FYS3520 - Problem set 4

Spring term 2017

v1 - corrections:

- Problem 3: improved notation on normalization coefficients and corrected which one should be 0!
- Sign in the last Lorentzian $-(\Gamma/2)^2 \to +(\Gamma/2)^2$

In this problem set we will run through the complete story of how to build up the deuteron, leading to two-state mixing in order to reproduce the magnetic dipole moment. This is a a quite powerful demonstration of the reality and complexity of the nuclear system and that it opens the door to many other discussions on nuclear forces and ground-state properties.

Additionally we will make the derive the relation between lifetime τ and line-width Γ of a state. **In class** we will discuss the strategies to solve different steps and try to relate observables actual measurements. It is recommendable to have read through Krane, ch. 4.1, and 4.4. Additionally you may briefly want to refresh your knowledge of the hyperfine-structure.

I can also strongly recommend B. Povh, *Particles and Nuclei* as an **additional resource** for the course. It is freely available online at UiO.

Problem 1 – the basics

a) What values could you expect for the ground state total angular momentum I of the deuteron? What is the experimentally observed total angular momentum I and parity π of the deuteron? What do we learn about L and S?

b) Name at least two ways to experimentally determine the total angular momenta.

c) What is the g factor related to the magnetic moment $\boldsymbol{\mu} = \mu_{\mathrm{N}} g \boldsymbol{s}/\hbar$ of a proton (neutron)? Why does the neutron have a non-zero g factor?

d) What is the magnetic moment of the deuteron? What is the implication for the ground state of the deuteron?

e) Sketch of the radial dependence of the nucleon-nucleon potential with $l = 0$ without regard to spin and isospin dependence.

f) List and briefly explain the most important contributions to the nuclear potential $V(\boldsymbol{r})$.

Problem 2 The deuteron

The deuteron wave function can be solved by assuming that the nuclear potential is a three-dimensional square well:

$$V(r) = \begin{cases} -V_0, & r < R \\ 0, & r > R \end{cases}$$

Where r is the separation of the proton and the neutron and R is the *radius* of the deuteron well. The radial part of the wave-function $\psi(\vec{r})$ can be defined as $u(r)/r$, resulting in following radial equation:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u(r) = Eu(r)$$

Where $m = \frac{m_p m_n}{m_p + m_n}$ is the reduced mass. In the following problems, assume $l = 0$ and $E = -E_{\mathrm{B}} < 0$.

a) Convince yourself that the solution for the radial equations is given by:

$$u_1(r) = A\sin(k_1 r), \quad \text{for } r < R$$
$$u_2(r) = Be^{-k_2 r}, \quad \text{for } r > R$$

where $k_1 = \frac{\sqrt{2m(E+V_0)}}{\hbar}$, and $k_2 = \frac{\sqrt{-2mE}}{\hbar}$.

b) The wave function and its derivative is required to be continuous in $r = R$. Use the conditions:

$$u_1(R) = u_2(R)$$
$$\left.\frac{du_1}{dr}\right|_R = \left.\frac{du_2}{dr}\right|_R$$

To show that:

$$\frac{k_1}{k_2} = -\tan(k_1 R) = -\frac{1}{\cot(k_1 R)} \tag{1}$$

c) On the previous sheet you have shown that the binding energy of the deuteron is $E \approx -2.225$ MeV. From elastic scattering the rms charge radius is known to be $\sqrt{\langle r^2 \rangle} \approx 2.1$ fm. Show that the depth V_0 of the nuclear potential is ≈ 35 MeV.
Hint: Equation (1) is transcendental and cannot be solved analytically. Write a script or draw a graph to find the value V_0 where the RHS and the LHS of (1) is equal.

d) Use Fig. 4.2 in Krane to approximate the probability to find the nucleons outside of the potential well? The analytical calculation is possible, but quite heavy and not necessarily much more insightful.

e) Show that a spin dependent term of the form $\boldsymbol{s}_1 \cdot \boldsymbol{s}_2$ gives different energies for singlet and triplet states. Which one will be energetically favored for the GS?

f) The g factor of the deuteron is slightly lower then sum of the g factors of proton and neutron. Follow the calculations of the mixing ratio of the different l states.

g) Why can the nuclear potential not include terms like $\boldsymbol{s}_1 + \boldsymbol{s}_2$?

h) Why does one need the tensor potential? Try to understand why the form of the potential favors the $l = 2$ state over the $l = 0$ state. How does this lead to mixing?

$$V_{12} = 3(\boldsymbol{s}_1 \cdot \boldsymbol{r})(\boldsymbol{s}_2 \cdot \boldsymbol{r})/r^2 - \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$$

Problem 3 Lifetime and natural line widths

Proof that the relationship between the lifetime τ of a decaying state and its (natural) line width Γ is given by $\Gamma\tau = \hbar$.

In first approximation we may again assume the nucleus to behave like quasi stationary states in a finite potential well. Note that this time we consider quasi-bound states where $V_0 \gg E > 0$, giving rise to particles that can penetrate outside the well. The solutions for $r > R$ are then given by:

$$u_2(r) = \mathcal{A}e^{ik_2' r} + \mathcal{B}e^{-ik_2' r},$$

where $k_2' = ik_2 = \frac{\sqrt{2mE}}{\hbar}$.

a) We obtain a quasi-stationary state if we postulate that for $r > R$ the solution consists of outgoing waves only. Show that this is equivalent to the condition $\mathcal{B} = 0$.

b) Show that the continuity condition leads to the equation

$$f(E) := k_1 R \cot(k_1 R) = i k_2' R, \tag{2}$$

c) This equation is quite alike Eq. (1), however now involves an imaginary magnitude. We can solve the problem by a linear approximation method. If we first impose a simpler condition where $f(E) = 0$, we find the solutions $E_s \in \mathbb{R}^{\geq 0}$. If we call W_s the solutions for which Eq. (2) is fulfilled, we use a Taylor expansion around E_s:

$$f(W_s) := \left(\frac{df}{dE}\right)\bigg|_{E_s} (W_s - E_s) + \dots . \tag{3}$$

Show that $W_s = E_s - i\Gamma/2$, where $\Gamma := -\frac{2k_2' R}{(df/dE))|_{E_s}}$

d) This tells us that W_s differs from E_s in first order only by addition of an imaginary part to the energy. However, what is the physical meaning of a complex energy? Hint: The wave function is given by $\psi(t) = \psi(t = 0)\exp(-iW_s t/\hbar)$ ($\rightarrow$why?). Calculate the time dependence of the probability density; remember that W_s has a real and an imaginary part.

e) We pave now prepared the way to show the dependence between Γ and τ. The time and energy (more exactly frequency) domain are just as space and momentum are related by a Fourier transformation. Show that

$$\psi(E) \propto \frac{1}{i(E - E_s) - \Gamma/2},$$

and that the energy distribution is given by a Lorentzian function:

$$|\psi(E)|^2 \propto \frac{1}{(E - E_s)^2 + (\Gamma/2)^2}.$$

f) Plot the resonance for the γ line of the ^{60}Co source (which is the ^{60}Ni nucleus) given on the last problem set: $E_s = 1333 keV$, $t_{1/2} = 0.7$ ps. You may verify that Γ is the half-width of the distribution.

You may read more on that subject, especially also on the generalization to an actual nucleus in one of the classical monographs in the field, Blatt and Weisskopf, *Theoretical Nuclear Physics* (1952). p. 412ff.