FYS-GEO 4500

Lecture Notes #9
Waves in Elastic Media

Where we are today

	date	Topic	Chapter in LeVeque
1	1.Sep 2011	introduction to conservation laws, Clawpack	1 & 2
2	15.Sep 2011	the Riemann problem, characteristics	3 & 5
3	22.Sep 2011	finite volume methods for linear systems, high resolution	4 & 6
4	29.Sep 2011	boundary conditions, accuracy, variable coeff.	7,8, part 9
5	6.Oct 2011	nonlinear conservation laws, finite volume methods	11 & 12
6	13.Oct 2011	nonlinear equations & systems	13 & 14
	20.Oct 2011	no lecture	
7	27.Oct 2011	finite volume methods for nonlinear systems	15,16,17
8	3.Nov 2011	multidimensional systems and source terms, etc.	18, 19, 20, 21
	10.Nov 2011	no lecture	
9	17.Nov 2011	waves in elastic media	22
10	24.Nov 2011	unfinished business: capacity functions, source terms, project plans	
11	1.Dec 2011	student presentations	
	8.Dec 2011	no lecture	
12	15.Dec 2011	FINAL REPORTS DUE	

11. Waves in Elastic Media (Chapter 22 in Leveque)

Linear acoustics

We linearise the barotropic system by examining only a perturbation to the (constant) background state. Let

$$q(x,t) = q_0 + \tilde{q}(x,t),\ \tilde{q} = \begin{bmatrix} \tilde{\rho} \\ \widetilde{\rho u} \end{bmatrix}$$

The conservation law $q_t + f(q)_x = 0$ becomes the constant-coefficient linear system $\tilde{q}_t + f'(q_0)\tilde{q}_x = 0$ since we discard powers of the perturbed quantity.

The Jacobian of the perturbed barotropic system is

$$A = f'(q) = \begin{bmatrix} \dfrac{\partial f^1}{\partial q^1} & \dfrac{\partial f^1}{\partial q^2} \\ \dfrac{\partial f^2}{\partial q^1} & \dfrac{\partial f^2}{\partial q^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 + P'(\rho) & 2u \end{bmatrix}$$

Linear acoustics

The system just obtained can be written out as:

$$\tilde{\rho}_t + (\widetilde{\rho u})_x = 0$$

$$(\widetilde{\rho u})_t + \left(-u_0^2 + P'(\rho_0)\right)\tilde{\rho}_x + 2u_0(\widetilde{\rho u})_x = 0$$

Then writing $P'(\rho_0)\tilde{\rho} = \tilde{p}$, $\widetilde{\rho u} = u_0\tilde{\rho} + \rho_0\tilde{u}$, and $K = \rho_0 P'(\rho_0)$

We obtain the linear acoustics equations

$$\tilde{p}_t + u_0\tilde{p}_x + K\tilde{u}_x = 0$$

$$\rho_0\tilde{u}_t + \tilde{p}_x + \rho_0 u_0\tilde{u}_x = 0$$

With $u_0 = 0$, this becomes the system (dropping the tildes):

$$q_t(x,t) + Aq_x(x,t) = 0$$

$$q = \begin{bmatrix} p \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & K \\ \frac{1}{\rho_0} & 0 \end{bmatrix}, \qquad \text{so} \quad \begin{aligned} p_t + Ku_x &= 0 \\ \rho_0 u_t + p_x &= 0 \end{aligned}$$

Linear acoustics

The acoustic equations are:

$$p_t(x,t) + K u_x(x,t) = 0$$

$$u_t(x,t) + \frac{1}{\rho} p_x(x,t) = 0$$

Express in matrix notation:

$$q_t(x,t) + A q_x(x,t) = 0 \qquad q = \begin{bmatrix} p \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & K \\ \frac{1}{\rho} & 0 \end{bmatrix}.$$

Resolve into the eigensystem: $Ar = \lambda r$,

with eigenvalues $\lambda^{1,2} = \pm c = \pm\sqrt{\frac{K}{\rho}}$ and eigenvectors $r^{1,2} = \begin{bmatrix} \pm\sqrt{K\rho} \\ 1 \end{bmatrix}$.

The eigenvalues are the wave speeds, and the eigenvectors express relations between the components of the solution q.

Elastic Media

Elastic media can support shear waves (S-waves) as well as pressure waves (P-waves), hence there will be two characteristic speeds, c_p and c_s, with

$$c_p > c_s$$

In one dimension (treated in Section 2.12), the P and S waves are uncoupled, and can be treated by two separate systems of two equations each.

In multidimensions, there is a coupling, so the whole system has to be treated together.

In Chapter 22, Leveque treats the general three-dimensional system, and later reduces to useful two-dimensional approximations.

Here follows a brief summary.

The tensors of elasticity

The (symmetric) matrices of strain and stress are:

$$\varepsilon = \begin{bmatrix} \varepsilon^{11} & \varepsilon^{12} & \varepsilon^{13} \\ \varepsilon^{21} & \varepsilon^{22} & \varepsilon^{23} \\ \varepsilon^{31} & \varepsilon^{32} & \varepsilon^{33} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{21} & \sigma^{22} & \sigma^{23} \\ \sigma^{31} & \sigma^{32} & \sigma^{33} \end{bmatrix}$$

Where the strains are given by the displacements

$$\varepsilon^{11} = \delta_x^1 \equiv \frac{\partial X}{\partial x} - 1, \quad \varepsilon^{22} = \delta_y^2 \equiv \frac{\partial Y}{\partial y} - 1, \quad \varepsilon^{33} = \delta_z^3 \equiv \frac{\partial Z}{\partial z} - 1,$$

$$\varepsilon^{12} = \tfrac{1}{2}\left(\delta_y^1 + \delta_x^2\right), \quad \varepsilon^{13} = \tfrac{1}{2}\left(\delta_z^1 + \delta_x^3\right), \quad \varepsilon^{23} = \tfrac{1}{2}\left(\delta_z^2 + \delta_y^3\right).$$

and a generalised Hooke's law gives the stress-strain relation

$$\sigma^{ij} = \sum_{k,l} C^{ijkl} \varepsilon^{kl}$$

where the tensor C can have up to 21 independent components.

Elasticity in isotropic media

For isotropic media, the 6 independent components of ε and σ are related by:

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{23} \\ \sigma^{13} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon^{11} \\ \varepsilon^{22} \\ \varepsilon^{33} \\ \varepsilon^{12} \\ \varepsilon^{23} \\ \varepsilon^{13} \end{bmatrix}$$

where μ is the *shear modulus*, and the *Lame parameter* $\lambda \equiv \dfrac{\nu E}{(1+\nu)(1-2\nu)}$ contains the *bulk modulus* E and the *Poisson ratio* ν *.*

Conservation laws for the elastic system

$$\varepsilon_t^{11} - u_x = 0$$

$$\varepsilon_t^{22} - v_y = 0$$

$$\varepsilon_t^{33} - w_z = 0$$

$$\varepsilon_t^{12} - \tfrac{1}{2}\left(v_x + u_y\right) = 0$$

$$\varepsilon_t^{23} - \tfrac{1}{2}\left(w_y + v_z\right) = 0$$

$$\varepsilon_t^{13} - \tfrac{1}{2}\left(u_z + w_x\right) = 0$$

$$\rho u_t - \sigma_x^{11} - \sigma_y^{12} - \sigma_z^{13} = 0$$

$$\rho v_t - \sigma_x^{12} - \sigma_y^{22} - \sigma_z^{23} = 0$$

$$\rho w_t - \sigma_x^{13} - \sigma_y^{23} - \sigma_z^{33} = 0$$

These involve both strains and stresses, but we can use the matrix equation on the previous slide to eliminate the strains.

Conservation laws for the elastic system using stresses alone

$$\sigma^{11}_t - (\lambda + 2\mu) u_x - \lambda v_y - \lambda w_z = 0$$

$$\sigma^{22}_t - \lambda u_x - (\lambda + 2\mu) v_y - \lambda w_z = 0$$

$$\sigma^{33}_t - \lambda u_x - \lambda v_y - (\lambda + 2\mu) w_z = 0$$

$$\sigma^{12}_t - \mu \left(v_x + u_y \right) = 0$$

$$\sigma^{23}_t - \mu \left(w_y + v_z \right) = 0$$

$$\sigma^{13}_t - \mu \left(u_z + w_x \right) = 0$$

$$\rho u_t - \sigma^{11}_x - \sigma^{12}_y - \sigma^{13}_z = 0$$

$$\rho v_t - \sigma^{12}_x - \sigma^{22}_y - \sigma^{23}_z = 0$$

$$\rho w_t - \sigma^{13}_x - \sigma^{23}_y - \sigma^{33}_z = 0$$

And then we write this system as

$$q_t + A q_x + B q_y + C q_z = 0$$

The matrices for the elastic wave equations

We write this system as $q_t + Aq_x + Bq_y + Cq_z = 0$

where

$$q = \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{23} \\ \sigma^{13} \\ u \\ v \\ w \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+2\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\ -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \end{bmatrix}$$

The matrices B and C have similar nonzero elements in different places.

These matrices do not commute! So there is always coupling among the dimensions.

Eigenvalues and eigenvectors for the elastic wave equations

This system of 9 equations has, as you would expect, 9 eigenvalues, but 3 of them are zero (waves with zero speed). Leveque orders them as follows:

$$s^1 = -c_p,\ s^2 = +c_p,\ s^3 = -c_s,\ s^4 = +c_s,\ s^5 = -c_s,\ s^6 = +c_s,\ s^7 = s^8 = s^9 = 0$$

and the eigenvectors for the A matrix are (similarly for the other two):

$$r^{1,2} = \begin{bmatrix} \lambda + 2\mu \\ \lambda \\ \lambda \\ 0 \\ 0 \\ 0 \\ \pm c_p \\ 0 \\ 0 \end{bmatrix},\ r^{3,4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mu \\ 0 \\ 0 \\ 0 \\ \pm c_s \\ 0 \end{bmatrix},\ r^{5,6} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mu \\ 0 \\ 0 \\ \pm c_s \end{bmatrix},\ r^{7} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ r^{8} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ r^{9} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

pressure waves in x

shear waves in y and z

stationary jumps in $\sigma^{23}, \sigma^{22}, \sigma^{33}$

Conservation laws for the elastic system using stresses alone

With no variation in the z direction, we can drop all z-derivatives.

$$\sigma_t^{11} - (\lambda + 2\mu) u_x - \lambda v_y - \lambda w_z = 0$$

$$\sigma_t^{22} - \lambda u_x - (\lambda + 2\mu) v_y - \lambda w_z = 0$$

$$\sigma_t^{33} - \lambda u_x - \lambda v_y - (\lambda + 2\mu) w_z = 0$$

$$\sigma_t^{12} - \mu \left(v_x + u_y \right) = 0$$

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Also, it turns out that σ^{33} can be written in terms of σ^{11} and σ^{22}.

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are related by:

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where μ is the *shear modulus*, and the *Lame parameter* $\lambda \equiv \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

contains the *bulk modulus* E and the *Poisson ratio* ν .

Note that $\sigma^{33} = \lambda\varepsilon^{11} + \lambda\varepsilon^{22} + (\lambda+2\mu)\varepsilon^{33}$

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Note that $\sigma^{33} = \lambda\varepsilon^{11} + \lambda\varepsilon^{22} + \cancel{(\lambda+2\mu)\varepsilon^{33}} = \nu\sigma^{11} + \nu\sigma^{22}$

Conservation laws for the elastic system using stresses alone

With no variation in the z direction, we can drop all z-derivatives.

$$\sigma_t^{11} - (\lambda + 2\mu) u_x - \lambda v_y - \lambda w_z = 0$$

$$\sigma_t^{22} - \lambda u_x - (\lambda + 2\mu) v_y - \lambda w_z = 0$$

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Also, it turns out that σ^{33} can be written in terms of σ^{11} and σ^{22}.

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Also, it turns out that σ^{33} can be written in terms of σ^{11} and σ^{22}.

Finally, note that all equations involving w do not couple with the rest.

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Finally, note that all equations involving w do not couple with the rest.

Two-Dimensional Elasticity: Plain-Strain

With no variation in the z direction, we can drop all z-derivatives.

Also, it turns out that σ^{33} can be written in terms of σ^{11} and σ^{22}.

The remaining 8 equations decouple into two sets, of 5 and 3 equations:

The "plane-strain equations"

$$\sigma_t^{11} - (\lambda + 2\mu)u_x - \lambda v_y = 0$$

$$\sigma_t^{22} - \lambda u_x - (\lambda + 2\mu)v_y = 0$$

$$\sigma_t^{12} - \mu\left(v_x + u_y\right) = 0$$

$$\rho u_t - \sigma_x^{11} - \sigma_y^{12} = 0$$

$$\rho v_t - \sigma_x^{12} - \sigma_y^{22} = 0$$

Shear waves propagating in z

$$\sigma_t^{23} - \mu w_y = 0$$

$$\sigma_t^{13} - \mu w_x = 0$$

$$\rho w_t - \sigma_x^{13} - \sigma_y^{23} = 0$$

This is essentially assuming the material is infinite in the z direction.

Solving the plane-strain equations

Again, we can write the system as

$$q_t + Aq_x + Bq_y = 0$$

But now we form the linear combination

$$\breve{A} = n^x A + n^y B$$

Where $\vec{n}$ is a unit vector with arbitrary direction. Then

$$q = \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{12} \\ u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 0 & n^x(\lambda+2\mu) & n^y\lambda \\ 0 & 0 & 0 & n^x\lambda & n^y(\lambda+2\mu) \\ 0 & 0 & 0 & n^y u & n^x u \\ n^x/\rho & 0 & n^y/\rho & 0 & 0 \\ 0 & n^y/\rho & n^x/\rho & 0 & 0 \end{bmatrix}$$

Eigenvalues and eigenvectors of the plane-strain system

The eigenvalues of $\breve{A}$ are: $\breve{s}^1 = -c_p,\ \breve{s}^2 = +c_p,\ \breve{s}^3 = -c_s,\ \breve{s}^4 = +c_s,\ \breve{s}^5 = 0$

and the eigenvectors are:

$$\breve{r}^1 = \begin{bmatrix} \lambda + 2\mu\left(n^x\right)^2 \\ \lambda + 2\mu\left(n^y\right)^2 \\ 2\mu n^x n^y \\ n^x c_p \\ n^y c_p \end{bmatrix},\ \breve{r}^2 = \begin{bmatrix} \lambda + 2\mu\left(n^x\right)^2 \\ \lambda + 2\mu\left(n^y\right)^2 \\ 2\mu n^x n^y \\ -n^x c_p \\ -n^y c_p \end{bmatrix},$$ for the P-waves

$$\breve{r}^3 = \begin{bmatrix} -2\mu n^x n^y \\ 2\mu n^x n^y \\ \mu\left[\left(n^x\right)^2 - \left(n^y\right)^2\right] \\ -n^y c_s \\ n^x c_s \end{bmatrix},\ \breve{r}^4 = \begin{bmatrix} -2\mu n^x n^y \\ 2\mu n^x n^y \\ \mu\left[\left(n^x\right)^2 - \left(n^y\right)^2\right] \\ n^y c_s \\ -n^x c_s \end{bmatrix},\ \breve{r}^5 = \begin{bmatrix} \left(n^y\right)^2 \\ \left(n^x\right)^2 \\ -n^x n^y \\ 0 \\ 0 \end{bmatrix}$$

for the S-waves (for $\breve{r}^3$, $\breve{r}^4$); for the stationary wave (for $\breve{r}^5$)

Another two-dimensional limit: Plain-Stress

If the domain is a thin plate (thin in the z direction), we can develop an approximation in which

$$\sigma^{13} = \sigma^{23} = \sigma^{33} = 0$$

With no velocity in the z direction, we reduce the system of equations to:

$$\sigma_t^{11} - \left(\hat{\lambda} + 2\mu\right)u_x - \hat{\lambda}v_y = 0$$

$$\sigma_t^{22} - \hat{\lambda}u_x - \left(\hat{\lambda} + 2\mu\right)v_y = 0$$

$$\sigma_t^{12} - \mu\left(v_x + u_y\right) = 0$$

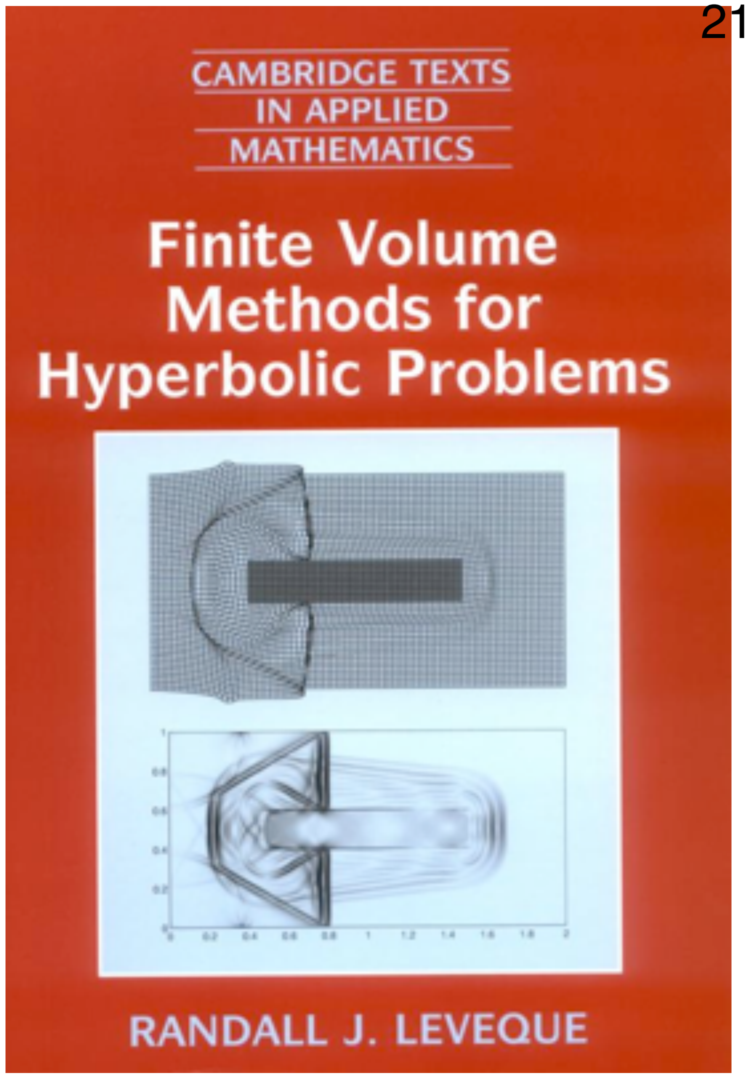
$$\rho u_t - \sigma_x^{11} - \sigma_y^{12} = 0$$

$$\rho v_t - \sigma_x^{12} - \sigma_y^{22} = 0$$

Which is almost the same as the plain-strain equations, with the substitution of

$$\hat{\lambda} = \frac{2\mu\lambda}{\lambda + 2\mu} \quad \text{for} \quad \lambda$$

Assignment for next time



Read Chapters 20 - 22. Pay careful attention to Chapter 21 for subtleties in the multidimensional implementation, especially Section 21.3.

Write first versions of the routines necessary for your project and test them.

Project recipe:

Write down all the equations of the problem in conservative form, including closure relations (equations of state, for example). You should also prepare an entropy equation that will be calculable should transonic conditions (or centred rarefactions) arise.

Find the Jacobian of the corresponding quasilinear system, and calculate its eigenvalues and eigenvectors.

For an arbitrary pair of right and left states, solve the Riemann problem, either exactly or approximately. Then write down general formulas for the waves, $\mathcal{W}^p_{i-1/2}$, the wave speeds $s^{\pm}$, the fluctuations $\mathcal{A}^{\pm}\Delta Q_{i-1/2}$ and the entropy fix for the transonic case.

Project recipe, continued

Under $CLAW/myclaw, prepare a new directory for the routines you must write. Write the Riemann solver (**rp1.f** or **rp2n.f** and **rp2t.f**) in Fortran in this directory. Copy over $CLAW/util/(**testrp1.f** or **testrp2n.f**), modify it and use it to test your Riemann solver.

Figure out what special work space you need, what boundary conditions, source terms, and other things that you want, and what special variables you need to input or initialise. Then write (or copy and modify) the appropriate routines (**driver**, **setprob**, setaux, **qinit**, bcN, b4stepN, srcN) in the same directory. Write (or copy and modify) a **Makefile** that points to these files, and construct **setrun.py** and **setplot.py** to fill the data files and make the plots.

Finally: compile, run, and check your results.