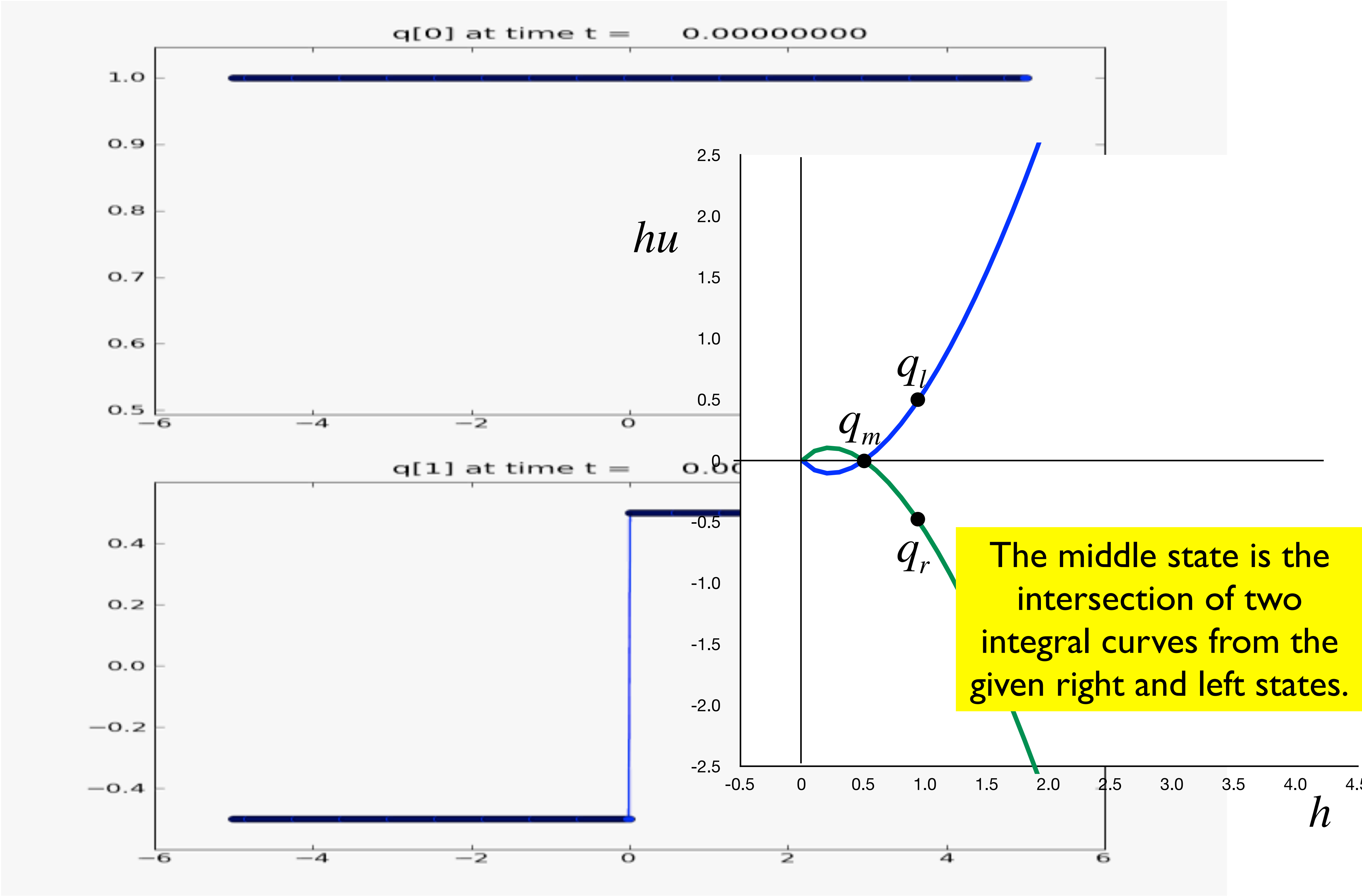


The two-rarefaction problem

The two-rarefaction problem



The two-rarefaction problem



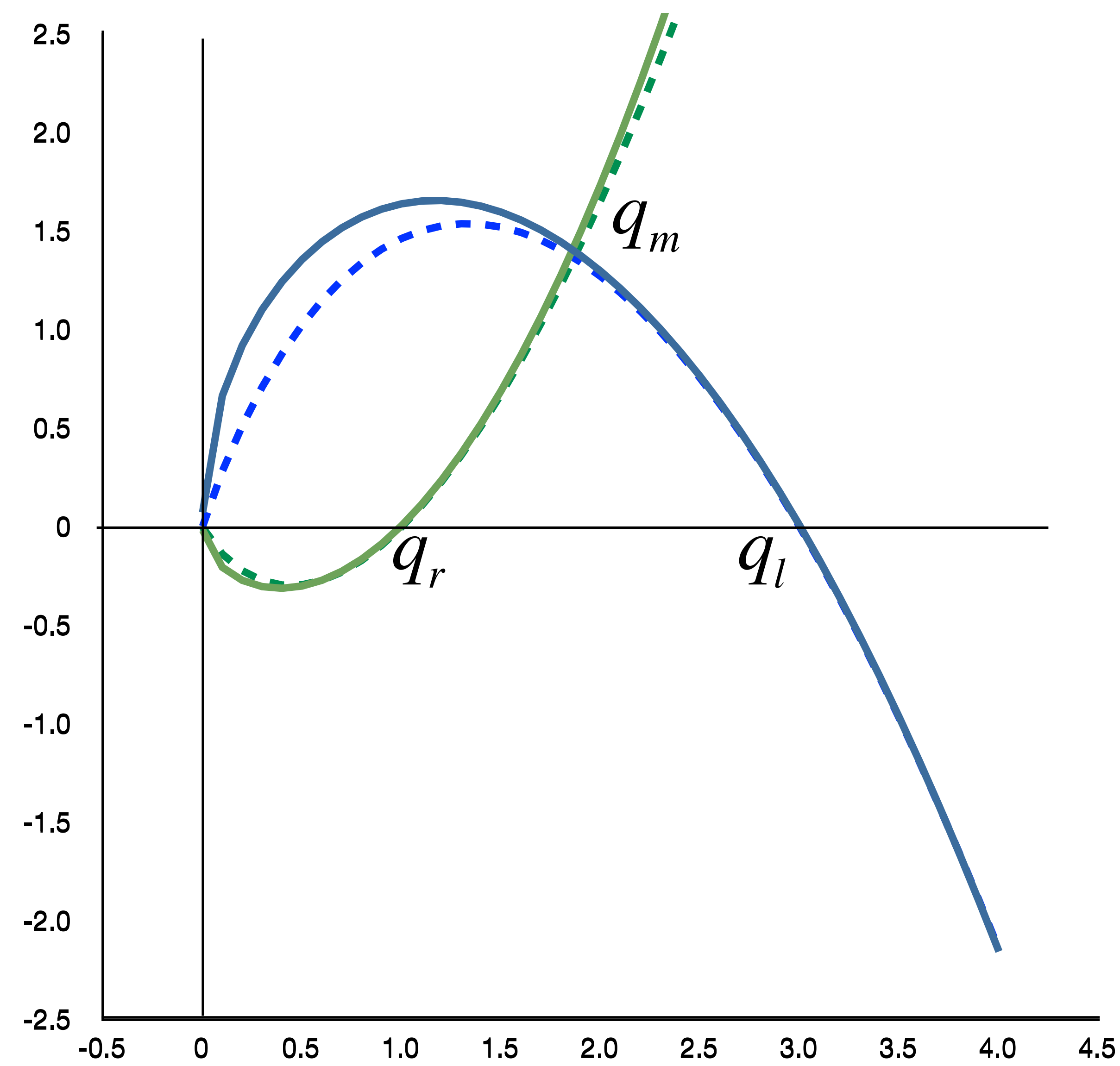
The correct solution for the dam-break problem

We have to connect q_l and q_m through a rarefaction wave via an integral curve, and at the same time connect q_m and q_r through a shock via a Hugoniot locus.

Here the Hugoniot loci through q_l and q_r are shown as solid lines (blue for p=1, green for p=2) and the integral curves are shown as dotted lines (blue for p=1, green for p=2). They are close together, but their difference is important.

The true middle state is thus given by the intersection of the green solid line and the blue dotted line. Again, this can be found by an iterative method.

$$h_l = 3, h_r = 1, u_l = u_r = 0, g = 1$$



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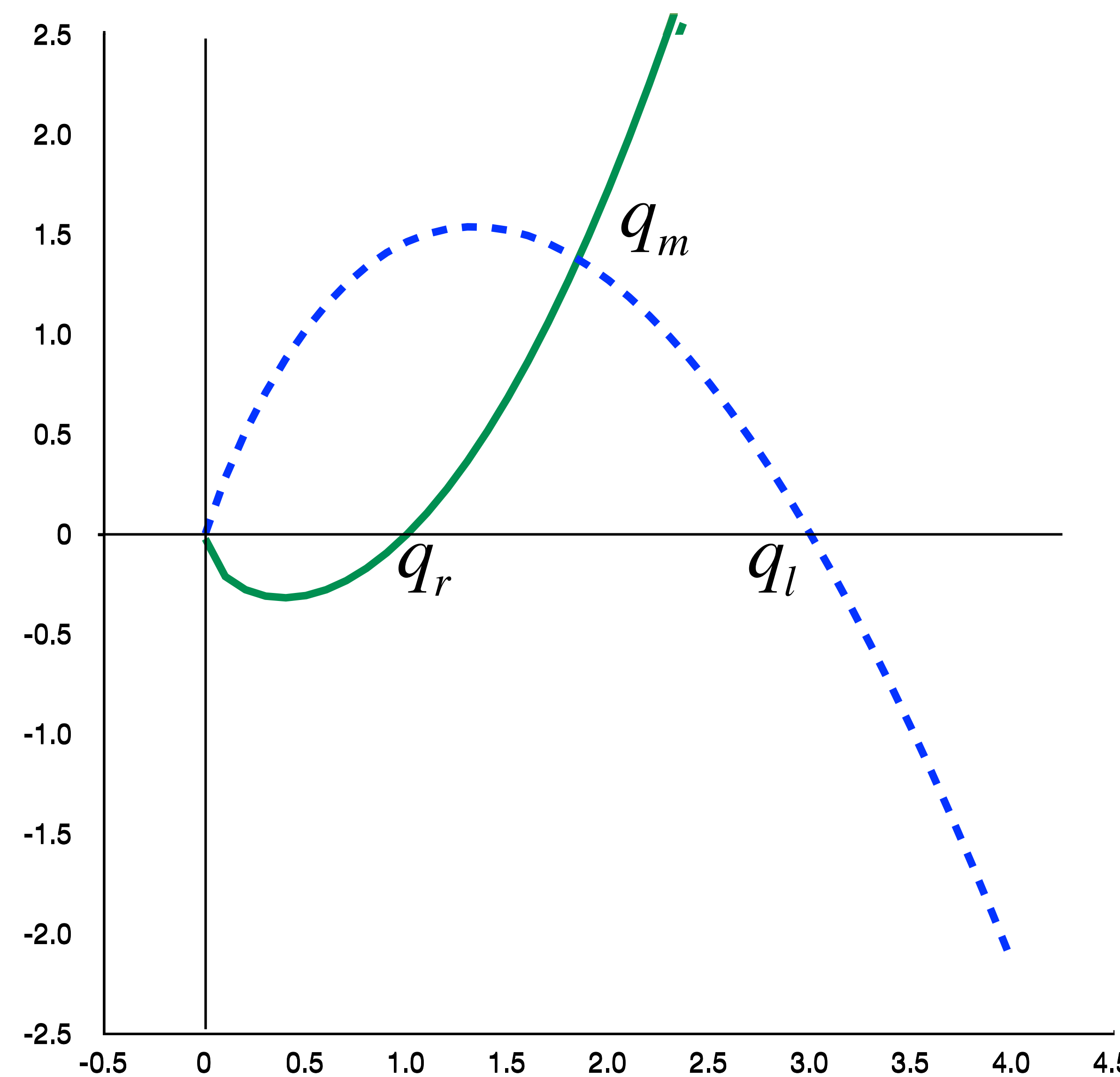
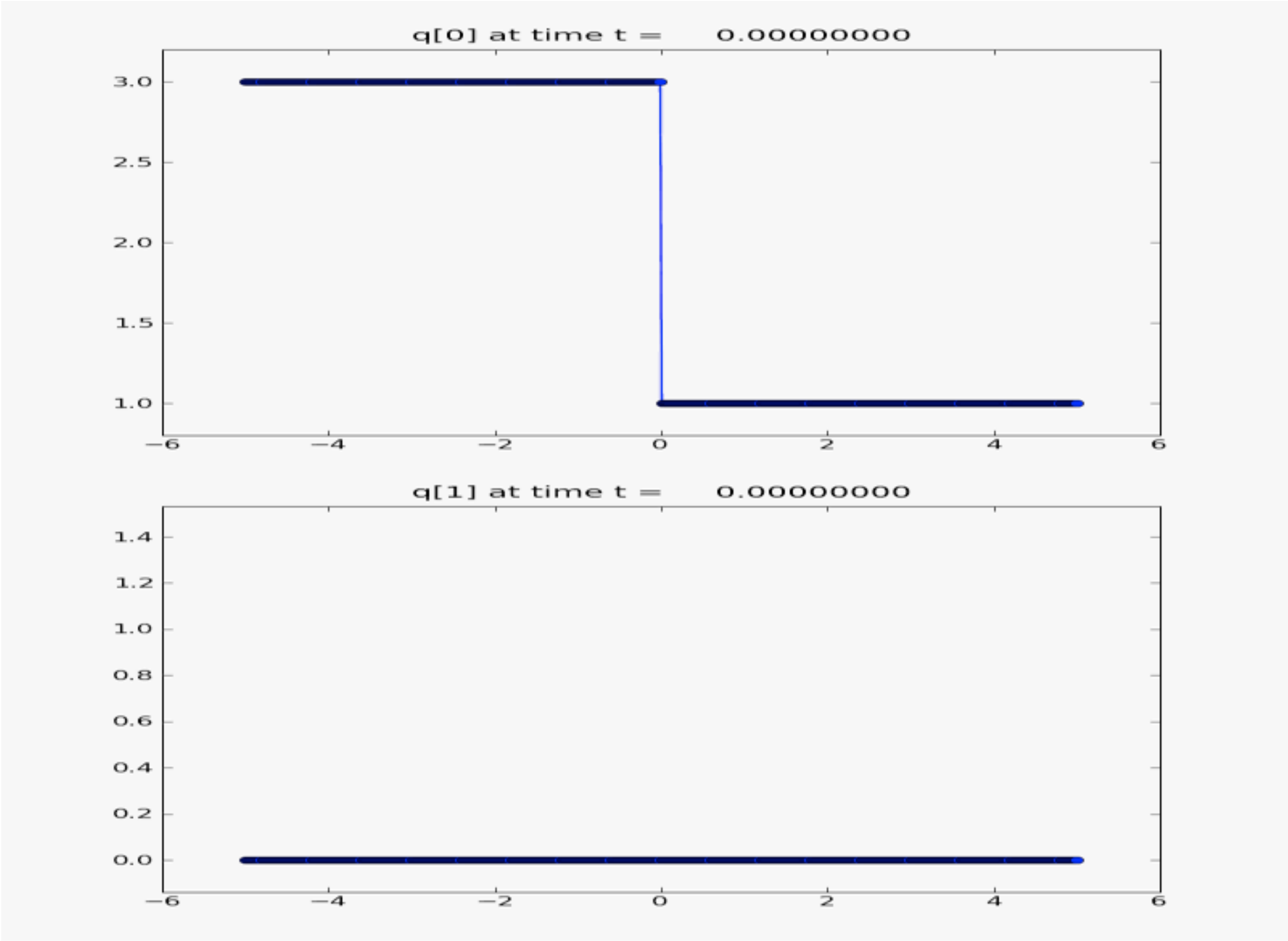


Shallow-water waves: the dam break

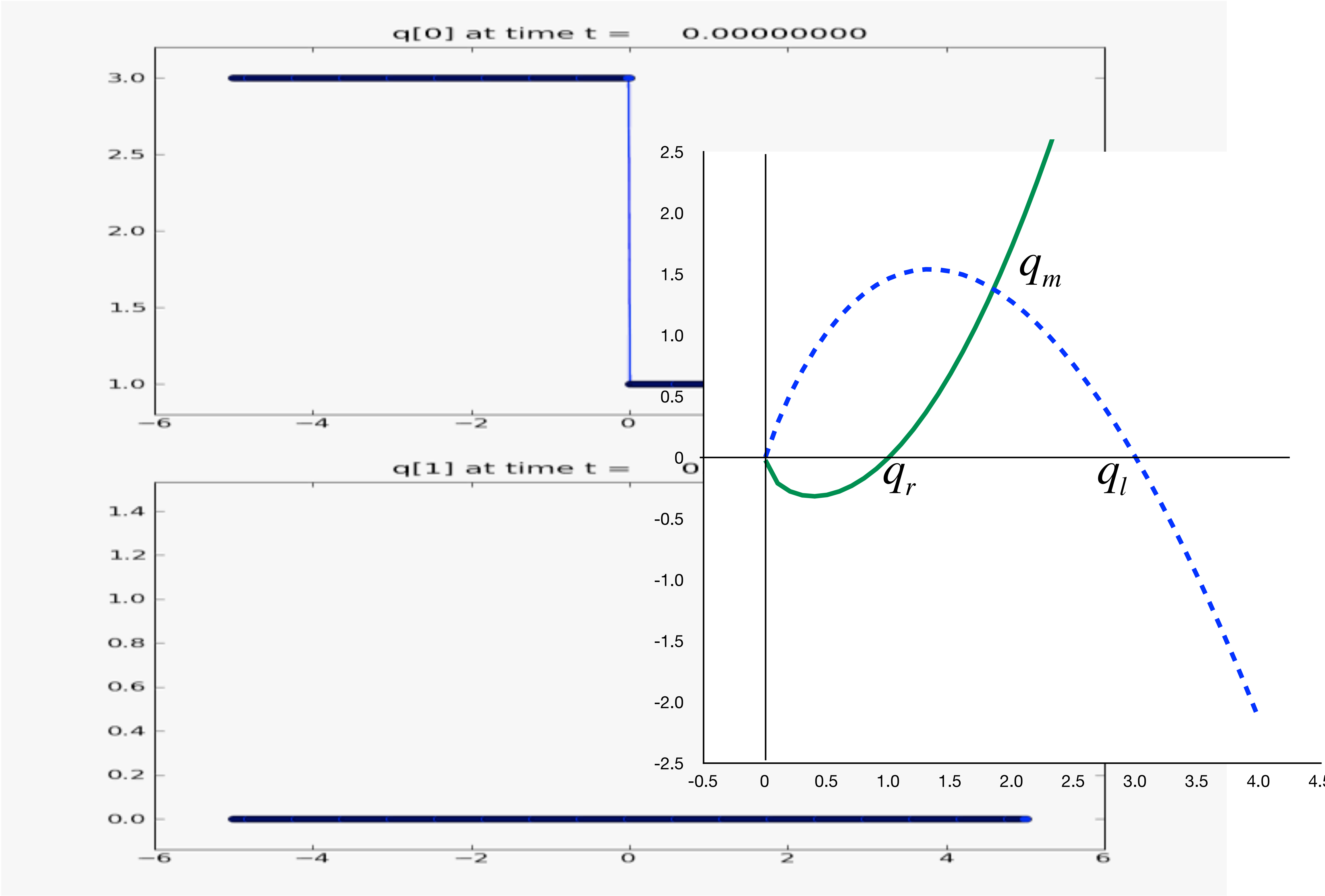
with both a shock and a rarefaction

Shallow-water waves: the dam break



with both a shock and a rarefaction

Shallow-water waves: the dam break



with both a shock and a rarefaction

The general (exact) Riemann solver for the shallow-water equations

For general values of q_l and q_r we could have a combination of shocks and rarefactions and we have to find out which are which.

We define two functions

$$u = \varphi_l(h) = \begin{cases} u_l + 2\left(\sqrt{gh_l} - \sqrt{gh}\right) & \text{if } h < h_l \\ u_l - (h - h_l)\sqrt{\dfrac{g}{2}\left(\dfrac{1}{h} + \dfrac{1}{h_l}\right)} & \text{if } h > h_l \end{cases}$$

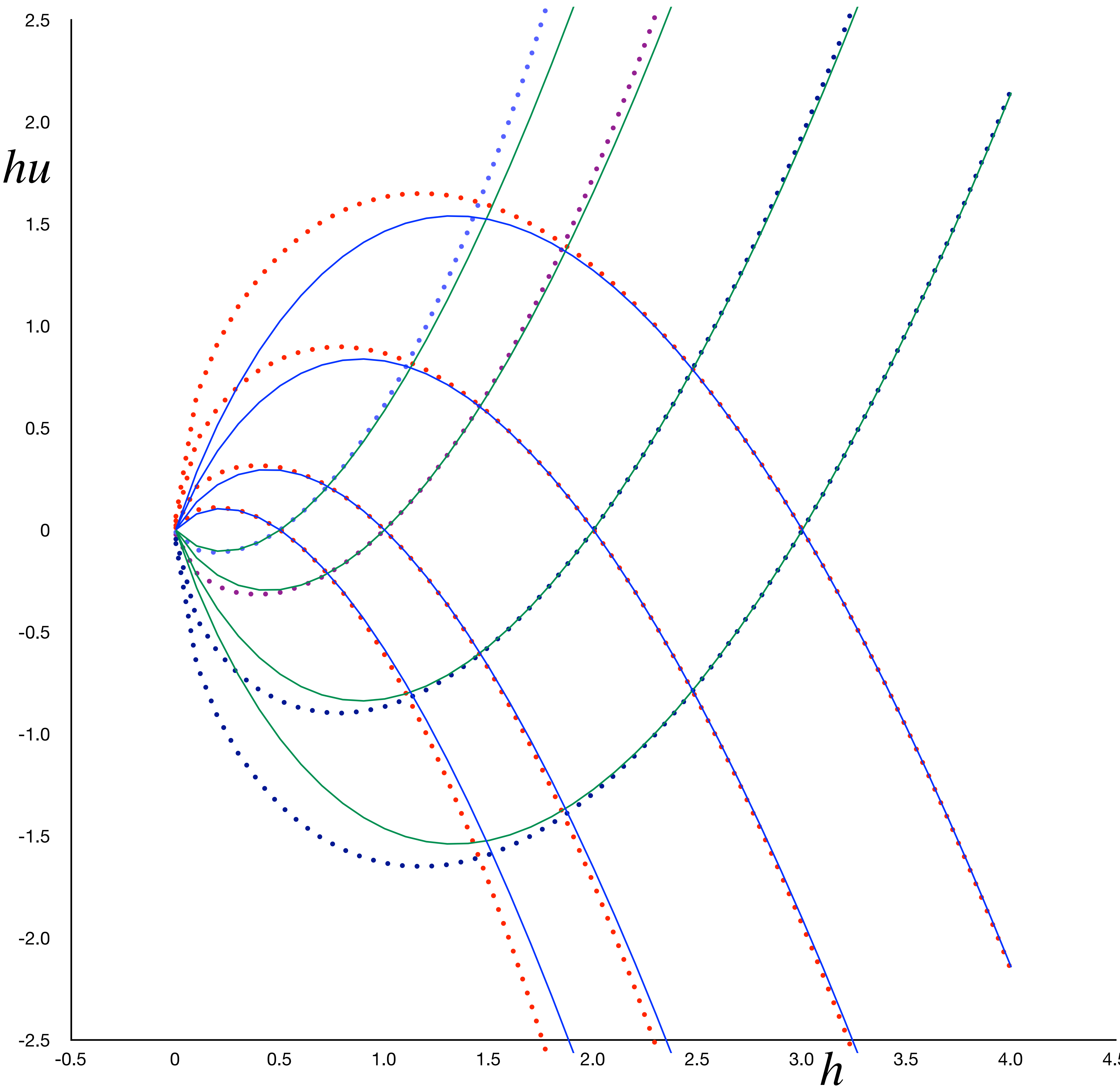
$$u = \varphi_r(h) = \begin{cases} u_r - 2\left(\sqrt{gh_r} - \sqrt{gh}\right) & \text{if } h < h_r \\ u_r + (h - h_r)\sqrt{\dfrac{g}{2}\left(\dfrac{1}{h} + \dfrac{1}{h_r}\right)} & \text{if } h > h_r \end{cases}$$

and then we find the state q_m for which $\varphi_l(h_m) = \varphi_r(h_m)$.

Some integral curves (solid) and Hugoniot loci (dotted) for the shallow-water equations. An iterative solver can start from the intersection of the integral curves, which can be obtained explicitly.

$$hu = h\left(u_* \pm 2\left(\sqrt{gh_*} - \sqrt{gh}\right)\right)$$

$$hu = h\left(u_* \pm (h - h_*)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_*}\right)}\right)$$



Gas Dynamics
(Chapter 14 in Leveque)

We've already encountered the barotropic set of equations

Conservation of

mass: ρ flux: ρu

momentum: ρu flux: $(\rho u)u + p$

(energy) - not needed in the barotropic case; we'll save this for later

The conservation laws in one dimension are:

$$\rho_t + (\rho u)_x = 0$$
$$\left(\rho u\right)_t + \left(\rho u^2 + p\right)_x = 0$$

Together with the barotropic equation of state: $p = P(\rho)$,

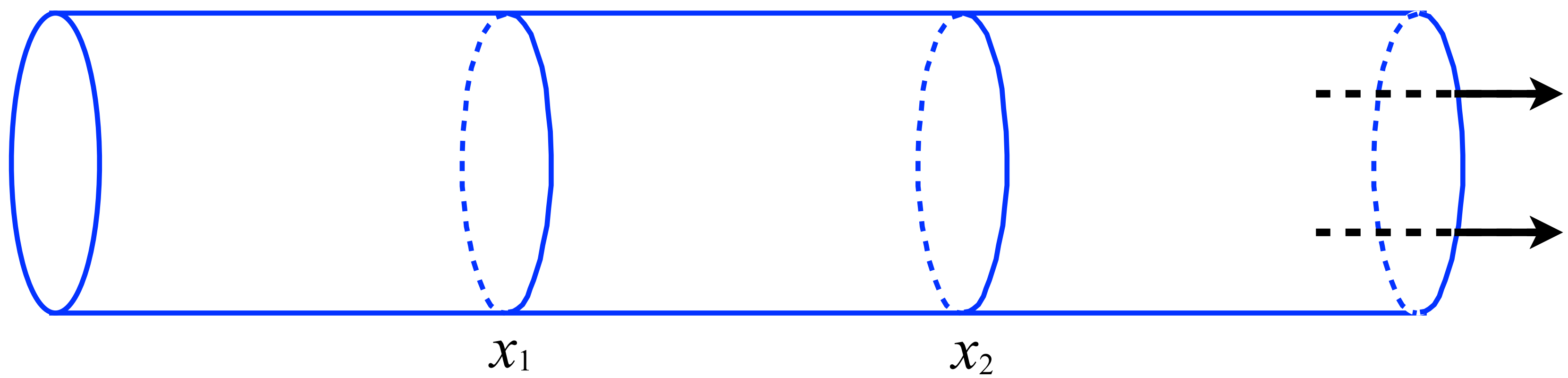
these define the system.

Compressible gas dynamics - flow in a pipe

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$

$$\left(\rho u\right)_t + \left(\rho u^2 + p\right)_x = 0$$



Let's first try to understand the momentum equation a little better.

$(\rho u)u$ is the advective momentum flux;

p , the pressure in a cell, (times the cell's cross-sectional area) is the force, which by Newton's second law changes the momentum, *if unbalanced*.

But p can also be understood more directly as a momentum flux due to the microscopic motion of gas molecules.

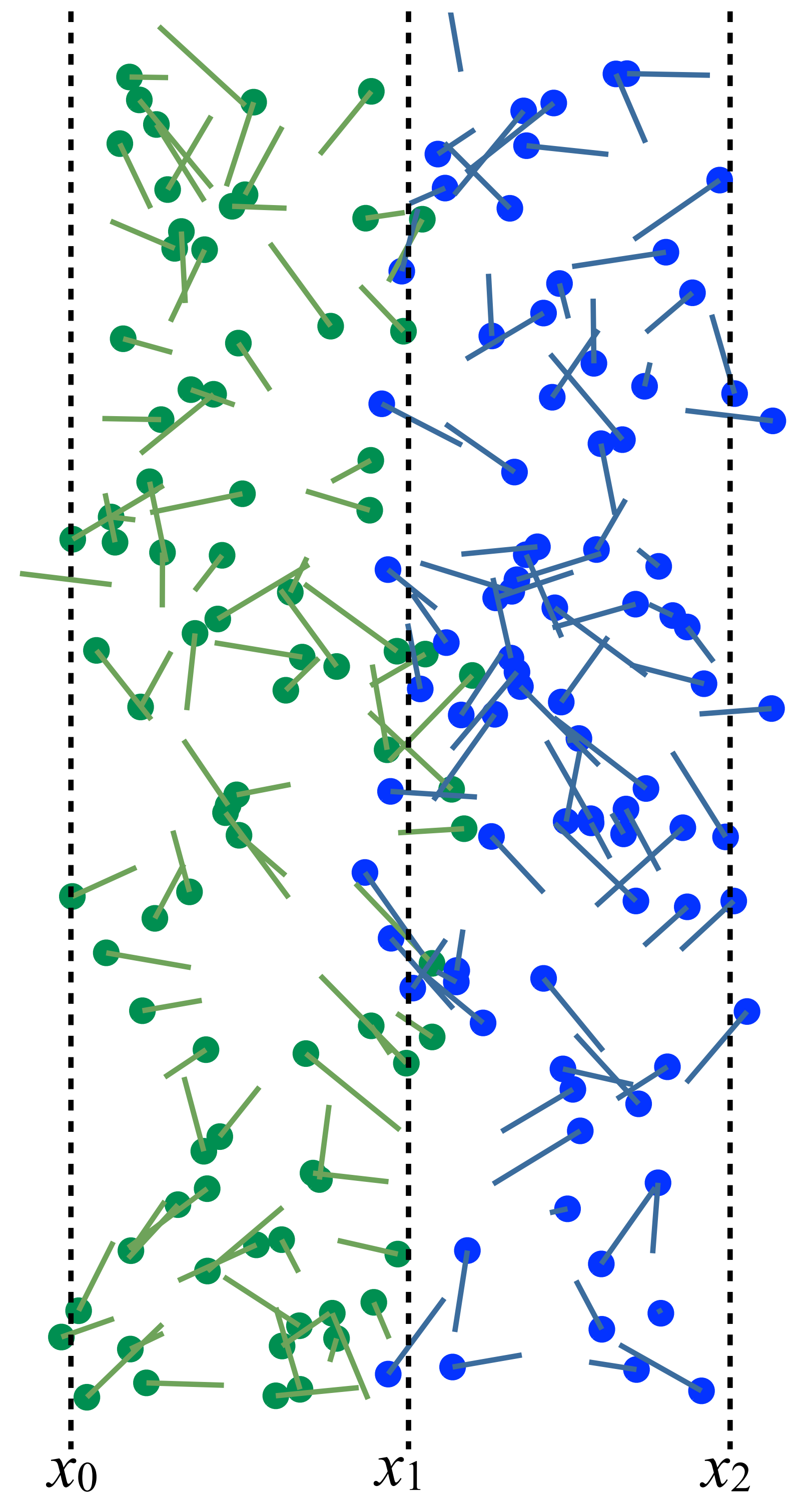
Momentum flux arising from pressure

The green and blue molecules are at the same temperature and pressure. Nevertheless the pressure contributes to a momentum flux in the following way:

Green molecules moving right across x_1 increase the positive momentum in $[x_1, x_2]$.

Blue molecules moving left across x_1 decrease the negative momentum in $[x_1, x_2]$ and therefore *also increase* the positive momentum.

If the pressure is uniform everywhere, however, there is no *net* increase in positive momentum in $[x_1, x_2]$ because the same considerations at x_2 lead to a decrease in the positive momentum in $[x_1, x_2]$ by exactly the same amount.



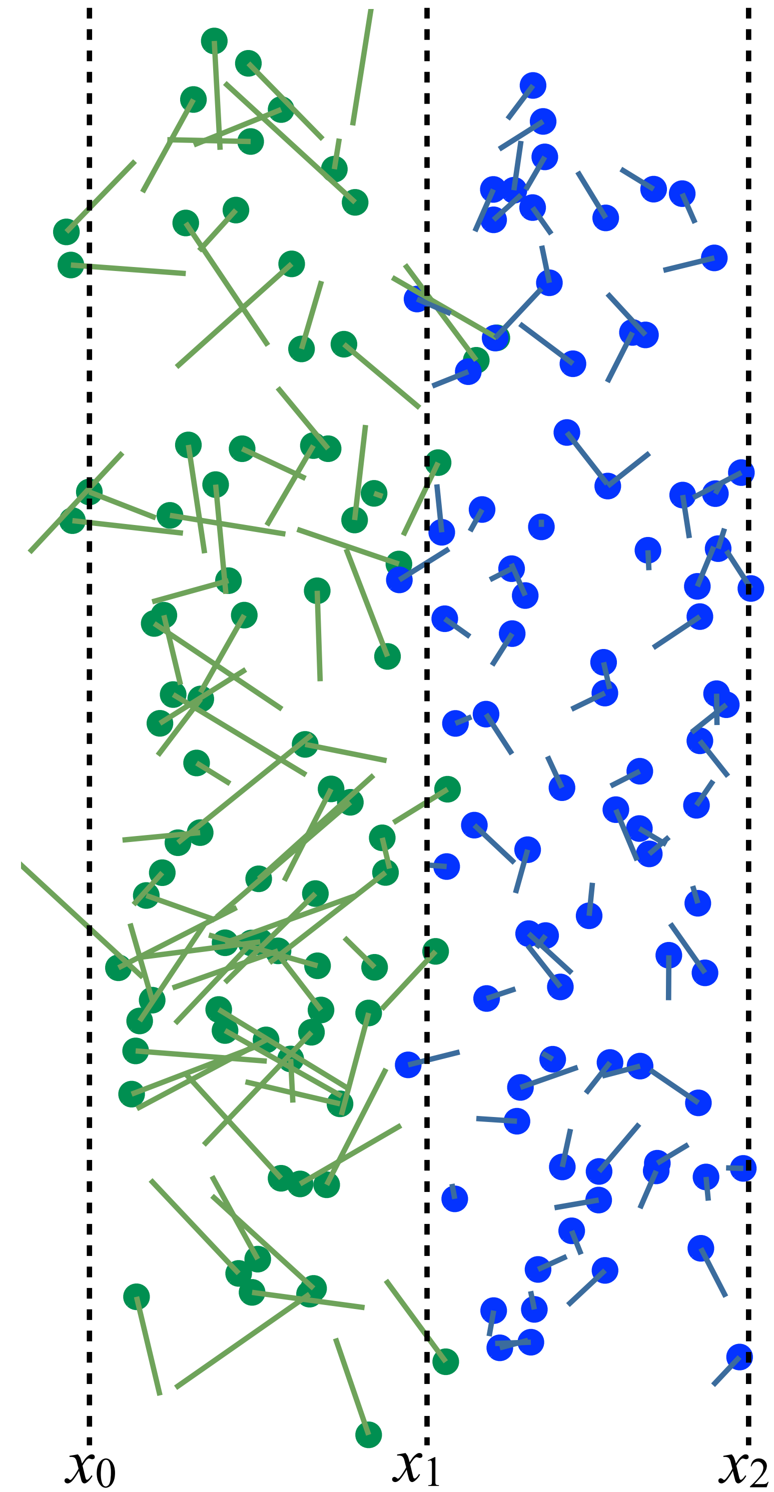
Acceleration arising from pressure gradient

The green molecules have the same density but higher temperature, therefore higher pressure, than the blue molecules.

Again, green molecules that cross to the right increase the momentum in $[x_1, x_2]$ and blue molecules that cross to the left do also.

But in this case this momentum flux is not exactly compensated at x_2 because the exchange of momentum there is less vigorous.

Hence there is a net positive momentum flux across x_1 due to the pressure gradient, which leads to a macroscopic acceleration, even though the individual molecules are not accelerated.



The barotropic equations and the shallow-water equations

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0$$

The conservation laws for the barotropic system (i.e. with $p = P(\rho)$)

are exactly like the shallow water equations if we identify ρ with h and use

the equation of state

$$p = P(\rho) = \frac{1}{2} g\rho^2$$

Other barotropic forms include the isothermal equation of state

$$p = P(\rho) = a^2 \rho,$$

and the polytropic (or gamma-law) equation of state

$$p = P(\rho) = K\rho^\gamma$$

But next we add the energy equation…

The Euler equations of gas dynamics

This is the full system of three conservation laws, for mass, momentum, and energy, for fully compressible gas dynamics:

$$q_t + f(q)_x = 0,$$

where

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}.$$

The total energy is composed of internal energy plus kinetic energy,

$$E = \rho e + \frac{1}{2}\rho u^2$$

and the system is completed by an equation of state $e = e(p,\rho)$.

The Jacobian $f'(q)$ has eigenvalues $u\text{–}c,\ c,\ u+c$ where the speed of sound is $c = \sqrt{\dfrac{dp}{d\rho}}$ at constant entropy.

The equation of state and associated relations for a polytropic gas:

The ideal gas law: $$\frac{p}{\rho} = nkT$$

The internal energy: $$e = c_v T = \frac{\alpha}{2} nkT = \frac{p}{(\gamma - 1)\rho}$$

The *enthalpy*: $$h = e + \frac{p}{\rho} = c_p T = \left(1 + \frac{\alpha}{2}\right) nkT$$

Relations between the specific heats: $$\begin{cases} c_p - c_v = nk \\ \gamma = \dfrac{c_p}{c_v} = \left(\dfrac{\alpha + 2}{\alpha}\right) \end{cases}$$

---

n : number of molecules per unit mass

k : Boltzmann's constant

c_v : specific heat at constant volume

α : number of degrees of freedom

c_p : specific heat at constant pressure

γ : adiabatic exponent

Entropy

In the system of Euler equations for gas dynamics, we have the advantage of having an explicit formula for entropy that we can use as an entropy condition.

The specific entropy s (i.e. entropy per unit mass) is given by the formula:

$$s = c_v \log\left(\frac{p}{\rho^\gamma}\right) + \text{constant}$$

The additive constant is unimportant and may be omitted, since the important thing to keep track of is changes in entropy. In smooth flow, entropy is constant; at shocks it jumps to a higher value.

Primitive variables

It is often useful to examine the equivalent equations in directly observable "primitive" variables, rather than the conserved variables.

The Euler equations in primitive form for a polytropic gas:

$$\rho_t + u\rho_x + \rho u_x = 0$$

$$u_t + uu_x + \frac{1}{\rho} p_x = 0$$

$$p_t + \gamma p u_x + u p_x = 0$$

in matrix notation:

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x + \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t = 0$$

Then the eigenvalues and eigenvectors are:

$$\lambda^1 = u - c \qquad \lambda^2 = u \qquad \lambda^3 = u + c$$

$$r^1 = \begin{bmatrix} -\rho / c \\ 1 \\ -\rho c \end{bmatrix} \quad r^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad r^3 = \begin{bmatrix} \rho / c \\ 1 \\ \rho c \end{bmatrix}$$

where

$$c = \sqrt{\frac{\gamma \rho}{p}}$$

is the speed of sound in the polytropic gas.

The Jacobian for the conservation laws

This (for the polytropic gas) is slightly more complex, though equivalent:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-1)u^3 - uH & H-(\gamma-1)u^2 & \gamma u \end{bmatrix}$$

where $H = \dfrac{E+p}{\rho} = h + \dfrac{1}{2}u^2$ is the total specific enthalpy.

And the eigenvalues and eigenvectors are:

$$\lambda^1 = u - c \qquad \lambda^2 = u \qquad \lambda^3 = u + c$$

$$r^1 = \begin{bmatrix} 1 \\ u-c \\ H-uc \end{bmatrix} \quad r^2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} \quad r^3 = \begin{bmatrix} 1 \\ u+c \\ H+uc \end{bmatrix}$$

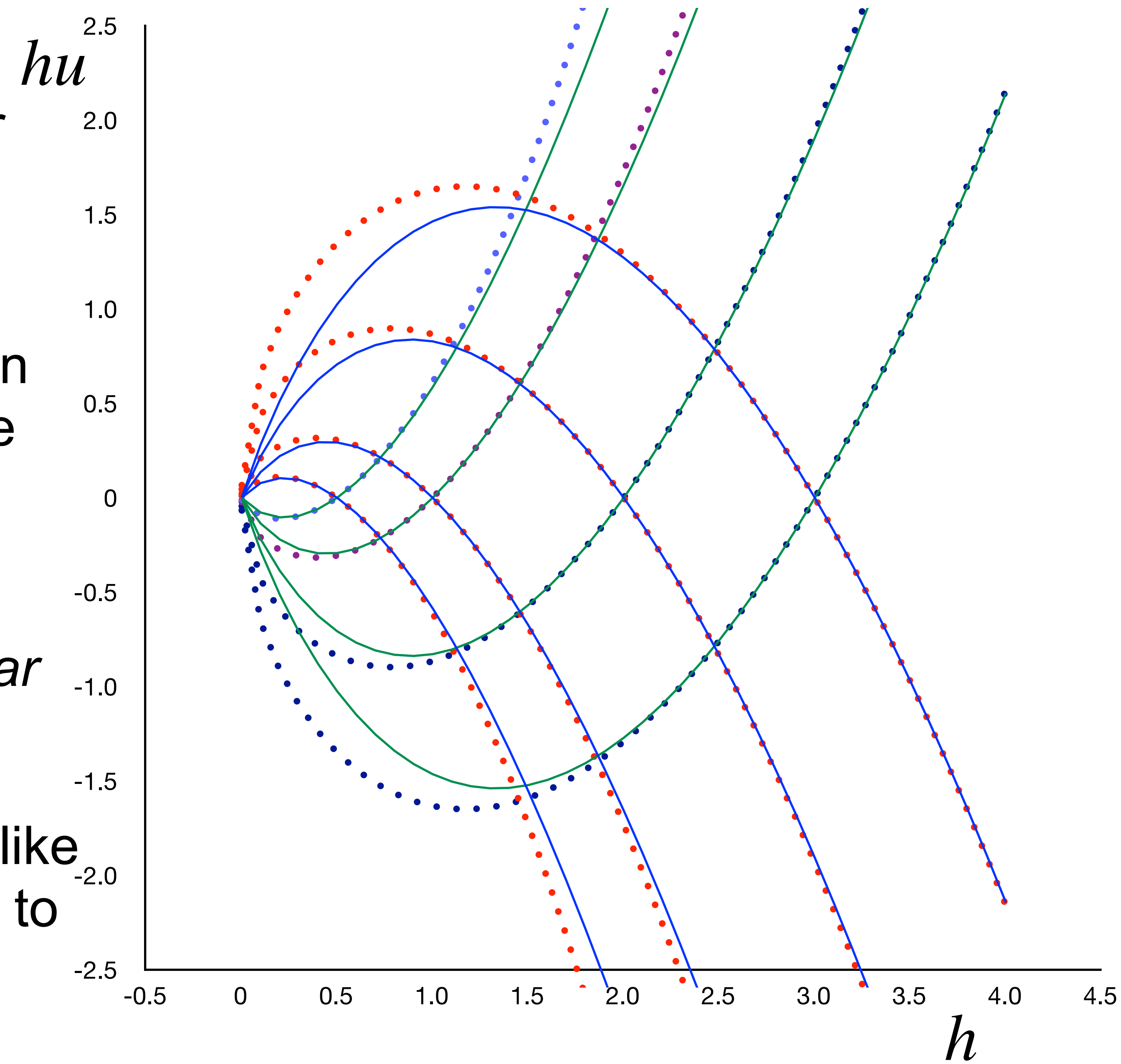
Genuine nonlinearity, linear degeneracy

Nonlinear systems of hyperbolic equations produce shocks and rarefaction waves; linear systems do not.

Nonlinear systems have integral curves and Hugoniot loci that diverge from one another; in linear systems these curves are identical. The stronger the nonlinearity, the greater the divergence.

Shocks and rarefactions are *genuine nonlinear* waves.

But nonlinear systems can also have waves, like contact discontinuities, that in fact act appear to be linear waves. Such waves are *linearly degenerate*.



Shallow water equations

Genuine nonlinearity, linear degeneracy

A wave (or field, we may say, referring to the collection of waves of the same family in all accessible space) is *genuinely nonlinear* if

$$\nabla\lambda^p \bullet r^p(q) \neq 0 \ \text{ for all } q$$

Physically this means that the characteristics are either compressing or expanding.

The opposite case is linear degeneracy,

$$\nabla\lambda^p \bullet r^p(q) = 0 \ \text{ for all } q$$

in this case the characteristics are parallel to one another.

The Euler equations have one linearly degenerate field

$$\lambda^1 = u - c \qquad \lambda^2 = u \qquad \lambda^3 = u + c$$

$$r^1 = \begin{bmatrix} -\rho / c \\ 1 \\ -\rho c \end{bmatrix} \quad r^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad r^3 = \begin{bmatrix} \rho / c \\ 1 \\ \rho c \end{bmatrix}$$

This is easiest to see in the eigensystem for the primitive equations. In this case the gradient operator is defined by

$$\nabla \equiv \begin{bmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial u} \\ \frac{\partial}{\partial p} \end{bmatrix},$$

so we find

$$\nabla \lambda^1 = \begin{bmatrix} +\frac{c}{2\rho} \\ 1 \\ -\frac{c}{2\rho} \end{bmatrix}; \nabla \lambda^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \nabla \lambda^1 = \begin{bmatrix} -\frac{c}{2\rho} \\ 1 \\ +\frac{c}{2\rho} \end{bmatrix}$$

And therefore

$$\nabla \lambda^1 \bullet r^1 = \tfrac{1}{2}(\gamma + 1)$$

$$\nabla \lambda^2 \bullet r^2 = 0$$

$$\nabla \lambda^3 \bullet r^3 = \tfrac{1}{2}(\gamma + 1)$$

The Riemann invariants for the polytropic gas

Thus, of the three eigenvectors, 1 and 3 represent waves that can become either rarefactions or shocks, while 2 is linearly degenerate and can only be a *contact discontinuity*.

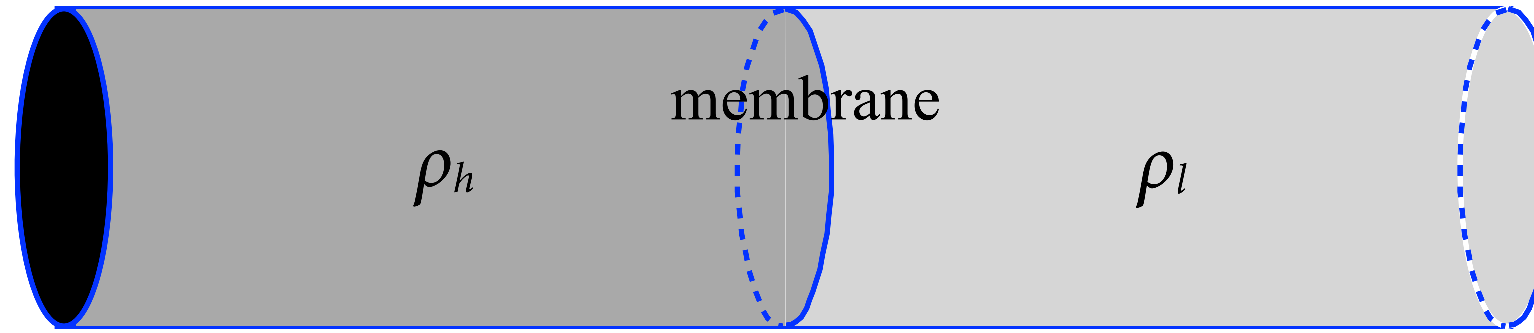
For any *simple wave* (not a rarefaction or a shock), the Riemann invariants are constant along particle paths through the wave. These are, for the 3 waves:

$$\text{1-wave: } s, \quad u + \frac{2c}{\gamma - 1}$$

$$\text{2-wave: } u, \quad p$$

$$\text{3-wave: } s, \quad u - \frac{2c}{\gamma - 1}$$

Now we can make more sense of the shock tube!

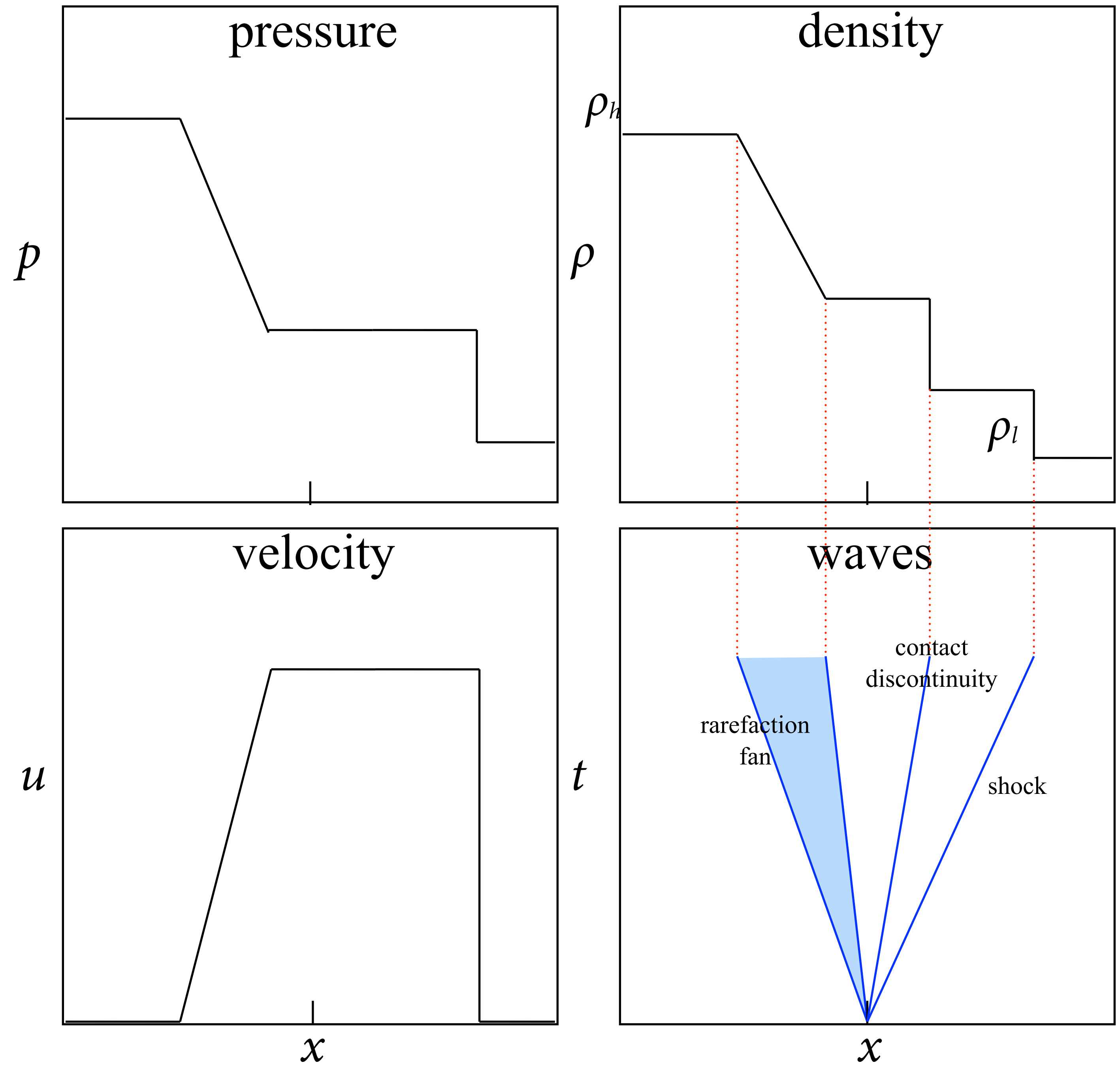


A closed tube filled with gas, separated by a membrane into sections with different densities.

The membrane is suddenly removed, and the gas starts moving from the high-density region into the lower density region.

Three waves develop: a *shock wave*, a *contact discontinuity*, and a *rarefaction wave* (or *fan*). The first two travel to the right, the third to the left.

At the shock, velocity, pressure and density are all discontinuous. At the contact, only density is discontinuous. In the rarefaction fan, all variables are continuous, but their derivatives are not.



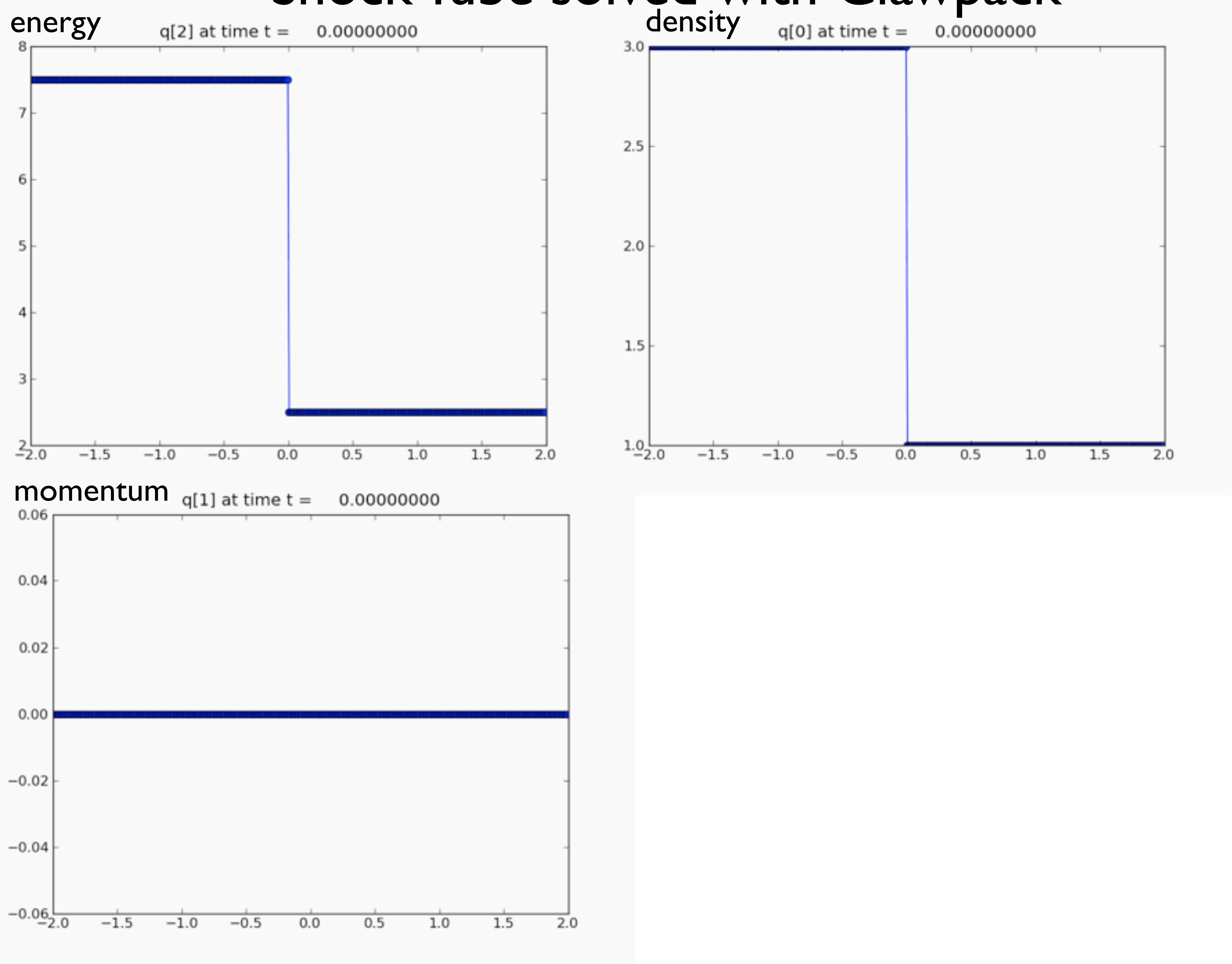
Shock Tube solved with Clawpack

energy

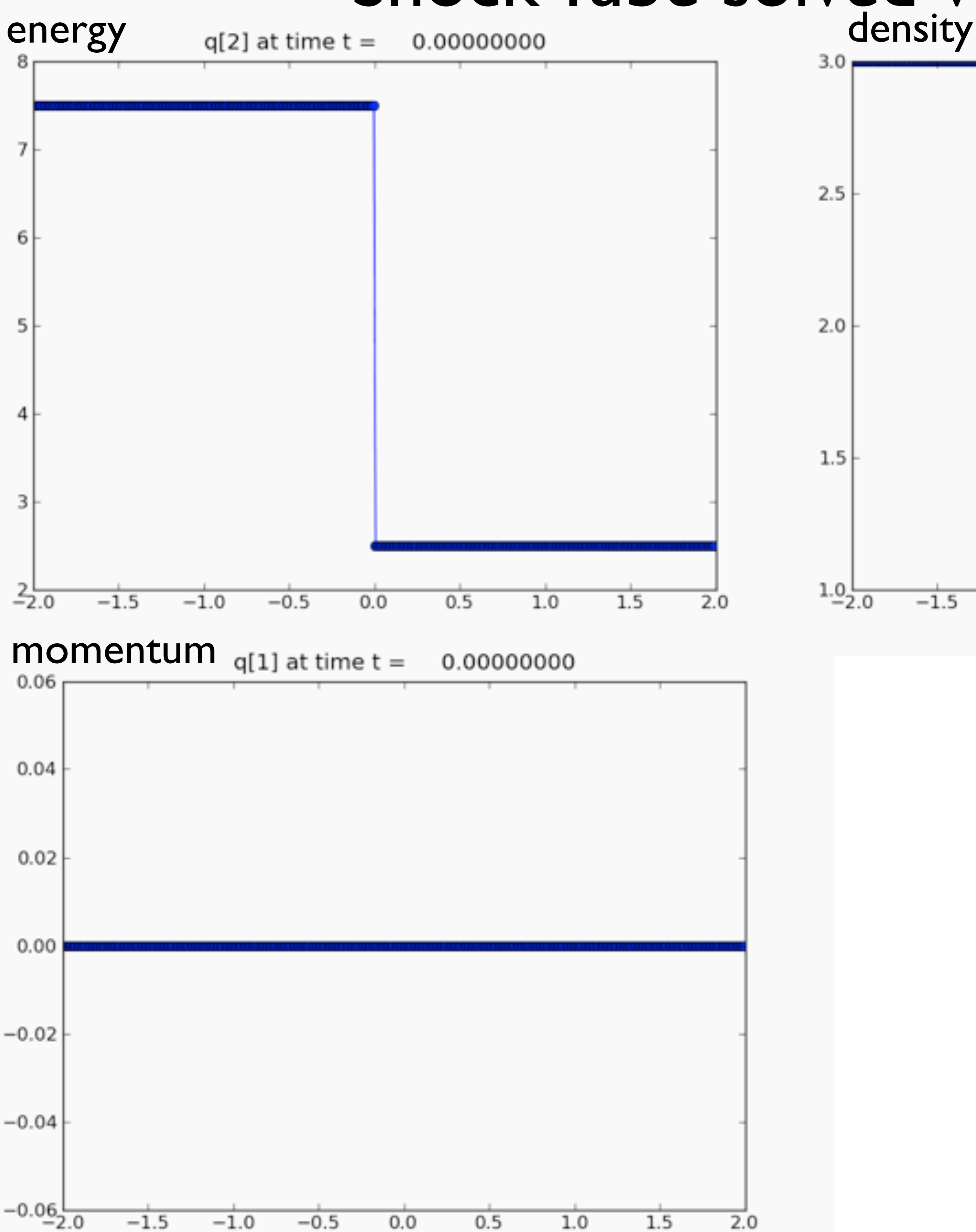
density

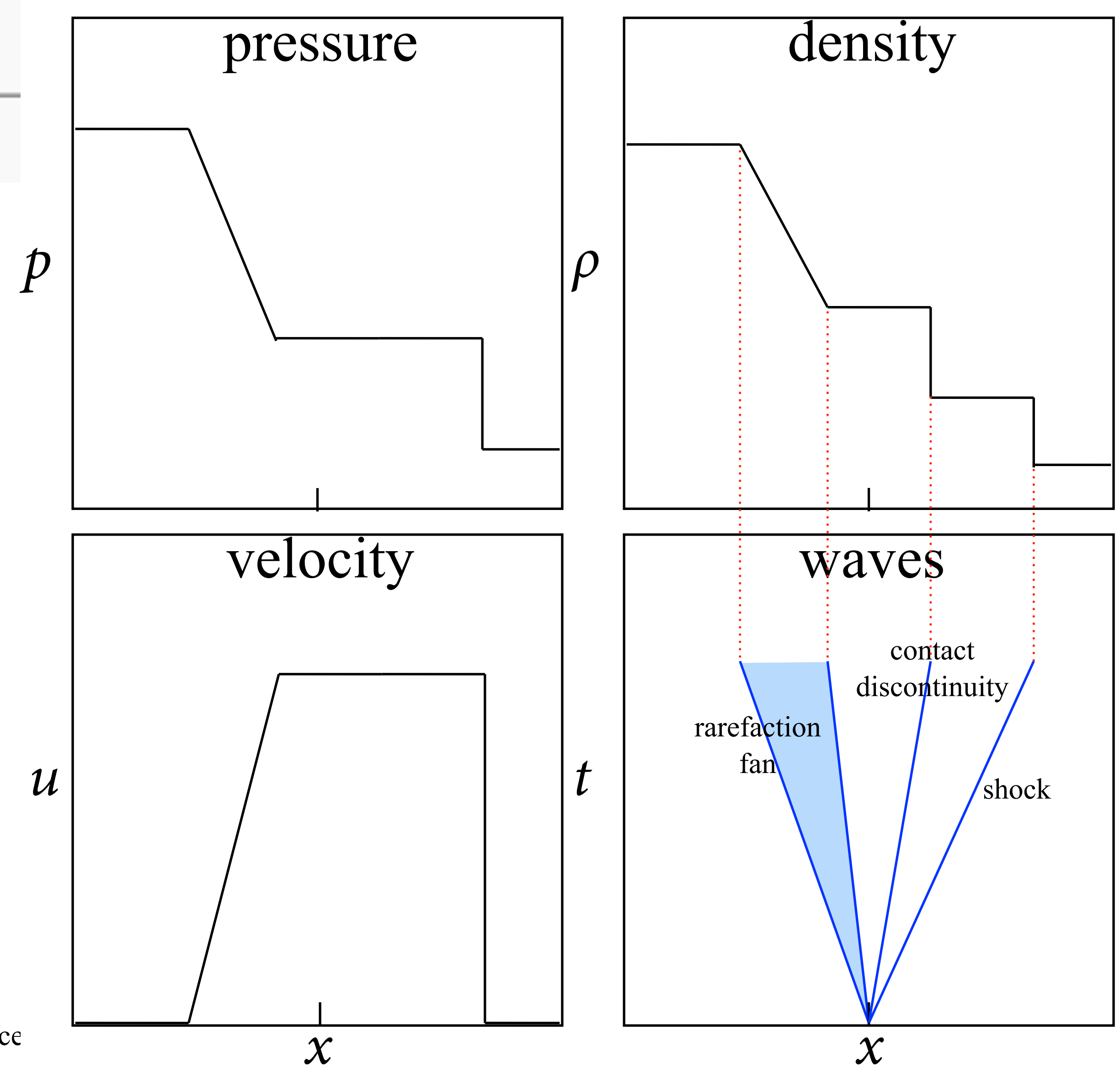
momentum

Shock Tube solved with Clawpack

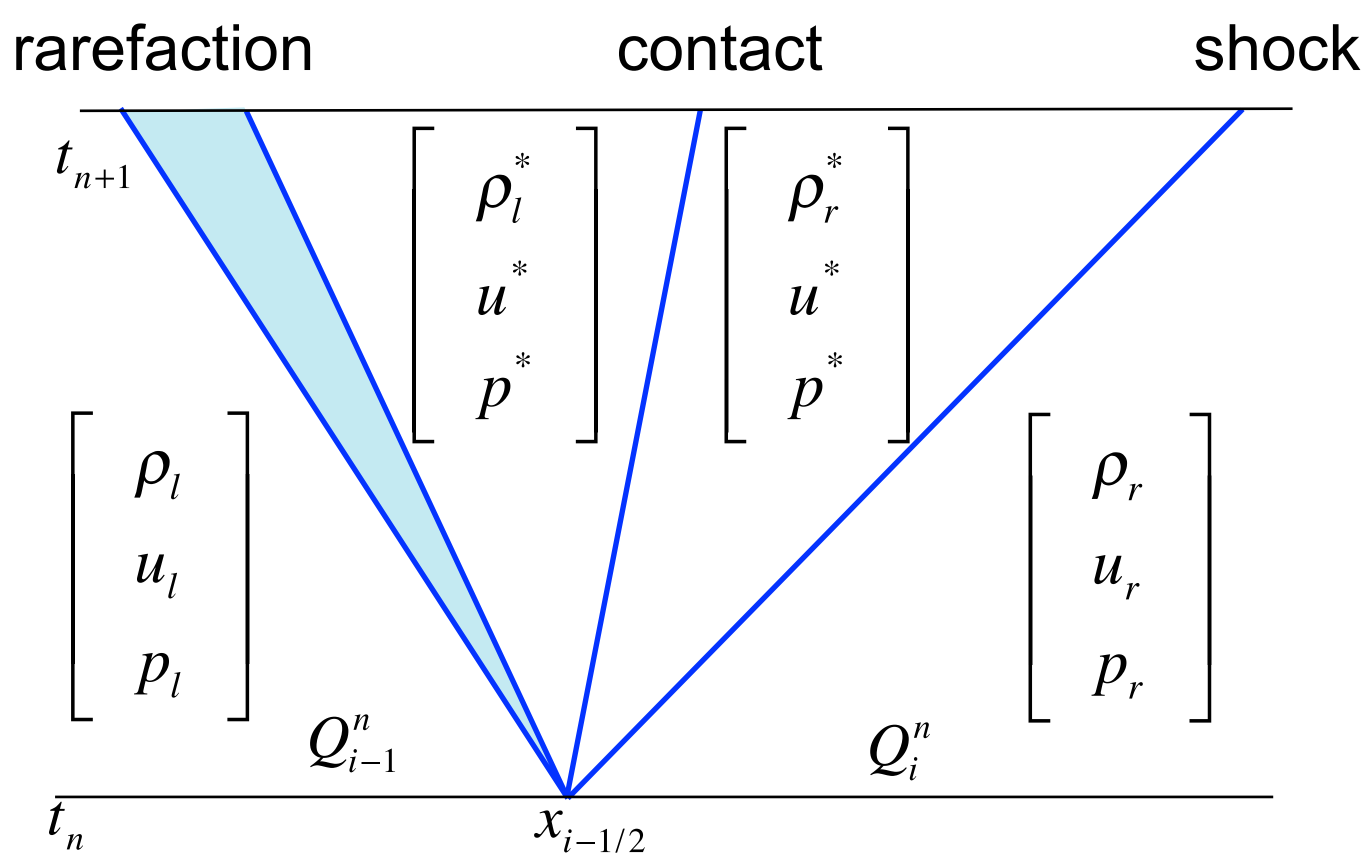


Shock Tube solved with Clawpack





The Riemann problem for 3 waves:

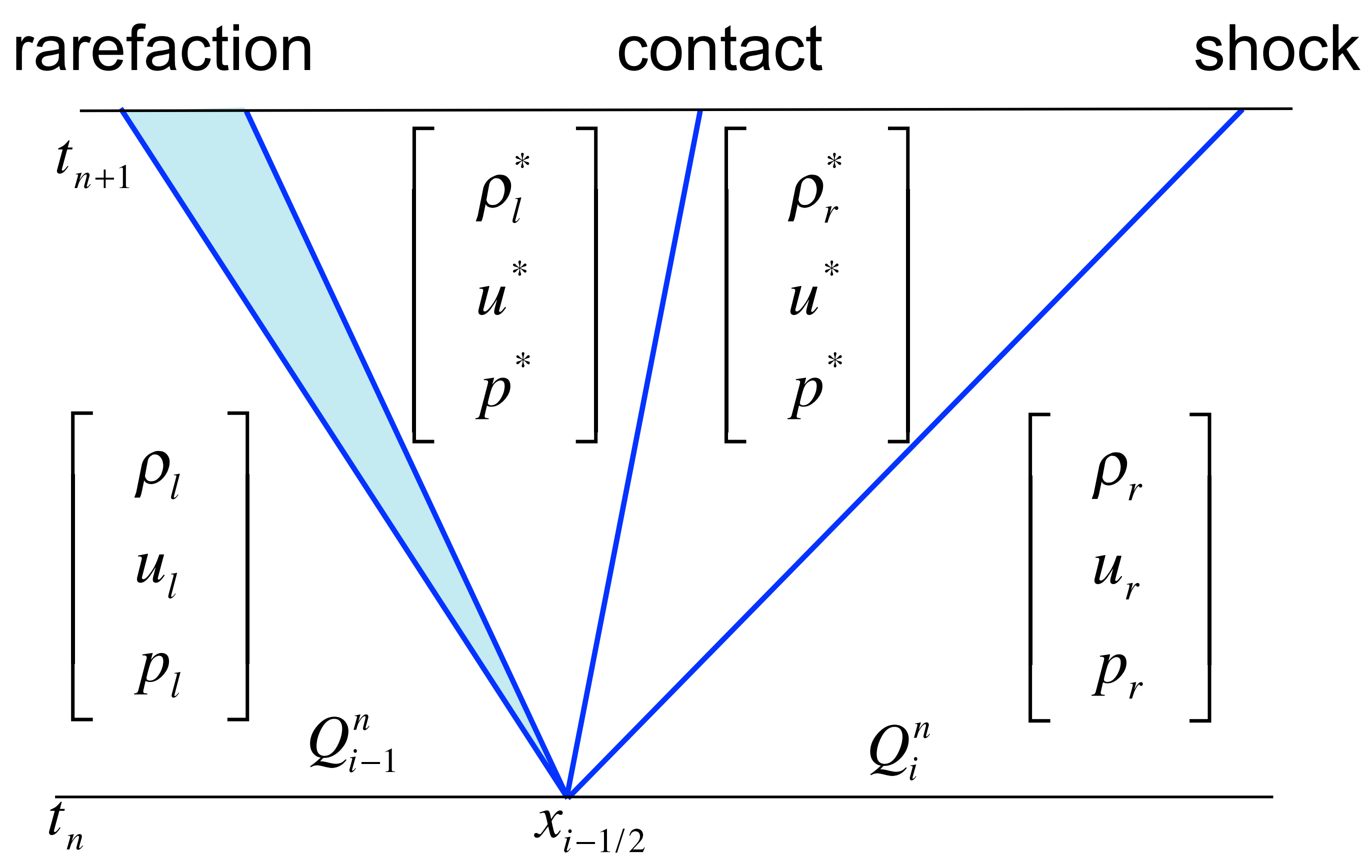


Now we have to solve for two intermediate states, not just one as before. And we have the rarefaction fan to deal with.

Note that only density changes across the contact discontinuity. This helps matters, in that we can use *almost* the same procedure as for the 2-equation shallow water set.

First we obtain (p^*, u^*) and then we use separate conditions to determine ρ_l^* and ρ_r^*.

The Riemann problem for 3 waves:



The 2-field is linearly degenerate. Across the contact u and p will be constant and only ρ will jump.

The strategy for solving the problem is to use the Hugoniot loci and integral curves for the 1-field and 3-field, in the phase plane of u and p, in the same way as for the shallow-water equations to obtain (p^*, u^*). Then we calculate the densities on either side of the contact. Finally we solve for the solution within the rarefaction fan.

The general (exact) Riemann solver for the Euler equations for a polytropic gas

As before, we define functions

$$u = \varphi_l(p) = \begin{cases} u_l + \dfrac{2c_l}{\gamma - 1}\left[1 - \left(p / p_l\right)^{\frac{\gamma-1}{2\gamma}}\right] & \text{if} \quad p \le p_l \\ u_l + \dfrac{2c_l}{\sqrt{2\gamma(\gamma-1)}}\left[\dfrac{1 - p / p_l}{\sqrt{1 + \beta p / p_l}}\right] & \text{if} \quad p \ge p_l \end{cases}$$

$$u = \varphi_r(p) = \begin{cases} u_r - \dfrac{2c_r}{\gamma - 1}\left[1 - \left(p / p_r\right)^{\frac{\gamma-1}{2\gamma}}\right] & \text{if} \quad p \le p_r \\ u_r - \dfrac{2c_r}{\sqrt{2\gamma(\gamma-1)}}\left[\dfrac{1 - p / p_r}{\sqrt{1 + \beta p / p_r}}\right] & \text{if} \quad p \ge p_r \end{cases}$$

where $\beta = \dfrac{\gamma + 1}{\gamma - 1}$.

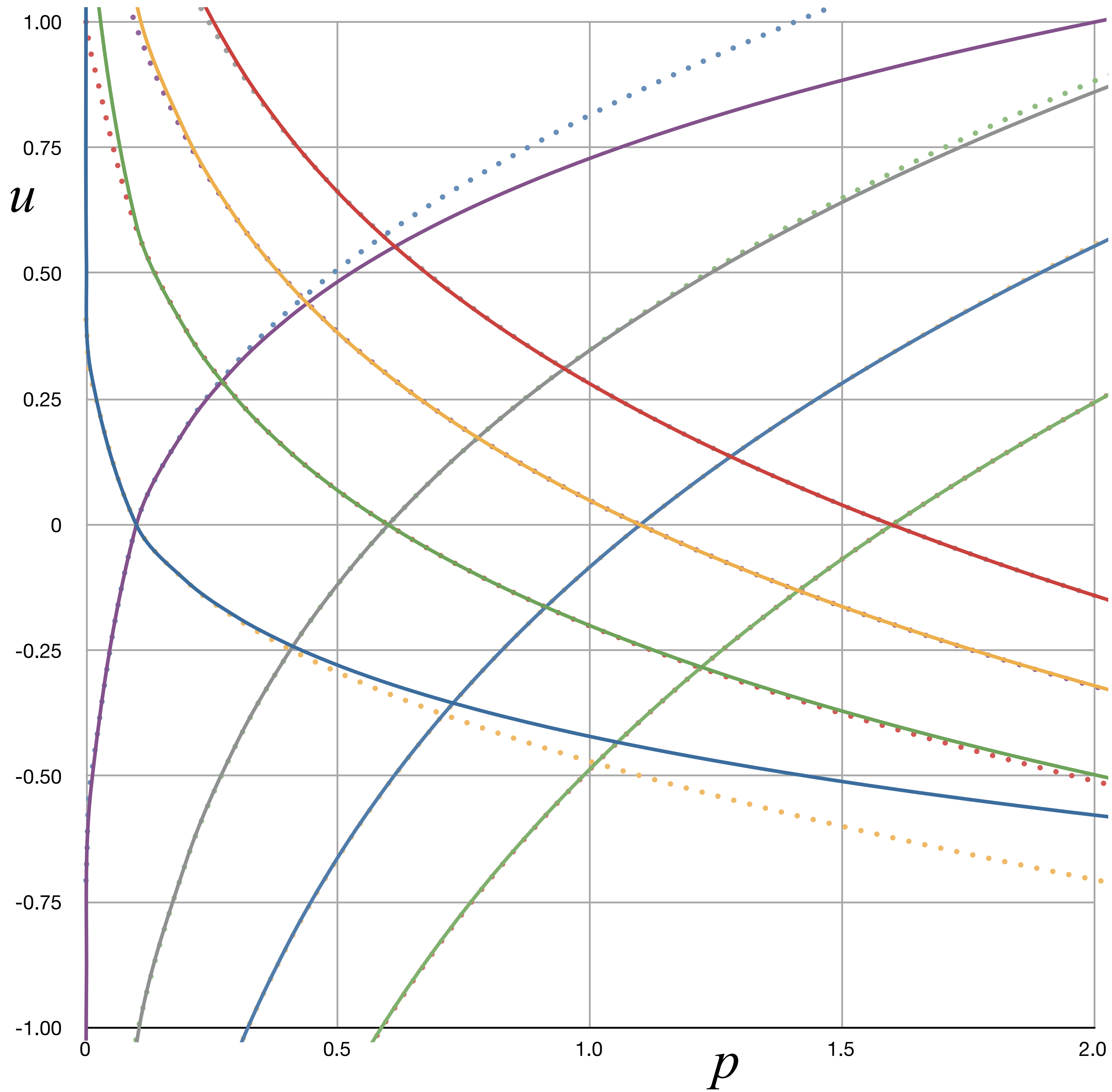
We then require that $\varphi_l(p_m) = \varphi_r(p_m)$, using an iterative procedure to find the intersection (p^*, u^*) of the curves. The densities on either side of the contact will then be given by

$$\rho_l^* = \left(\frac{1 + \beta p^* / p_l}{p^* / p_l + \beta}\right)\rho_l; \quad \rho_r^* = \left(\frac{1 + \beta p^* / p_r}{p^* / p_r + \beta}\right)\rho_r$$

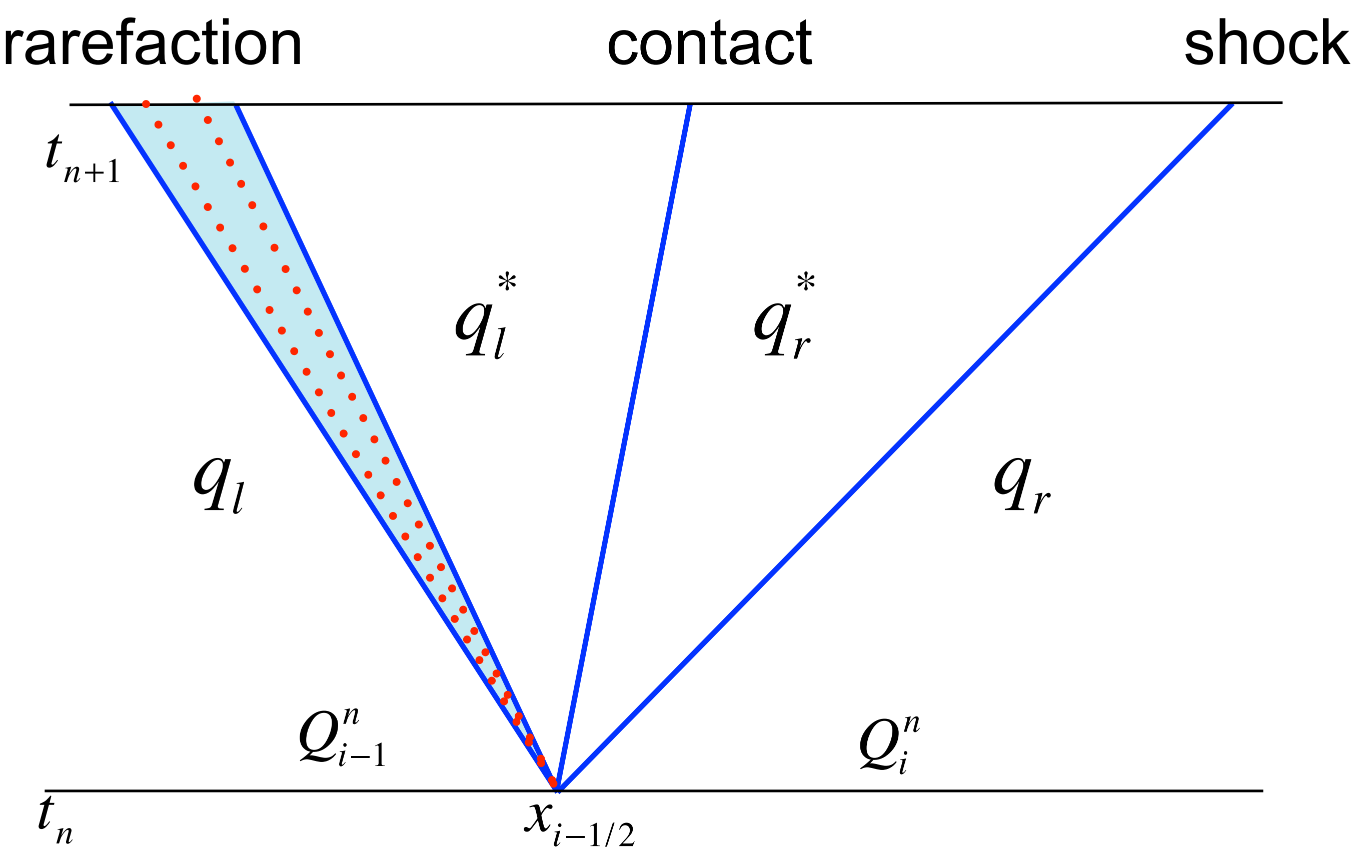
Hugoniot loci and integral curves for the Euler Equations (polytropic gas)

Some integral curves (solid) and Hugoniot loci (dotted) for the Euler equations. Just as in the shallow water equations, these curves are close together in many places. An iterative solver can start from the intersection of the integral curves, which can be obtained explicitly.

These curves are computed for γ=1.4 and densities ρ_l= 3 and ρ_r=1. For lower γ and higher density contrasts, the curves spread further apart.



We now have everything except the rarefaction:



The 1-rarefaction to connect the states q_l and q_l^*. Again we take advantage of the fact that the solution is a similarity solution, constant along the rays $\xi = x/t$. And we also know that the 1-Riemann invariant is constant through a rarefaction wave:

$$u + \frac{2c}{\gamma - 1} = u_l + \frac{2c_l}{\gamma - 1}$$

Then we have that $\xi = \lambda^1 = u - c$ within the rarefaction wave, so we can rewrite the Riemann invariant as

$$u + \frac{2(u - \xi)}{\gamma - 1} = u_l + \frac{2c_l}{\gamma - 1},$$

and solve for u as a function of ξ.

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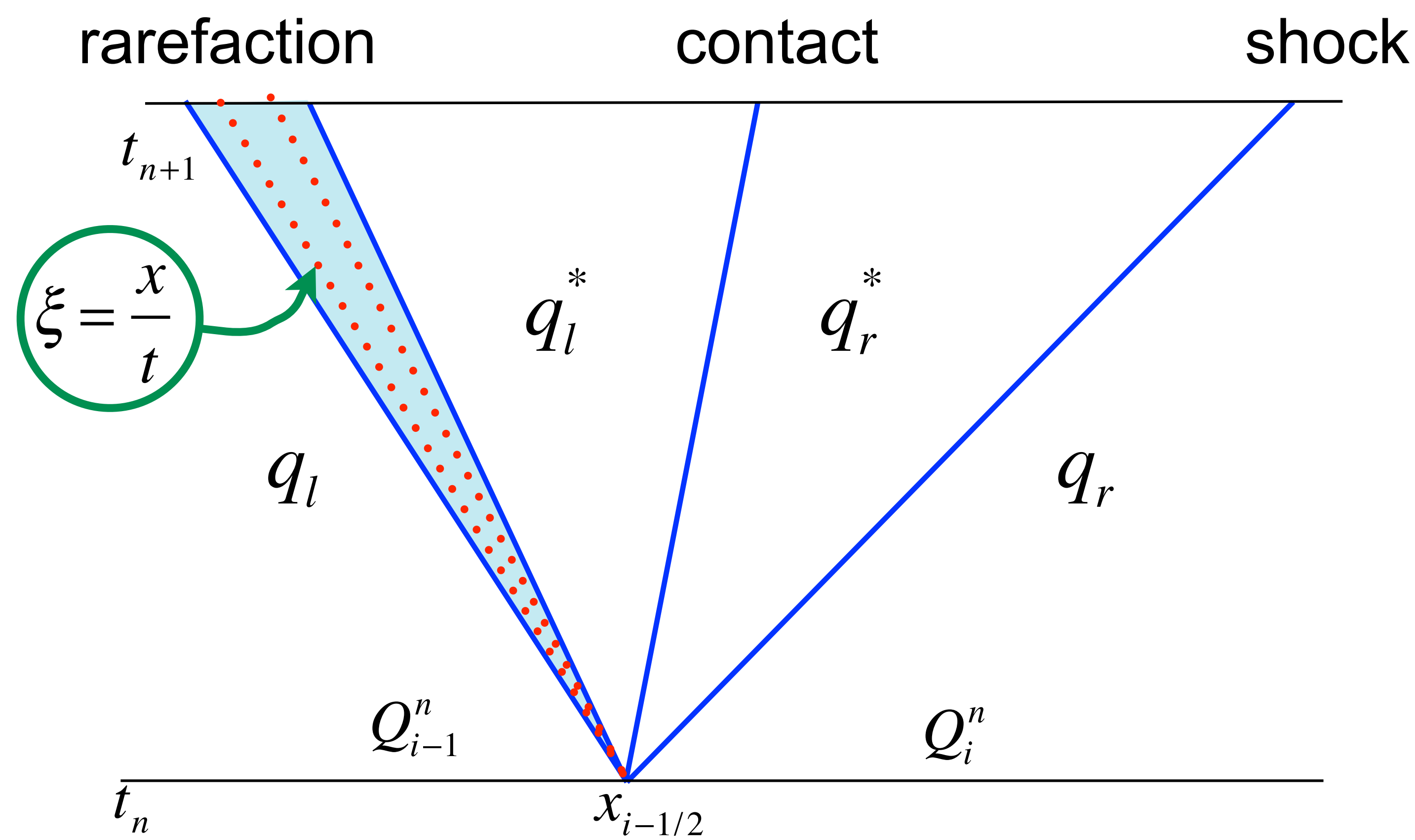

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Continuing the solution within the rarefaction

With the Riemann invariant

$$u + \frac{2(u-\xi)}{\gamma - 1} = u_l + \frac{2c_l}{\gamma - 1},$$

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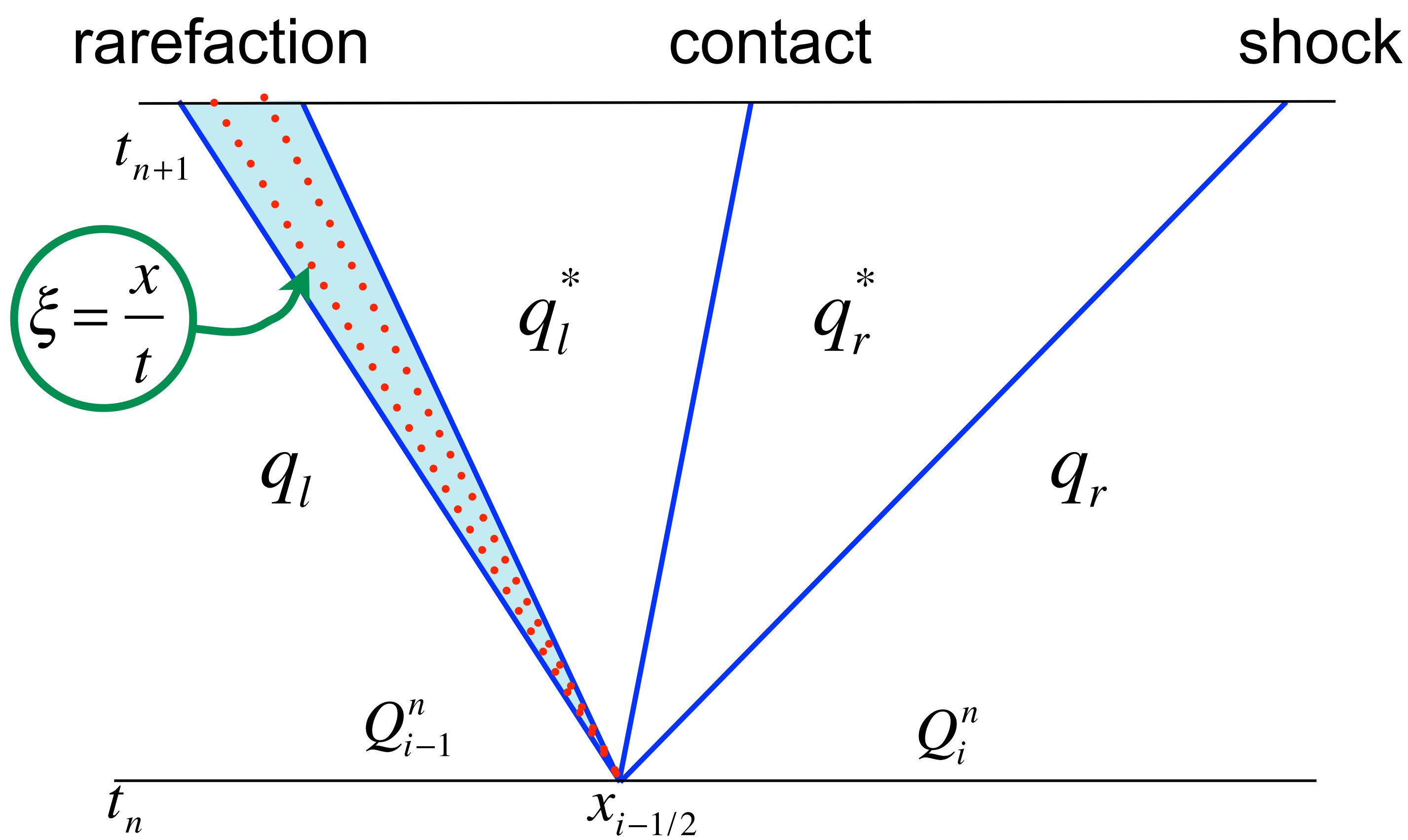
$$u(\xi) = \frac{(\gamma-1)u_l + 2(c_l + \xi)}{\gamma + 1}.$$



Then since p/ρ^γ is constant,

$$c^2 = \gamma\left(\frac{p_l}{\rho_l^\gamma}\right)\rho^{\gamma-1} = \left(u(\xi) - \xi\right)^2$$

and

$$\rho(\xi) = \left(\frac{\rho_l^\gamma}{\gamma p_l}\left(u(\xi) - \xi\right)^2\right)^{1/(\gamma-1)}$$

$$p(\xi) = \left(\frac{p_l}{\rho_l^\gamma}\right)\rho(\xi)^\gamma$$

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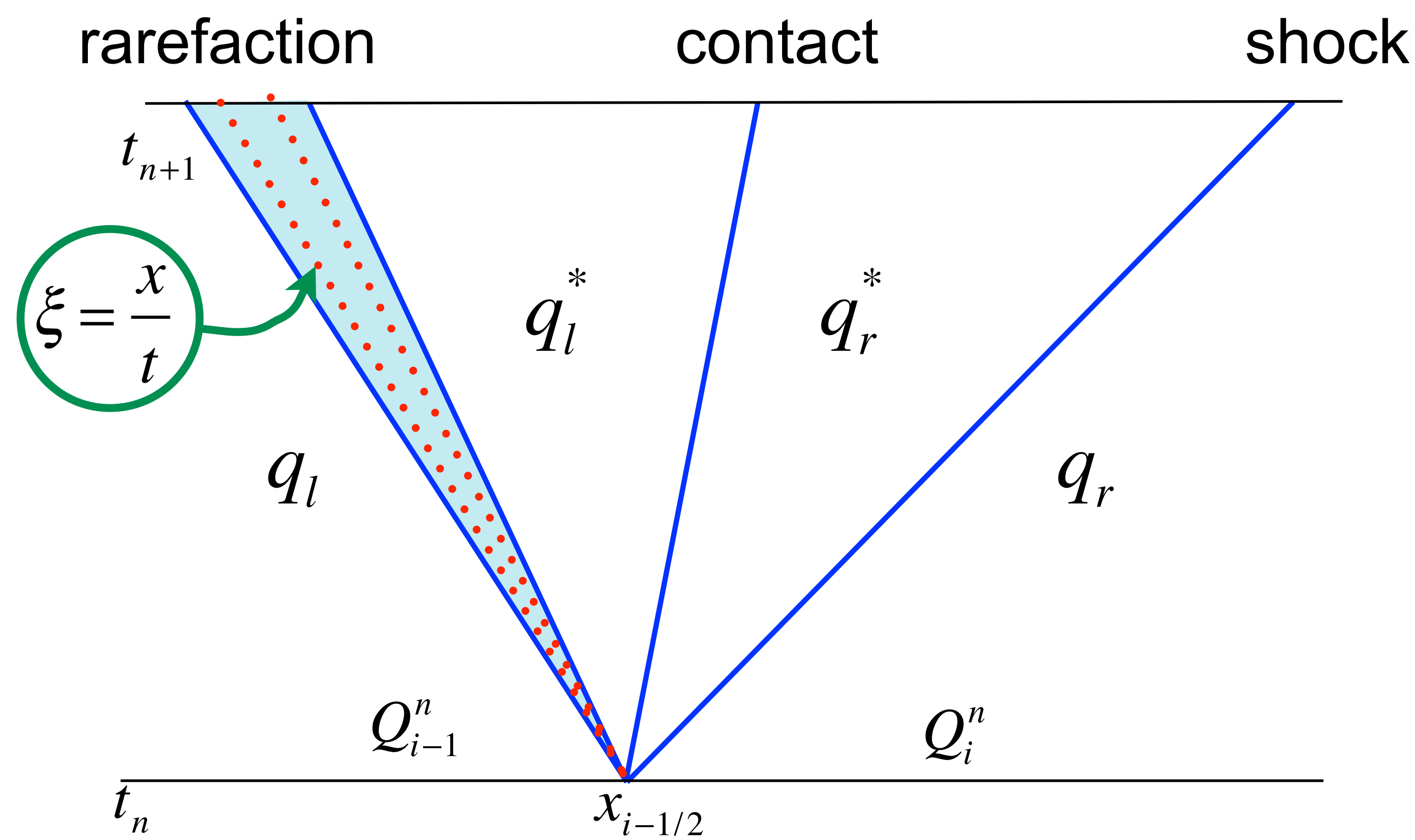

The exact solution is a perfect and complete structure and can be solved using iterative methods.

Wednesday, 26 October 2011

Continuing the solution within the rarefaction

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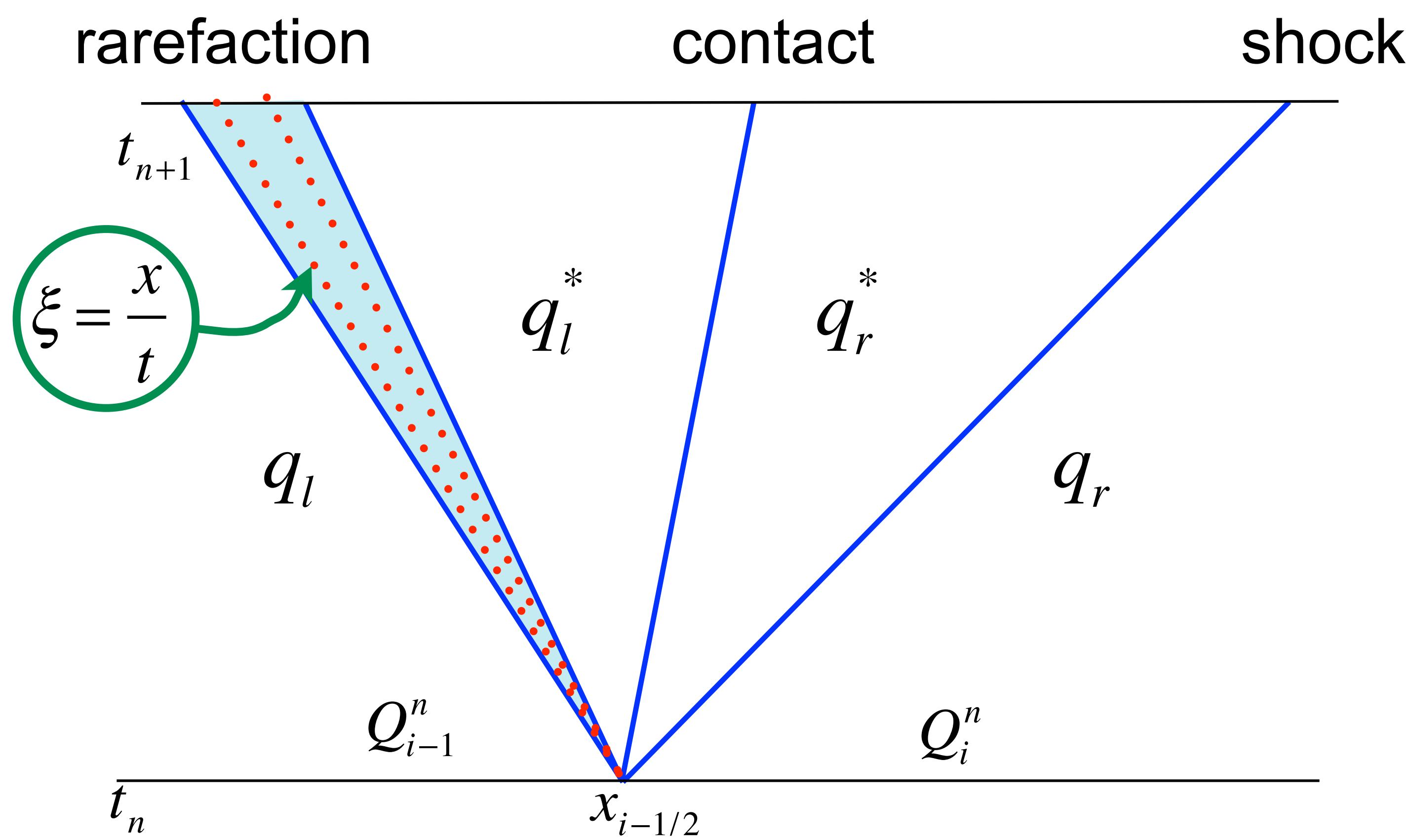

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However, in practical computation all of this is **not** used; instead we will use approximate Riemann solvers.

Continuing the solution within the rarefaction

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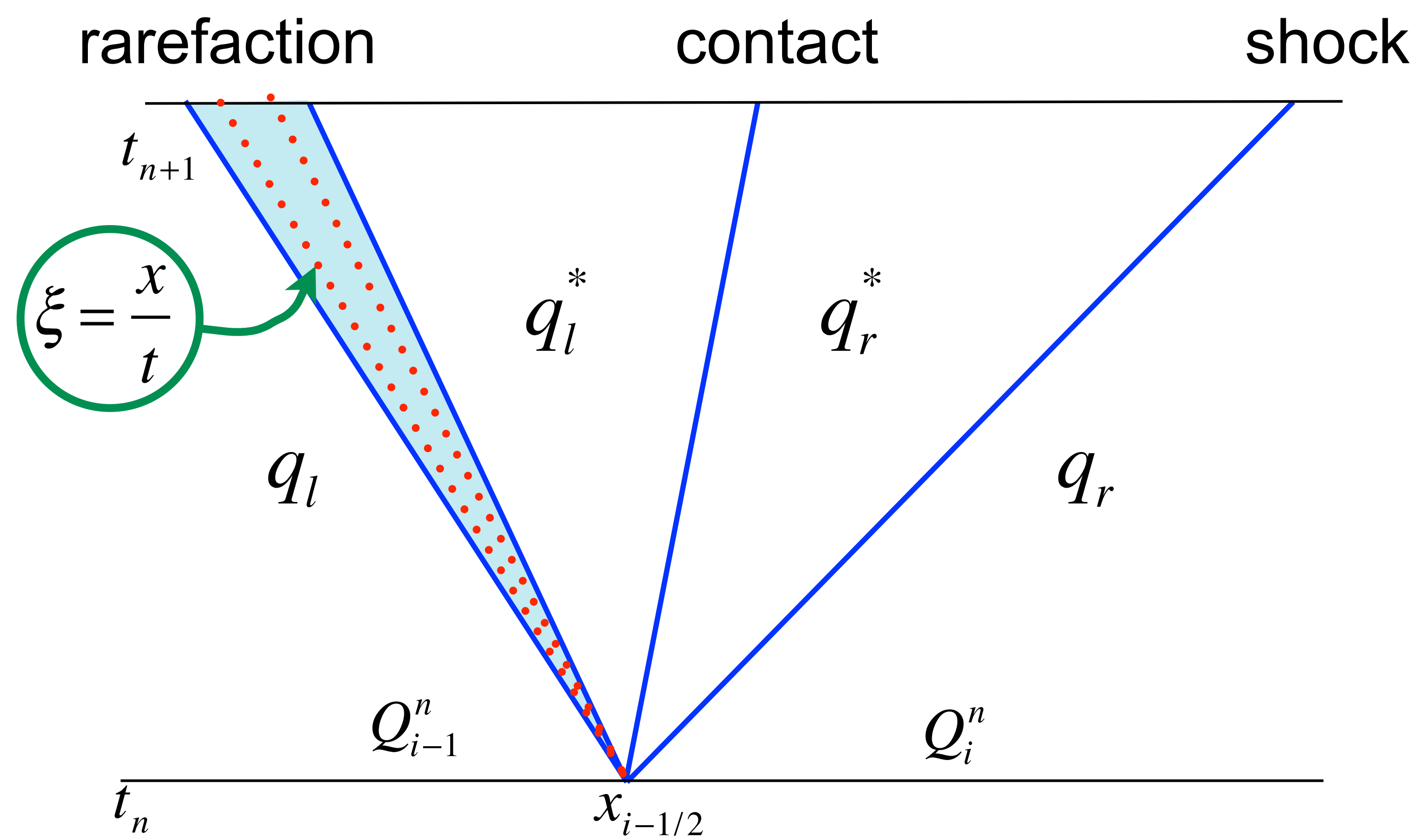
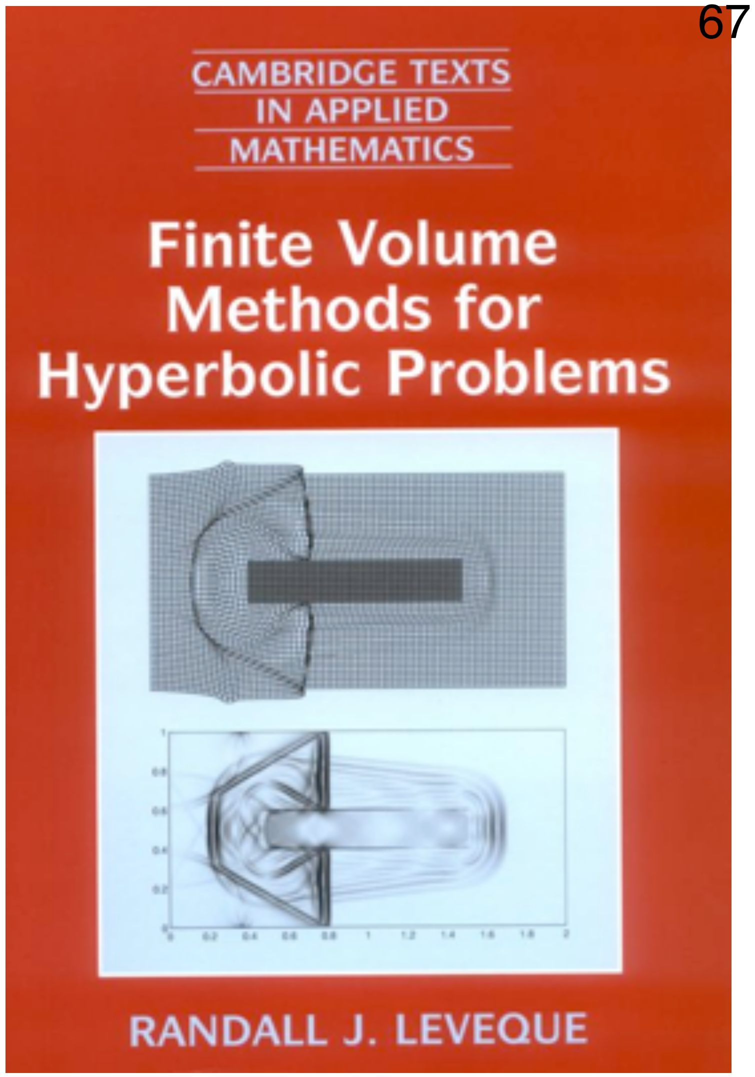


The exact solution is a perfect and complete structure and can be solved using iterative methods.

However, in practical computation all of this is **not** used; instead we will use approximate Riemann solvers.

The exact solution is useful for verification, however.

Assignment for next time



Read Chapter 13 and Chapter 14.

Work problems 13.2 and 13.4.

Write a program (in Fortran or Python or any other suitable language) that uses an iterative root finder (like Newton's method) to find the intermediate state q_m from the Hugoniot loci and integral curves for the shallow-water equation as described in section 13.10.

Next: Finite Volume Methods for Nonlinear Systems (Ch 15)