1 Solution

1.1 Derive 1D induction equation

3D induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B} \tag{1}$$

Do the vector products and write out the Laplacian:

$$\frac{\partial}{\partial t} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} v_y B_z - v_z B_y \\ v_z B_x - v_x B_z \\ v_x B_y - v_y B_x \end{pmatrix} + \eta \begin{pmatrix} \frac{\partial^2}{\partial x^2} B_x + \frac{\partial^2}{\partial y^2} B_x + \frac{\partial^2}{\partial z^2} B_x \\ \frac{\partial^2}{\partial x^2} B_y + \frac{\partial^2}{\partial y^2} B_y + \frac{\partial^2}{\partial z^2} B_y \\ \frac{\partial^2}{\partial x^2} B_z + \frac{\partial^2}{\partial y^2} B_z + \frac{\partial^2}{\partial z^2} B_z \end{pmatrix} \tag{2}$$

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = & \begin{pmatrix} \frac{\partial}{\partial y}(v_x B_y - v_y B_x) - \frac{\partial}{\partial z}(v_z B_x - v_x B_z) \\ \frac{\partial}{\partial z}(v_y B_z - v_z B_y) - \frac{\partial}{\partial x}(v_x B_y - v_y B_x) \\ \frac{\partial}{\partial x}(v_z B_x - v_x B_z) - \frac{\partial}{\partial y}(v_y B_z - v_z B_y) \end{pmatrix} \\ & + \eta \begin{pmatrix} \frac{\partial^2}{\partial x^2} B_x + \frac{\partial^2}{\partial y^2} B_x + \frac{\partial^2}{\partial z^2} B_x \\ \frac{\partial^2}{\partial x^2} B_y + \frac{\partial^2}{\partial y^2} B_y + \frac{\partial^2}{\partial z^2} B_y \\ \frac{\partial^2}{\partial x^2} B_z + \frac{\partial^2}{\partial y^2} B_z + \frac{\partial^2}{\partial z^2} B_z \end{pmatrix} \end{aligned} \tag{3}$$

Assume $\vec{B} = (0, 0, B_z(x,t))^T$ and $\vec{v} = (v_x(x,t), 0, 0)^T$, then $\partial/\partial y = \partial/\partial z = 0$

$$\frac{\partial}{\partial t} \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial x}(-v_x B_z) \end{pmatrix} + \eta \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2}{\partial x^2} B_z \end{pmatrix} \tag{4}$$

or

$$\frac{\partial B_z}{\partial t} = -\frac{\partial}{\partial x}(v_x B_z) + \eta \frac{\partial^2 B_z}{\partial x^2} \tag{5}$$

1.2 Find time dependent solution

Initial profile

$$B_z(x, t=0) = f(x) = A_0 \exp\left(-\frac{x^2}{L^2} \right) \tag{6}$$

Time dependent solution is given by

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-\lambda) \exp\left(-\frac{\lambda^2}{4\eta^2 t} \right) d\lambda \tag{7}$$

such that we get

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} A_0 \exp\left(-\frac{(x-\lambda)^2}{L^2}\right) \exp\left(-\frac{\lambda^2}{4\eta^2 t}\right) d\lambda \tag{8}$$

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} A_0 \exp\left(-\left[\left(\frac{1}{L^2}+\frac{1}{4\eta^2 t}\right)\lambda^2 - \frac{2x}{L^2}\lambda + \frac{x^2}{L^2}\right]\right) d\lambda \tag{9}$$

$$B_z(x,t) = \frac{1}{2\eta\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} A_0 \exp\left(-\left[r^2(\lambda - s)^2 + q\right]\right) d\lambda \tag{10}$$

$$B_z(x,t) = \frac{A_0}{2\eta\sqrt{\pi t}} \exp(-q) \int\limits_{-\infty}^{\infty} \exp\left(-\alpha\tau^2\right) d\lambda \tag{11}$$

with

$$\tau = \lambda - s \tag{12}$$

$$\alpha = r^2 = \frac{1}{L^2} + \frac{1}{4\eta^2 t} \tag{13}$$

$$2r^2 s = \frac{2x}{L^2} \leftrightarrow s = \frac{x}{L^2 r^2} \tag{14}$$

$$s = \frac{x}{L^2\left(\frac{1}{L^2}+\frac{1}{4\eta^2 t}\right)} \tag{15}$$

$$r^2 s^2 + q = \frac{x^2}{L^2} \leftrightarrow q = \frac{x^2}{L^2} - r^2 s^2 \tag{16}$$

$$\begin{aligned} q &= \frac{x^2}{L^2} - \left(\frac{1}{L^2}+\frac{1}{4\eta^2 t}\right)\left(\frac{x}{L^2\left(\frac{1}{L^2}+\frac{1}{4\eta^2 t}\right)}\right)^2 \\ &= \frac{x^2}{L^2}\left(1 - \frac{1}{L^2\left(\frac{1}{L^2}+\frac{1}{4\eta^2 t}\right)}\right) \\ &= \frac{x^2}{4\eta^2 t + L^2} \end{aligned} \tag{17}$$

Executing the integration and simplifying yields

$$B_z(x,t) = \frac{A_0}{2\eta\sqrt{\pi t}} \exp\left(-\frac{x^2}{4\eta^2 t + L^2}\right) \sqrt{\frac{\pi}{\frac{1}{L^2} + \frac{1}{4\eta^2 t}}} \tag{18}$$

or

$$B_z(x,t) = \frac{A_0 L}{\sqrt{4\eta^2 t + L^2}} \exp\left(-\frac{x^2}{4\eta^2 t + L^2}\right) \tag{19}$$

It describes a Gaussian distribution centered around $x = 0$. For $t \to \infty$ the amplitude of the distribution slowly decreases while the width increases (conservation of total B!). This is analog to the dispersion of a wave packet. Instead of being concentrated at one location, B disperses over the entire x axis – it diffuses!

Test that total B_z is conserved ($\chi = 1/\sqrt{4\eta^2 t + L^2}$):

$$\int_{-\infty}^{\infty} B_z(x,t)dx = A_0 L\chi \int_{-\infty}^{\infty} \exp\left(-\chi^2 x^2\right) dx = A_0 L\chi\sqrt{\frac{\pi}{\chi^2}} = A_0 L\sqrt{\pi} \tag{20}$$

The result is independent of t, hence the total B_z is conserved.

The resistivity η is the time constant of the diffusion process. If the conductivity is very large, the resistivity η becomes very small, such that the time dependence of $B_z(x,t)$ becomes very weak. Hence the diffusion process is very slow and the dispersion takes a very long time, the shape of the B-field distribution is conserved over long times (frozen- in theorem!).

1.3 Find velocity profile such that B_z profile is constant in time

Initial time *in*dependent profile:

$$B_z(x) = A_0 \exp\left(-\frac{x^2}{L^2}\right) \tag{21}$$

Then

$$\frac{\partial B_z}{\partial x} = -\frac{2x}{L^2} B_z \tag{22}$$

and

$$\frac{\partial^2 B_z}{\partial x^2} = -\frac{2}{L^2} B_z + \frac{4x^2}{L^4} B_z = \frac{2}{L^2}\left(\frac{2x^2}{L^2} - 1\right) B_z \tag{23}$$

For $\partial B_z/\partial t = 0$:

$$-v_x\frac{\partial B_z}{\partial x} - B_z\frac{\partial v_x}{\partial x} + \eta\frac{\partial^2 B_z}{\partial x^2} = 0 \tag{24}$$

Inserting derivatives:

$$v_x\frac{2x}{L^2}B_z - B_z\frac{\partial v_x}{\partial x} + \eta\left(\frac{2}{L^2}\left(\frac{2x^2}{L^2} - 1\right)B_z\right) = 0 \tag{25}$$

B_z cancels (it can never be 0) and rearranging gives:

$$\frac{\partial}{\partial x}v_x - \frac{2x}{L^2}v_x = \frac{2\eta}{L^2}\left(\frac{2x^2}{L^2} - 1\right) \tag{26}$$

That is a linear first order differential equation, i.e., a differential equation of the form

$$\frac{\partial}{\partial x}y + p(x)y = g(x) \tag{27}$$

with

$$p(x) = -\frac{2x}{L^2} \tag{28}$$

and

$$g(x) = \frac{2\eta}{L^2}\left(\frac{2x^2}{L^2} - 1\right). \tag{29}$$

We can solve by finding the integrating factor $\mu(x)$

$$\mu(x) = \exp\left(\int p(x)dx\right) \tag{30}$$

$$\int p(x)dx = -\frac{x^2}{L^2} \tag{31}$$

$$\mu(x) = \exp\left(-\frac{x^2}{L^2}\right) \tag{32}$$

Multiplying the differential equation with the integrating factor yields:

$$\exp\left(-\frac{x^2}{L^2}\right)\frac{\partial}{\partial x}v_x - \exp\left(-\frac{x^2}{L^2}\right)\frac{2x}{L^2}v_x = \exp\left(-\frac{x^2}{L^2}\right)\frac{2\eta}{L^2}\left(\frac{2x^2}{L^2} - 1\right) \tag{33}$$

The left hand side can be rearranged using the product rule:

$$\frac{\partial}{\partial x}\left(\exp\left(-\frac{x^2}{L^2}\right)v_x\right) = \frac{2\eta}{L^2}\left(\frac{2x^2}{L^2} - 1\right)\exp\left(-\frac{x^2}{L^2}\right) \tag{34}$$

Integrate both sides

$$\exp\left(-\frac{x^2}{L^2}\right) v_x = \int \left[\frac{2\eta}{L^2}\left(\frac{2x^2}{L^2} - 1\right) \exp\left(-\frac{x^2}{L^2}\right)\right] dx \tag{35}$$

yields, after realizing that $\int (2x^2 - 1)\exp(-x^2)dx = -x\exp(-x^2)$:

$$\exp\left(-\frac{x^2}{L^2}\right) v_x = -\frac{2\eta x}{L^2} \exp\left(-\frac{x^2}{L^2}\right) + C \tag{36}$$

or the final solution

$$v_x = -\frac{2\eta x}{L^2} + C\exp\left(\frac{x^2}{L^2}\right). \tag{37}$$

Test solution by putting it into the differential equation. First:

$$\frac{\partial}{\partial x} v_x = -\frac{2\eta}{L^2} + \frac{2xC}{L^2}\exp\left(\frac{x^2}{L^2}\right) \tag{38}$$

then substitute:

$$\left(C\exp\left(\frac{x^2}{L^2}\right) - \frac{2\eta x}{L^2}\right)\frac{2x}{L^2}B_z + B_z\left(\frac{2\eta}{L^2} - \frac{2xC}{L^2}\exp\left(\frac{x^2}{L^2}\right)\right) + \frac{2\eta}{L^2}\left(\frac{2x^2}{L^2} - 1\right)B_z \stackrel{?}{=} 0 \tag{39}$$

B_z cancels and indeed the result is zero:

$$\frac{2xC}{L^2}\exp\left(\frac{x^2}{L^2}\right) - \frac{4\eta x^2}{L^4} + \frac{2\eta}{L^2} - \frac{2xC}{L^2}\exp\left(\frac{x^2}{L^2}\right) + \frac{4\eta x^2}{L^4} - \frac{2\eta}{L^2} \stackrel{!}{=} 0. \tag{40}$$

However, the $C\exp(x^2/L^2)$ part of the solution is unphysical such that the only physical solution is $C = 0$. We expect, because $B_z(x)$ is symmetric around $x = 0$, that the velocity field satisfies $v_x(-x) = -v_x(x)$ for all x, i.e., at a certain distance x from $x = 0$ the velocity is equal but of opposite direction, depending on its location relative to $x = 0$. This is fulfilled only if $C = 0$.

Hence the physical solution for this problem is

$$\boxed{v_x = -\frac{2\eta x}{L^2}} \tag{41}$$