5 The solar wind

If we include the effects of gravity but neglect electromagnetic forces (i.e., the $\vec{j} \times \vec{B}$-term), then the motion of plasma is described by the following equation:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla p + \rho \vec{g}. \tag{5.1}$$

Of course, mass is conserved:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{5.2}$$

5.1 Hydrostatic solution

Consider a gas layer around a celestial object, like the Sun's or the Earth's atmosphere. For the gas itself we can assume the ideal gas law, i.e.,

$$p = nk_BT = \frac{\rho k_B T}{\langle m \rangle} \tag{5.3}$$

where $\langle m \rangle$ is the average mass per particle. Further assume that the system is in static (i.e., $\vec{v} = 0$) equilibrium (i.e., $\partial/\partial t = 0$). Then the continutiy equation is trivial and of no great help in learning somehting about the system. The EoM reduces to

$$\nabla p = \rho \vec{g} \tag{5.4}$$

which in sperical coordinates assuming spherical symmetry (all parameters are function of the radial coordinate r only) becomes

$$\frac{\partial p}{\partial r} = -\rho g(r) = -\frac{\langle m \rangle g(r)}{k_B T} p. \tag{5.5}$$

Already we can deduce a few things about the altitude dependence of the pressure. First, the pressure is neccessarily decreasing because neither ρ nor the acceleration due to gravity g can ever become negative. Furthermore we see that the rate of change of the pressure at any altitude r is dependent on the factor $\xi = \langle m \rangle g(r)/k_B T$; for a small value of ξ the change with altitude is slow, for a large value the pressure decreases rapidly.
At the surface of Earth ξ is about 7×10^{-2} per km which turns out to be a relatively large value, meaning that the pressure changes quickly with altitude. Therefore, we can use the thin atmosphere limit which states that the atmospheric pressure decreases rapidly with altitude, allowing us to assume that, over the thin atmosphere, the acceleration due to gravity is constant. Then we can separate the variables

$$\frac{\partial p}{p} = -\frac{\langle m \rangle g}{k_B T} \partial r. \tag{5.6}$$

and integrate from the surface of the Earth at R_s to any greater radial distance r and find the solution for the altitude dependence of the atmospheric pressure:

$$p(r) = p_s \exp\left\{-\frac{\langle m \rangle g}{k_B T}\left[r - R_s\right]\right\}. \tag{5.7}$$

We can introduce the factor $\hat{h} = 1/\xi$ and rewrite this in terms of the altitude above the surface $h = r - R_s$ to

$$p(r) = p_s \exp\left\{-\frac{h}{\hat{h}}\right\}. \tag{5.8}$$

The scale factor $\hat{h}$ is called the scale height and gives the altitude over which the pressure decreases by a factor of $e^{-1} \approx 0.36$ of its original value. For Earth, the scale height is of the order of 20 km, again a posteriori supporting our assumption of a thin atmosphere. Also not that in this solution the pressure goes toward zero as $h \to \infty$ which is physically sensible.
In the case of the Sun's upper atmosphere, the corona, the case is different. ξ is about 3-4 order of magnitude smaller than at Earth, indicating a much slower rate of decreasing atmospheric pressure with altitude. We can

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therefore not use the thin atmosphere limit but have to solve

$$\frac{\partial p}{\partial r} = -\frac{\langle m\rangle g(r)}{k_B T} p = -\frac{\langle m\rangle GM}{k_B T}\frac{p}{r^2} \tag{5.9}$$

where G is the gravitational constant and M is the solar mass. Again separating the variables and integrating from the solar surface we find:

$$p(r) = p_s \exp\left\{-\frac{\langle m\rangle GM}{k_B T}\left[\frac{1}{R_s} - \frac{1}{r}\right]\right\}. \tag{5.10}$$

In this solution, $p(r) \neq 0$ as $r \to \infty$, indicating a finite pressure at large distance from the Sun. But what should balance this pressure? And more importantly, this finite pressure $p_\infty \approx 1.7 \times 10^{-5}$ Pa whereas we measure about 10^{-13} Pa. The only solution to this problem is accepting that the upp solar atmosphere is not in static equilibrium.

5.2 Parker's solar wind model

E.N. Parker was the first to assume a non-static solar atmosphere but instead assume a radial velocity profile with $v = v(r) \neq 0$. The system was still assumed to be in equilibrium such that $\partial/\partial t = 0$. Then the continuity equation gives

$$\nabla \cdot (\rho\vec{v}) = 0 \tag{5.11}$$

or, in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho v(r) r^2) = 0 \tag{5.12}$$

and hence $\rho v(r) r^2$ representing the mass flux through a spherical surface is constant.
The EoM in spherical coordinates gives

$$\rho v\frac{\partial v}{\partial r} = -\frac{\partial p}{\partial r} + \rho g \tag{5.13}$$

$$v\frac{\partial v}{\partial r} = -\frac{k_B T}{\langle m\rangle}\frac{1}{\rho}\frac{\partial \rho}{\partial r} - \frac{GM}{r^2} \tag{5.14}$$

acknowledging that for an ideal gas $\partial p/\partial r = k_B T/\langle m\rangle \partial\rho/\partial r$. Further evaluating the continuity equation we get

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\left(\frac{2}{r} + \frac{1}{v}\frac{\partial v}{\partial r}\right) \tag{5.15}$$

which, substituted into eq. (5.14) gives

$$\frac{\partial v}{\partial r} = \frac{v}{r}\frac{\left(\dfrac{2k_B T}{\langle m\rangle} - \dfrac{GM}{r}\right)}{v^2 - \dfrac{k_B T}{\langle m\rangle}} \tag{5.16}$$

We can evaluate eq. (5.16) qualitatively to gain some insight into the possible solutions. The numerator on the RHS becomes zero when $r = r_c = GM\langle m\rangle/2k_B T$, at which point $\partial v/\partial r = 0$. If $r < r_c$, then the numerator is negative such that $\partial v/\partial r$ decreases. On the other hand, if $r > r_c$, then the numerator is positive such that $\partial v/\partial r$ increases.
Turning to the denominator we see that if the velocity approaches the speed of sound, i.e., $v = v_c = \sqrt{k_B T/\langle m\rangle} = c_s$, then the rate of change of v goes toward infinity. This divides the solutions of eq. (5.16) into four quadrants:

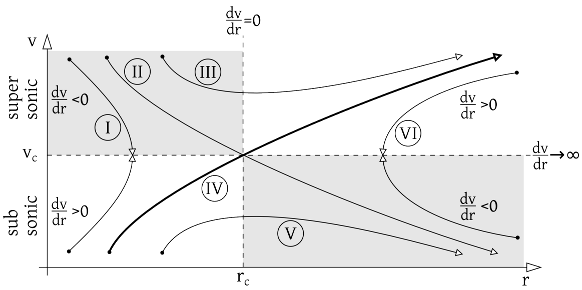

Figure 1: The possible solutions of Parker's solar wind model.

Clearly, families I and VI are unphysical. Before addressing which of the remaining family of solutions is the one that describes the actual solar wind best, let's establish that for a hydrogen plasma at 10^6 K the speed of sound is about $c_s \approx 180$ km/s and the critical radius is $r_c \approx 3R_s$. Earth is located at about 210 R_s and we measure solar wind speeds of about 400 km/s, i.e., it is supersonic. That only leaves famillies IV and III. We can however exclude III because close to the Sun the outflow velocities are generally lower than at Earth. Therefore, solution IV best describes the radial variation of the solar wind speed.
In fact, you can separate the variables of eq. (5.16) and integrate the resulting differential

equation to obtain an analytical solution for $v(r)$ in the case of a isothermal solar wind:

$$\frac{v^2}{c_s^2} - 2\ln\frac{v}{c_s} = 4\ln\frac{r}{r_c} + 4\frac{r_c}{r} - 3. \qquad (5.17)$$

Does this solution also ensure that the pressure decreases to 0 as $r \to \infty$? Because the Parker solution is isothermal, the pressure is only dependent on the mass density ($p = \rho RT$). The behaviour of ρ can be found by looking at the continuity equation eq. (5.12)

$$2r\rho v + r^2\frac{\partial\rho}{\partial r}v + r^2\rho\frac{\partial v}{\partial r} = 0. \qquad (5.18)$$

For larger r, i.e., $r \gg r_c$ we can estimate

$$v = 2c_s\sqrt{\ln\frac{r}{r_c}}. \qquad (5.19)$$

Insert this solution and its derivative into eq. (5.18) to find

$$\rho = \frac{1}{r^2\sqrt{\ln\frac{r}{r_c}}}. \qquad (5.20)$$

Therefore the pressure $p \to 0$ as $r \to \infty$ as required for a physical solution.

5.3 Typical values at 1 AU

The solar wind is usually divided into two categories: fast and slow. Fast solar wind originates from so called coronal holes where the solar field lines are near vertical, allowing for an efficient escape of the solar wind (frozen-in!).

	fast	slow
speed	700 km/s	400 km/s
density	3 cm^{-3}	8 cm^{-3}
T	5×10^5 K	0.5×10^5 K
c_s	~50 km/s	~20 km/s