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Interaction theory – charged particles

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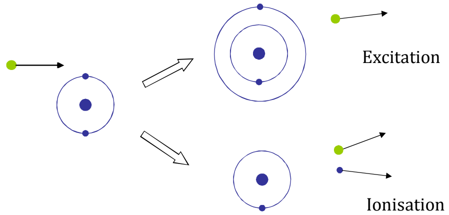
Excitation / ionization

- Incoming charged particle interacts with atom / molecule:



- An ion pair is created

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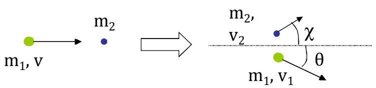
Elastic collision 1

- Interaction between two particles where kinetic energy is preserved:



- Classical mechanics:

$$T_0 = \frac{1}{2}m_1v^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$
$$m_1v = m_1v_1\cos\theta + m_2v_2\cos\chi$$
$$0 = m_1v_1\sin\theta - m_2v_2\sin\chi$$

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Elastic collision 2

$$\Rightarrow v_2 = \frac{2m_1v\cos\chi}{m_1+m_2} \quad , \quad v_1 = v\sqrt{1-\frac{4m_1m_2\cos^2\chi}{(m_1+m_2)^2}}$$

$$\tan\theta = \frac{\sin 2\chi}{\frac{m_1}{m_2} - \cos 2\chi}$$

- Equations give, among others, maximum energy transferred:

$$E_{max} = \frac{1}{2}m_2v_{2,max}^2 = 4\frac{m_1m_2}{(m_1+m_2)^2}T_0$$

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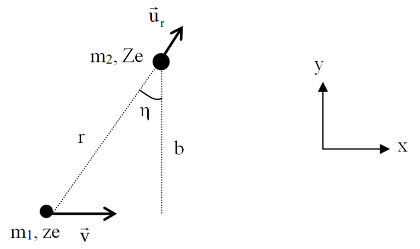
Elastic collision 3

a) $m_1 >> m_2$	b) $m_1 = m_2$	c) $m_1 << m_2$
$0 \le \chi \le \pi/2$ $0 \le \theta \le \tan^{-1}(\frac{m_2}{m_1}\sin 2\chi)$ $E_{max} = 4\frac{m_2}{m_1}T_0$	$0 \le \chi \le \pi/2$ $0 \le \theta \le \pi/2$ $E_{max} = T_0$	$0 \le \chi \le \pi/2$ $0 \le \theta \le \pi$ $E_{max} = 4\frac{m_1}{m_2}T_0$

- Proton-electron collision:
  $\theta_{max} = 0.03°$, $E_{max} = 0.2$ %
- Electron-electron (or e.g. proton-proton) coll.:
  $\theta_{max} = 90°$, $E_{max} = 100$ %

Elastic collision – cross section 1



Force exerted on particle 2:

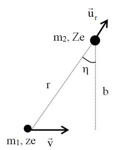
$$\vec{F} = \frac{zZe^2}{4\pi\varepsilon_0 r^2}\vec{u}_r$$

$$F_x = F\sin\eta \quad , \quad F_y = F\cos\eta$$

Elastic collision – cross section 2

Momentum of particle 2: $d\vec{p}_{tr} = \vec{F}dt$

$$\frac{dx}{dt} = v \quad , \quad \tan\eta = \frac{x}{b}$$



$$\Rightarrow \quad \frac{d}{d\eta}\tan\eta = \frac{1}{\cos^2\eta} = \frac{dx}{bd\eta} \quad \Rightarrow \quad dt = \frac{bd\eta}{v\cos^2\eta}$$

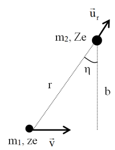
Total momentum transfer in interaction:

$$\vec{p}_{tr} = \int_{-\pi/2}^{\pi/2} F\cos\eta \frac{bd\eta}{v\cos^2\eta}\vec{j} = \frac{zZe^2 b}{4\pi\varepsilon_0 v}\int_{-\pi/2}^{\pi/2}\frac{d\eta}{r^2\cos\eta}\vec{j} \quad , \quad r = \frac{b}{\cos\eta}$$

$$\Rightarrow \quad \vec{p}_{tr} = \frac{zZe^2}{4\pi\varepsilon_0 bv}\int_{-\pi/2}^{\pi/2}\cos\eta d\eta\vec{j} = \frac{2zZe^2}{4\pi\varepsilon_0 bv}\vec{j}$$

Elastic collision – cross section 3

Energy transfer: $E = \frac{p_{tr}^2}{2m_2} = \frac{2}{m_2}\left(\frac{zZe^2}{4\pi\varepsilon_0 bv}\right)^2$



Cross section: σ=πb² → dσ=2πbdb. Thus:

$$b^2 = \frac{2}{m_2}\left(\frac{zZe^2}{4\pi\varepsilon_0 v}\right)^2\frac{1}{E} \quad \Rightarrow \quad |2\pi bdb| = d\sigma = \frac{2\pi}{m_2}\left(\frac{zZe^2}{4\pi\varepsilon_0 v}\right)^2\frac{1}{E^2}dE$$

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}$$

$$\Rightarrow \quad \frac{d\sigma}{dE} = \frac{2\pi r_e^2 (zZ)^2 (m_e c^2)^2}{m_2 v^2}\frac{1}{E^2} = \underline{\underline{2\frac{m_e}{m_2}(zZ)^2\frac{\pi r_e^2 m_e c^2}{\beta^2}\frac{1}{E^2}}}$$

Elastic collision – cross section 4

- Consider z=1 og $m_1=m_e<<m_2$

$$m_1 << m_2 \quad \Rightarrow$$

$$E = \frac{1}{2}m_2 v_2^{\,2} \approx \frac{1}{2}m_2\left(\frac{2m_1 v\cos\chi}{m_2}\right)^2 = 2\frac{m_1^{\,2}}{m_2}v^2\cos^2\chi$$

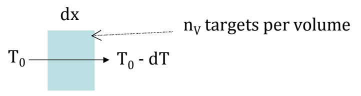
$$\tan\theta \approx -\frac{\sin 2\chi}{\cos 2\chi} = -\tan 2\chi \quad \Rightarrow \quad \chi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{d\sigma}{d\theta} = \frac{d\sigma}{dE}\frac{dE}{d\theta}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{2\pi\sin\theta}\frac{d\sigma}{d\theta} = \underline{\underline{\frac{Z^2}{4}\frac{r_e^{\,2}m_e c^2}{\beta^2}\frac{1}{\sin^4(\theta/2)}}}$$

Stopping power

- S=dT/dx; expected energy loss per unit lenght



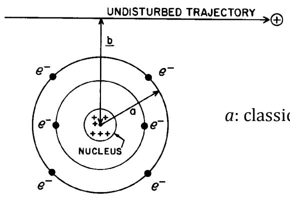
$$dT = \langle E n_V dx\sigma\rangle = n_V dx\int_{E_{min}}^{E_{max}}\frac{d\sigma}{dE}EdE = \rho\left(\frac{N_A Z}{A}\right)dx\int_{E_{min}}^{E_{max}}\frac{d\sigma}{dE}EdE$$

$$\left(\frac{dT}{\rho dx}\right) = \frac{S}{\rho} = \left(\frac{N_A Z}{A}\right)\int_{E_{min}}^{E_{max}}\frac{d\sigma}{dE}EdE$$

- S=dT/ρdx : *mass* stopping power

Impact parameter

- Charged particles: Coulomb interactions
- Most important: interactions with electrons
- Impact parameter *b*:



a: classical atomic radius

Soft collisions 1

- *b* >> *a* : incoming particle passes atom at long distance
- Weak forces, small energy transfers to the atom
- Inelastic collisions: Predominantly excitations, some ionizations
- Energy transfer range from "*Emin*" to "*H*"
- Hans Bethe: Quantum mechanical considerations
- Theory for heavy charged particles in the following

Soft collisions 2

$$\frac{S_{c,soft}}{\rho} = \left(\frac{dT_{soft}}{\rho dx}\right)_c = \frac{N_A Z}{A}\frac{2\pi r_0^2 m_e c^2 z^2}{\beta^2}\ln\left[\frac{2m_e c^2\beta^2 H}{I^2(1-\beta^2)} - \beta^2\right]$$

- r_0: classical electron radius = $e^2/4\pi\varepsilon_0 m_e c^2$
- I: mean excitation potential
- $\beta = v/c$
- z: charge of incoming particle
- ρ: density of medium
- $N_A Z/A$: numbers of electron per gram
- H: maksimum energy transferred by soft collisions

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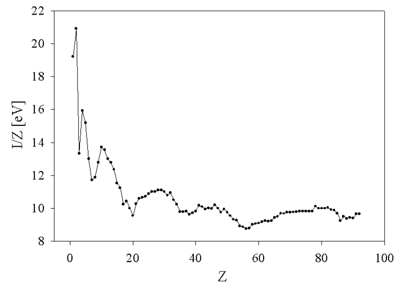
Soft collisions 2

- Quantum mechanics (atomic structure) is reflected in the mean excitation potential



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Hard collisions 1

- $b \sim a$: charged particle pass 'through' atom
- Large (but few) energy transfers
- Energy transfers from H to E_{max}
- May be considered as an elastic collision between free particles (binding energy is negligible)

$$\frac{S_{c,hard}}{\rho} = \left(\frac{dT}{\rho dx}\right)_{hard} = \frac{N_A Z}{A}\frac{2\pi r_0^2 m_e c^2 z^2}{\beta^2}\left[ln\left(\frac{E_{max}}{H}\right) - \beta^2\right]$$

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Collision stopping power

- For inelastic collisions, the total cross section is thus:

$$\frac{S_c}{\rho} = \frac{S_{c,soft}}{\rho} + \frac{S_{c,hard}}{\rho}$$

$$= 4\pi r_0^2 m_e c^2\left(\frac{N_A Z}{A}\right)\left(\frac{z}{\beta}\right)^2\left[\ln\left(\frac{2m_e c^2\beta^2}{(1-\beta^2)I}\right) - \beta^2\right]$$

- Important: Increases with z^2, decreases with v^2 and I, not dependent on particle mass

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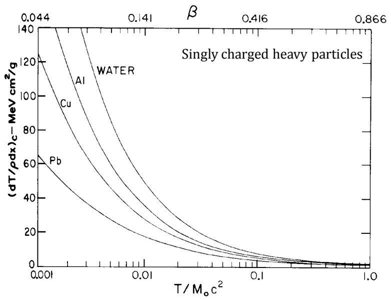
S_c/ρ, different substances



- *I* and electron density *(ZN_A/A)* give differences

S_c/ρ, electrons and positrons

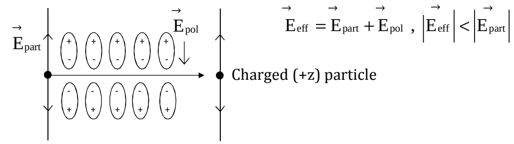
- Electron-electron scattering is more complicated; scattering between two identical particles
- $S_{c,\,hard}/\rho$ (el-el) is described by the Möller cross section
- $S_{c,\,hard}/\rho$ (pos-el) is described by the Bhabha c.s.
- $S_{c,\,soft}/\rho$ was given by Bethe, as for heavy particles
- Characteristics similar to that for heavy charged particles

Shell correction

- Derivation of S_c assumes $v >> v_{\text{atomic electrons}}$
- When $v \sim v_{\text{atomic electrons}}$, no ionizations
- Most important for K-shell electrons
- Shell correction C/Z takes this into accout, and thus *reduces* S_c/ρ
- C/Z depends on particle energy and medium

Density correction

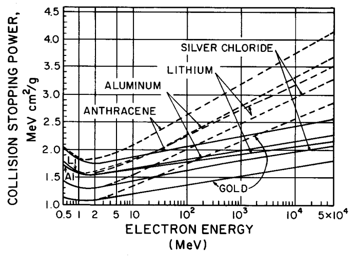
- Charged particles polarizes medium which is being traversed



- Weaker interactions with remote atoms due to reduction in electromagnetic field strenght
- Polarization increases with energy and density
- Most important for electrons and positrons

Density correction

- Density correction δ *reduces* S_c/ρ for liquids and solids
- S_c/ρ (water vapor) > S_c/ρ (water)



Dashed line: S_c/ρ without δ

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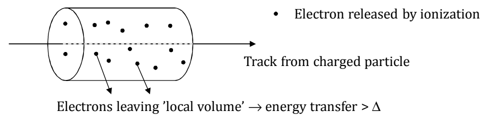
Linear Energy Transfer 1

- LET_Δ is denoted the *restricted stopping power*
- dT/dx: mean energy loss per unit lenght – but how much is deposited 'locally'?



- S_c: energy transfers from E_{min} to E_{max}
- How much energy per unit lenght is deposited within the range of an electron given energy Δ?

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Linear Energy Transfer 2

- Energy loss (soft + hard) per unit lenght for $E_{min} < E < \Delta$:

$$L_\Delta = \left(\frac{dT}{dx}\right)_\Delta = \rho\left(\frac{N_A Z}{A}\right)\int_{E_{min}}^{\Delta}\frac{d\sigma}{dE}EdE$$

$$= \rho 2\pi r_0^2 m_e c^2\left(\frac{N_A Z}{A}\right)\left(\frac{z}{\beta}\right)^2\left[\ln\left(\frac{2m_e c^2\beta^2\Delta}{(1-\beta^2)I}\right) - 2\beta^2\right]$$

- For $\Delta = E_{max}$, we have $L_\infty = S_c$; *unrestricted* LET
- LET_Δ is often given in [keV/µm]
- 30 MeV protons in water: $LET_{100\ eV} / L_\infty = 0.53$

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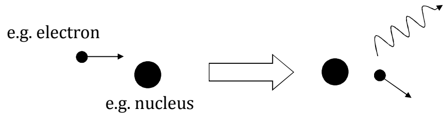
Brehmsstrahlung 1

- Photon may be emitted from charged particle accelerated in the field from an electron or nucleus



- Larmor's formula (classical electromagnetism) for radiated effect from accelerated charged particle:

$$P = \frac{(ze)^2 a^2}{6\pi\varepsilon_0 c^3}$$

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Brehmsstrahlung 2

- For particle accelerated in nuclear field:

$$F = ma = \frac{zZe^2}{4\pi\varepsilon_0 r^2} \quad \Rightarrow \quad a = \frac{zZe^2}{4\pi\varepsilon_0 m r^2}$$

$$\Rightarrow P \propto \left(\frac{Z}{m}\right)^2$$

- Comparison of protons and electrons:

$$\frac{P_p}{P_e} = \left(\frac{m_e}{m_p}\right)^2 \approx \frac{1}{1836^2}$$

- Brehmsstrahlung not important for heavy charged particles

Brehmsstrahlung 3

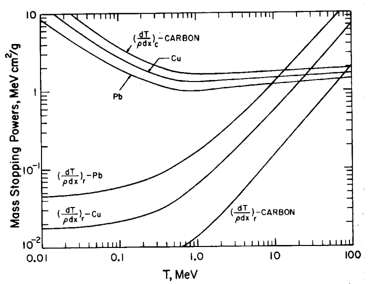
- Energy loss by brehmsstrahlung is called *radiative loss*
- Maksimum energy loss is the total kinetic energy *T*
- Radiative loss per unit lenght: *radiative stopping power*:

$$\left(\frac{S}{\rho}\right)_r = \left(\frac{dT}{\rho dx}\right)_r \approx \alpha r_0^2 \frac{N_A Z^2}{A}(T + m_e c^2)\overline{B}_r(T, Z)$$

- $\overline{B}_r(T, Z)$ weakly dependent on T and Z
- Brehmsstrahlung increases with energy and atomic number

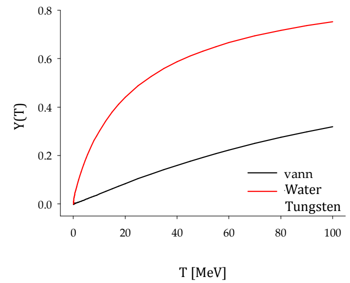
Total stopping power, electrons

$$\left(\frac{dT}{\rho dx}\right)_{tot} = \left(\frac{dT}{\rho dx}\right)_c + \left(\frac{dT}{\rho dx}\right)_r$$

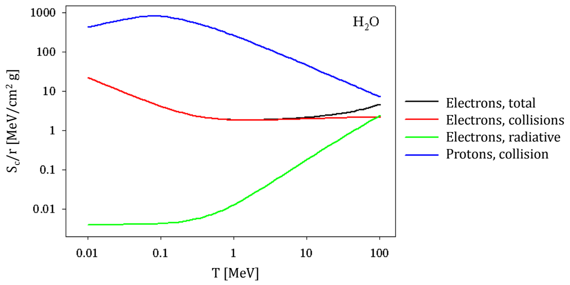


Radiation yield

$$Y(T) = \frac{(dT/\rho dx)_r}{(dT/\rho dx)_c + (dT/\rho dx)_r} = \frac{S_r}{S} \approx \frac{TZ}{n} \qquad \text{n=750 MeV}$$

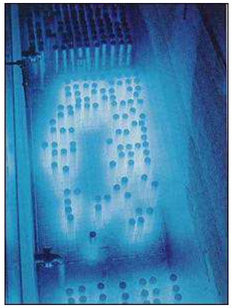


S/r, protons and electrons



Cerenkov effect

- High energy electrons (v > c/n) polarizes medium (e.g. water) and blueish light (+ UV) is emitted
- Low energy loss



Other interactions

- *Nuclear interactions*: Inelastic process where charged particle (e.g. proton) excites nucleus →
  - Scattering of charged particle
  - Emission of neutron, photon, or α-particle ($^4_2$He)
- Not important below ~10 MeV (protons)

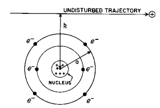
- *Positron annihilation*: Positron interacts with electron → a pair of photons with energy ≥ 2 x 0.511 MeV is created. Photons are emitted in opposite directions.
- Probability decreases as ~ 1/v

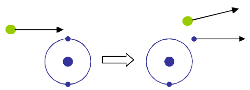
Charged particle interactions, summary

Collisions

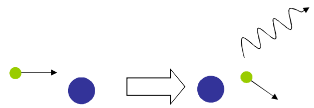
Soft

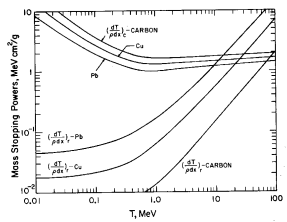
Hard





Radiative loss



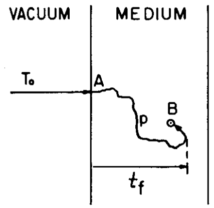


Range 1

- *The range* $\mathfrak{R}$ of a charged particle in matter is the (expectation value) of it's total pathlenght *p*
- *The projected range <t>* er is the (expectation value) of the largest depth t_f a charged particle can reach along it's incident direction
- Electrons:

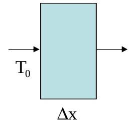
  $\langle t \rangle < \mathfrak{R}$
- Heavy charged particles:

  $\langle t \rangle \approx \mathfrak{R}$



CSDA-range

- The range may be approximated by $\mathfrak{R}_{CSDA}$ (continuous slowing down approximation)
- Energy loss per unit lenght dT/dx – gives implicitly a measure of the range:



$$T_0 - \Delta T = T_0 - \frac{dT}{dx}\Delta x$$

$$\Delta x = \frac{dx}{dT}\Delta T \ , \quad \Rightarrow \mathfrak{R} = \sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} \left(\frac{dx}{dT}\right)_i \Delta T$$

$$\Rightarrow \mathfrak{R}_{CSDA} = \int_0^{T_0} \left(\frac{dT}{dx}\right)^{-1} dT$$

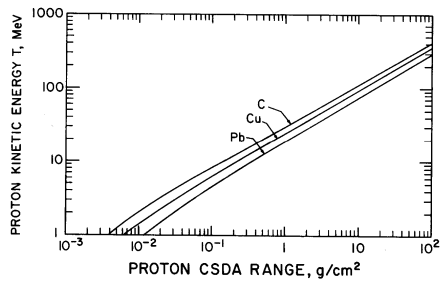
Range 3

- The range is often given multiplied by the density:

$$\mathfrak{R}_{CSDA} = \int_0^{T_0} \left(\frac{dT}{\rho dx}\right)^{-1} dT$$

- Unit thus becomes [cm] [g/cm^3] = [g/cm^2]
- Range of charged particle depends on:
  - Charge and kinetic energy
  - Density, electron density and mean excitation potential of absorber

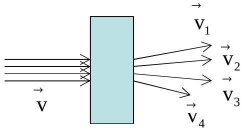
Range 4



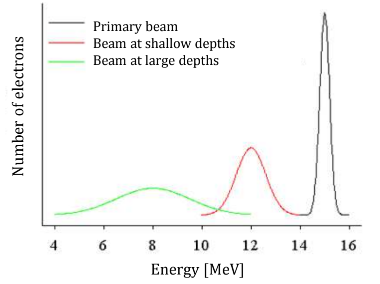
Multiple scattering and straggling

- In a beam of charged particles, one has:
  - Variations in energy deposition (straggling)
  - Variations in angular scattering

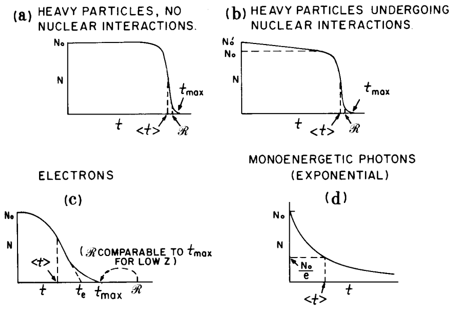
→ The beam, where all particles originally had the same velocity, will be smeared out as the particles traverses matter



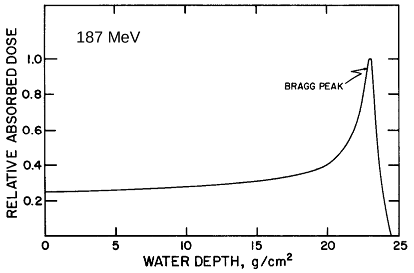
Straggling



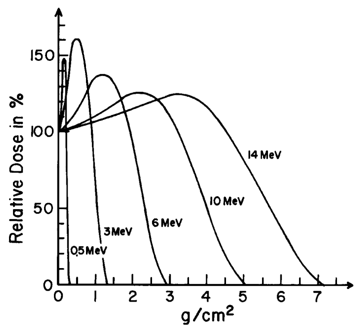
Range issues



Energy deposition, protons

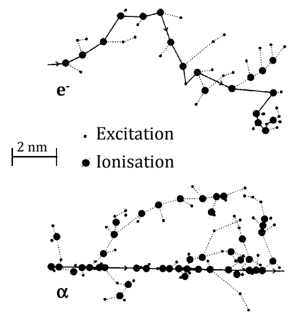


Energy deposition, electrons



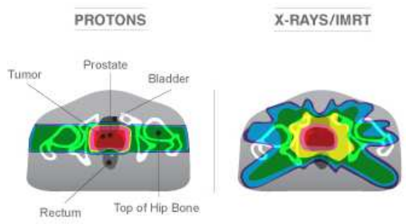
Monte Carlo simulations

- Monte Carlo simulations of the track of an electron (0.5 keV) and an α-particle (4 MeV) in water
- Note:
  e^- is most scattered
  α has the highest dT/dx



Hadron therapy

- Heavy charged particles may be used for radiation therapy – conforms better to the target than photons or electrons



Web pages

- For stopping powers:

http://www.nist.gov/pml/data/star

- For attenuation coefficients:

http://www.nist.gov/pml/data/xraycoef