PROBLEM 1 (15 points)

In a Cartesian coordinate system, assume the magnetic flux density $\vec{B}(z)$ varies as

$$\vec{B}(z) = B_L \tanh\left(\frac{z}{h}\right) \vec{e}_x$$

where h is a constant, $\vec{e}_x$ is the unit vector in x direction.

a) Sketch the magnetic flux density $\vec{B}(z)$ and the magnetic pressure $p_B(z)$. Indicate the direction of $-\nabla p_B(z)$. What is the significance of $-\nabla p_B(z)$? (5 points)

b) Calculate the current density $\vec{j}(z)$ needed to support the spatial change in $\vec{B}(z)$. (2 points)

c) Calculate the thermal plasma pressure $p_T(z)$ assuming that the given situation is in static equilibrium. (4 points)

d) Sketch $p_T(z)$. Indicate the direction of $-\nabla p_T(z)$. Discuss your results! (3 points)

e) The configuration is called the Harris current sheet. What region of the magnetosphere which can be described by it? (1 point)

PROBLEM 2 (15 points)

Given are the following equations:

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and

$$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

a) What are the well-known names of the equations above? Define the parameters involved (3 points).

b) Eliminate $\vec{E}$ and $\vec{j}$ to derive an expression for $\frac{\partial \vec{B}}{\partial t}$. (4 points)

c) Use your result from b) to show that the magnetic Reynolds number is given by $R_M = \mu_0 \sigma v L$. Discuss the physical implications of $R_M \gg 1$ and $R_M \ll 1$. (3 points)

d) Briefly explain the concept of frozen-in magnetic flux. How does this concept relate to R_M? (2 points)

e) Define the plasma parameter β. In what way is it important when considering the consequences of frozen-in magnetic flux? (2 points)

f) Give an example of when the frozen-in magnetic flux concept breaks down. (1 point)

PROBLEM 3 (15 points)

a) Sketch the structure of the Sun, from its deepest interior to its upper atmosphere. Describe briefly the main physical processes and characteristics of each layer. (6 points)

b) What is the solar wind and how is it formed? What are typical values for solar wind speed, particle density and temperature at a radial distance of 1 AU from the Sun? (2 points)

c) Describe the interaction between the solar wind and the terrestrial magnetic field on the dayside. What are the names of the boundary layers/regions? (3 points)

d) What controls the stand-off distance of the dayside boundary layers? Give typical values for the stand-off distances. (2 points)

e) How is the interplanetary magnetic field formed? What are typical values of the IMF at a radial distance of 1 AU from the Sun? (2 points)

PROBLEM 4 (15 points)

In the appendix you will find figures that give the densities of various particle species between 100 km and 700 km altitude at local noon in the mid-latitudes (latitude of 60°N) during average solar activity, followed by altitude profiles of the neutral, ion and electron temperature and absorption cross sections for some species. Furthermore, you will find some ionization frequencies and reaction rates of some important chemical reactions. Using this data:

a) What is the electron density at 500 km altitude? What atom/molecule forms most of the neutral atmosphere at 500 km altitude and what is its density? What ion species is most abundant at 500 km altitude and what is its density? (3 points)

b) The ion production rate of ion X^+ at altitude h due to photoionization is given by

$$q_{X^+}(h) = J_X n_X(h) \mathrm{e}^{-\tau(h)}$$

where J_X is the ionization frequency of species X (see table below), $n_X(h)$ is the density of species X at altitude h, and $\tau(h)$ is the optical depth at altitude h. The optical depth is given by

$$\tau(h) \cong \sec\chi \, \sigma^A n(h) H(h)$$

with the angle of incidence χ, the absorption cross section σ^A, and $n(h)$ and $H(h)$ being the (neutral) density and scale height $k_B T_X(h)/m_X g(h)$ at altitude h, respectively. m_X is the mass of species X and $g(h)$ is the gravitational acceleration at altitude h. Using the production time constant

$$\tau_p = \frac{n_e(h)}{q(h)}$$

estimate the time needed to build up the electron density $n_e(h)$ found in a). (4 points)

c) How large is the chemical loss rate $l(h) = \sum k^j_{X,Y} n_X(h) n_Y(h)$ (where k is the reaction coefficient from the appendix and the summation is over all possible chemical reactions) at this altitude for the dominant ion species upon accounting for both charge exchange (ignore reaction (3)) and radiative recombination? (4 points)

d) Using the loss rate from c), calculate the electron density assuming photo-chemical equilibrium. By what factor does the photo-chemical equilibrium density exceed the observed value? What is the reason for the discrepancy? (4 points)

APPENDIX

Some information about hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\int \tanh x \operatorname{sech}^2 x = -\frac{1}{2}\operatorname{sech}^2 x$$

Vector identities

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

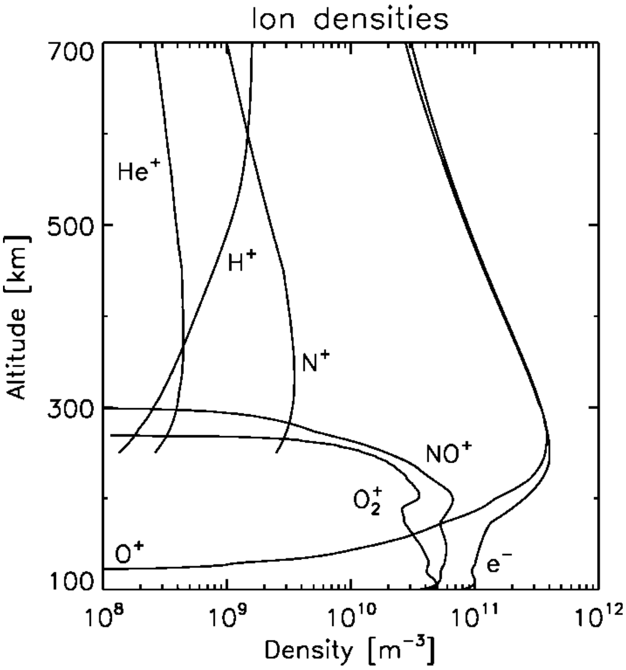
Some constants

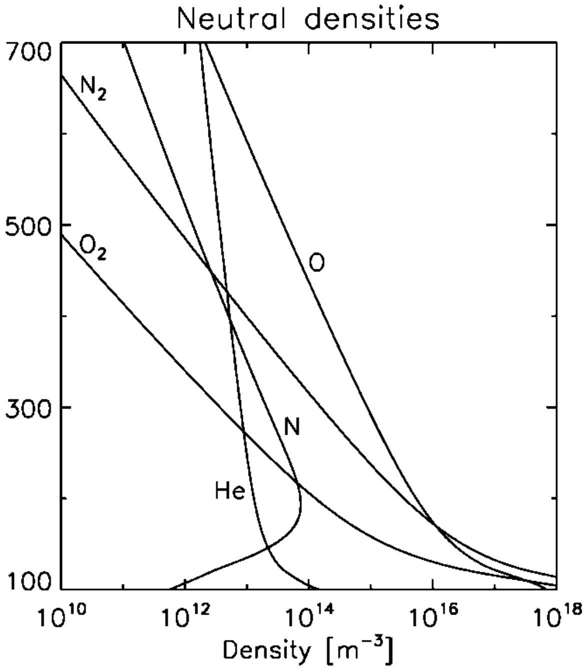
Mass of proton:	$m_p = 1.66 \times 10^{-27}\ kg$
Boltzmann constant	$k_B = 1.38 \times 10^{-23}\ J\ K^{-1}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}$
Mass of Earth	$m_E = 5.97 \times 10^{24}\ kg$
Radius of Earth	$r_E = 6.37 \times 10^6\ m$

Ionization frequencies averaged over all relevant wavelengths for different species. The range of values reflects the variations in solar activity. Also given are the particle masses in terms of proton mass

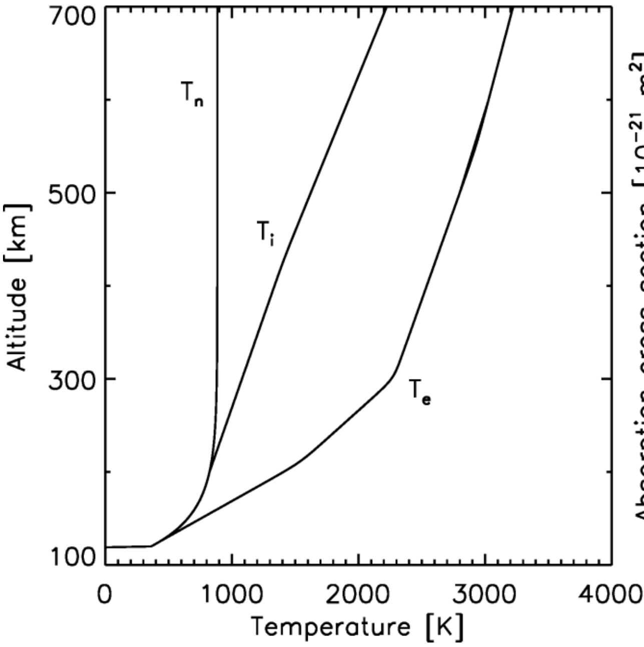
Ion species X^+	$J_X\ [10^{-7} s^{-1}]$	$m_{X^+} [m_p]$
O^+	$2 - 7$	16
N_2^+	$3 - 9$	28
O_2^+	$5 - 14$	32
He^+	$0.4 - 1$	4
H^+	$0.8 - 3$	1

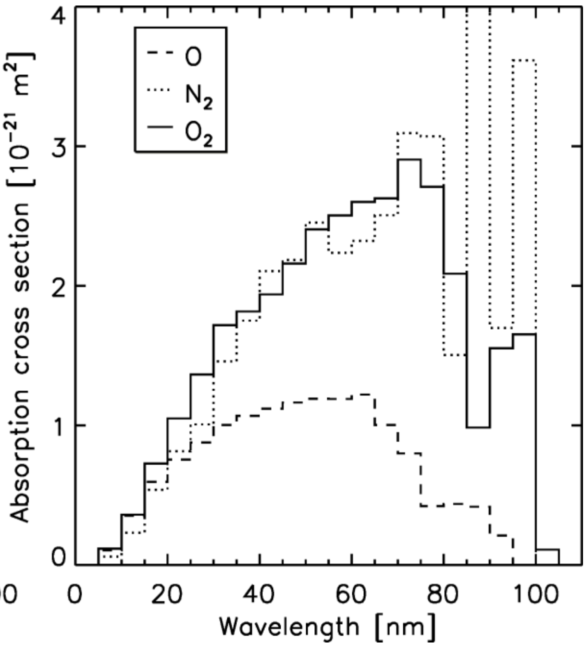
Ion and neutral densities as function of altitude during average solar activity in the mid-latitudes at about 60°N





Neutral, ion and electron temperature as a function of altitude (left) and absorption cross section for various species as a function of wavelength (right)





Important chemical reactions in the ionosphere and their reaction constant

Table 4.3. Important chemical reactions in the ionosphere (adapted from Schunk, 1983)

(1)	$O^+ + N_2$	$\longrightarrow$	$NO^+ + N,$	
	$k_1 = 1.533 \cdot 10^{-18} - 5.92 \cdot 10^{-19}\ (T/300) + 8.60 \cdot 10^{-20}\ (T/300)^2$; $300 \leq T \leq 1700$ K			
	$k_1 = 2.73 \cdot 10^{-18} - 1.155 \cdot 10^{-18}\ (T/300) + 1.483 \cdot 10^{-19}\ (T/300)^2$; $1700 < T \leq 6000$ K			
(2)	$O^+ + O_2$	$\longrightarrow$	$O_2^+ + O,$	
	$k_2 = 2.82 \cdot 10^{-17} - 7.74 \cdot 10^{-18}\ (T/300) + 1.073 \cdot 10^{-18}\ (T/300)^2 - 5.17 \cdot 10^{-20}\ (T/300)^3 + 9.65 \cdot 10^{-22}\ (T/300)^4$; $300 \leq T \leq 6000$ K			
(3)	$O^+ + H$	$\rightleftharpoons$	$H^+ + O,$	$\overrightarrow{k}_3 = 2.5 \cdot 10^{-17} \sqrt{T_n}$ $\overleftarrow{k}_3 = 2.2 \cdot 10^{-17} \sqrt{T_i}$
(4)	$N_2^+ + O_2$	$\longrightarrow$	$O_2^+ + N_2,$	$k_4 = 5 \cdot 10^{-17}\ (300/T)$
(5)	$N_2^+ + O$	$\longrightarrow$	$O^+ + N_2,$	$k_5 = 1 \cdot 10^{-17}\ (300/T)^{0.23}$; $T \leq 1500$ K
(6)	$N_2^+ + O$	$\longrightarrow$	$NO^+ + N,$	$k_6 = 1.4 \cdot 10^{-16}\ (300/T)^{0.44}$; $T \leq 1500$ K
(7)	$N^+ + O_2$	$\longrightarrow$	$NO^+ + O,$	$k_7 = 2.6 \cdot 10^{-16}$
(8)	$N^+ + O_2$	$\longrightarrow$	$O_2^+ + N,$	$k_8 = 3.1 \cdot 10^{-16}$
(9)	$He^+ + N_2$	$\longrightarrow$	$N^+ + He + N,$	$k_9 = 9.6 \cdot 10^{-16}$
(10)	$He^+ + N_2$	$\longrightarrow$	$N_2^+ + He,$	$k_{10} = 6.4 \cdot 10^{-16}$
(11)	$He^+ + O_2$	$\longrightarrow$	$O^+ + He + O,$	$k_{11} = 1.1 \cdot 10^{-15}$
(12)	$N_2^+ + e$	$\longrightarrow$	$N + N,$	$k_{12} = 1.8 \cdot 10^{-13}\ (300/T_e)^{0.39}$
(13)	$O_2^+ + e$	$\longrightarrow$	$O + O,$	$k_{13} = 1.6 \cdot 10^{-13}\ (300/T_e)^{0.55}$
(14)	$NO^+ + e$	$\longrightarrow$	$N + O,$	$k_{14} = 4.2 \cdot 10^{-13}\ (300/T_e)^{0.85}$
(15)	$O^+ + e$	$\longrightarrow$	$O^{(*)} + h\nu,$	$k_{15} \simeq 1.4 \cdot 10^{-18}\ (1160/T_e)^{0.5}$

where the reaction constants k_i are in $[\mathrm{m^3 s^{-1}}]$, $T \simeq T_n$ for small ion drift velocities and $T_i \simeq T_n$. In the presence of polar electric fields, the temperature in the F region increases as $T[\mathrm{K}] \simeq T_n[\mathrm{K}] + 0.33\,\mathcal{E}_{eff}^2[\mathrm{mV/m}]$, where $\vec{\mathcal{E}}_{eff} = \vec{\mathcal{E}}_\perp + \vec{u}_n \times \vec{B}$ ($\vec{\mathcal{E}}_\perp$ = externally applied electric field component perpendicular to the magnetic field $\vec{B}$, $\vec{u}_n$ = neutral gas velocity and $\vec{B}$ = geomagnetic field vector; see also Section 7.5.2).

PROBLEM 1

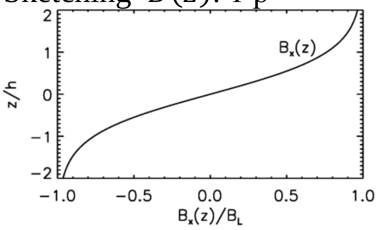
a) The magnetic pressure is given by

$$p_B(z) = \frac{B_L^2}{2\mu_0}\tanh^2\left(\frac{z}{h}\right)$$

The negative gradient of the magnetic pressure gives the direction and magnitude of the force acting on the plasma due to $\vec{j}\times\vec{B}$ forces according to equation of motion:

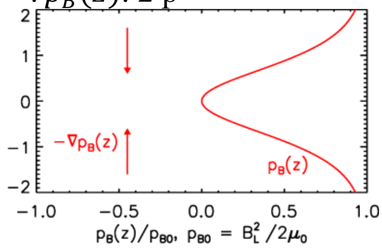
$$\rho\frac{d\vec{v}}{dt} = -\nabla(p_T + p_B)$$

Sketching $\vec{B}(z)$: 1 p



Writing down pressure definition:1 p

Sketching $p_B(z)$ and indicating $-\nabla p_B(z)$: 2 p



Explaining $-\nabla p_B(z)$: 1 p

b) Use Ampere's Law

$$\nabla\times\vec{B} = \mu_0\vec{j}$$

To obtain

$$\vec{j} = \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{1}{\mu_0}\begin{pmatrix} \frac{dB_z}{dy} - \frac{dB_y}{dz} \\ \frac{dB_x}{dz} - \frac{dB_z}{dx} \\ \frac{dB_y}{dx} - \frac{dB_x}{dy} \end{pmatrix}$$

$$= \frac{1}{\mu_0}\begin{pmatrix} 0 \\ \frac{B_L}{h}\mathrm{sech}^2\left({z}/{h}\right) \\ 0 \end{pmatrix}$$

The resulting current flows in the y-direction and is strongest around $z = 0$.

Writing down Ampere's Law: 1p

Calculating j: 1p

c) For static ($\vec{v} = 0$) equilibrium (${\partial}/{\partial t} = 0$) we have

$$\rho\frac{d\vec{v}}{dt} = -\nabla p_T + \vec{j}\times\vec{B} = 0$$

$$\frac{dp_T}{dz} = -j_y B_x$$

$$\int_0^{p_T(z)} dp_T' = \int_{-\infty}^{z} -j_y B_x dz'$$

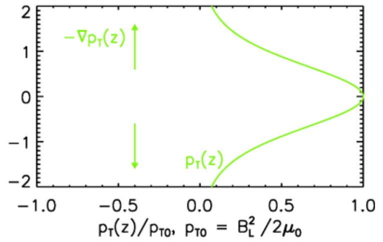
$$p_T(z) - 0$$

$$= -\int_{-\infty}^{z}\frac{B_L}{\mu_0 h}\mathrm{sech}^2\left({z'}/{h}\right) B_L\tanh\left({z'}/{h}\right)dz'$$

Substitute $k' = z'/h$, then $dz' = hdk'$

Writing down equilibrium condition: 1p

Calculating $p_T(z)$: 3p

$p_T(z) = -\frac{B_L^2}{\mu_0}\int_{-\infty}^{k} \mathrm{sech}^2(k')\tanh(k')\,dk'$ $p_T(z) = -\frac{B_L^2}{2\mu_0}[-\mathrm{sech}^2(k')]_{-\infty}^{k}$ $p_T(z) = \frac{B_L^2}{2\mu_0}\mathrm{sech}^2\left({}^{z}/_{h}\right)$	
d) The situation is stable because the force due to the magnetic pressure pointing towards the center of the current sheet is balanced by the force due to the thermal pressure gradient acting away from the current sheet.	Sketching $p_T(z)$ and indicating $-\nabla p_T(z)$: 2p  Mentioning pressure balance: 1p
e) The cross tail current can be approximated by such a configuration.	Mentioning cross tail current: 1p

PROBLEM 2	
a) Ampere's Law, Faraday's Law (induction law), Ohm's Law, magnetic field (or flux density), electric field, current density, conductivity, velocity, permeability of free space	Each gives 1/3 p
b) Rearranging Ohm's Law	
$\vec{E} = \frac{\vec{j}}{\sigma} - \vec{v}\times\vec{B}$	1p
And Ampere's Law	
$\vec{j} = \frac{1}{\mu_0}\nabla\times\vec{B}$	1p
Inserting into Faraday's Law	
$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\frac{1}{\sigma\mu_0}\nabla\times\vec{B} - \vec{v}\times\vec{B}\right)$	1p
Gives	
$\frac{\partial\vec{B}}{\partial t} = \frac{1}{\sigma\mu_0}\nabla^2\vec{B} + \nabla\times\left(\vec{v}\times\vec{B}\right)$	1p
c) Relative strength of convection to diffusion term, on typical length scale L	Realize it's the relative strength: 1p Do the calculation: 1p

$$\frac{\nabla \times (\vec{v} \times \vec{B})}{\frac{1}{\sigma\mu_0}\nabla^2\vec{B}} \approx \frac{vB/_L}{B/_{\sigma\mu_0 L^2}} = \mu_0\sigma v L$$ Convection ($R_M \gg 1$) vs diffusion ($R_M \ll 1$), convection means B is transported by plasma movement, diffusion means B can move through plasma without moving the plasma.	Realize one is diffusion, the other conv., and mention that convection involves plasma movement, diffusion not: 1p
d) If $\sigma \gg 0$, then $R_M \gg 1$ and convection dominates, i.e. plasma tied to field lines, magnetic field cannot move without dragging plasma along, plasma cannot move without taking magnetic field with it.	Make connection between σ and R_M: 1p Explain movement of plasma and magnetic field: 1p
e) Plasma parameter is ratio of thermal to magnetic pressure $$\beta = \frac{p_T}{p_B} = \frac{2\mu_0 p_T}{B^2}$$	Writing down definition: 1p
If $\beta \ll 1$, then magnetic pressure dominates, motion of plasma is dominated by motion of magnetic field, if $\beta \gg 1$, then plasma pressure is much larger than magnetic pressure, motion of plasma takes the embedded magnetic field with it.	Explaining who wins, magnetic field or plasma: 1p
f) Reconnection	Writing down that word: 1p

PROBLEM 3	
a) Core (nuclear fusion), radiative zone (energy is transported in form of EM radiation), convection zone (temperature gradient is too large, heat is transported by movement of plasma), photosphere (surface of Sun, source of EM radiation at Earth), chromosphere (temperature increases), corona (temperature increases even more, solar wind is formed)	Point per each zone with name and main mechanism right: 6p
b) Electrons and ions (mostly protons) are streaming off the Sun because of the high temperatures in the corona. ~5/ccm, 500km/s, 10^5K	Composition: 1p Per value 1/3p
c) Supersonic solar wind is shocked at bow shock, in the magnetosheath the flow is diverted around Earth. Magnetopause is boundary between IMF and the terrestrial magnetic field. Electrons and ions are reflected at sharp increase in magnetic field, in opposite directions, forms Chapman-Ferraro current. Dayside terrestrial magnetic field is compressed.	Bow shock: shock wave that slows down the supersonic wind: 1p Magnetosheath, region in which the subsonic solar wind flows around Earth's magnetic field: 1p Magnetopause: solar wind particles are reflected, Chapman-Ferraro current: 1p
d) Solar wind dynamic pressure, bow shock 15 Re,	Dynamic pressure: 1p

magnetopause 10 Re e) Solar magnetic field is frozen into the solar wind ($\beta > 1$) and fills the interplanetary space. $B_r \sim 5nT$, $B_\lambda \sim 5nT$.	Distances: each 1/2p Solar magnetic field frozen to solar wind: 1p Values: 1p

PROBLEM 4 a) Electron density: 10^11 m^-3, O+ density 10^11 m^-3, O density 5 10^13 m^-3	Per value: 1p
b) First collect all values. Realize that O and O+ are the dominant species, so we only need to concern us with them. χ is 60°. σ^A (for O) is roughly 10^-21 m^2. The neutral density (which is the one to use in the optical depth calculation) is 5 10^13 m^-3, the neutral temperature is about 900 K and the gravitational acceleration at 500 km altitude is about 8.4 m/s^2. That makes the scale height $k_B T_n / m_O g(h)$ about 57 km. Together the optical depth is about 0.06 and exp $-\tau$ = 0.94. Again, O/O+ dominates so $q_{O^+}(h) = J_O n_O(h) \mathrm{e}^{-\tau(h)}$ is about 2 10^7 m^-3 s^-1. This gives for τ_p about 4700 s, i.e., 1 h 20 min.	Scale height: 1p Optical depth: 1p Production rate: 1p Time constant: 1p
c) Again, we are only concerned with the values that include O+/O, so for charge exchange the only relevant reactions are (1) and (2) in the long list. $k^{CC}_{O^+,N_2}$ and $k^{CC}_{O^+,O_2}$ which for this altitudes (using again the neutral temperature of 900 K) work out to be 5.3 10^-19 m^3/s and 1.3 10^-17 m^3/s, respectively. The reaction constant for radiative $k^{RR}_{O^+}$ is 8.7 10^-19 m^3/s. These values need to multiplied with the molecular oxygen/molecular nitrogen/electron densities and the oxygen ion density to get the loss rates $l_c = (k^{RR}_{O^+} n_e + k^{CC}_{O^+,O_2} n_{O_2} + k^{CC}_{O^+,N_2} n_{N_2})\, n_{O^+}$ is about 7.5 10^4 m^-3 s^-1, i.e. three orders of magnitude smaller than the production rate.	Per reaction rate: 1p Loss rate: 1p
d) In photo-chemical equilibrium $q = l_c$. Use the value of q from b) and use the formula for l_c from c), solve for an unknown $n_{eq} = n_{O^+} = n_e$ $$q = k^{RR}_{O^+} n_{eq}^2 + k^{CC}_{O^+,O_2} n_{O_2} n_{eq} + k^{CC}_{O^+,N_2} n_{N_2} n_{eq}$$ Such that $$n_{eq} = -\frac{k^{CC}_{O^+,O_2} n_{O_2} + k^{CC}_{O^+,N_2} n_{N_2}}{2k^{RR}_{O^+}} \mp \sqrt{\left(\frac{k^{CC}_{O^+,O_2} n_{O_2} + k^{CC}_{O^+,N_2} n_{N_2}}{2k^{RR}_{O^+}}\right)^2 + \frac{q}{k^{RR}_{O^+}}}$$	Photo-chemical equilibrium: 1p Rearranging formula: 1p Correct value: 1p Mentioning transport: 1p

Which is about 4.5 10^12 m^-3, so about 45 times higher than the observed value. Transport is not considered.	