FYS3520 - Problem set 3

Spring term 2017

v2:

1. Corrected/clarified Problem 3 d $\rightarrow$ now: d and f
2. Hopefully improved labeling in Problem 6

v3:

1. Homogeneous charge distribution is now homogeneous ($1/r^3 \rightarrow 1/R^3$)

Problem 1 – in class

When we describe nuclear γ-ray resonances, we usually give the energy E_γ of an emitted photon is the difference E_0 between the excited state with energy E_x and the ground state (GS), $E_\gamma = E_0 = E_\mathrm{x} - E_\mathrm{GS}$. This is not exact for free atoms or molecules, as it neglects the recoil energy of the nucleus.

a) Suppose that the nucleus was at rest before γ-ray emission and calculate the exact gamma-ray energy. Can we neglect the recoil effect?

b) ^{60}Co is one of the most important γ-ray calibration sources. By β-decay it feeds excited levels in ^{60}Ni. The 1332 keV level (with direct decay into ground state) has a half-life $t_{1/2}$ below 1ps ($< 10^{-12}$ s). What does this imply for the energy of the γ-ray emitted in the emission? (Remember to convert half-life to lifetime)

c) What are the implication for the nuclear resonance absorption of γ-ray photons? Argue qualitatively how these results would be effected if the nucleus was not at rest during decay.

d) (At home?) What is the Mössbauer effect and how does this combine with these results?

Problem 2 – in class

a) A neutron and a proton undergo radioactive capture at rest: $p + n \rightarrow d + \gamma$. Find the energy of the photon emitted in this capture with and without recoil of the deutron. How does the recoil affect the energy of the photon? Use the following masses in this problem: $m_p = 938.280$ MeV, $m_n = 939.573$ MeV and $m_d = 1875.628$ MeV.

b) A deutron with binding energy B is disintigrated into a neutron and a proton by a gamma ray of energy E_γ. Find the minimum value of $(E_\gamma - B)$ for which the reaction can occur.
Hint: Don't forget to conserve total energy and momentum.

Problem 3

In Figure 1 the differential cross-sections $\mathrm{d}\sigma/\mathrm{d}\Omega$ for scattering of high energy electrons on ^{40}Ca and ^{48}Ca are displayed.

a) Explain the general behavior of the cross-section as a function of the scattering angle. What is the source for the (local)minima?

b) The form factor $F(\boldsymbol{q}^2)$ is defined as the Fourier transformation of the charge density $\rho(\boldsymbol{r})$, with the momentum transfer $\boldsymbol{q} = \boldsymbol{k}_\mathrm{f} - \boldsymbol{k}_\mathrm{i}$ and the initial and final momentum $\boldsymbol{k}_\mathrm{i,f}$. Under the usual assumption of the Born approximation and negligible recoil, $F(\boldsymbol{q}^2)$ can be calculated by

$$F(\boldsymbol{q}^2) = \int e^{i\boldsymbol{q}\boldsymbol{r}/\hbar}\rho(\boldsymbol{r})\mathrm{d}^3\boldsymbol{r}. \tag{1}$$

Calculate $F(\boldsymbol{q}^2)$ for a *homogeneous* spherical charge distribution, $\rho(r) = 3Ze/(4\pi R^3)$ for $r < R$, and 0 elsewhere. Show that the result is given by

$$F(\boldsymbol{q}^2) = 3(\frac{\hbar}{qR})^3[(\frac{qR}{\hbar})\cos(\frac{qR}{\hbar}) - \sin(\frac{qR}{\hbar})] \tag{2}$$

c) Plot the result. (It is important to choose a reasonable scaling for q. Recall that $|\boldsymbol{q}| = 2|\boldsymbol{p}|\sin(\frac{\theta}{2})$)

d) Use this and Figure 1 to compare the radius of ^{40}Ca and ^{48}Ca. Hint: How does the location of the minima depend on R for given angle.

e) Some extra calculations ("bonus"): Develop eq.(1) for a general spherical potential in powers of $|\boldsymbol{q}|$ (first 2 non-vanishing terms) using the mean square radius $\langle r^2 \rangle$. This will lead you to

$$F(\boldsymbol{q}) = 1 - \frac{1}{6}\frac{\boldsymbol{q}\,\langle r^2\rangle}{\hbar^2} + \ldots. \tag{3}$$

How can you solve the expression for the mean square charge radius $\langle r^2 \rangle$. What momentum transfers are most important in order to measure $\langle r^2 \rangle$? What does this mean for the experiment (for example, would you measure at certain angles, rather high or low energies,...?) Hint: a) The mean square charge radius is defined as $\langle r^2 \rangle = \int r^2 \rho(r)\, d^3r$. b) Remember that the form factor is normed such that $F(\boldsymbol{q}^2 = 0) = 0$.

f) *For further thought ;P*: Experimentally only a restricted range of momentum transfers is accessible, as it is limited by the beam intensity. In addition, the cross-section drops quickly with increasing momentum transfers. Think about a method to determine the charge distribution despite these problems.

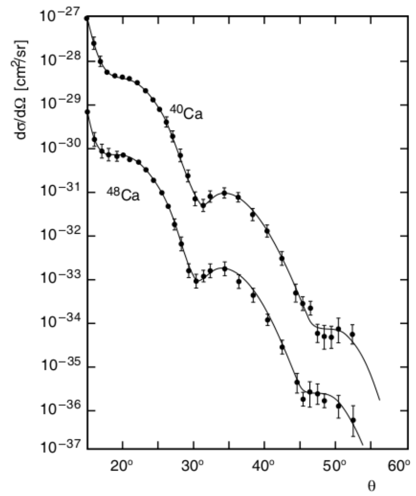

Figure 1: Differential cross-section for scattering of 750 MeV electrons on ^{40}Ca and ^{48}Ca. The cross-sections have been multiplied by 10 for ^{40}Ca and by 10^{-1} for ^{48}Ca for displaying purposes. Source: J. B. Bellicard, et al. Phys. Rev. Lett. **19** (1967) 527

Problem 4

A proton and a neutron can form a bound state but two protons or two neutrons cannot. What does this tell about the nuclear force?

Problem 5

a) What is the difference between bosons and fermions? Are nucleons bosons or fermions? What are the condition on the total wavefunction under exchange of particles?

b) In a potential $V(|\vec{r}_1 - \vec{r}_2|)$ that only depends on the distance between two particles it is often useful to rewrite the Shrödinger equation in terms of $\vec{r} = \vec{r}_1 - \vec{r}_2$. Explain the connection between the exchange operator and the parity operator for that case.

c) An example for the above potential is the infinite spherical well. If two identical fermions in such a potential form a bond with $l = 1$, what sort of combinations can the spins of the two fermions be?

Problem 6   Special hand-in for this part: In 2 weeks

In order to get the correct magic numbers in the nuclear shell model we need to include a spin-orbit coupling. This leads to a Hamiltonian similar to:

$$\hat{H} = \hat{H}_0 - V_{\text{SO}}\hat{\vec{L}} \cdot \hat{\vec{S}} \tag{4}$$

Where $\hat{H}_0$ is a Hamiltonian with eigenstates $\hat{H}_0 \left|N, l\right\rangle = \hbar\omega(N + 3/2) \left|N, l\right\rangle$ with $l = N, N - 2, N - 4, \ldots 1$ or $0, l \geq 0$ and $V_{\text{SO}}\hat{\vec{L}} \cdot \hat{\vec{S}}$ is the spin-orbit coupling. V_{SO} is the strength of the coupling and can be regarded as a constant in this problem. In this problem we only look at spin-1/2 fermions.

a) The operator for the total spin is $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$. Where $\hat{\vec{L}}$ is the angular momentum operator and $\hat{\vec{S}}$ is the spin operator. Show that:

$$\hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{1}{2}(\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \hat{\vec{S}}^2)$$

b) For a given angular momentum l the total spin state $\left|j, m_j\right\rangle$ can be $j = |l - 1/2|, l + 1/2$. The two possible states can be written as a superposition of $\left|l, m_l = m_j \pm 1/2\right\rangle \otimes \left|s = 1/2, m_s \mp 1/2\right\rangle$ states:

$$\begin{aligned}\left|j = l \pm 1/2, m_j\right\rangle = & C^l_{1/2} \left|l, m_j - 1/2\right\rangle \otimes \left|\uparrow\right\rangle \\ & + C^l_{-1/2} \left|l, m_j + 1/2\right\rangle \otimes \left|\downarrow\right\rangle,\end{aligned}$$

where the Clebsch-Gordan coefficients $C^l_{m_s} = 0$ if $|m_j - m_s| > l$ or $m_j + m_s > l$; and $\left|\uparrow\right\rangle = \left|1/2, 1/2\right\rangle$, $\left|\downarrow\right\rangle = \left|1/2, -1/2\right\rangle$. Find the energy of the state $\left|N, l, j = l \pm 1/2, m_j\right\rangle$ using (4).