FYS 3610

Solution Week 35

a) The zeroth order gives the particle density n. Calculate the zeroth moment:

$$\iiint_{-\infty}^{\infty} f(v_x, v_y, v_z) dv_x dv_y dv_z = n = A \iiint_{-\infty}^{\infty} \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T}\right) dv_x dv_y dv_z$$

Substitute $x = \sqrt{m/2k_B T} v_x$, then $dx = \sqrt{m/2k_B T} dv_x$ and equivalent for v_y and v_z

$$n = A \sqrt{\frac{2k_B T}{m}} \sqrt{\frac{2k_B T}{m}} \sqrt{\frac{2k_B T}{m}} \iiint_{-\infty}^{\infty} \exp(-x^2 - y^2 - z^2)\, dx dy dz = A \left(\sqrt{\frac{2\pi k_B T}{m}}\right)^3$$

And therefore $A = n \left(\frac{m}{2\pi k_B T}\right)^{3/2}$.

b) Calculate first moment in x direction:

$$v_{b,x} = \frac{1}{n} \iiint_{-\infty}^{\infty} v_x f(v_x, v_y, v_z) dv_x dv_y dv_z = \frac{1}{n} \iiint_{-\infty}^{\infty} v_x \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T}\right) dv_x dv_y dv_z$$

Substitute $x = \sqrt{m/2k_B T} v_x$, then $dx = \sqrt{m/2k_B T} dv_x$ and equivalent for v_y and v_z

$$v_{b,x} = \frac{1}{n} \int_{-\infty}^{\infty} x \exp(-x^2)\, dx \int_{-\infty}^{\infty} \exp(-y^2)\, dy \int_{-\infty}^{\infty} \exp(-z^2)\, dz$$

Because of symmetry:

$$\int_{-\infty}^{\infty} x \exp(-x^2)\, dx = \int_{-\infty}^{0} x \exp(-x^2)\, dx + \int_{0}^{\infty} x \exp(-x^2)\, dx$$

$$= -\int_{0}^{\infty} x \exp(-x^2)\, dx + \int_{0}^{\infty} x \exp(-x^2)\, dx = 0$$

Therefore $v_{b,x} = 0$ and by symmetry also $v_{b,y}$ and $v_{b,z}$.

a) Induction equation looks like:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B}\right) + \eta \nabla^2 \vec{B}$$

Assume $B = B_z(x,t)$ and $v = v_x(x,t)$:

$$\frac{\partial B_z}{\partial t} = -\frac{\partial}{\partial x}(v_x B_z) + \frac{\partial^2}{\partial x^2} B_z$$

b) With

$$B_z(x, t = 0) = f(x) = A_0 \exp\left(-x^2/_{L^2}\right)$$

and

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-\lambda) \exp\left(-\frac{\lambda^2}{4\eta^2 t}\right) d\lambda$$

we get

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int_{-\infty}^{\infty} A_0 \exp\left(-\frac{(x-\lambda)^2}{L^2}\right) \exp\left(-\frac{\lambda^2}{4\eta^2 t}\right) d\lambda$$

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int_{-\infty}^{\infty} A_0 \exp\left(-\left[\left(\frac{1}{L^2} + \frac{1}{4\eta^2 t}\right)\lambda^2 - \frac{2x}{L^2}\lambda + \frac{x^2}{L^2}\right]\right) d\lambda$$

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int_{-\infty}^{\infty} A_0 \exp(-[r^2(\lambda - s)^2 + q]) d\lambda$$

$$B_z(x,t) = \frac{A_0}{2\eta\sqrt{\pi t}} \exp(-q) \int_{-\infty}^{\infty} \exp(-\alpha\tau^2) d\tau$$

with

$$\tau = \lambda - s$$

$$\alpha = r^2 = \frac{1}{L^2} + \frac{1}{4\eta^2 t}$$

$$2r^2 s = \frac{2x}{L^2}$$

$$s = \frac{x}{L^2\left(\frac{1}{L^2} + \frac{1}{4\eta^2 t}\right)}$$

$$r^2 s^2 + q = \frac{x^2}{L^2}$$

$$q = \frac{x^2}{L^2} - \left(\frac{1}{L^2} + \frac{1}{4\eta^2 t}\right)\left(\frac{x}{L^2\left(\frac{1}{L^2} + \frac{1}{4\eta^2 t}\right)}\right)^2 = \frac{x^2}{L^2}\left(1 - \frac{1}{L^2\left(\frac{1}{L^2} + \frac{1}{4\eta^2 t}\right)}\right) = \frac{x^2}{4\eta^2 t + L^2}$$

Executing the integration and simplifying yields

$$B_z(x,t) = \frac{A_0}{2\eta\sqrt{\pi t}} \exp\left(-\frac{x^2}{4\eta^2 t + L^2}\right)\sqrt{\frac{\pi}{\frac{1}{L^2} + \frac{1}{4\eta^2 t}}}$$

$$B_z(x,t) = \frac{A_0 L}{\sqrt{4\eta^2 t + L^2}} \exp\left(-\frac{x^2}{4\eta^2 t + L^2}\right)$$

c) It describes a Gaussian distribution centered around $x = 0$. For $t \to \infty$ the amplitude of the distribution slowly decreases while the width increases (conservation of total B!). This is analog to the dispersion of a wave packet. Instead of being concentrated at one location, B disperses over the entire x axis – it diffuses! The resistivity η is the time constant of the diffusion process. If the conductivity is very large, the resistivity η becomes very small, such that the time dependence of $B_z(x,t)$ becomes very weak. Hence the diffusion process is very slow and the dispersion takes a very long time, the shape of the B-field distribution is conserved over long times (frozen-in theorem!).

d) Constant in time: $\frac{\partial B_z}{\partial t} = 0$. Do the derivation:

$$\left[-\frac{A_0 L \frac{4\eta^2}{\sqrt{4\eta^2 t + L^2}}}{4\eta^2 t + L^2} + \frac{A_0 L}{\sqrt{4\eta^2 t + L^2}}\left(\frac{4\eta^2 x^2}{(4\eta^2 t + L^2)^2}\right)\right] \exp\left(-\frac{x^2}{4\eta^2 t + L^2}\right) = 0$$

Because the exponential function never can become zero:

$$-\frac{A_0 L \frac{4\eta^2}{\sqrt{4\eta^2 t + L^2}}}{4\eta^2 t + L^2} + \frac{A_0 L}{\sqrt{4\eta^2 t + L^2}}\left(\frac{4\eta^2 x^2}{(4\eta^2 t + L^2)^2}\right) = 0$$

Or

$$\frac{4\eta^2 A_0 L}{(4\eta^2 t + L^2)^{3/2}}\left(1 - \frac{x^2}{4\eta^2 t + L^2}\right) = 0$$

And with the same argument

$$1 - \frac{x^2}{4\eta^2 t + L^2} \Leftrightarrow t = \frac{x^2 - L^2}{4\eta^2}$$

With this result of $B_z(x)$ becomes (do not resolve $\sqrt{x^2}$ because this insures that $B_z(x)$ is symmetric with respect to negative and positive x):

$$B_z(x) = \frac{A_0 L}{\sqrt{x^2}} e^{-1}$$

Back to the differential equation we get

$$0 = -\frac{\partial}{\partial x}(v_x B_z) + \frac{\partial^2}{\partial x^2} B_z$$

Integrate over x once

$$-v_x B_z + \frac{\partial}{\partial x} B_z = C$$

Differentiate solution of B_z once:

$$\frac{\partial}{\partial x} B_z = -\frac{A_0 L}{x^2} e^{-1} = -\frac{1}{x} B_z$$

Inserting and rearranging gives

$$v_x = -\frac{C + \frac{1}{x}B_z}{B_z} = -\frac{1}{x} - \frac{C}{A_0 L e^{-1}}\sqrt{x^2}$$

The integration constant C must be zero because for any $C \neq 0$ we see

$$\lim_{x\to\infty} v_x = \pm\infty$$

which is an unphysical solution. The interpretation is then that, while the B-field "wave packet" wants to diffuse toward larger x, a $-1/x$ velocity profile constantly pushing from left and right can hold the B-field profile together.