Atmosphere-Ocean Dynamics

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1 Equations

The motion in the atmosphere and ocean is governed by a set of equations, known as the *Navier-Stokes* equations. These equations are used to produce our forecasts, for the weather and also for ocean currents. While there are details about these equations which are uncertain (for example, how we parametrize processes smaller than the grid size of the models), they are for the most part accepted as fact. Let's consider how these equations come about.

1.1 Derivatives

A fundamental aspect is how various fields (temperature, wind, density) change in time and space. Thus we must first specify how to take derivatives.

Consider a scalar, ψ, which varies in both time and space, i.e. $\psi = \psi(x, y, z, t)$. This could be the wind speed in the east-west direction, or the ocean density. By the chain rule, the total change in the ψ is:

$$d\psi = \frac{\partial}{\partial t}\psi\, dt + \frac{\partial}{\partial x}\psi\, dx + \frac{\partial}{\partial y}\psi\, dy + \frac{\partial}{\partial z}\psi\, dz \tag{1}$$

so:

$$\frac{d\psi}{dt} = \frac{\partial}{\partial t}\psi + u\,\frac{\partial}{\partial x}\psi + v\,\frac{\partial}{\partial y}\psi + w\,\frac{\partial}{\partial z}\psi \tag{2}$$

or, in short form:

$$\frac{d\psi}{dt} = \frac{\partial}{\partial t}\psi + \vec{u}\cdot\nabla\psi \tag{3}$$

Here (u, v, w) are the components of the velocity in the (x, y, z) directions. On the left side, the derivative is a total derivative. That implies that ψ on the left side is only a function of time. This the case when ψ is observed *following the flow*. For instance, if you measure temperature in a balloon, moving with the winds, you only see changes in time. We call this the *Lagrangian* formulation. The derivatives on the right side though are partial derivatives. These are relevant for an observer at *a fixed location*. This person records temperature as a function of time, but her information also depends on her position. An observer at a different location will generally have a different records (depending on how far away she is). We call the right side the *Eulerian* formulation.

Exercise 1.1: There are two observers, one at a weather station at a point x and another passing by in a balloon. The observer at the station notices that the temperature is falling at rate of 1^oC/day, while the balloonist doesn't observe any change at all. If the balloon is moving east at a constant rate of 10 m/sec, what can you conclude about the background temperature field?

1.2 Continuity equation

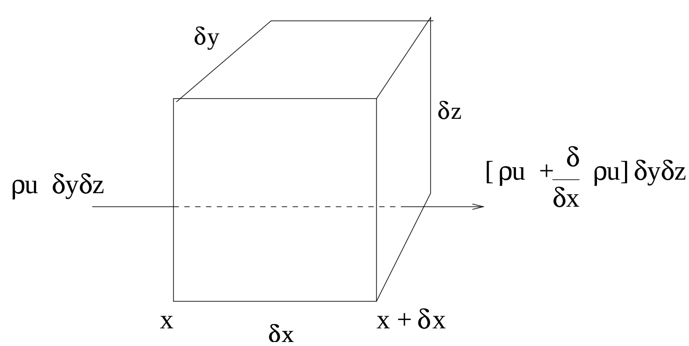


Figure 1: A infinitesimal element of fluid, with volume δV.

Consider a box fixed in space, with fluid (either wind or water) flowing through it. The flux of density through the left side is:

$$F_l = \rho u \times (area) = (\rho u)\,\delta y\,\delta z \tag{4}$$

Using a Taylor expansion, we can write the flux through the right side as:

$$F_r = [\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\,\delta y\,\delta z \tag{5}$$

If these density fluxes differ, then the box's mass will change. The net rate of change in mass is:

$$\frac{\partial}{\partial t}M = \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = F_l - F_r$$

$$= (\rho u)\,\delta y\,\delta z - [\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\,\delta y\,\delta z = -\frac{\partial}{\partial x}(\rho u)\delta x\,\delta y\,\delta z \tag{6}$$

The volume of the box is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u) \tag{7}$$

Taking into account all the other sides of the box we have:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = -\nabla \cdot (\rho \vec{u}) \tag{8}$$

We can rewrite the RHS as follows:

$$\nabla \cdot (\rho \vec{u}) = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho \tag{9}$$

Thus the continuity equation can be written:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = 0 \tag{10}$$

This is the continuity equation in its Eulerian form. Alternately we can write:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0 \tag{11}$$

which is the equation in its Lagrangian form. This says that the density of a parcel of fluid advected by the flow will change if the flow is divergent, i.e. if:

$$\nabla \cdot \vec{u} \neq 0 \tag{12}$$

Exercise 1.2: Derive the continuity equation a different way, by considering a balloon advected by the flow. The balloon has a fixed *mass*, i.e. it contains a fixed number of molecules (of, say, helium). Say the balloon is cubic, with sides δx, δy and δz. The balloon's volume is then:

$$V = \delta x \, \delta y \, \delta z$$

and its mass is ρV. If the mass is conserved following the flow, so is this quantity:

$$\frac{1}{M}\frac{d}{dt}M = 0 \tag{13}$$

Use this to re-derive the continuity equation (10). Take the limit as $\delta \to 0$.

Again, the density changes in proportion to the velocity divergence; the divergence determines whether the box shrinks or grows. If the box expands/shrinks, the density decreases/increases, to preserve the box's mass.

1.3 Momentum equations

The continuity equation pertains to mass. Now we consider the fluid velocities. We can derive expressions for these from Newton's second law:

$$\vec{a} = \vec{F}/m \tag{14}$$

The forces acting on a fluid parcel (a vanishingly small box) are:

- pressure gradients: $\frac{1}{\rho}\nabla p$
- gravity: $\vec{g}$
- friction: $\vec{F}$

For a parcel with density ρ, we can write:

$$\frac{d}{dt}\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{15}$$

This is the *momentum equation*, written in its Lagrangian form. Under the influence of the forcing terms, on the RHS, the air parcel will accelerate.

The equation is actually three equations in one, one for each spatial direction. Gravity, which acts only in the vertical, appears in just one of the equations. The pressure gradient terms exist in all three (this term can be derived in a similar way to the continuity equation of the previous section). Friction too can act in all three directions.

In fact, this is the momentum equation for a non-rotating earth. There are additional acceleration terms which come about due to rotation. As opposed to the *real* forces shown in (15), rotation introduces *apparent* forces. A stationary parcel on the earth will rotate with the planet. From the perspective of an observer in space, that parcel is traveling in circles, completing a circuit once a day. Since circular motion represents an acceleration (the velocity is changing direction), there is a corresponding force.

Consider such a stationary parcel, on a rotating sphere, with its position represented by a vector, $\vec{A}$ (Fig. 2). During the time, δt, the vector rotates through an angle:

$$\delta\Theta = \Omega\delta t \tag{16}$$

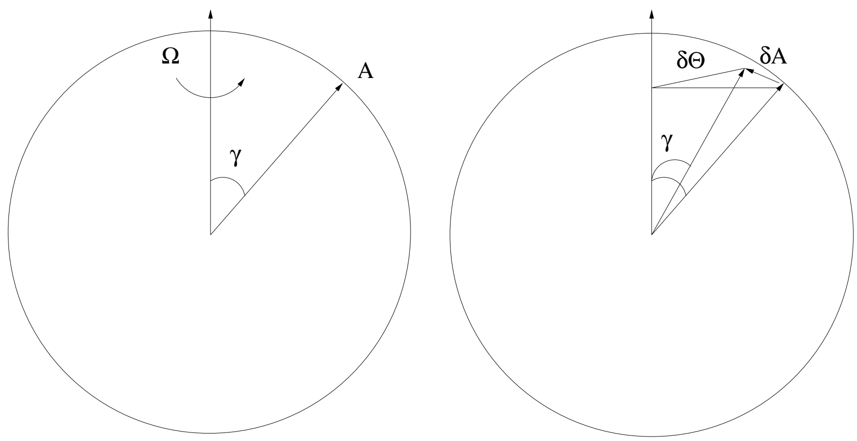


Figure 2: The effect of rotation on a vector, A, which is otherwise stationary. The vector rotates through an angle, $\delta\Theta$, in a time δt.

where:

$$\Omega = \frac{2\pi}{86400}\ sec^{-1}$$

is the sphere's rotation rate. The change in A is δA, the arc-length:

$$\delta\vec{A} = |\vec{A}| sin(\gamma)\delta\Theta = \Omega|\vec{A}| sin(\gamma)\delta t = (\vec{\Omega} \times \vec{A})\,\delta t \tag{17}$$

So we can write:

$$lim_{\delta\to 0}\ \frac{\delta\vec{A}}{\delta t} = \frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A} \tag{18}$$

If the vector is not stationary but moving in the rotating frame, one can show that:

$$(\frac{d\vec{A}}{dt})_F = (\frac{d\vec{A}}{dt})_R + \vec{\Omega} \times \vec{A} \tag{19}$$

The F here refers to the fixed frame and R to the rotating one. If $\vec{A} = \vec{r}$, the position vector, then:

$$(\frac{d\vec{r}}{dt})_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{20}$$

So the velocity in the fixed frame is just that in the rotating frame plus the velocity associated with the rotation.

Exercise 1.3: Say that a car is driving at 50 km/hr, in Oslo. What is the car's speed when viewed from space?

Now consider that $\vec{A}$ is velocity in the fixed frame, $\vec{u}_F$. Then:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_F}{dt})_R + \vec{\Omega} \times \vec{u}_F \tag{21}$$

Substituting in the previous expression for u_F, we get:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d}{dt}[\vec{u}_R + \vec{\Omega} \times \vec{r}])_R + \vec{\Omega} \times [\vec{u}_R + \vec{\Omega} \times \vec{r}] \tag{22}$$

Collecting terms, we get:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} \tag{23}$$

We now have two additional terms: the *Coriolis* and *centrifugal* accelerations. Plugging these into the momentum equation, we obtain:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{24}$$

Consider the centrifugal acceleration. This is the negative of the centripetal acceleration and acts perpendicular to the axis of rotation (Fig. 3). The force projects onto both the radial and the N-S directions. This suggests that a parcel in the Northern Hemisphere would accelerate upward and southward. But these accelerations are balanced by gravity, which acts to pull the parcel toward the center *and* northward. The latter occurs because rotation changes the shape of the earth itself, making it ellipsoidal rather than spherical. The change in shape results in an exact cancellation of the N-S component of the centrifugal force.

The radial component on the other hand is overcome by gravity. If this weren't true, the atmosphere would fly off the earth. So the centrifugal

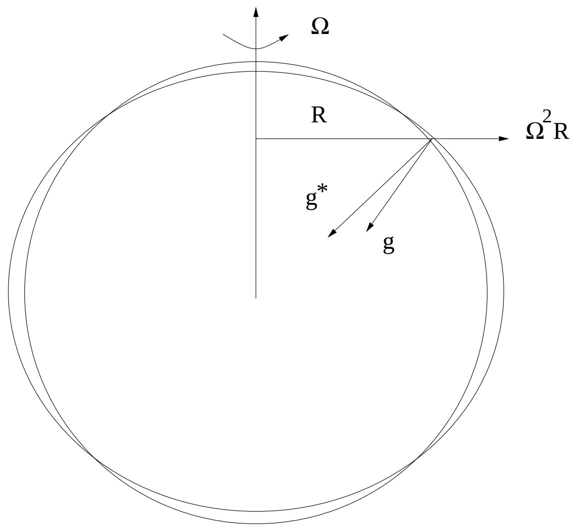


Figure 3: The centrifugal force and the deformed earth. Here is g is the gravitational vector for a spherical earth, and g^* is that for the actual earth. The latter is an *oblate spheroid*.

force *modifies gravity*, reducing it over what it would be if the earth were stationary. Thus we can absorb the centrifugal force into gravity:

$$g' = g - \vec{\Omega} \times \vec{\Omega} \times \vec{r} \tag{25}$$

Exercise 1.4: How much does rotation alter gravity? Figure out how large the acceleration is at the equator. How large is this compared to $g = 9.8$ m/sec^2?

The correction is so small in fact that we will ignore it (and drop the prime on g hereafter). So the momentum equation can be written:

$$(\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{26}$$

There is only one rotational term to worry about, the Coriolis acceleration. We'll say more about this in a minute.

There are three spatial directions and each has a corresponding momentum equation. In what follows, we will assume that we are in a localized region of the atmosphere, centered at a latitude, θ. Then we can define local coordinates (x, y, z) such that:

$$dx = acos(\theta)\, d\phi, \quad dy = a\, d\theta, \quad dz = dR$$

where ϕ is the longitude, a is the earth's radius and R is the radius. Thus x is the east-west coordinate, y the north-south coordinate and z the vertical coordinate. We define the corresponding velocities:

$$u \equiv \frac{dx}{dt}, \quad v \equiv \frac{dy}{dt}, \quad w \equiv \frac{dz}{dt}$$

The momentum equations will determine the accelerations in (x,y,z).

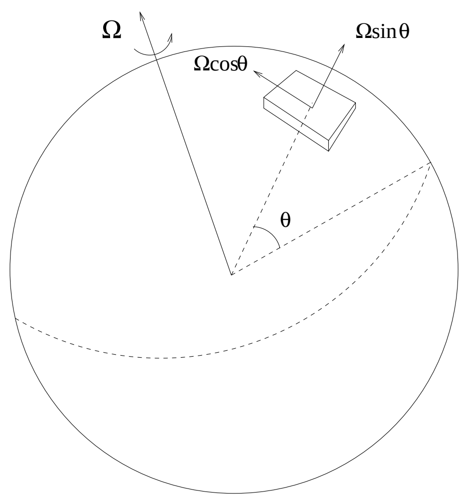


Figure 4: A region of the atmosphere at latitude θ. The earth's rotation vector projects onto the local latitudinal and radial coordinates.

The Coriolis term (which is a vector itself) projects onto both the y and z directions:

$$2\vec{\Omega} \times \vec{u} = (0, 2\Omega_y, 2\Omega_z) \times (u, v, w) =$$
$$2\Omega(w\, cos\theta - v\, sin\theta,\; u\, sin\theta, -u\, cos\theta) \tag{27}$$

Adding terms, we have:[1]

[1]If we had used spherical coordinates instead, we would have several additional *curvature* terms. However, these terms are generally small at the scales of interest and so are left out here.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + 2\Omega w\, cos\theta - 2\Omega v\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \quad (28)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \quad (29)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\Omega u\, cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \quad (30)$$

where F_i is the frictional force acting in the i-direction.

Exercise 1.5: Consider a car again, driving eastward at 50 km/hr in Oslo. What is the Coriolis acceleration acting on the car? Which direction is it pointing? And how does it compare to gravity? Now imagine the car is driving the same speed and direction, but in Wellington, New Zealand. What is the Coriolis acceleration?

The above is a general result. The Coriolis force acts to the right of the motion in the Northern Hemisphere and to the left in the Southern Hemisphere. Because it acts perpendicular to the motion, it does no work—that means it doesn't change the speed of a parcel, just its direction of motion. We'll see that the Coriolis force is one of the dominant terms at weather scales.

Lastly, there is the friction force, $\vec{F}$. For synoptic scale motions, this is meant to represent the action of small scale eddies. If, for example, our weather model has 10 km resolution, the frictional terms represent the effects of eddies smaller than 10 km on the motion.

We represent the frictional force as the gradient of a "stress tensor". The latter represents correlations between the various velocity components of the small scale eddies.[2] If there are gradients in this stress, the fluid will accelerate or decelerate. So for example we can write:

$$\frac{du}{dt} = \frac{\partial}{\partial x}\frac{\tau_{xx}}{\rho} + \frac{\partial}{\partial y}\frac{\tau_{yx}}{\rho} + \frac{\partial}{\partial z}\frac{\tau_{zx}}{\rho} \tag{31}$$

where τ_{zx} is the stress component in the x-direction which varies in the z-direction and so on.

We won't concern ourselves further with the details about friction, as it is relatively unimportant at synoptic scales in the atmosphere and ocean. Where it *is* significant is in the vertical boundary layers, at the bottom of the atmosphere and ocean and at the surface of the ocean. We consider those in sec. (1.11).

The momentum equations are complex and *nonlinear*, involving products of velocities. As such, they are essentially unsolvable in this form. However, not all the terms are equally important. To see which ones dominate, we *scale* the equations. This means we will estimate the sizes of the various terms in the equation by using reasonable values for the variables at the scales we're interested in.

1.4 Equations of state

In addition to the continuity and the three momentum equations, we have an "equation of state" which relates the density to the temperature and, for the ocean, the salinity. In the atmosphere, the density and temperature are linked via the *Ideal Gas Law*:

$$p = \rho R T \tag{32}$$

[2]The details can be found, for example, in Holton's book, *An Introduction to Dynamic Meteorology*.

where $R = 287\, Jkg^{-1}K^{-1}$ is the gas constant for dry air. The law is thus applicable for a dry gas, i.e. one without moisture. But a similar equation applies in the presence of moisture if one replaces the temperature with the so-called "virtual temperature".[3]

In the ocean, both salinity and temperature affect the density. The dependence is expressed:

$$\rho = \rho(T, S) = \rho_c(1 - \alpha_T T + \alpha_S S) + h.o.t. \tag{33}$$

where ρ_c is a constant, T and S are the temperature and salinity and where $h.o.t.$ means "higher order terms". Increasing the temperature or decreasing the salinity reduces the density (makes lighter water). An important point is that the temperature and salinity corrections are much less than one, so that the density is dominated by the first term, ρ_c, which is constant. We exploit this in section (1.7) in making the so-called Boussinesq approximation.

1.5 Thermodynamic equation

We require one additional equation for the atmosphere. This is the *thermodynamic energy* equation:

$$c_v \frac{dT}{dt} + p\frac{d}{dt}(\frac{1}{\rho}) = c_p \frac{dT}{dt} - (\frac{1}{\rho})\frac{dp}{dt} = J \tag{34}$$

This expresses how the fluid responds to heating. The equation derives from the First Law of Thermodynamics, which states that the heat added to a volume minus the work done by the volume equals the change in its internal energy. Here c_v and c_p are the specific heats at constant volume and

[3] See, e.g. Holton, *An Introduction to Dynamic Meteorology*.

pressure, respectively, and J represents the heating. So heating changes the temperature and also the pressure and density of air.

We will find it convenient to use a different, though related, equation pertaining to the *potential temperature*. The potential temperature is defined:

$$\theta = T\,(\frac{p_s}{p})^{R/c_p} \tag{35}$$

This is the temperature a parcel would have if it were moved *adiabatically* (with zero heating) to a reference pressure, usually taken to be the pressure at the earth's surface. The advantage is that we can write the thermodynamic energy equation in terms of only one variable:

$$c_p \frac{d(ln\theta)}{dt} = \frac{J}{T} \tag{36}$$

This relation is simpler than (34) because it doesn't involve the pressure. It implies that the potential temperature is conserved on an air parcel if there is no heating ($J = 0$), i.e.:

$$\frac{d\theta}{dt} = 0 \tag{37}$$

1.6 The Geostrophic Relations

Not all the terms in the horizontal momentum equations are equally important. To see which ones dominate, we scale the equations. Take the x-momentum equation, neglecting the frictional term for the moment:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + 2\Omega w\,cos\theta - 2\Omega v\,sin\theta = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad 2\Omega W \qquad 2\Omega U \qquad \frac{\triangle_H p}{\rho L}$$

$$\frac{1}{2\Omega T} \qquad \frac{U}{2\Omega L} \qquad \frac{U}{2\Omega L} \qquad \frac{W}{2\Omega D} \qquad\qquad \frac{W}{U} \quad 1 \quad \frac{\triangle_H p}{2\Omega\rho U L}$$

In the second line we have *scaled* the equation by assuming typical values for the variables. In the third line, we have divided through by the scaling of the second Coriolis acceleration, $2\Omega U$ (which we have assumed will be important). The resulting parameters are all *dimensionless*, i.e. they have no units.

To estimate these parameters, we use values typical of weather systems:

$$U \approx 10\, m/sec, \quad 2\Omega = \frac{4\pi}{86400\, sec} \approx 10^{-4} sec^{-1},$$

$$L \approx 10^6\, m, \quad D \approx 10^4\, m, \quad T = L/U \approx 10^5\, sec$$

$$\triangle_H P/\rho \approx 10^3\, m^2/sec^2, \quad W \approx 1\, cm/sec, \tag{38}$$

The horizontal scale, 1000 km, is the *synoptic scale*. Notice that we assume the scale is the same in the x and y directions. Similarly we use a single velocity scale for both u and v; the vertical velocity though has a different scale, as vertical motion is much weaker at these horizontal scales.

The time scale, proportional to the length scale divided by the velocity scale, is the *advective* time scale. With an advective time scale, we have:

$$\frac{1}{2\Omega T} = \frac{U}{2\Omega L} \equiv \epsilon$$

So the first term is the same size as the second and third terms. This parameter is the *Rossby number*. At synoptic scales it is approximately:

$$\frac{U}{2\Omega L} = 0.1$$

So the first three terms are smaller than the second Coriolis term.

However, the other terms are even smaller:

$$\frac{W}{2\Omega D} = 0.01, \quad \frac{W}{U} = .001$$

and so can be neglected. Lastly, the pressure gradient term scales as:

$$\frac{\triangle p_H}{2\Omega \rho U L} = 1$$

and thus is comparable in size to the second Coriolis term.

The scalings given above are applicable to the atmosphere, but using values relevant to the ocean yields similar results (see problem 1.1). Furthermore, the scaling of the y-momentum equation is identical to that of the x-momentum equation. The dominant balances are thus:

$$-fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p \quad (39)$$

$$fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p \quad (40)$$

where:

$$f \equiv 2\Omega sin\theta$$

is the vertical component of the Coriolis parameter. These are the *geostrophic relations*, the primary balance in the horizontal direction at synoptic scales. They imply that if we know the pressure field, we can deduce the velocities.

Consider the flow in Fig. (5). The pressure is high to the south and low to the north. In the absence of rotation, this pressure difference would force the air to move north. But under the geostrophic balance, the air flows *parallel* to the pressure contours. Because $\frac{\partial}{\partial y}p < 0$, we have that $u > 0$ (eastward), from (40). The Coriolis force is acting to the right of the motion, exactly balancing the pressure gradient force. Furthermore, because the two forces are balanced, the motion is constant in time.

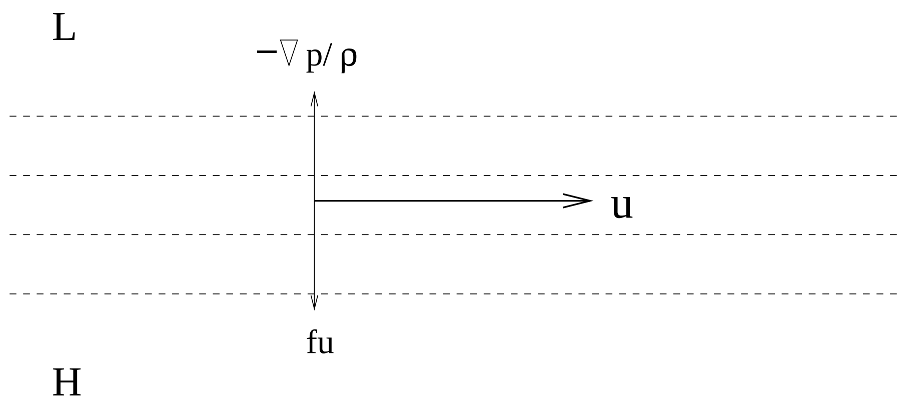


Figure 5: The geostrophic balance.

Note that $f = 2\Omega sin\theta$ is *negative* in the southern hemisphere. So the flow in Fig. (5) would be westward, with the Coriolis force acting to the left. In addition, the Coriolis force is *zero* at the equator. In fact, the geostrophic balance cannot hold there and one must invoke other terms in the momentum equations.

Exercise 1.6: Scale the x-momentum equation for parameters typical of the ocean. Assume:

$$U = 10cm/sec, \quad W = .01cm/sec, \quad L = 100km, \quad D = 5km$$

Also use the advective time scale, $T \propto L/U$ and that $sin(\theta) \approx 1$. Show that the geostrophic balance also applies with these scales. Note that I haven't given you the pressure scale, $\triangle p/\rho$. Can you estimate what it is, given the above scaling? What if it were actually much less than this—what could you say about the motion?

Exercise 1.7: Scale the y-momentum equation. Assume:

$$U = 1m/sec, \quad W = 1cm/sec, \quad L = 100m, \quad D = 0.5km$$

Also use the advective time scale, $T \propto L/U$ and that $sin(\theta) \approx 1$.

Which are the dominant terms? How big is $\triangle p/\rho$? Finally, write the approximate equation.

1.7 The Hydrostatic Balance

Now we scale the vertical momentum equation. For this, we need an estimate of the vertical variation in pressure. This is actually different than the horizontal variation:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Thus we have:

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - 2\Omega u cos\theta = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g \qquad (41)$$

$$\frac{WU}{L} \quad \frac{UW}{L} \quad \frac{UW}{L} \quad \frac{W^2}{D} \quad 2\Omega U \quad \frac{\triangle_V P}{\rho D} \quad g$$

$$\frac{UW}{gL} \quad \frac{UW}{gL} \quad \frac{UW}{gL} \quad \frac{W^2}{gD} \quad \frac{2\Omega U}{g} \quad \frac{\triangle_V P}{g\rho D} \quad 1$$

$$10^{-8} \quad 10^{-8} \quad 10^{-8} \quad 10^{-9} \quad 10^{-4} \quad 1 \quad 1$$

Again we have neglected the frictional term, which is small at these scales. Notice too that we divided through by g, assuming that gravity will be a large term. Indeed this is the case, as the gravity and pressure gradient terms are *much* larger than any of the others. So the vertical momentum equation can be replaced by:

$$\frac{\partial}{\partial z}p = -\rho g \tag{42}$$

This is the *hydrostatic* relation. This is a tremendous simplification over the full vertical momentum equation.

However, notice that the hydrostatic balance also applies if there is *no motion at all*. If we set $u = v = w = 0$ in the vertical momentum equation, we obtain the same balance. In fact, this is where the name comes from— ”hydro” meaning water and “static” meaning not moving. So the balance may not be that relevant for the dynamic (moving) part of the flow.

But it is. Let’s separate the pressure and density into static and dynamic components:

$$\begin{aligned} p(x,y,z,t) &= p_0(z) + p'(x,y,z,t) \\ \rho(x,y,z,t) &= \rho_0(z) + \rho'(x,y,z,t) \end{aligned} \tag{43}$$

The static components are only functions of z (so that they possess a vertical gradient). The dynamic components are usually much smaller than the static components, so that:

$$|p'| \ll |p_0|, \quad |\rho'| \ll |\rho_0|, \tag{44}$$

Thus we can write:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0+\rho'}\frac{\partial}{\partial z}(p_0+p') - g \approx -\frac{1}{\rho_0}\,(1-\frac{\rho'}{\rho_0})\,\frac{\partial}{\partial z}(p_0+p') - g$$

$$\approx -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + (\frac{\rho'}{\rho_0^2})\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g \tag{45}$$

Note we neglect terms proportional to the product of the dynamical variables, like $p'\rho'$.

How do we scale these dynamical pressure terms? Measurements suggest the vertical variation of p' is comparable to the horizontal variation:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2 \ .$$

The perturbation density, ρ', is roughly $1/100$ as large as the static density, so:

$$\frac{\rho'}{\rho_0} g \approx 10^{-1} m/sec^2 \ .$$

To scale these, we again divide by g, so that both terms are of order 10^{-2}. Thus while they are smaller than the static terms, they are still *two orders of magnitude larger* than the next largest term in (41). The approximate vertical momentum equation is still the hydrostatic balance, except now with the perturbation pressure and density:

$$\frac{\partial}{\partial z} p' = -\rho' g \tag{46}$$

The hydrostatic approximation is so good that it is used in most numerical models instead of the full vertical momentum equation. Models which use the latter are rarer and are called “non-hydrostatic” models.

While the values given above are for the atmosphere, a scaling using oceanic values produces the same result. The hydrostatic balance is an excellent approximation, in either system.

Exercise 1.8: The surface pressure in the atmosphere is due to the weight of all the air in the atmospheric column above the surface. Use the hydrostatic relation to estimate how large the surface pressure is. Assume that the atmospheric density decays exponentially with height:

$$\rho(z) = \rho_0 \, exp(-z/H)$$

where $\rho_0 = 1.2\, kg/m^3$ and the scale height, $H = 8.6$ km. Assume too that the pressure at $z = \infty$ is zero.

1.8 Approximations

We have greatly simplified the equations of motion. Instead of eight terms, the approximate x-momentum equation has only two terms. But the geostrophic relations are nevertheless *nonlinear*, because the pressure gradient term involves a product with the density. But we can employ a few more approximations which will allow us to further simplify the equations.

1.8.1 The β-plane approximation

After scaling, we see that the horizontal component of the Coriolis term, $2\Omega cos\theta$, vanishes from the momentum equations. The term which remains is the vertical component, $2\Omega sin\theta$. We will call this f. However, while all the other terms in the momentum equations are in Cartesian coordinates, f is a function of latitude.

To remedy this, we focus on a limited range of latitudes. We can Taylor-expand f about the central latitude, θ_0:

$$f(\theta) = f(\theta_0) + \frac{df}{d\theta}(\theta_0)\;(\theta - \theta_0) + \frac{1}{2}\,\frac{d^2 f}{d\theta^2}(\theta_0)\;(\theta - \theta_0)^2 + ... \qquad (47)$$

We will neglect the higher order terms, so that:

$$f \approx f(\theta_0) + \frac{df}{d\theta}(\theta_0)\;(\theta - \theta_0) \equiv f_0 + \beta y \qquad (48)$$

where:

$$f_0 = 2\Omega sin(\theta_0)$$

$$\beta = \frac{1}{a}\frac{df}{d\theta}(\theta_0) = \frac{2\Omega}{a}cos(\theta_0)$$

and

$$y = a(\theta - \theta_0)$$

Here a is the radius of the earth. We call (48) the β-plane approximation. Notice f is only a function of y; it varies only in the North-South direction. In order for the Taylor expansion to hold, the beta term must be much smaller than f_0, which implies:

$$\frac{\beta L}{f_0} \ll 1$$

This constrains the latitude range, L, since:

$$L \ll \frac{f_0}{\beta} = \frac{2\Omega sin(\theta)}{2\Omega cos(\theta)/a} = a\, tan(\theta_0) \approx a \tag{49}$$

So L must be smaller than the earth's radius, which is roughly 6400 km.

We can take advantage of the small β term in the geostrophic relations. Specifically, we replace f with f_0 and write:

$$v_g = \frac{1}{\rho f_0}\frac{\partial p}{\partial x} \tag{50}$$

$$u_g = -\frac{1}{\rho f_0}\frac{\partial p}{\partial y} \tag{51}$$

Despite this simplication though, the geostrophic relations remain non-linear, because density is a variable. We remedy that in the following two sections.

1.8.2 The Boussinesq approximation

In the atmosphere, the background density ρ_0 varies significantly with height. In the ocean however, the density barely changes at all. This allows us to make the *Boussinesq* approximation. In this, we take the density to be constant except in the "buoyancy term" on the RHS of the hydrostatic relation in (46).

Making this approximation, the geostrophic relations become:

$$
\begin{aligned}
v_g &= \frac{1}{\rho_c f_0} \frac{\partial}{\partial x} p && (52) \\
u_g &= -\frac{1}{\rho_c f_0} \frac{\partial}{\partial y} p && (53)
\end{aligned}
$$

where ρ_c is the constant density term in (33). Now the terms on the RHS are linear.

This simplification has an important effect because it makes the geostrophic velocities *horizontally non-divergent*. In particular:

$$
\frac{\partial}{\partial x} u_g + \frac{\partial}{\partial y} v_g = -\frac{1}{\rho_c f_0} \frac{\partial^2 p}{\partial y \partial x} + \frac{1}{\rho_c f_0} \frac{\partial^2 p}{\partial x \partial y} = 0 \qquad (54)
$$

We'll exploit this later on. The non-divergence comes about because the geostrophic velocities, which are horizontal, are much greater than the vertical velocities.

Under the Boussinesq approximation, the continuity equation is also much simpler. If we set $\rho_0 = \rho_c$ in (10), we obtain:

$$
\nabla \cdot \vec{u} = 0 \qquad (55)
$$

So the total velocities are non-divergent, i.e. the flow is *incompressible*. This assumption is frequently made in oceanography.

It may seem odd that the geostrophic velocities are horizontally non-divergent and that at the same time the total velocities are non-divergent. That would apparently imply that the vertical velocities don't vary with z(!) But in fact, these two facts can be made consistent with each other, while still allowing the vertical velocities to vary. We'll see this later on, when we consider "ageostrophic" velocities.

1.8.3 Pressure coordinates

We cannot responsibly use the Boussinesq approximation with the atmosphere, except possibly in the planetary boundary layer (this is often done, for example, when considering the surface boundary layers, as in sec. 1.11). But it is possible to achieve the same simplifications if we change the vertical coordinate to pressure instead of height.

We do this by exploiting the hydrostatic balance. Consider a pressure surface in two dimensions, (x, z). Applying the chain rule, we have:

$$\triangle p(x,z) = \frac{\partial p}{\partial x} \triangle x + \frac{\partial p}{\partial z} \triangle z = 0 \tag{56}$$

on the surface. Substituting the hydrostatic relation, we get:

$$\frac{\partial p}{\partial x} \triangle x - \rho g \triangle z = 0 \tag{57}$$

so that:

$$\frac{\partial p}{\partial x}|_z = \rho g \frac{\triangle z}{\triangle x}|_p \tag{58}$$

The left-hand side is the pressure gradient in x along a surface of constant height (hence the z subscript). The right-hand side is proportional to the *height gradient* along a surface of constant pressure—i.e. how much the pressure surface tilts in x. The gradient on the RHS thus has a p subscript, indicating pressure coordinates.

If we furthermore define the *geopotential*:

$$\Phi = gz \tag{59}$$

then we have:

$$\frac{\partial p}{\partial x}|_z = \rho \frac{\partial \Phi}{\partial x}|_p \tag{60}$$

This alteration removes the density from momentum equation, because:

$$-\frac{1}{\rho} \nabla p|_z \quad \rightarrow \quad -\nabla \Phi|_p$$

So the geostrophic balance in pressure coordinates is simply:

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi \tag{61}$$

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi \tag{62}$$

As with the Boussinesq approximation, the terms on the RHS are linear. So in pressure coordinates too, the horizontal velocities are horizontally non-divergent.

In addition, the change to pressure coordinates simplifies the continuity equation. We could show this by applying a coordinate transformation directly to (10), but it is even simpler to do it as follows. Consider a Lagrangian box (filled with a fixed number of molecules). The box has a volume:

$$\delta V = \delta x\, \delta y\, \delta z = -\delta x\, \delta y\, \frac{\delta p}{\rho g} \tag{63}$$

after substituting from the hydrostatic balance. The mass of the box is:

$$\delta M = \rho\, \delta V = -\frac{1}{g}\delta x\, \delta y\, \delta p$$

Since the number of molecules is fixed, the box's mass is also fixed. Conservation of mass implies:

$$\frac{1}{\delta M}\frac{d}{dt}\delta M = \frac{-g}{\delta x \delta y \delta p}\frac{d}{dt}(-\frac{\delta x \delta y \delta p}{g}) = 0 \tag{64}$$

Rearranging:

$$\frac{1}{\delta x}\delta(\frac{dx}{dt}) + \frac{1}{\delta y}\delta(\frac{dy}{dt}) + \frac{1}{\delta p}\delta(\frac{dp}{dt}) = 0 \tag{65}$$

If we let $\delta \to 0$, we get:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{66}$$

where ω (called "omega" in meteorology) is the velocity perpendicular to the pressure surface (like w is perpendicular to a z-surface). As with the Boussinesq approximation, the flow is incompressible in pressure coordinates.

The hydrostatic equation also takes a different form under pressure coordinates. It can be written:

$$\frac{d\Phi}{dp} = -\frac{RT}{p} \tag{67}$$

after invoking the Ideal Gas Law.

Pressure coordinates simplifies the equations considerably, but they are nonetheless awkward to work with in theoretical models. The lower boundary in the atmosphere (the earth's surface) is most naturally represented in z-coordinates, e.g. as $z = 0$. As the pressure varies at the earth surface, it is less obvious what boundary value to use for p. So we will use z-coodinates primarily hereafter. But the solutions in p-coordinates are often very similar.

Exercise 1.9: Derive (67), using the Ideal Gas Law.

1.9 Thermal wind

If we combine the geostrophic and hydrostatic relations, we get the thermal wind relations. These tell us about the velocity shear. Take, for instance, the p-derivative of the geostrophic balance for v:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f_0}\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{pf_0}\frac{\partial T}{\partial x} \tag{68}$$

after using (67). Note that the p passes through the x-derivative because it is constant on an isobaric (p) surface, i.e. they are independent variables.

Likewise:

$$\frac{\partial u_g}{\partial p} = \frac{R}{pf_0}\frac{\partial T}{\partial y} \tag{69}$$

after using the hydrostatic relation (67). Thus the vertical shear is proportional to the lateral gradients in the temperature.

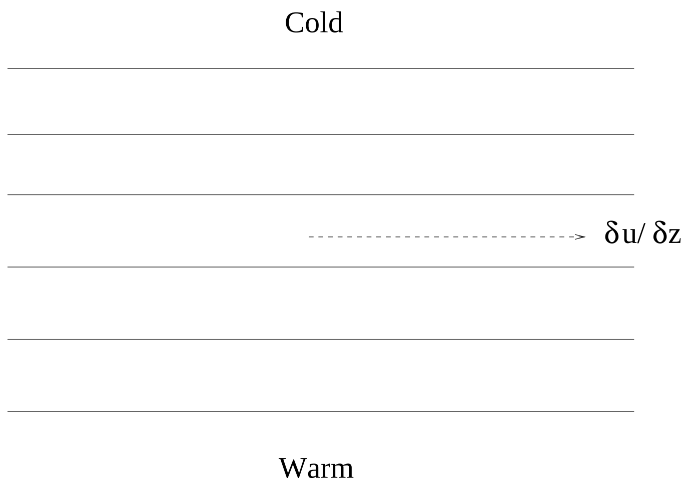


Figure 6: The thermal wind shear associated with a temperature gradient in the y-direction.

The thermal wind is parallel to the temperature contours, with the warm air/light water on the right. To see this, consider Fig. (6). There is a temperature gradient in y, meaning the thermal wind is oriented in the x-direction. The temperature is decreasing to the north, so the gradient is negative. From (69) we have then that $\partial u_g/\partial p$ is also negative. This implies that $\partial u_g/\partial z$ is *positive*, because the pressure decreases going up. So the zonal velocity is increasing going up, i.e. with the warm air to the right.

Using thermal wind, we can derive the geostrophic velocities on a nearby pressure surface, if we know the velocities on an adjacent surface and the temperature in the layer between the two levels. Consider the case shown in Fig. (7). The geopotential lines for the lower surface of the layer are

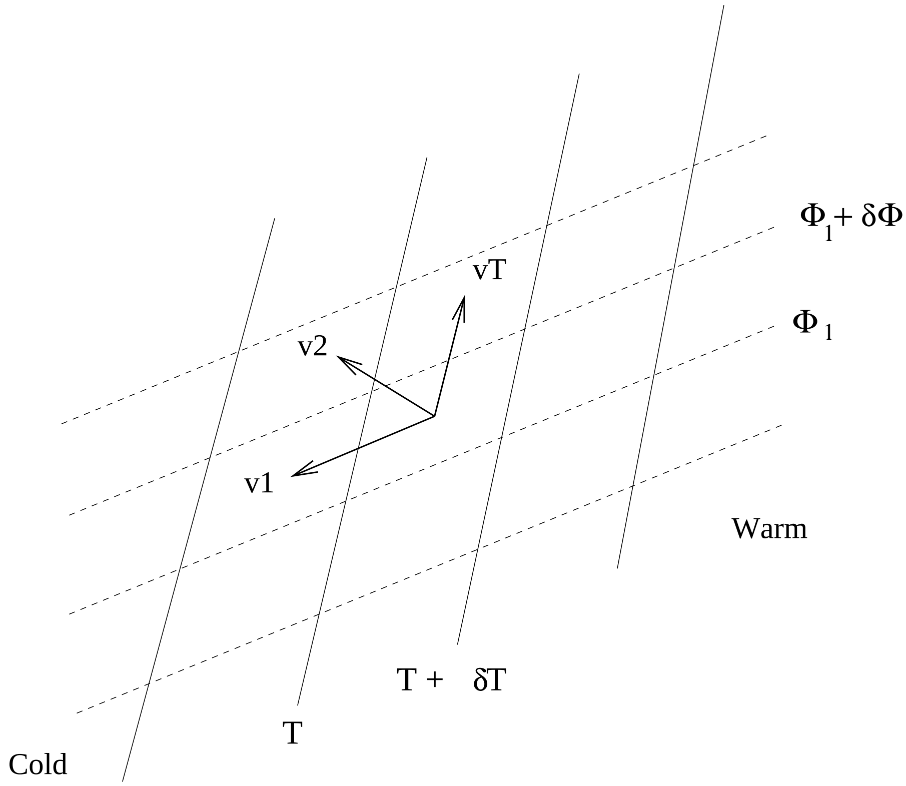


Figure 7: Thermal wind between two layers (1 and 2). The geopotential height contours for the lower layer, Φ_1, are the dashed lines and the temperature contours are the solid lines.

indicated by dashed lines. The wind at this level is parallel to these lines, with the larger values of Φ_1 to the right. The temperature contours are the solid lines, with the temperature increasing to the right. The thermal wind vector is parallel to these contours, with the larger temperatures on the right. We add the vectors v_1 and v_T to obtain the vector v_2, which is the wind at the upper surface. This is to the northwest, advecting the warm air towards the cold.

Notice that the wind vector turns clockwise with height. This is called *veering* and is typical of warm advection. Cold advection produces counter-clockwise turning, called *backing*.

Thus the geostrophic wind is parallel to the geopotential contours with larger values to the right of the wind (in the Northern Hemisphere). The thermal wind on the other hand is parallel to the mean temperature contours, with larger values to the right. Recall though that the thermal wind is not an actual wind, but the *difference* between the lower and upper level winds.

The thermal wind relations for the ocean derive from taking z-derivatives of the Boussinesq geostrophic relations (52-53), and then invoking the hydrostatic relation. The result is:

$$\frac{\partial v_g}{\partial z} = -\frac{g}{\rho_c f_0}\frac{\partial \rho}{\partial x} \tag{70}$$

$$\frac{\partial u_g}{\partial z} = \frac{g}{\rho_c f_0}\frac{\partial \rho}{\partial y} \tag{71}$$

Thus the shear in the ocean depends on lateral gradients in *density*, which can result from changes in either temperature or salinity.

Relations (70) and (71) are routinely used to estimate ocean currents from density measurement made from ships. Ships collect *hydrographic* measurements of temperature and salinity, and these are then used to determine $\rho(x, y, z, t)$, from the equation of state (33). Then the thermal wind relations are integrated upward from chosen level to determine (u, v) above the level, for example:

$$u_g(x, y, z) - u_g(x, y, z_0) = \int_{z0}^{z} \frac{1}{\rho_c f_0}\frac{\partial \rho(x, y, z)}{\partial y}\, dz \tag{72}$$

If (u, v, z_0) is set to zero at the lower level, it is known as a "level of no motion".

Exercise 1.10: Say the temperature at the South Pole is -20C and it's 40C at the Equator. Assuming the average wind speed is zero at the Earth's surface (1000 hPa), what is the mean zonal speed at 250 hPa at 45S?

Assume the temperature gradient is constant with latitude and pressure. Use the thermal wind relations in pressure coordinates and integrate them with respect to pressure to find the velocity difference between the surface and 250 hPa.

1.10 The vorticity equation

A central quantity in dynamics is the vorticity, which is the curl of the velocity:

$$\vec{\zeta} \equiv \nabla \times \vec{u} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \tag{73}$$

The vorticity resembles angular momentum in that it pertains to "spinning" motion. A tornado has significant vorticity, with its strong, counterclockwise swirling motion.

The rotation of the earth alters the vorticity because the earth itself is rotating. As noted in sec. (1.3), the velocity seen by a fixed observer is the sum of the velocity seen in the rotating frame (earth) and a rotational term:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{74}$$

As such, the vorticity is altered by the planet's rotation as well:

$$\vec{\zeta}_a = \nabla \times (\vec{u} + \vec{\Omega} \times \vec{r}) = \vec{\zeta} + 2\vec{\Omega} \tag{75}$$

We call $\vec{\zeta}_a$ the *absolute vorticity*. It is the sum of the *relative vorticity*, $\vec{\zeta} = \nabla \times \vec{u}$, and the *planetary vorticity*, $2\vec{\Omega}$.

Because synoptic scale motion is dominated by the horizontal velocities, the most important component of the vorticity is the vertical component:

$$\zeta_a \cdot \hat{k} = (\frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u) + 2\Omega sin(\theta) \equiv \zeta + f \tag{76}$$

This is the only component we will be considering.

We can derive an equation for ζ directly from the horizontal momentum equations. For this, we use the approximate equations that we obtained after scaling, retaining the terms to order Rossby number—the geostrophic terms, plus the time derivative and advective terms. We will use the Boussinesq equations; the same equation obtains if one uses pressure coordinates.

The equations are:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - fv = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p \qquad (77)$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + fu = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \qquad (78)$$

where

$$f = f_0 + \beta y$$

To obtain the vorticity equation, we *cross-differentiate* the equations: we take the x derivative of the second equation and subtract the y derivative of the first. The result, after some re-arranging, is:

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta + v\frac{df}{dy} + (\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \qquad (79)$$

or, alternately:

$$\frac{d_H}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \qquad (80)$$

where:

$$\frac{d_H}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \qquad (81)$$

is the Lagrangian derivative based on the horizontal velocities. Note that we can write the equation this way because f is only a function of y.

A useful feature of the vorticity equation is that the pressure term has dropped out. This follows from the Boussinesq approximation—if we hadn't made that, then there would be terms involving derivatives of the density. Likewise, the geopotential drops out when using pressure coordinates. This is left for an exercise.

The vorticity equation is related to a result known as *Kelvin's theorem*, derived in Appendix A. This is of fundamental importance in rotating fluid dynamics. It concerns how the vorticity and area of a fluid parcel is related to its latitude.

Exercise 1.11: Derive equation (80). Now derive the equivalent equation using pressure coordinates instead of z-coordinates.

1.11 Boundary layers

The vorticity equation (80 applies in the absence of friction, which we've seen is weak at synoptic scales. However, without friction there would be nothing to remove energy supplied by the sun (to the atmosphere) and by the winds (to the ocean), and the velocities would accelerate to infinity. Where friction *is* important is in boundary layers at the earth's surface in the atmosphere, and at the surface and bottom of the ocean. How do these layers affect the interior motion?

A central feature of the boundary layer is that the geostrophic balance is *broken by friction*. As noted in sec. (1.3), we represent friction as the gradient of a tensor, τ. A general feature of boundary layers is that the vertical extent is much less than their horizontal; so it will suffice to consider the vertical derivative of the stress. Thus the geostrophic relations (52) and

(53) are modified thus:

$$-f_0 v = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p \; + \; \frac{\partial}{\partial z}\frac{\tau_x}{\rho_c} \qquad (82)$$

$$f_0 u = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \; + \; \frac{\partial}{\partial z}\frac{\tau_y}{\rho_c} \qquad (83)$$

where τ_x and τ_y are stresses acting in the x and y directions. We can rewrite these relations thus:

$$-f_0(v - v_g) = -f_0 v_a = \frac{\partial}{\partial z}\frac{\tau_x}{\rho_c} \qquad (84)$$

$$f_0(u - u_g) = f_0 u_a = \frac{\partial}{\partial z}\frac{\tau_y}{\rho_c} \qquad (85)$$

where (u_a, v_a) are ageostrophic velocities (the departures from pure geostrophic flow). The ageostrophic velocities in the boundary layer are proportional to the stresses; if we know the frictional stresses, we can find these velocities.

We are only concerned with how the boundary layer affects the motion in the interior. To see this, consider the vorticity equation (80). We can rewrite this as:

$$(\zeta + f)^{-1}\frac{d}{dt}(\zeta + f) = \frac{d}{dt}ln(\zeta + f) = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \qquad (86)$$

Using the continuity equation (55), this is:

$$\frac{d}{dt}ln(\zeta + f) = \frac{\partial w}{\partial z} \qquad (87)$$

To get an indication about how the interior reponds to the boundary layers, we can integrate this in the vertical, between the upper and lower layers (lying at $z = a$ and $z = b$, for example):

$$\int_a^b \frac{d}{dt}ln(\zeta + f)\, dz = w(b) - w(a) \qquad (88)$$

This implies that the *vertical velocity* from the boundary layers act to force the flow in the interior. If there is flow out of the boundary layer, it will affect the interior flow by generating vorticity.

Consider the boundary layer at the surface of the ocean first. Let's say the surface is at $z = 0$ and the layer extends down to $z = -\delta$. To obtain w, we will again use the continuity equation (55):

$$\frac{\partial}{\partial z}w = -\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u_a - \frac{\partial}{\partial y}v_a \tag{89}$$

The horizontal divergence involves only the ageostrophic velocities because the geostrophic velocities are horizontally non-divergent. Integrating this over the layer yields:

$$w(0) - w(-\delta) = -\int_{-\delta}^{0}(\frac{\partial}{\partial x}u_a + \frac{\partial}{\partial y}v_a)\, dz \tag{90}$$

Since there is no flow out of the ocean surface, we can write $w(0) = 0$. Then we have, at the base of the layer:

$$w(-\delta) = \frac{\partial}{\partial x}U_s + \frac{\partial}{\partial y}V_s \tag{91}$$

where (U_s, V_s) are the horizontal *transports* in the surface layer:

$$U_s \equiv \int_{-\delta}^{0} u_a\, dz, \qquad V_s \equiv \int_{-\delta}^{0} v_a\, dz \tag{92}$$

We obtain these by integrating (84) and (85) vertically.

The stress at the surface ($z = 0$) is due to the wind:

$$\vec{\tau}^w = (\tau_x^w, \tau_y^w)$$

The stress at the base of the Ekman layer is zero—because the stress only acts in the layer itself. So we obtain:

$$U_s = \frac{\tau_y^w}{\rho_c f_0}, \qquad V_s = -\frac{\tau_x^w}{\rho_c f_0}$$

Thus the transport in the layer is *90 degrees to the right of the wind stress*. If the wind is blowing to the north, the transport is to the east. This is Ekman's (1905) famous result. Nansen had noticed that icebergs don't move downwind, but drift to the right of the wind. This simple model explains why this happens.

To get the vertical velocity, we take the divergence of these transports:

$$w(\delta) = \frac{\partial}{\partial x}\frac{\tau_y^w}{\rho_c f_0} + \frac{\partial}{\partial y}(-\frac{\tau_x^w}{\rho_c f_0}) = \frac{1}{\rho_c f_0}\hat{k}\cdot\nabla\times\vec{\tau}^w \tag{93}$$

So the vertical velocity is *proportional to the curl of the wind stress*. It is the curl, not the stress itself, which is most important for the interior flow in the ocean at synoptic scales.

Notice we made no assumptions about the stress in the surface layer itself to obtain this result. By integrating over the layer, we only need to know the stress at the surface. So the result (93) is *independent* of the stress distribution, $\tau(z)/\rho_c$, in the layer.

Then there is the bottom boundary layer, which exists in both the ocean and atmosphere. Let's assume the bottom is flat and that the Ekman layer goes from $z = 0$ to $z = \delta$. The integral of the continuity equation is:

$$w(\delta) - w(0) = w(\delta) = -(\frac{\partial}{\partial x}U_B + \frac{\partial}{\partial y}V_B) \tag{94}$$

where now U_B, V_B are the integrated (ageostrophic) transports in the bottom layer. Note that the vertical velocity vanishes at the *top* of the layer this time. Again we integrate (84) and (85) to find the transports. However, we don't know the stress at the bottom. All we know is that the bottom boundary isn't moving.

The formal way to proceed is to solve for the velocities in the layer. This is what Ekman (1905) did, assuming a simplified representation of the vertical mixing. In fact, you get the same result if you assume that the bottom stress is simply proportional to the interior flow. So we will present the simpler solution here and leave the full solution for Appendix B.

The interior flow is nearly geostrophic, so we take that to be:

$$(u, v) = (u_g, v_g) \tag{95}$$

We represent the bottom stress as:

$$\vec{\tau}^B = (-Au_g, -Av_g) \tag{96}$$

where A is a constant. The linear (or "Rayleigh") drag acts to de-accelerate the velocities. Thus the transports are:

$$U_B = \frac{\tau_y}{\rho_c f_0}|_0^\delta = -\frac{A}{\rho_c f_0} v_g \tag{97}$$

and:

$$V_B = -\frac{\tau_x}{\rho_c f_0}|_0^\delta = \frac{A}{\rho_c f_0} u_g \tag{98}$$

The stress vanishes at the top of the layer, at the boundary with the interior. Thus the vertical velocity from the layer is:

$$w(\delta) = -(\frac{\partial}{\partial x} U_B + \frac{\partial}{\partial y} V_B) = \frac{A}{\rho_c f_0}(\frac{\partial v_g}{\partial x} - \frac{\partial u}{\partial y}) = \frac{A}{\rho_c f_0} \zeta_g \tag{99}$$

In other words, the vertical velocity from the bottom Ekman layer is *proportional to the relative vorticity in the interior*. So there will be strong vertical motion beneath a strong vortex, like a hurricane.

Notice that the term $A/(\rho_c f_0)$ has units of length (because ζ_g has units of inverse time and w is in m/sec). In the Ekman derivation (Appendix B), you find that this is proportional to the depth of the boundary layer:

$$\frac{A}{\rho_c f_0} \equiv \frac{\delta}{2} \tag{100}$$

So:

$$w(\delta) = \frac{\delta}{2}\zeta_g \tag{101}$$

These two results represent a tremendous simplification. We can include the boundary layers without actually worrying about what is happening in the layers themselves. We will see that the bottom layers cause relative vorticity to decay in time (sec. 2.7), and the stress at the ocean surface forces the ocean (e.g. sec. 2.9). We can include these two effects and then neglect explicit friction hereafter.

2 Barotropic flows

Now we will examine specific solutions to the vorticity equation. In this chapter we assume the fluid is *barotropic*. This implies that there is no vertical shear in the horizontal velocities. While this may seem like a gross over-simplification, many of the phenomena seen in the barotropic case carry over to stratified (baroclinic) flows.

2.1 Vertical shear in a barotropic fluid

The fact that there is no shear follows from the thermal wind relations (68) and (69). If the temperature is constant on pressure surfaces, so that $T = T(p)$, then:

$$\frac{\partial v_g}{\partial p} = \frac{\partial u_g}{\partial p} = 0 \tag{102}$$

So the geostrophic velocities don't change with height. The velocities at the top of the atmosphere are the same as those at the surface.

The corresponding condition in the ocean, from (70- 71), is that the density is constant on z-surfaces. Thus if $\rho = \rho(z)$, we have:

$$\frac{\partial v_g}{\partial z} = \frac{\partial u_g}{\partial z} = 0 \tag{103}$$

Then the currents are the same at the surface and bottom of the ocean.

The lack of vertical shear implies that fluid moves in *columns* in barotropic flows. Parcels which are vertically aligned stay aligned. This simplification greatly simplifies the solutions, because the motion is really two dimensional rather than three dimensional.

2.2 Barotropic PV equation

Now we will derive the equation of motion for barotropic flows. This comes from the quasi-horizontal vorticity equation given in (80):

$$\frac{d_H}{dt}(\zeta + f) = -(f + \zeta)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) = (f + \zeta)\frac{\partial}{\partial z}w \tag{104}$$

after invoking the incompressibility condition (55) again. Note that this involves three unknowns, u, v and w.

Assume we have a layer of fluid (atmosphere or ocean) which is bounded by two surfaces, the lower one at z_0 and the upper at z_1. Define the total depth to be $D = z_1 - z_0$. Because the velocities don't vary with height, it is simple to integrate (104) in the vertical direction:

$$\int_{z0}^{z1} \frac{d_H}{dt}(\zeta + f)\, dz = D\frac{d_H}{dt}(\zeta + f) = (f + \zeta)[w(z_1) - w(z_0)] \tag{105}$$

The terms involving ζ pass through the integrals, since they are independent of height.

Three effects can induce vertical motion at the boundaries. If the boundary is irregular (not flat), this will cause fluid parcels to move vertically. For example, when the wind blows over a mountain range, the parcels must go up and then come down again. Second, if the boundary moves (like the ocean surface), this will also yield vertical motion. An friction also can induce vertical motion, as we have seen with Ekman layers if there is convergence or divergence in the layer. We'll neglect friction for the moment, then replace it in section (2.6).

The vertical velocity is actually the Lagrangian derivative of the height:

$$w = \frac{d}{dt}z$$

where here, d/dt is the full Lagrangian derivative. If the parcel is confined to a surface, say at the bottom, then we'd have:

$$w = \frac{d}{dt} z_0(x, y) = u \frac{\partial}{\partial x} z_0 + v \frac{\partial}{\partial y} z_0$$

ooo

In the absence of friction, we can write:

$$w(z_1) - w(z_0) = \frac{d}{dt}(z_1 - z_0) = \frac{d_H}{dt} D \tag{106}$$

The last derivative is a horizontal one because D is a function of (x, y, t). Note that

So the integrated vorticity equation is:

$$D \frac{d_H}{dt}(\zeta + f) = (f + \zeta) \frac{d_H}{dt} D \tag{107}$$

which implies:

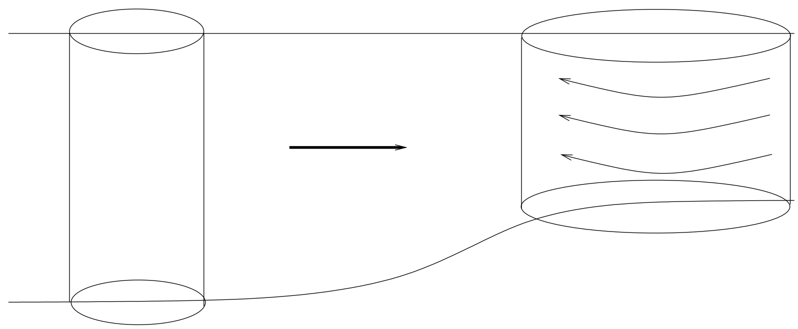
$$\frac{d_H}{dt}[\frac{\zeta + f}{D}] = 0 \tag{108}$$

Equation (108) expresses the *conservation of potential vorticity* for a barotropic fluid. In the absence of friction, we have that:

$$\frac{\zeta + f}{D} = const. \tag{109}$$

on fluid parcels.

Consider the fluid column shown in Fig. (2.2), initially with no vorticity. As it moves to the right, it encounters ridge. Thus the depth, D, is decreasing. In order for the PV to remain constant, the vorticity must also decrease, becoming negative in this case. So the column acquires a clockwise spin when surmounting the ridge.



The conservation of PV is similar to Kelvin's theorem (Appendix A). This is because the volume of the fluid column is conserved, due to incompressibility, which implies the product of the height, D, and the column area are conserved. So if D decreases, the column's area increases, and Kelvin's theorem demands that the vorticity decrease to offset that.

Exercise 2.1: Show that (108) follows from (107). Say we have a cyclone with $\zeta = f/2$ and 2 km high. What is the cyclone's vorticity if it is compressed to 1 km over a mountain range? Assume it stays at the same latitude.

2.2.1 The quasi-geostrophic vorticity equation

The PV equation (108) can be derived directly from the *shallow water equations*, which are the equations which govern a constant density fluid with topography. Interestingly, the shallow water equations apply to flows with a fully varying Coriolis parameter and steep topography. They are the equations that we solve for predicting the global tides.

But a significant drawback with equation (108) is that it has two unknowns, u and v. So we can't solve it by itself. In the shallow water context, we have to supplement the equation with an additional one (derived from continuity).

But by making several approximations, valid at synoptic scales, we can obtain an alternate form of the equation which has only one unknown. This is *quasi-geostrophic vorticity equation*. The approximations are as follows:

- The Rossby number is small
- $|\beta y| \ll f_0$
- The bottom topography is weak

Consider the first condition. From sec. (1.6) we know that when the Rossby number, ϵ, is small, the horizontal velocities are approximately in geostrophic balance. So if:

$$\vec{u} = \vec{u}_g + \vec{u}_a \tag{110}$$

where $\vec{u}_a$ is the ageostrophic velocity, then:

$$\frac{|\vec{u}_a|}{|\vec{u}_g|} \propto \epsilon$$

Likewise, the vorticity is much less than f_0, because:

$$\frac{|\zeta|}{f_0} \propto \frac{U}{f_0 L} = \epsilon$$

To satisfy the second condition, we assume:

$$\frac{|\beta y|}{|f_0|} \propto \epsilon$$

Of course we could demand that the β term be even smaller, but assuming a Rossby number scaling will preserve the variation of f.

Lastly, there is the condition on the bottom topography. Assume we can write the depth as:

$$D = D_0 - h(x, y)$$

Here D_0 is a constant reference depth (like 5 km for the interior ocean). Then, to satisfy the last condition above, we assume:

$$\frac{|h|}{D_0} \propto \epsilon$$

So the bottom variations are small compared to the reference depth. We don't allow for mountains which project upward through the entire fluid. The tallest ones can only extend to say 10 % of the total depth (Fig. 8).

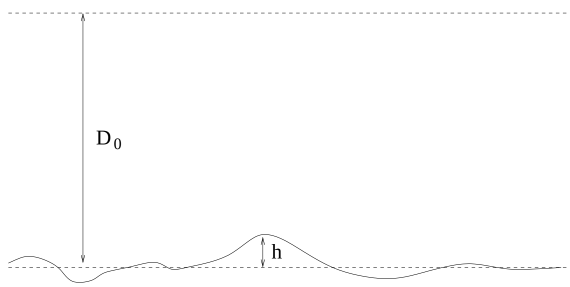


Figure 8: The geometry of our fluid layer. The topographic height, h, is much less than the depth of the layer.

We now use these assumptions to write a simpler version of the vorticity equation. First, we replace the horizontal velocities with their geostrophic equivalents in the Lagrangian derivative:

$$\frac{d_H}{dt} \rightarrow \frac{d_g}{dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \tag{111}$$

Similarly, we replace the vorticity with its geostrophic version:

$$\zeta \rightarrow \zeta_g = \frac{\partial}{\partial x} v_g - \frac{\partial}{\partial y} u_g \tag{112}$$

So the PV equation is:

$$\frac{d_H}{dt} \frac{\zeta + f}{D} = \frac{d_g}{dt} \frac{\zeta_g + f_0 + \beta y}{D_0 - h} = 0 \tag{113}$$

Using our three assumptions, we can simplify the PV as follows:

$$\frac{\zeta_g + f_0 + \beta y}{D_0 - h} = \frac{f_0}{D_0}\left(\frac{1 + \zeta_g/f_0 + \beta y/f_0}{1 - h/D_0}\right) \tag{114}$$

$$\approx \frac{f_0}{D_0}(1 + \frac{\zeta_g}{f_0} + \frac{\beta y}{f_0})(1 + \frac{h}{D_0}) \tag{115}$$

$$\approx \frac{f_0}{D_0} + \frac{\zeta_g}{D_0} + \frac{\beta y}{D_0} + \frac{f_0 h}{D_0^2} \tag{116}$$

Each of the last three terms are of order Rossby number compared to the first term. Moreover, the terms which we've dropped involve the *products* of the small terms and are hence of order Rossby number squared. Substituting this into (113) yields:

$$\frac{d_g}{dt}(\zeta_g + \beta y + \frac{f_0}{D_0}h) = 0 \tag{117}$$

after multiplying through by the constant, D_0, and dropping the constant f_0/D_0 (which has zero derivative). This the quasi-geostrophic PV equation without forcing or friction.

The great advantage of this is that it has only one unknown: the pressure. From the geostrophic relations, we have:

$$u_g = -\frac{1}{\rho_c f_0}\frac{\partial}{\partial y}p\ , \quad v_g = \frac{1}{\rho_c f_0}\frac{\partial}{\partial x}p \tag{118}$$

The relative vorticity can also be expressed solely in terms of the pressure:

$$\zeta_g = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = \frac{1}{\rho_c f_0}\nabla^2 p \tag{119}$$

We can simplify this somewhat by defining a *streamfunction*:

$$\psi = \frac{p}{\rho_c f_0} \tag{120}$$

Then we have:

$$u = -\frac{\partial}{\partial y}\psi, \qquad v = \frac{\partial}{\partial x}\psi, \qquad \zeta_g = \nabla^2\psi \tag{121}$$

Using these, the vorticity equation is:

$$(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y})(\nabla^2\psi + \beta y + \frac{f_0}{D_0}h) = \tag{122}$$

$$(\frac{\partial}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y})(\nabla^2\psi + \beta y + \frac{f_0}{D_0}h) = 0 \tag{123}$$

Exercise 2.2: Use scaling to figure out how big the ageostrophic velocities typically are. Use z-coordinates and assume the Boussinesq approximation. First show the horizontal divergence of the ageostrophic velocities is the same size as the vertical derivative of the vertical velocity. Then scale the result. Use typical oceanic values for W, L and D (see exercise 1.6). Does the result make sense with regards to the Rossby number?

Exercise 2.3: Consider a barotropic layer between z_0 and z_1, where z_0 is a flat surface. What happens if the upper surface can *move*? Assume that $z_1 = D_0 + \eta(x, y, t)$. Let the bottom be at $z_0 = 0$. Write the quasi-geostrophic PV equation for this case.

2.3 Geostrophic contours

The PV equation (122) states that the PV is conserved on fluid parcels, where the PV is:

$$q = \nabla^2\psi + \beta y + \frac{f_0}{D_0}h$$

This is a strong constraint. The PV is comprised of a time-varying portion (the vorticity) and a time-independent part (due to β and the bottom topography). So we can rewrite equation (122) this way:

$$\frac{d_g}{dt}\nabla^2\psi + \vec{u}_g \cdot \nabla q_s = 0 \tag{124}$$

where the function:

$$q_s \equiv \beta y + \frac{f_0}{D_0} h$$

defines the *geostrophic contours*, the stationary (unchanging) part of the potential vorticity.

If a parcel crosses the geostrophic contours, its relative vorticity must change to conserve the total PV. Consider the example in figure (9). Here there is no topography, so the contours are just latitude lines ($q_s = \beta y$). Northward motion is accompanied by a *decrease* in relative vorticity: as y increases, ζ_g must decrease. If the parcel has zero vorticity initially, it acquires negative vorticity (clockwise circulation) in the northern hemisphere. Southward motion likewise generates positive vorticity. This is just Kelvin's theorem again.

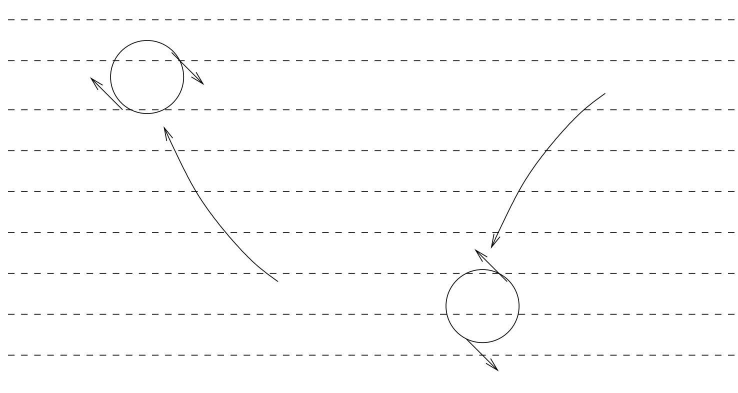


Figure 9: The change in relative vorticity due to northward or southward motion relative to βy.

Topography generally distorts the geostrophic contours. If it is large enough, it can overwhelm the βy term locally, even causing *closed* contours (near mountains or basins). But the same principle holds, as shown in Fig. (10). Motion towards larger values of q_s generates negative vorticity and motion to lower values of q_s generates positive vorticity.

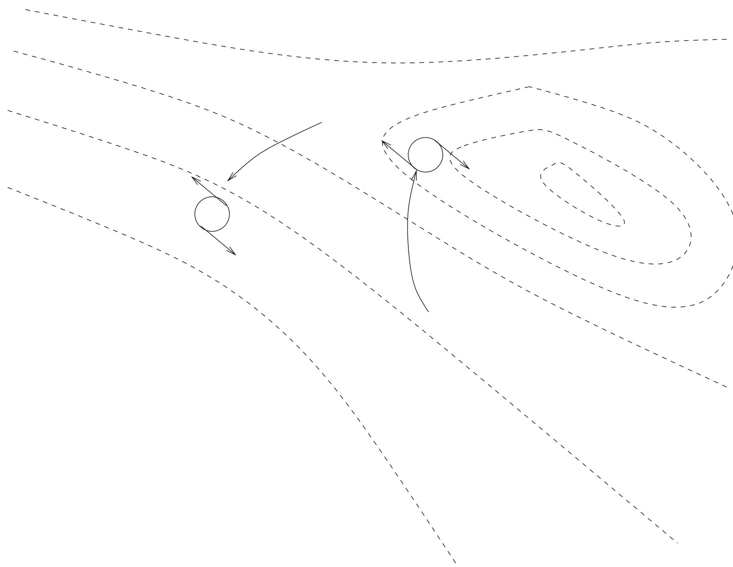


Figure 10: The change in relative vorticity due to motion across geostrophic contours with topography.

If the flow is steady, then (166) is just:

$$\vec{u}_g \cdot \nabla(\zeta_g + q_s) = 0 \tag{125}$$

Thus for a steady flow the geostrophic flow is *parallel to the total PV contours*, $q = \zeta_g + q_s$. If the relative vorticity is weak, so that $\zeta_g \ll q_s$, then:

$$\vec{u}_g \cdot \nabla q_s = 0 \tag{126}$$

So the flow follows the geostrophic contours.

Take the case again of no topography. Then:

$$\vec{u}_g \cdot \nabla \beta y = \beta v_g = 0 \tag{127}$$

So the steady flow is purely *zonal*. This is because meridional motion necessarily implies a changing relative vorticity. An example are the Jet Streams in the atmosphere. These is approximately zonal flows.

Alternately if the region is small enough so that we can ignore changes in the Coriolis parameter, then:

$$\vec{u}_g \cdot \nabla h = 0 \tag{128}$$

(after dropping the constant f_0/D_0 factor). Then the flow follows the topographic contours. This is why many major currents in the ocean are parallel to the isobaths.

Whether such steady flows actually exist depends in addition on the boundary conditions. The atmosphere is a *re-entrant domain*, so a zonal wind can simply wrap around the earth (Fig. 11, left). But most ocean basins have lateral boundaries (continents), and these block the flow. As such, steady, along-contour flows in a basin can occur *only where topography causes the contours to close* (Fig. 11, right). This can happen in basins.

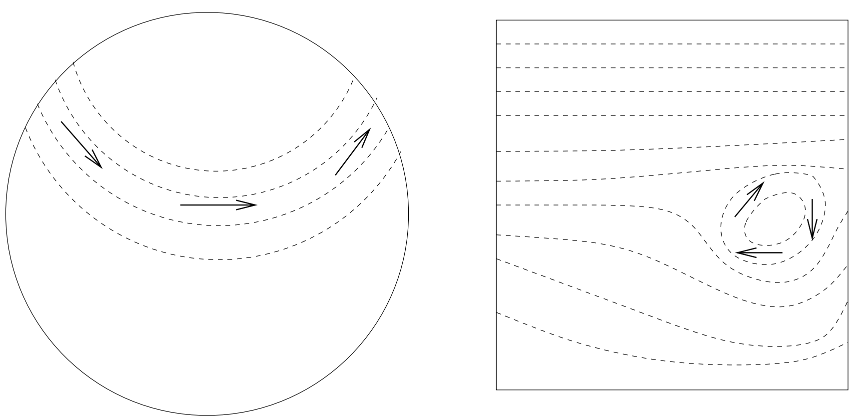


Figure 11: Steady, along-geostrophic contour flow in the atmosphere (left) and in the ocean (right).

Consider Fig. (12). This is a plot of the mean surface velocities, derived from surface drifters, in and near the Lofoten Basin off the west coast of Norway. The strong current on the right hand side is the Norwegian

Atlantic Current, which flows in from the North Atlantic and proceeds toward Svalbard. Notice how this follows the continental slope (the steep topography between the continental shelf and deeper ocean). In the basin itself, the flow is more variable, but there is a strong, clockwise circulation in the deepest part of the basin, where the topographic contours are closed. Thus both closed and open geostrophic contour flows are seen here.

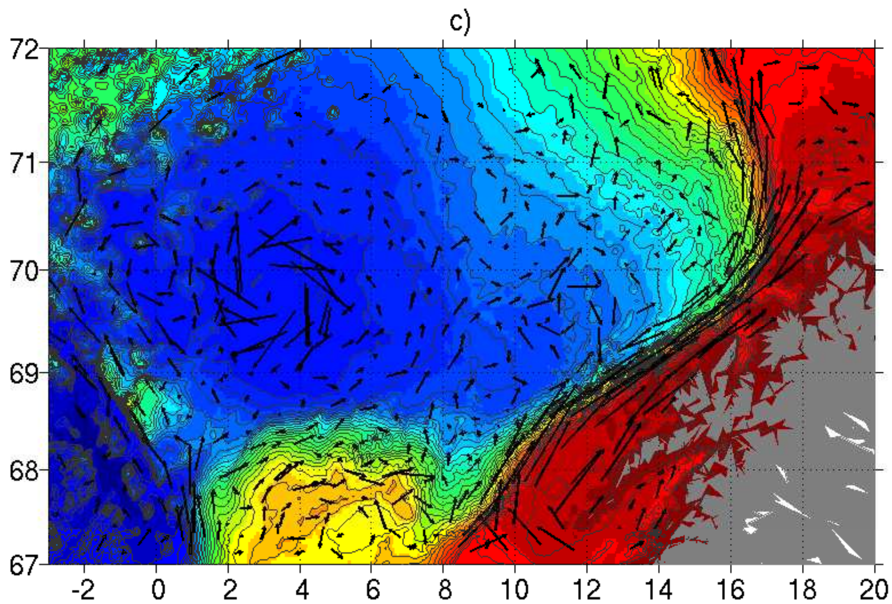


Figure 12: Mean velocities estimated from surface drifters in the Lofoten Basin west of Norway. The color contours indicate the water depth. Note the strong flow along the continental margin and the clockwise flow in the center of the basin, near 2° E. From Koszalka et al. (2010).

If the relative vorticity is not small compared to q_s, the flow will deviate from the latter contours. This can be seen for example with the Gulf Stream, which crosses topographic contours as it leaves the east coast of the U.S. If the relative vorticity is much stronger than q_s, then we have:

$$\vec{u}_g \cdot \nabla \zeta_g \approx 0 \tag{129}$$

as a condition for a steady flow. Then the flow follows contours of constant vorticity. An example is flow in a vortex. The vorticity contours are circular or ellipsoidal and the streamlines have the same shape. The vortex persists for long times precisely because it is near a steady state.

2.4 Barotropic Rossby waves

2.4.1 Linearization

The barotropic PV equation (122) is still a nonlinear equation, so analytical solutions are difficult to find. But we can make substantial progress by *linearizing* the equation.

Consider the case with no topography. As we found in the previous section, the only steady flow we could expect is a zonal one. So we could write:

$$u = U + u', \quad v = v'$$

Here, U is a constant zonal velocity which is assumed to be much greater than the primed velocities. In the atmosphere, U would represent the Jet Stream. Because U is constant, the relative vorticity is just:

$$\zeta = \frac{\partial}{\partial x}v' - \frac{\partial}{\partial y}u' = \zeta'$$

We substitute the velocities and vorticity into the PV equation to get:

$$\frac{\partial}{\partial t}\zeta' + (U + u')\frac{\partial}{\partial x}\zeta' + v'\frac{\partial}{\partial y}\zeta' + \beta v' = 0 \qquad (130)$$

Because the primed variables are small, we neglect their products. That leaves an equation with only linear terms. Written in terms of the streamfunction (and dropping the primes), we have:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = 0 \qquad (131)$$

This is the *barotropic Rossby wave equation*. Again, this has only one unknown: the streamfunction, ψ.

2.4.2 Wave solutions

Equation (131) is a first order *wave equation*. There are standard methods to solve such equations. One of the most common is the *Fourier transform*, in which we write the solution as an infinite series of sinusoidal waves. Exactly which type of wave one uses depends on the boundary conditions. To illustrate the method, we assume an infinite plane. Although this is not very realistic for the atmosphere, the results are very similar to those in a east-west re-entrant channel.

Thus we will write:

$$\psi = Re\{\sum_k \sum_l A(k,l) e^{ikx+ily-i\omega t}\} \tag{132}$$

where:

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{133}$$

is a complex number. The amplitude, A, can also be complex, i.e.

$$A = A_r + iA_i \tag{134}$$

However, since the wavefunction, ψ, is real, we need to take the real part of the product of A and $e^{i\theta}$. This is signified by the $Re\{x\}$ operator.

Now because the Rossby wave equation is linear, we can consider the solution for a *single* wave. This is because with a linear equation, we can add individual wave solutions together to obtain the full solution. So we consider the following solution:

$$\psi = Re\{Ae^{ikx+ily-i\omega t}\} \tag{135}$$

Here k and l are *wavenumbers* in the x and y directions, and ω is the wave *frequency*.

Consider the simpler case of a one-dimensional wave (in x), with a unit amplitude:

$$\psi = Re\{e^{ikx-i\omega t}\} = cos(kx - \omega t) \qquad (136)$$

The wave has a *wavelength* of $2\pi/k$. If $\omega > 0$, the wave propagates toward larger x (Fig. 13). This is because as t increases, $-\omega t$ decreases, so x must increase to preserve the phase of the wave (the argument of the cosine).

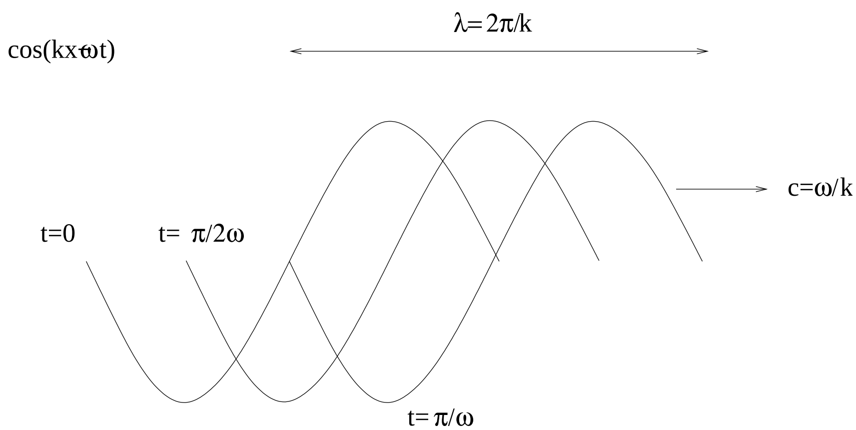


Figure 13: A one-dimensional wave, propagating toward the right.

In other words, if we define the phase:

$$\theta = kx - \omega t$$

then the position of, say, the wave crest at $\theta = 2\pi$ is:

$$x(\theta = 2\pi) = \frac{2\pi}{k} + \frac{\omega}{k}t$$

Thus, so long as ω and k are positive, the crest moves to the right, because x is increasing. The velocity of the crest is just:

$$c = \frac{\omega}{k} \tag{137}$$

This is known as the wave's *phase speed*. We often incorporate the phase speed by writing the wave form thus:

$$\psi = cos\, k(x - ct) \tag{138}$$

Notice that c has units of length over time, as expected for a velocity.

If the phase speed depends on the wavelength (wavenumber), i.e. if $c = c(k)$, we say the wave is *dispersive*. This is because different size waves will move at different speeds. Thus a packet of waves, originating from a localized region, will separate in time. Waves that are *non-dispersive* move at the same speed regardless of wavelength. A packet of such waves would move away from their region of origin together.

2.4.3 Rossby wave phase speed

Now we return to the linearized barotropic PV equation (131) and substitute in our general wave solution in (135). We get:

$$(-i\omega + ikU)(-k^2 - l^2)\, Ae^{ikx+ily-i\omega t} + i\beta k\, Ae^{ikx+ily-i\omega t} = 0 \tag{139}$$

(We will drop the $Re\{x\}$ operator, but remember that in the end, it is the real part we're interested in). Notice that both the wave amplitude and the exponential term drop out. This is typical of linear wave problems: we get no information about the amplitude from the equation itself (that requires specifying initial conditions). Solving for ω, we get:

$$\omega = kU - \frac{\beta k}{k^2 + l^2} \tag{140}$$

This is the *Rossby wave dispersion relation*. It relates the frequency of the wave to its wavenumbers. The corresponding phase speed (in the x-direction) is:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2} \equiv U - \frac{\beta}{\kappa^2} \tag{141}$$

where $\kappa = (k^2 + l^2)^{1/2}$ is the total wavenumber.

There are a number of interesting features about this. First, the phase speed depends on the wavenumbers, so the waves are dispersive. The largest speeds occur when k and l are small, corresponding to long wavelengths. Thus large waves move faster than small waves.

Second, all waves propagate *westward* relative to the mean velocity, U. If $U = 0$, $c < 0$ for *all* (k, l). This is a distinctive feature of Rossby waves. Satellite observations of Rossby waves in the Pacific Ocean show that the waves, originating off of California and Mexico, sweep westward toward Asia (as seen hereafter).

The phase speed also has a meridional component, and this can be either towards the north or south:

$$c_y = \frac{\omega}{l} = \frac{Uk}{l} - \frac{\beta k}{l(k^2 + l^2)} \tag{142}$$

The sign of c_y thus depends on the signs of k and l. So Rossby waves can propagate northwest, southwest or west—but not east.

With a mean flow, the waves can be swept eastward, producing the appearance of eastward propagation. This happens frequently in the atmosphere, where the mean westerlies advect Rossby waves (pressure systems) eastward. If

$$\kappa > \kappa_s \equiv (\frac{\beta}{U})^{1/2}$$

the wave moves eastward. Longer waves move westward, opposite to the mean flow, and short waves are advected eastward. If $\kappa = \kappa_s$, the wave is *stationary* and the crests don't move at all—the wave is propagating west at exactly the same speed that the background flow is going east. Stationary waves can only occur if the mean flow is eastward, because the waves propagate westward.

Example: How big is the stationary wave if the mean flow is 20 m/sec to the east? Assume we are at 45 degrees N and that $k = l$.

At 45N:

$$\beta = \frac{1}{6.3 \times 10^6} \frac{4\pi}{86400} cos(45) = 1.63 \times 10^{-11} \, m^{-1} sec^{-1}$$

so:

$$\kappa_s = \frac{\beta}{U} = (\frac{1.63 \times 10^{-11} \, m^{-1} sec^{-1}}{20 \, m/sec})^{1/2} = 9.03 \times 10^{-7} \, m^{-1}$$

Assuming $\lambda_x = \lambda_y$, we have that:

$$\kappa_s = \frac{2\sqrt{2}\pi}{\lambda_s}$$

so:

$$\lambda_s = 9.84 \times 10^6 \, m \approx 9000 \, km$$

Remember that this is a wavelength, so it includes positive and negative pressure anomalies. But it still is larger than our typical storm scale of 1000 km.

Exercise 2.4: Bottom topography, like the β-effect, can support Rossby-like waves, called *topographic waves*. To see this, use the linearized version of the barotropic PV equation (123) with β=0 (a constant

Coriolis parameter). Assume the bottom slopes uniformly to the east:

$$H = H_0 - \alpha x \tag{143}$$

Derive the phase speed (in the y-direction) for the waves, assuming no background flow ($U = V = 0$). Which way do the waves propagate, relative to the shallower water? What if $\alpha < 0$? What about in the southern hemisphere?

Exercise 2.5: We solved the Rossby wave problem on an infinite plane. Now consider what happens if there are solid walls. Start with the linear vorticity equation, with no mean flow ($U = 0$). Assume the variations in y are weak, so that you can approximate the vorticity by $\frac{\partial}{\partial x}v$. For the boundary conditions, let $\psi = 0$ at $x = 0$ and $x = L$—this ensures that there is no flow into the walls. What are the solutions for ω and k?

Hint 1: Assume $\psi = A(x)cos(kx - \omega t)$

Hint 2: Impose the boundary conditions on A.

Hint 3: The coefficients of the sine and cosine terms should both be zero.

Hint 4: The solutions are *quantized* (have discrete values).

2.4.4 Westward propagation: mechanism

We have discussed how motion across the mean PV contours, q_s, induces relative vorticity. The same is true with a Rossby wave. Fluid parcels which are advected north in the wave acquire negative vorticity, while those advected south acquire positive vorticity (Fig. 14). Thus one can

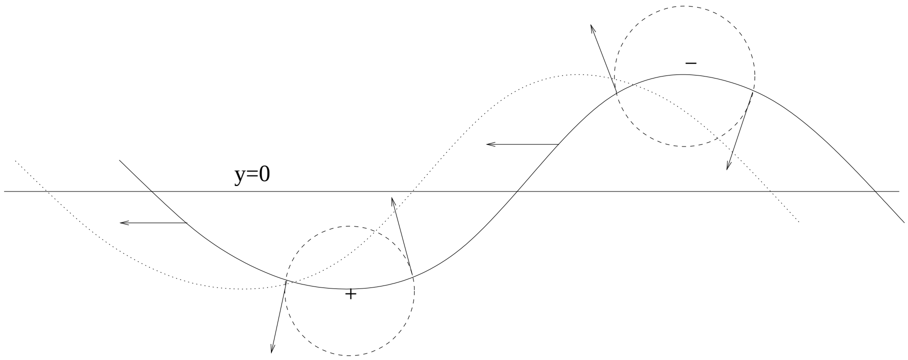


Figure 14: Relative vorticity induced in a Rossby wave. Fluid advected northwards acquires negative vorticity and fluid advected southwards positive vorticity.

think of a Rossby wave as a string of negative and positive vorticity anomalies (Fig. 15).

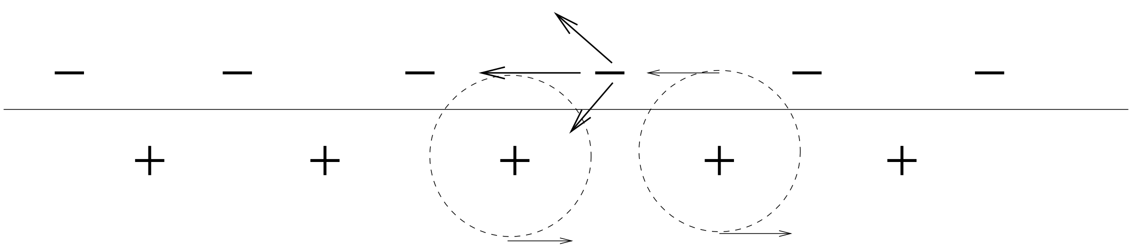


Figure 15: The Rossby wave as a string of vorticity anomalies. The cyclone in the right hand circle advects the negative anomaly to the southwest, while the left cyclone advects it toward the northwest. The net effect is westward motion.

Now the negative anomalies to the north will act on the positive anomalies to the south, and vice versa. Consider the two positive anomalies shown in Fig. (15). The right one advects the negative anomaly between them southwest, while the left one advects it northwest. Adding the two velocities together, the net effect is a westward drift for the anomaly. Similar reasoning suggests the positive anomalies are advected westward by the negative anomalies.

What does a Rossby wave look like? Recall that ψ is proportional to the geopotential, or the pressure in the ocean. So a sinusoidal wave is a sequence of high and low pressure anomalies. An example is shown in Fig. (16). This wave has the structure:

$$\psi = cos(x - \omega t)sin(y) \tag{144}$$

(which also is a solution to the wave equation, as you can confirm). This appears to be a grid of high and low pressure regions.

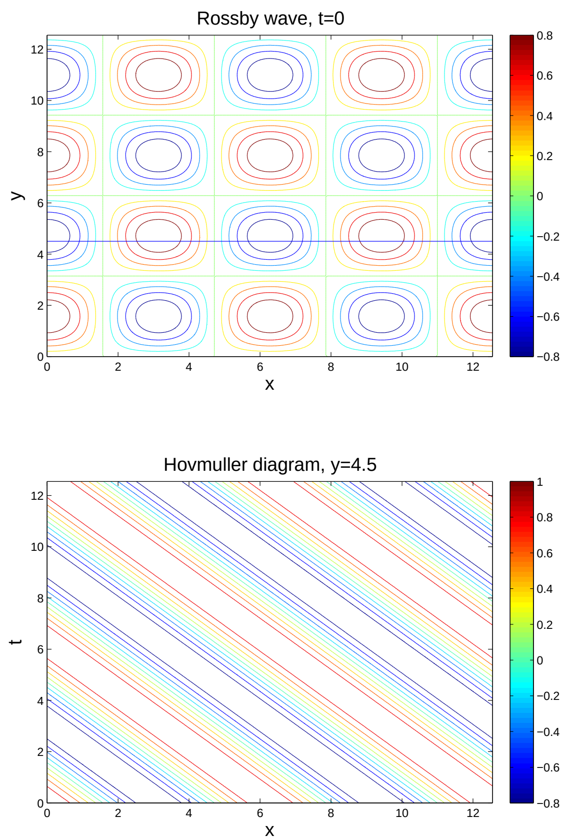


Figure 16: A Rossby wave, with $\psi = cos(x - \omega t)sin(y)$. The red corresponds to high pressure regions and the blue to low. The lower panel shows a "Hovmuller" diagram of the phases at $y = 4.5$ as a function of time.

The whole wave in this case is propagating westward. Thus if we take a cut at a certain latitude, here $y = 4.5$, and plot $\psi(x, 4.5, t)$, we get the plot in the lower panel. This shows the crests and troughs moving westward at a constant speed (the phase speed). This is known as a "Hovmuller" diagram.

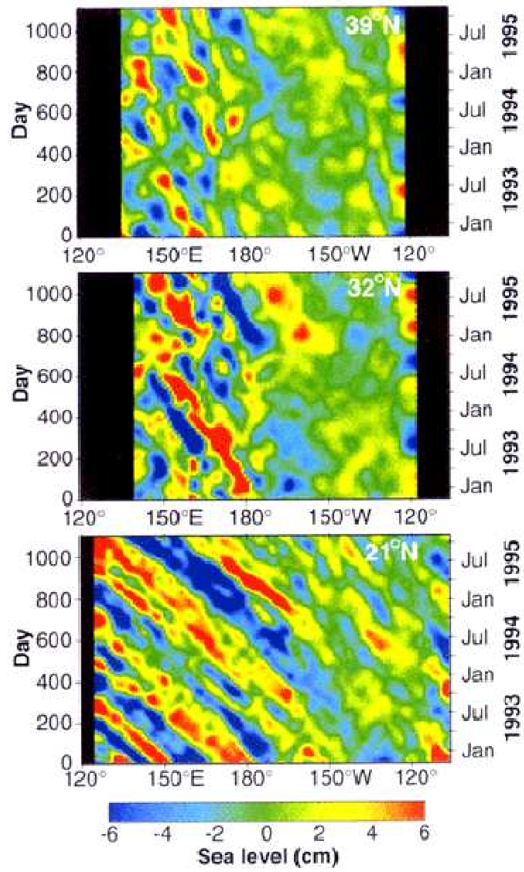


Figure 17: Three Hovmuller diagrams constructed from sea surface height in the North Pacific. From Chelton and Schlax (1996).

Three examples from the ocean are shown in Fig. (17). These are Hovmuller diagrams constructed from sea surface height in the Pacific, at three different latitudes. We see westward phase propagation in all three cases. Interestingly, the phase speed (proportional to the tilt of the lines) differs in the three cases. To explain this, we will need to take stratification into account, as discussed later on. In addition, the waves are more pronounced west of 150-180 W. The reason for this however is still unknown.

2.4.5 Group Velocity

Thus Rossby waves propagate westward. But this actually poses a problem. Say we are in an ocean basin, with no mean flow ($U = 0$). If there is a disturbance on the eastern wall, Rossby waves will propagate westward into the interior. Thus changes on the eastern wall are *communicated* to the rest of the basin by Rossby waves. Because they propagate westward, the whole basin will soon know about these changes. But say the disturbance is on the *west wall*. If the waves can go only toward the wall, the energy would necessarily be trapped there. How do we reconcile this?

The answer is that the phase velocity tells us only about the motion of the crests and troughs—it does not tell us how the energy is moving. To see how energy moves, it helps to consider a *packet* of waves with different wavelengths. If the Rossby waves were initiated by a localized source, say a meteor crashing into the ocean, they would start out as a wave packet. Wave packets have both a phase velocity and a "group velocity". The latter tells us about the movement of packet itself, and this reflects how the energy is moving. It is possible to have a packet of Rossby waves which are moving eastwards, while the crests of the waves in the packet move westward.

Consider the simplest example, of two waves with different wavelengths and frequencies, but the same (unit) amplitude:

$$\psi = cos(k_1 x + l_1 y - \omega_1 t) + cos(k_2 x + l_2 y - \omega_2 t) \qquad (145)$$

Imagine that k_1 and k_2 are almost equal to k, one slightly larger and the other slightly smaller. We'll suppose the same for l_1 and l_2 and ω_1 and ω_2. Then we can write:

$$\psi = cos[(k + \delta k)x + (l + \delta l)y - (\omega + \delta\omega)t]$$

$$+cos[(k-\delta k)x+(l-\delta l)y-(\omega-\delta\omega)t] \tag{146}$$

From the cosine identity:

$$cos(a \pm b)=cos(a)cos(b) \mp sin(a)sin(b) \tag{147}$$

So we can rewrite the streamfunction as:

$$\psi = 2\, cos(\delta kx+\delta ly-\delta\omega t)\, cos(kx+ly-\omega t) \tag{148}$$

The combination of waves has two components: a plane wave (like we considered before) multiplied by a *carrier wave*, which has a longer wavelength and lower frequency. The carrier wave has a phase speed of:

$$c_x = \frac{\delta\omega}{\delta k} \approx \frac{\partial\omega}{\partial k} \equiv c_{gx} \tag{149}$$

and

$$c_y = \frac{\delta\omega}{\delta l} \approx \frac{\partial\omega}{\partial l} \equiv c_{gy} \tag{150}$$

The phase speed of the carrier wave is the *group velocity*, because this is the speed at which the group (in this case two waves) moves. While the phase velocity of a wave is ratio of the frequency and the wavenumber, the group velocity is the *derivative* of the frequency by the wavenumber.

This is illustrated in Fig. (18). This shows two waves, $cos(1.05x)$ and $cos(0.095x)$. Their sum yields the wave packet in the lower panel. The smaller ripples propagate with the phase speed, $c=\omega/k=\omega/1$, westward. But the larger scale undulations move with the group velocity, and this can be either west *or* east.

The group velocity concept applies to any type of wave. For Rossby waves, we take derivatives of the Rossby wave dispersion relation for ω. This yields:

$$c_{gx} = \frac{\partial\omega}{\partial k} = \beta\frac{k^2-l^2}{(k^2+l^2)^2}\,, \quad c_{gy} = \frac{\partial\omega}{\partial l} = \frac{2\beta kl}{(k^2+l^2)^2} \tag{151}$$

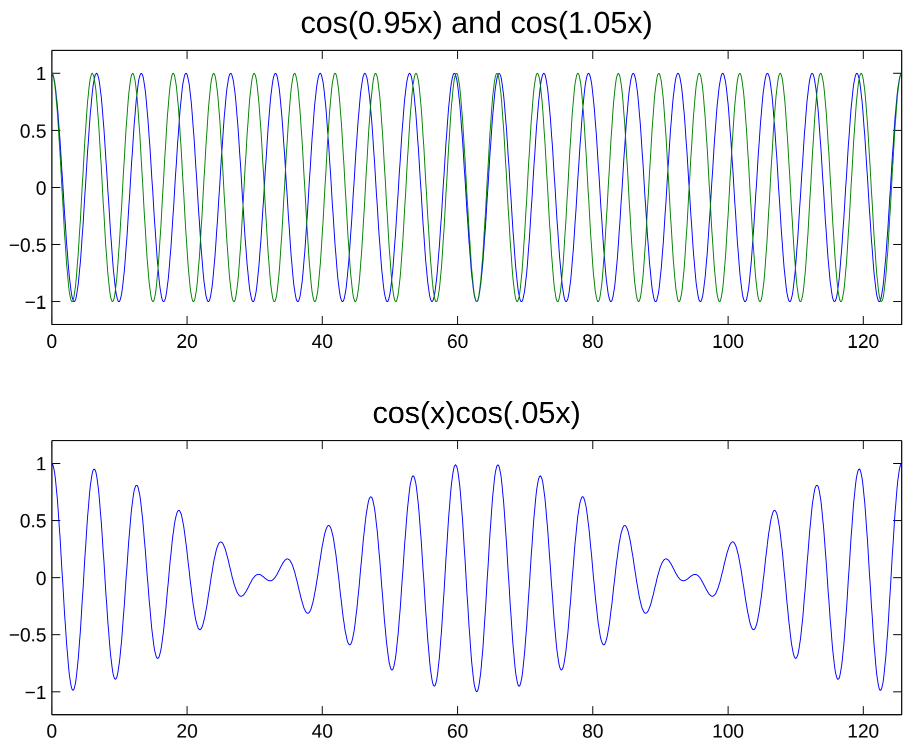


Figure 18: A wave packet of two waves with nearly the same wavelength.

Consider for example the group velocity in the zonal direction, c_{gx}. The sign of this depends on the relative sizes of the zonal and meridional wavenumbers. If

$$k > l$$

the wave packet has a positive (eastward) zonal velocity. Then the energy is moving in the *opposite* direction to the phase speed. This answers the question about the disturbance on the west wall. Energy can indeed spread eastward into the interior, if the zonal wavelength is shorter than the meridional one. Note that for such waves, the phase speed is still westward. So the crests will move toward the west wall while energy is carried eastward!

Another interesting aspect is that the group velocity in the y-direction is *always* in the opposite direction to the phase speed in y, because:

$$\frac{c_{gy}}{c_y} = -\frac{2l^2}{k^2 + l^2} < 0\,. \tag{152}$$

So northward propagating waves have southward energy flux!

The group velocity can also be derived by considering the energy equation for the wave. This is shown in Appendix C.

2.5 Rossby wave reflection

A good illustration of these Rossby wave properties is the case of a wave reflecting off a solid boundary. Consider what happens to a westward propagating plane Rossby wave which encounters a straight wall, oriented along $x = 0$. The incident wave can be written:

$$\psi_i = A_i\, e^{ik_i x + il_i y - i\omega_i t}$$

where:

$$\omega_i = \frac{-\beta k_i}{k_i^2 + l_i^2}$$

The incident wave has a westward group velocity, so that

$$k_i < l_i$$

Let's assume too that the group velocity has a northward component (so that the wave is generated somewhere to the south). As such, the phase velocity is oriented toward the *southwest*.

The wall will produce a reflected wave. If this weren't the case, all the energy would have to be absorbed by the wall. We assume instead that all the energy is reflected. The reflected wave is:

$$\psi_r = A_r\, e^{ik_r x + il_r y - i\omega_r t}$$

The total streamfunction is the sum of the incident and reflected waves:

$$\psi = \psi_i + \psi_r \tag{153}$$

In order for there to be no flow into the wall, we require that the zonal velocity vanish at $x = 0$, or:

$$u = -\frac{\partial}{\partial y}\psi = 0 \quad at \;\; x = 0 \tag{154}$$

This implies:

$$-il_i\, A_i\, e^{il_i y - i\omega_i t} - il_r\, A_r\, e^{il_r y - i\omega_r t} = 0 \tag{155}$$

In order for this condition to hold at all times, the frequencies must be equal:

$$\omega_i = \omega_r = \omega \tag{156}$$

Likewise, if it holds for all values of y along the wall, the meridional wavenumbers must also be equal:

$$l_i = l_r = l \tag{157}$$

Note that because the frequency and meridional wavenumbers are preserved on reflection, the meridional phase velocity, $c_y = \omega/l$, remains unchanged. Thus (155) becomes:

$$il\, A_i\, e^{ily - i\omega t} + il\, A_r\, e^{ily - i\omega t} = 0 \tag{158}$$

which implies:

$$A_i = -A_r \equiv A \tag{159}$$

So the amplitude of the wave is preserved, but the phase is changed by 180°.

Now let's go back to the dispersion relations. Because the frequencies are equal, we have:

$$\omega = \frac{-\beta k_i}{k_i^2 + l^2} = \frac{-\beta k_r}{k_r^2 + l^2} \,. \tag{160}$$

This is possible because the dispersion relation is quadratic in k and thus admits two different values of k. Solving the Rossby dispersion relation for k, we get:

$$k = -\frac{\beta}{2\omega} \pm \frac{\sqrt{\beta^2 - 4\omega^2 l^2}}{2\omega} \tag{161}$$

The incident wave has a smaller value of k because it has a westward group velocity; so it is the additive root. The reflected wave thus comes from the difference of the two terms.

This implies the wavenumber *increases* on reflection, by an amount:

$$|k_r - k_i| = 2\sqrt{\frac{\beta^2}{4\omega^2} - l^2} \tag{162}$$

In other words, the incident waves are long but the reflected waves are *short*.

We can also show that the meridional velocity, v, *increases* upon reflection and also that the mean energy (Appendix C) increases on reflection. The reflected wave is more energetic because the energy is squeezed into a shorter wave. However, the *flux* of energy is conserved; the amount of energy going in equals that going out. So energy does not accumulate at the wall.

Thus Rossby waves change their character on reflection. Interestingly, the change depends on the *orientation* of the boundary. A tilted boundary (e.g. northwest) will produce different results. In fact, the case with a zonally-oriented boundary (lying, say, along $y = 0$) is *singular*; you must introduce other dynamics, like friction, to solve the problem. Rossby waves, in many ways, are strange.

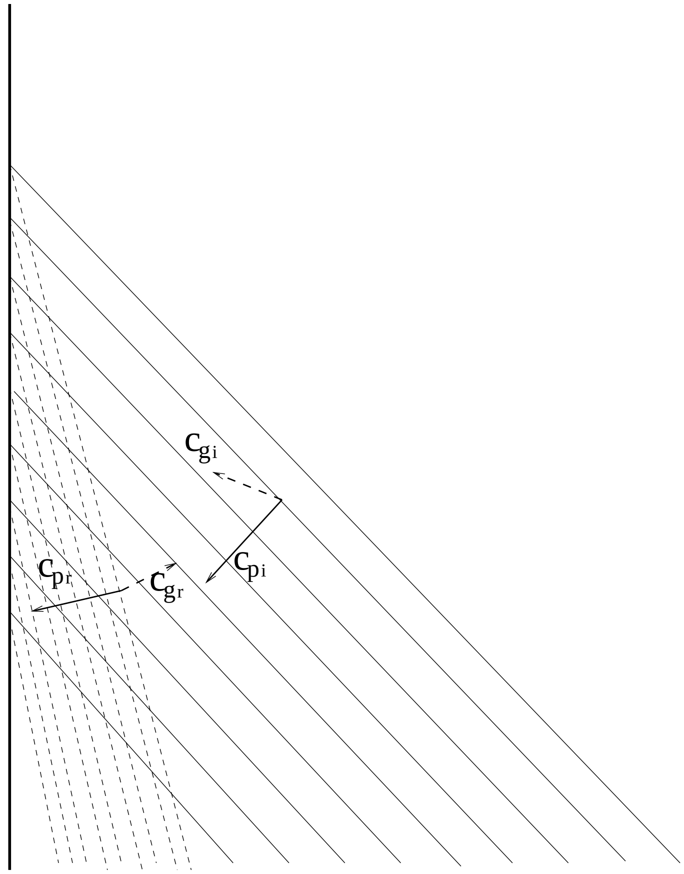


Figure 19: A plane Rossby wave reflecting at a western wall. The incident wave is shown by the solid lines and the reflected wave by the dashed lines. The phase velocities are indicated by the solid arrows and the group velocities by the dashed arrows. Note the wavelength in y doesn't change, but the reflected wavelength in x is much shorter. Note too the reflected wave has a phase speed directed toward the wall, but a group velocity away from the wall.

Exercise 2.6: Consider Rossby waves incident on a northern wall, i.e. oriented east-west, located at $y = 0$. Proceed as before, with one incident and one reflected wave. What can you say about the reflected wave?

Hint: there are two possibilities, depending on the sign of l_r.

2.6 The PV equation with forcing

Up until now, we have considered solutions of the inviscid PV equation—that is, without any forcing. Now we will consider what happens with friction included.

As noted in Chapter 1, friction is unimportant for synoptic scale motion. Where it is important is in the boundary layers. As we saw in section (1.11), the ageostrophic flow in these layers can generate vertical velocities, and these in turn can influence motion in the interior. We cannot simply include Ekman layers in our barotropic formalism, because the vertical shear in the layers is not zero. What we can do is to assume that the *interior* of the fluid is barotropic and that that is sandwiched between two Ekman layers, one on the upper boundary and one on the lower.

We can include these Ekman layer by adding two additional terms on the RHS of the integrated vorticity equation (117), thus:

$$\frac{d_g}{dt}(\zeta + \beta y + \frac{f_0}{D_0}h) = \frac{f_0}{D_0}[w_e(z_1) - w_e(z_0)] \tag{163}$$

The first term on the RHS is the vertical velocity associated with the boundary layer on the upper surface and the second term is that with the layer on the lower surface.

In the atmosphere, we would set the vertical velocity at the top boundary to zero (there is no Ekman layer on the tropopause). The ocean is different though, because the wind is causing divergence at the upper surface. So we include the wind stress term from (93):

$$w_e(z_1) = \frac{1}{\rho_0 f_0}\hat{k} \cdot \nabla \times \tau^w \tag{164}$$

The bottom Ekman layer exists in both the atmosphere and ocean. This exerts a drag proportional to the relative vorticity. From (101), we have:

$$w_e(z_0) = \frac{\delta}{2}\zeta_g \tag{165}$$

The Ekman layers thus affect the motion in the interior when there is vorticity.

Combining all the terms, we arrive at the forced barotropic PV equation:

$$\frac{d_g}{dt}(\nabla^2\psi + \beta y + \frac{f_0}{D_0}h) = \frac{1}{\rho_0 D_0}\hat{k} \cdot \nabla \times \vec{\tau}_w - r\nabla^2\psi \tag{166}$$

The constant, r, is called the "Ekman drag coefficient" and is defined:

$$r = \frac{f_0\delta}{2D_0}$$

An important point about this is that the forcing terms exert themselves over the entire depth of the fluid, because there is no vertical shear.

2.7 Spin down

Both the atmosphere and ocean have a bottom boundary layer. Bottom friction damps the velocities, causing the winds to slow. The simplest example of this is with no bottom topography and a constant f. Then the barotropic vorticity equation is:

$$\frac{d_g}{dt}\zeta = -r\zeta \tag{167}$$

This is a nonlinear equation. However it is easily solved in the Lagrangian frame. Following a parcel, we have that:

$$\zeta(t) = \zeta(0)e^{-rt} \tag{168}$$

So the vorticity decreases exponentially. The e-folding time scale is known as the Ekman *spin-down time*:

$$T_e = r^{-1} = (\frac{2}{A_z f_0})^{1/2} D \tag{169}$$

Typical atmospheric values are:

$$D = 10km, \;\; f = 10^{-4} sec^{-1}, \;\; A_z = 10\, m^2/sec$$

assuming the layer covers the entire troposphere. Then:

$$T_e \approx \; 4 \; days$$

If all the forcing (including the sun) were suddenly switched off, the winds would slow down, over this time scale. After about a week or so, the winds would be weak.

If we assume that the barotropic layer does not extend all the way to the tropopause but lies nearer the ground, the spin-down time will be even shorter. This is actually what happens in the stratified atmosphere, with the winds near the ground spinning down but the winds aloft being less affected. So bottom friction favors flows intensified further up. The same is true in the ocean.

2.8 Mountain waves

Barotropic Rossby waves have been used to study the mean surface pressure distribution in the atmosphere. This is the pressure field you get when averaging over long periods of time (e.g. years). The central idea is that the mean wind, U, blowing over topography can excite stationary waves ($c_x = 0$). As demonstrated by Charney and Eliassen (1949), one can find a reasonable first estimate of the observed distribution using the linear, barotropic vorticity equation.

We start with the vorticity equation as applied to the atmosphere. First we will neglect any frictional forcing:

$$\frac{d_g}{dt}(\zeta + \beta y + \frac{f_0}{D}h) = 0 \tag{170}$$

We will linearize about a mean zonal flow:

$$u = U + u', \quad v = v', \quad \zeta = \zeta'$$

We will also assume the topography is weak:

$$h = h'$$

in keeping with QG. Then the Rossby wave equation becomes:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\zeta' + \beta v' + U\frac{\partial}{\partial x}\frac{f_0}{D}h' = 0 \tag{171}$$

Substituting in the streamfunction, we have:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = -\frac{f_0}{D}U\frac{\partial}{\partial x}h' \tag{172}$$

We put the topographic term on the RHS because it does not involve the streamfunction, and so acts instead like a forcing term. So the winds blowing over the mountains generate the response.

The homogeneous solution to this equation are just the Rossby waves we discussed earlier. These are called “free Rossby waves”. If we were to suddenly “turn on” the wind, we would excite free waves. The particular solution, or the “forced wave”, is the part generated by the topographic term on the RHS. This is the portion of the flow that will remain after the free waves have propagated away.

So the forced wave is the portion that will determine the time mean flow. To find that, we can ignore the time derivative:

$$U\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = -\frac{f_0}{D}U\frac{\partial}{\partial x}h' \tag{173}$$

All the terms involve a derivative in x, so we can simply integrate the equation once in x to get rid of that. We can ignore the constant of integration, which would amount to adding a constant to the streamfunction. The latter would have no effect on the velocity field (why?).

In line with our previous derivations, we write the topography as a sum of Fourier modes:

$$h'(x,y) = Re\{\sum_k \sum_l h(k,l)\, e^{ikx+ily}\} \tag{174}$$

For simplicity, we will focus on the response to a single wave mode:

$$h' = h_0 cos(kx) cos(ly) \tag{175}$$

We can always construct the response to more complicated topography by adding the solutions for different (k,l), because the Rossby wave equation is linear. Substituting this in yields:

$$U\nabla^2\psi + \beta\psi = -\frac{f_0 h_0}{D} U cos(kx) cos(ly) \tag{176}$$

For the reasons given, we focus on the particular solution. This has the general form:

$$\psi = A cos(kx) cos(ly) \tag{177}$$

Plugging in:

$$(U(-k^2 - l^2) + \beta)\, A cos(kx) cos(ly) = -\frac{f_0 h_0}{D} U cos(kx) cos(ly) \tag{178}$$

or:

$$A = \frac{f_0 h_0}{D(\kappa^2 - \beta/U)} = \frac{f_0 h_0}{D(\kappa^2 - \kappa_s^2)} \tag{179}$$

where:

$$\kappa_s \equiv (\frac{\beta}{U})^{1/2}$$

is the wavenumber of the stationary Rossby wave with a background velocity, U (sec. 2.4.3). Notice with forcing that we obtain an expression for the amplitude, A—it doesn't drop out. So the forced solution is:

$$\psi = \frac{f_0 h_0}{D(\kappa^2 - \kappa_s^2)} cos(kx) cos(ly) \tag{180}$$

The pressure field thus resembles the topography. If the wavenumber of the topography, κ, is greater than the stationary wavenumber, the amplitude is positive. Then the forced wave is *in phase* with the topography. If the topographic wavenumber is smaller, the atmospheric wave is 180° out of phase with the topography. The latter case applies to large scale topography, for which the wavenumber is small. So we expect negative pressures over mountains and positive pressures over valleys. With small scale topography, the pressure over the mountains will instead be positive.

What happens though when $\kappa = \kappa_S$? Then the streamfunction is infinite! This is a typical situation with forced oscillations. If the forcing is at the natural frequency of the system, the response is infinite (we say the response is *resonant*). Having infinite winds is not realistic, so we must add additional dynamics. In particular, we can add friction.

We do this as follows. We must go back to the barotropic vorticity equation, but with a bottom Ekman layer:

$$\frac{d_g}{dt} (\zeta + \beta y + \frac{f_0}{D} h) = -r\zeta \tag{181}$$

Linearizing as before, we obtain:

$$U \frac{\partial}{\partial x} \nabla^2 \psi + \beta \frac{\partial}{\partial x} \psi = -\frac{f_0}{D} U \frac{\partial}{\partial x} h' - r \nabla^2 \psi \tag{182}$$

Using the same topography, we get:

$$(U \frac{\partial}{\partial x} + r) \nabla^2 \psi + \beta \frac{\partial}{\partial x} \psi = \frac{k f_0 h_0}{D} U sin(kx) cos(ly) \tag{183}$$

The equation is exactly as before, except that we have an additional factor in front of the relative vorticity. This prevents us from integrating the equation in x, like we did before. It also means that the cosine/cosine particular solution will no longer work. Instead, we use the following complex expression:

$$\psi = Re\{Ae^{ikx}\}cos(ly) \tag{184}$$

Remember that the amplitude, A, may also be a complex number. To be consistent, we write the topography in the same way, i.e.:

$$h' = Re\{h_0 e^{ikx}\}cos(ly) \tag{185}$$

(even though we know that h_0 is real). So we have:

$$(U\frac{\partial}{\partial x} + r)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = -\frac{ikf_0h_0}{D}Ue^{ikx}cos(ly) \tag{186}$$

Substituting in the wave solution, we get:

$$\left[(ikU + r)(-k^2 - l^2) + ik\beta\right]A = -\frac{ikf_0h_0}{D}U \tag{187}$$

after canceling the sinusoidal terms. Solving for A, we get:

$$A = \frac{f_0h_0}{D(\kappa^2 - \kappa_s^2 - iR)} \tag{188}$$

where:

$$R \equiv \frac{r\kappa^2}{kU} \tag{189}$$

As promised, the amplitude is complex.

The amplitude is as before, except for the additional term in the denominator proportional to the Ekman drag, r. This term does two things. First, it removes the singularity. At $\kappa = \kappa_s$, we have:

$$A = i\frac{f_0h_0}{DR} \tag{190}$$

So the response is no longer infinite. However, the response is still greatest at this wavenumber. Having $\kappa \neq \kappa_s$ produces a weaker amplitude.

Second, friction causes a *phase shift* in the pressure field relative to the topography. Consider the response at $\kappa = \kappa_S$. Then the amplitude is purely imaginary, as seen above. Putting this into the full solution, we get:

$$\psi = Re\{Ae^{ikx}\}cos(ly) = -\frac{f_0 h_0}{DR}sin(kx)cos(ly) \tag{191}$$

The topography on the other hand is proportional to $cos(kx)$. So the streamfunction is 90° out of phase with the mountains. In this case, the low pressure is downstream of the mountain. The extent of the phase shift depends on the difference between κ and κ_s. The larger the difference, the more aligned the pressure field is with the topography (either in phase, or 180° out of phase).

We summarize the results with sinusoidal topography and Ekman friction graphically in Fig. (20). When the topographic wavenumber is much less than the stationary wavenumber for the velocity, U, the pressure field is aligned but anti-correlated with the topography. When the wavenumber is much greater than κ_s, the pressure is aligned and correlated. When $\kappa = \kappa_s$, the pressure is 90 ° out of phase with the mountains.

Charney and Eliassen (1949) applied the barotropic equation to the actual atmosphere. But instead of using a sinusoidal topography, they used the observed topographic profile at 45 N. The result of their calculation is shown in Fig. (21). The topography is indicated by the dotted lines. The two maxima come from the Himalayas and the Rocky Mountains. The solution, with U=17 m/sec and r=1/6 day^{-1}, is indicated by the solid line. The dashed line shows the observed mean pressure at 500 mb. We see the model exhibits much of the same structure as the observed pressure

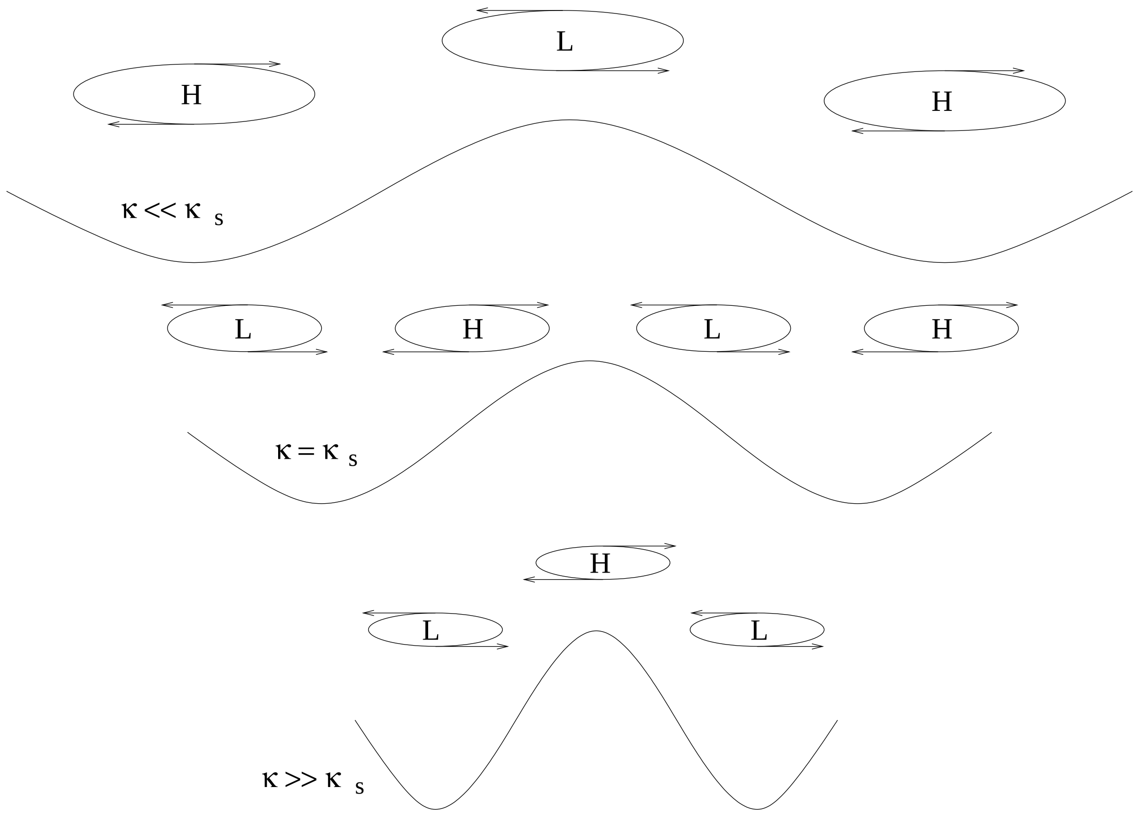


Figure 20: The mean pressure distribution over a sinusoidal mountain range. The topographic wavenumber is less than (upper), greater than (bottom) and equal to (middle) the stationary wavenumber.

field. Both have low pressure regions down wind from the mountains, and a marked high pressure upwind of the Rockies.

The agreement between the model and observations is remarkably good, given the simplicity of the model. In fact, it is probably too good. Charney and Eliassen used a meridional channel for their calculation (as one would do with a QG β-plane.), but if one redoes the calculation on a sphere, the Rossby waves can disperse meridionally and the amplitude is decreased (Held, 1983). Nevertheless, the relative success of the model demonstrates the utility of Rossby wave dynamics in understanding the low frequency atmospheric response.

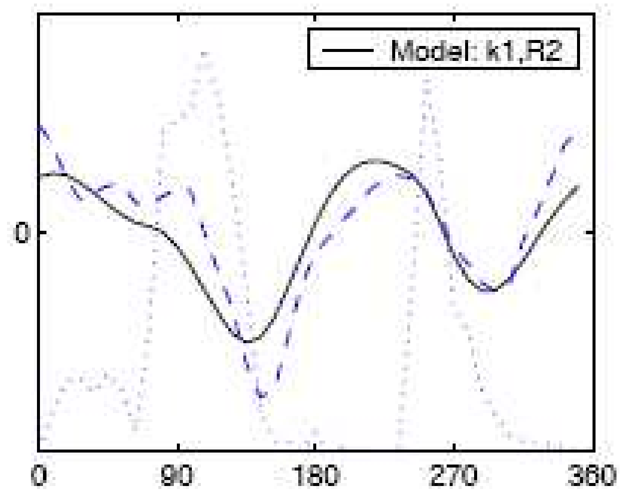


Figure 21: Charney and Eliassen's (1949) solution of the barotropic mountain wave problem at 45N. The dotted line indicates the topographic profile, the solid line is the model solution and the dashed line is the observed mean pressure at 500 mb. From Vallis (2007).

Exercise 2.7: Consider Rossby waves with an isolated mountain range. A purely sinusoidal mountain range is not very realistic. A more typical case is one where the mountain is localized. Consider a mountain "range" centered at $x = 0$ with:

$$h(x, y) = h_0\, e^{-x^2/L^2} \tag{192}$$

Because the range doesn't vary in y, we can write $\psi = \psi(x)$.

Write the wave equation, without friction. Transform the streamfunction and the mountain using the Fourier cosine transform. Then solve for the transform of ψ, and write the expression for $\psi(x)$ using the inverse transform (it's not necessary to evaluate the inverse transform).

Where do you expect the largest contribution to the integral to occur (which values of k)?

2.9 The Gulf Stream

Figure 22: Benjamin Franklin's map of the Gulf Stream. From Wikipedia.

The next example is one of the most famous in dynamical oceanography. It was known at least since the mid 1700's, when Benjamin Franklin mapped the principal currents of the North Atlantic (Fig. 22), that the Gulf Stream is an intense current which lies on the *western* side of the basin, near North America. The same is true of the Kuroshio Current, on the western side of the North Pacific, the Agulhas Current on the western side of the Indian Ocean, and numerous other examples. Why do these currents lie in the west? A plausible answer came from a work by Stommel (1948), based on the barotropic vorticity equation. We will consider this problem, which also illustrates the technique of *boundary layer analysis*.

We retain the β-effect and bottom Ekman drag, but neglect topography (the bottom is flat). We also include the surface Ekman layer, to allow for wind forcing. The result is:

$$\frac{d_g}{dt}(\zeta + \beta y) = \frac{d_g}{dt}\zeta + \beta v = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}_w - r\zeta \tag{193}$$

We will search for steady solutions, as with the mountain waves. Moreover, we will not linearize about a mean flow—it is the mean flow itself we're after. So we neglect the first term in the equation entirely. Using the streamfunction, we get:

$$\beta\frac{\partial}{\partial x}\psi = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}_w - r\nabla^2\psi \tag{194}$$

For our "ocean", we will assume a square basin. The dimensions of the basin aren't important, so we will just use the region $x = [0, L]$ and $y = [0, L]$ (L might be 5000 km).

It is important to consider the geostrophic contours in this case:

$$q_s = \beta y \tag{195}$$

which are just latitude lines. In this case, all the geostrophic contours intersect the basin walls. From the discussion in sec. (2.3), we know that there can be no steady flows without forcing, because such a flow would be purely zonal and would have to continue through the walls. However, with forcing there can be steady flow; we will see that this flow *crosses* the geostrophic contours.

Solutions to (194) can be obtained in a general form, once the wind stress is specified. But Stommel used a more elegant method. The main idea is as follows. Since the vorticity equation is linear, we can express the solution as the sum of two components:

$$\psi = \psi_I + \psi_B \tag{196}$$

The first part, ψ_I, is that driven by the wind forcing. We assume that this part is present in the whole domain. We assume moreover that the friction is weak, and does not affect this interior component. Then the interior component is governed by:

$$\beta \frac{\partial}{\partial x}\psi_I = \frac{1}{\rho_0 D}\nabla \times \vec{\tau} \tag{197}$$

This is the *Sverdrup relation*, after H. U. Sverdrup. It is perhaps the most important dynamical balance in oceanography. It states that vertical flow from the base of the surface Ekman layer, due to the wind stress curl, drives meridional motion. This is the motion across the geostrophic contours, mentioned above.

We can solve (197) if we know the wind stress and the boundary conditions. For the wind stress, Stommel assumed:

$$\vec{\tau} = -\frac{L}{\pi}\,cos(\frac{\pi y}{L})\,\hat{i}$$

The wind is purely zonal, with a cosine dependence. The winds in the northern half of the domain are eastward, and they are westward in the southern half. This roughly resembles the situation over the subtropical North Atlantic. Thus the wind stress curl is:

$$\nabla \times \vec{\tau} = -\frac{\partial}{\partial y}\tau^x = -sin(\frac{\pi y}{L})$$

Again, this is the vertical component of the curl. From the Sverdrup relation, this produces southward flow over the whole basin, with the largest velocities occurring at the mid-basin ($y = L/2$). We then integrate the Sverdrup relation (197) to obtain the streamfunction in the interior.

However, we can do this in two ways, either by integrating from the western wall or *to* the eastern wall (the reason why these produce different results will become clear). Let's do the latter case first. Then:

$$\int_x^L \frac{\partial}{\partial x}\psi_I \, dx = \psi_I(L,y) - \psi_I(x,y) = -\frac{1}{\beta\rho_0 D} \, sin(\frac{\pi y}{L})(L-x) \quad (198)$$

To evaluate this, we need to know the value of the streamfunction on the eastern wall, $\psi_I(L,y)$.

Now ψ_I must be a constant. If it weren't, there would be flow into the wall, because:

$$u(L,y) = -\frac{\partial}{\partial y}\psi_I(L,y) \quad (199)$$

If psi_I were constant, there would be flow into the wall. But what is the constant? We can simply take this to be zero, because using any other constant would not change the velocity field. So we have:

$$\psi_I(x,y) = \frac{1}{\beta\rho_0 D} \, sin(\frac{\pi y}{L})(L-x) \quad (200)$$

Notice though that this solution has flow *into* the western wall, because:

$$u_I(0,y) = -\frac{\partial}{\partial y}\psi_I(0,y) = -\frac{\pi}{\beta\rho_0 D} cos(\frac{\pi y}{L}) \neq 0 \quad (201)$$

This can't occur.

To fix the flow at the western wall, we use the second component of the flow, ψ_B. Let's go back to the vorticity equation, with the interior and boundary streamfunctions substituted in:

$$\beta\frac{\partial}{\partial x}\psi_I + \beta\frac{\partial}{\partial x}\psi_B = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}_w - r\nabla^2\psi_B \quad (202)$$

We have ignored the term $r\nabla^2\psi_I$; specifically, we assume this term is much smaller than $r\nabla^2\psi_B$. The reason is that ψ_B has rapid variations near the wall, so the second derivative will be much larger than that of ψ_I, which has a large scale structure. Using (197), the vorticity equation reduces to:

$$\beta\frac{\partial}{\partial x}\psi_B = -r\nabla^2\psi_B \quad (203)$$

ψ_B is assumed to be vanishingly small in the interior. But it will not be small in a boundary layer. We expect that boundary layer to occur in a narrow region near the western wall, because ψ_B must cancel the zonal interior flow at the wall.

This boundary layer will be narrow in the x-direction. The changes in y on the other hand should be more gradual, as we expect the boundary layer to cover the entire west wall. Thus the derivatives in x will be much greater than in y. So we have:

$$\beta \frac{\partial}{\partial x}\psi_B = -r\nabla^2\psi_B \approx -r\frac{\partial^2}{\partial x^2}\psi_B \tag{204}$$

This has a general solution:

$$\psi_B = Aexp(-\frac{\beta x}{r}) + B$$

In order for the boundary correction to vanish in the interior, the constant B must be zero. We then determine A by making the zonal flow vanish at the west wall (at $x = 0$). This again implies that the streamfunction is constant. That constant must be zero, because we took it to be zero on the east wall. If it were a different constant, then ψ would have to change along the northern and southern walls, meaning $v = \frac{\partial}{\partial x}\psi$ would be non-zero. Thus we demand:

$$\psi_I(0, y) + \psi_B(0, y) = 0 \tag{205}$$

Thus:

$$A = -\frac{L}{\beta\rho_0 D}\, sin(\frac{\pi y}{L}) \tag{206}$$

So the total solution is:

$$\psi = \frac{1}{\beta\rho_0 D}\, sin(\frac{\pi y}{L})\ [L - x - Lexp(-\frac{\beta x}{r})] \tag{207}$$

We examine the character of this solution below. But first let's see what would have happened if we integrated the Sverdrup relation (197) from the *western* wall instead of to the eastern. Then we would get:

$$\beta \int_0^x \frac{\partial}{\partial x}\psi \, dx = \beta\psi(x,y) - \beta\psi(0,y) = -x \, sin(\frac{\pi y}{L}) \tag{208}$$

Setting $\psi(0,y) = 0$, we get:

$$\psi(x,y) = -\frac{x}{\beta\rho_0 D} \, sin(\frac{\pi y}{L}) \tag{209}$$

This solution has flow into the eastern wall, implying we must have a boundary layer there. Again the boundary layer should have more rapid variation in x than in y, so the appropriate boundary layer equation is (204), with a solution:

$$\psi_B = Aexp(-\frac{\beta x}{r}) + B$$

We take B to be zero again, so the solution vanishes in the interior.

But does it? To satisfy the zero flow condition at $x = L$, we have:

$$\psi_I(L,y) + \psi_B(L,y) = 0 \tag{210}$$

or:

$$-\frac{L}{\beta\rho_0 D} \, sin(\frac{\pi y}{L}) + Aexp(-\frac{\beta L}{r}) = 0 \tag{211}$$

Solving for A, we get:

$$A = \frac{L}{\beta\rho_0 D} \, exp(\frac{\beta L}{r}) \, sin(\frac{\pi y}{L}) \tag{212}$$

So the total solution in this case is:

$$\psi = \frac{1}{\beta\rho_0 D} \, sin(\frac{\pi y}{L}) \, [-x + Lexp(\frac{\beta(L-x)}{r})] \tag{213}$$

Now there is a problem. The exponential term in this case does not decrease moving away from the eastern wall. Rather, it grows exponentially.

So the boundary layer solution *isn't confined* to the eastern wall! Thus we reject the possibility of an eastern boundary layer. The boundary layer must lie on the western wall. This is why, Stommel concluded, the Gulf Stream lies on the western boundary of the North Atlantic.

Another explanation for the western intensification was proposed by Pedlosky (1965). Recall that Rossby waves propagate to the west as long waves, and reflect off the western wall as short waves. The short waves move more slowly, with the result that the energy is intensified in the region near the west wall (sec. 2.5). Pedlosky showed that in the limit of low frequencies (long period waves), the Rossby wave solution converges to the Stommel solution. So western intensification occurs because Rossby waves propagate to the west.

Let's look at the (correct) Stommel solution. Shown in figure (23) is the Sverdrup solution (upper panel) and two full solutions with different r (lower panels). The Sverdrup solution has southward flow over the whole basin. So the mean flow crosses the geostrophic contours, as suggested earlier. There is, in addition, an eastward drift in the north and a westward drift in the south.

With the larger friction coefficient, the Stommel solution has a broad, northward-flowing western boundary current. With the friction coefficient 10 times smaller, the boundary current is ten times narrower and the northward flow is roughly ten times stronger. This is the Stommel analogue of the Gulf Stream.

Consider what is happening to a fluid parcel in this solution. The parcel's potential vorticity decreases in the interior, due to the negative wind stress curl, which causes the parcel to drift southward. We know the parcel needs to return to the north to complete its circuit, but to do that it must

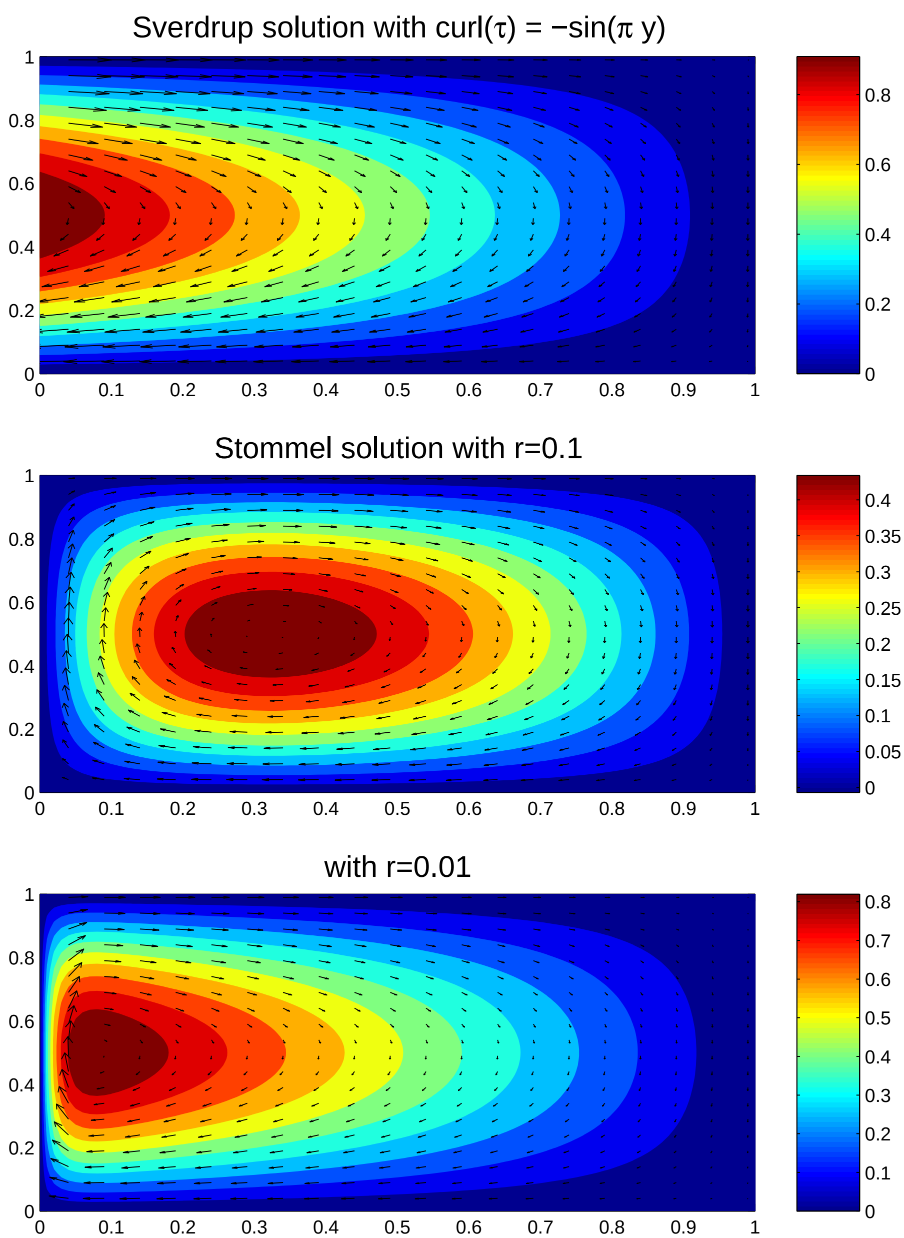


Figure 23: Solutions of Stommel's model for two different values of the friction coefficient, r.

somehow acquire vorticity. Bottom friction permits the parcel to acquire vorticity in the western layer. You can show that if the parcel were in an eastern boundary layer, it's vorticity would *decrease* going northward. So the parcel would not be able to re-enter the northern interior.

The Stommel boundary layer is like the bottom Ekman layer (sec. 1.11), in several ways. In the Ekman layer, friction, which acts only in a boundary layer, brings the velocity to zero to satisfy the no-slip condition. This yields a strong vertical shear in the velocities. In the Stommel layer, friction acts to satisfy the no-normal flow condition and causes strong *lateral* shear. Both types of boundary layer also are passive, in that they do not force the interior motion; they simply modify the behavior near the boundaries.

Shortly after Stommel's (1948) paper came another (Munk, 1950) appeared which also modelled the barotropic North Atlantic. The model is similar, except that Munk used lateral friction rather than bottom friction. The lateral friction was meant to represent horizontal stirring by oceanic eddies. The details of Munk's model are given in Appendix D.

Exercise 2.8: Is there really western intensification? To convince ourselves of this, we can solve the Stommel problem in 1-D, as follows. Let the wind stress be given by:

$$\vec{\tau} = y\hat{i} \tag{214}$$

Write the vorticity equation following Stommel (linear, U=V=0, steady). Ignore variations in y, leaving a 1-D equation. Assume the domain goes from $x = 0$ to $x = L$, as before. Solve it.

Note that you should have two constants of integration. This will allow you to satisfy the boundary conditions $\psi = 0$ at $x = 0$ and $x = L$. Plot the meridional velocity $v(x)$. Assume that $(\beta\rho_0 D)^{-1} = 1$ and $L(r\rho_0 D)^{-1} = 10$. Where is the jet?

2.10 Closed ocean basins

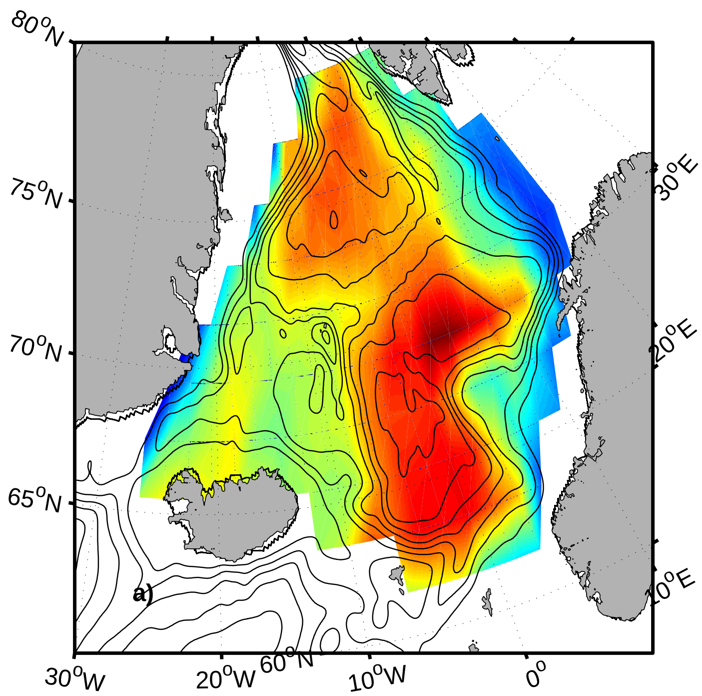


Figure 24: Geostrophic contours (solid lines) in the Nordic seas. Superimposed are contours showing the first EOF of sea surface height derived from satellite measurements. The latter shows strong variability localized in regions of closed q_s contours. From Isachsen et al. (2003).

Next we consider an example with bottom topography. As discussed in sec. (2.3), topography can cause the geostrophic contours to close on themselves. This is an entirely different situation because mean flows can exist on the closed contours (they do not encounter boundaries; Fig. 11). Such mean flows can be excited by wind-forcing and can be very strong.

There are several regions with closed geostrophic contours in the Nordic Seas (Fig. 24), specifically in three basins: the Norwegian, Lofoten and Greenland gyres. The topography is thus steep enough here as to overwhelm the β-effect. Isachsen et al. (2003) examined how wind-forcing could excite flow in these gyres.

This time we take equation (166) with wind forcing and bottom topography:

$$\frac{d_g}{dt}(\zeta + \beta y + \frac{f_0}{D}h) = \frac{1}{\rho_0 D}\nabla \times \vec{\tau} - r\zeta \tag{215}$$

We will linearize the equation, without a mean flow. We can write the result this way:

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla q_s = \frac{1}{\rho_0 D}\nabla \times \vec{\tau} - r\zeta \tag{216}$$

where

$$q_s \equiv \beta y + \frac{f_0}{D}h$$

defines the geostrophic contours (sec.2.3). Recall that these are the so-called "f/H" contours in the shallow water system. As noted, the q_s contours can close on themselves if the topography is strong enough to overwhelm the βy contribution to q_s (Fig. 11). This is the case in the Nordic Seas (Fig. 24).

As in the Gulf Stream model, we will assume the bottom friction coefficient, r, is small. In addition, we will assume that the wind forcing and the time derivative terms are as small as the bottom friction term (of order r). Thus the first, third and fourth terms in equation (216) are of comparable size. We can indicate this by writing the equation this way:

$$r\frac{\partial}{\partial t'}\zeta + \vec{u} \cdot \nabla q_s = r\frac{1}{\rho_0 D}\nabla \times \vec{\tau}' - r\zeta \tag{217}$$

where $t' = rt$ and $\tau' = \tau/r$ are the small variables normalized by r, so that they are order one.

Now we use a *perturbation expansion* and expand the variables in r. For example, the vorticity is:

$$\zeta = \zeta_0 + r\zeta_1 + r^2\zeta_2 + ...$$

Likewise, the velocity is:

$$\vec{u} = \vec{u}_0 + r\vec{u}_1 + r^2\vec{u}_2 + ...$$

We plug this into the vorticity equation and then collect terms which are multiplied by the same factor of r. The largest terms are those multiplied by one. These are just:

$$\vec{u}_0 \cdot \nabla q_s = 0 \tag{218}$$

So the first order component *follows the q_s contours*. In other words, the first order streamfunction is everywhere parallel to the q_s contours. Once we plot the q_s contours, we know what the flow looks like.

But this only tells us the *direction* of $\vec{u}_o$, not its strength or structure (how it varies from contour to contour). To find that out, we go to the next order in r:

$$\frac{\partial}{\partial t'}\zeta_0 + \vec{u}_1 \cdot \nabla q_s = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}' - \zeta_0 \tag{219}$$

This equation tells us how the zeroth order field changes in time. However, there is a problem. In order to solve for the zeroth order field, we need to know the first order field because of the term with u_1. But it is possible to eliminate this, as follows. First, we can rewrite the advective term thus:

$$\vec{u}_1 \cdot \nabla q_s = \nabla \cdot (\vec{u}_1 q_s) - q_s(\nabla \cdot \vec{u}_1) \tag{220}$$

The second term on the RHS vanishes by incompressibility. In particular:

$$\nabla \cdot \vec{u} = 0 \tag{221}$$

This implies that the velocity is incompressible at each order. So the vorticity equation becomes:

$$\frac{\partial}{\partial t'}\zeta_0 + \nabla \cdot (\vec{u}_1 q_s) = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}' - r\zeta_0 \tag{222}$$

Now, we can eliminate the second term if we integrate the equation over an area bounded by a closed q_s contour. This follows from Gauss's Law, which states:

$$\iint \nabla \cdot \vec{A}\, dx\, dy = \oint \vec{A} \cdot \hat{n}\, dl \tag{223}$$

Thus:

$$\iint \nabla \cdot (\vec{u} q_s)\, dA = \oint q_s \vec{u} \cdot \hat{n}\, dl = q_s \oint \vec{u} \cdot \hat{n}\, dl = 0 \tag{224}$$

We can take the q_s outside the line integral because q_s is constant on the bounding contour. The closed integral of $\vec{u} \cdot \hat{n}$ vanishes because of incompressibility:

$$\oint \vec{u} \cdot \hat{n}\, dl = \iint \nabla \cdot \vec{u}\, dA = 0$$

Thus the integral of (225) in a region bounded by a q_s contour is:

$$\frac{\partial}{\partial t'} \iint \zeta_0\, dxdy = \frac{1}{\rho_0 D} \iint \nabla \times \vec{\tau}'\, dxdy - \iint \zeta_0\, dxdy \tag{225}$$

Notice this contains only zeroth order terms. We can rewrite (225) by exploiting Stoke's Law, which states:

$$\iint \nabla \times \vec{A}\, dx\, dy = \oint \vec{A} \cdot\, \vec{dl} \tag{226}$$

So (225) can be rewritten:

$$\frac{\partial}{\partial t'} \oint \vec{u} \cdot \vec{dl} = \frac{1}{\rho_0 D} \oint \vec{\tau}' \cdot \vec{dl} - \oint \vec{u} \cdot \vec{dl} \qquad (227)$$

We have dropped the zero subscripts, since this is the only component we will consider. In terms of the real time and wind stress, this is:

$$\frac{\partial}{\partial t} \oint \vec{u} \cdot \vec{dl} = \frac{1}{\rho_0 D} \oint \vec{\tau} \cdot \vec{dl} - r \oint \vec{u} \cdot \vec{dl} \qquad (228)$$

Isachsen et al. (2003) solved (228) by decomposing the velocity into Fourier components in time:

$$\vec{u}(x, y, t) = \sum \tilde{u}(x, y, \omega)\, e^{i\omega t}$$

Then it is easy to solve (228) for the velocity integrated around the contour:

$$\oint \vec{u} \cdot \vec{dl} = \frac{1}{r + i\omega} \frac{1}{\rho_0 D} \oint \vec{\tau} \cdot \vec{dl} \qquad (229)$$

Note the solution is actually for the integral of the velocity around the contour (rather than the velocity at every point). We can divide by the length of the contour to get the average velocity on the contour:

$$< u > \equiv \frac{\oint \vec{u} \cdot \vec{dl}}{\oint dl} = \frac{1}{r + i\omega} \frac{1}{\rho_0 D} \frac{\oint \vec{\tau} \cdot \vec{dl}}{\oint dl} \qquad (230)$$

Isachsen et al. (2003) derived a similar relation using the shallow water equations. Their expression is somewhat more complicated but has the same meaning. They tested this prediction using various types of data from the Nordic Seas. One example is shown in figure (24). This shows the principal Empirical Orthogonal Function (EOF) of the sea surface height variability measured from satellite. The EOF shows that there are regions with spatially coherent upward and downward sea surface motion. These

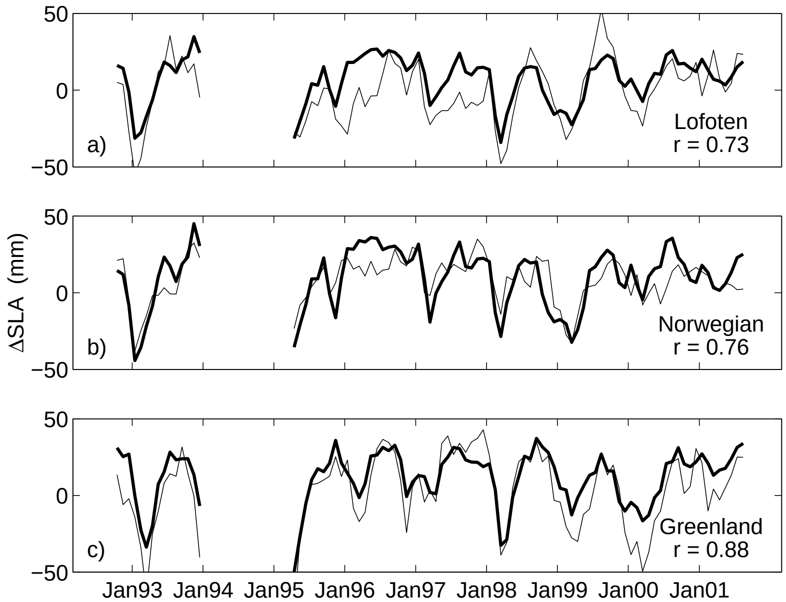


Figure 25: Time series of observed (thin line) and predicted (thick line) sea surface height displacements between the outer rim and the center of each of the principal gyres in the Nordic seas. The linear bottom drag coefficient was $R = 5 \times 10^{-4}$ m/sec. From Isachsen et al. (2003).

regions are exactly where the q_s contours are closed. This height variability reflects strong gyres which are aligned with the q_s contours.

Isachsen et al. took wind data, the actual bottom topography and an approximate value of the bottom drag to predict the transport in the three gyres (corresponding to the Norwegian, Lofoten and Greenland basins). The results are shown in figure (25). The simple model does astonishingly well, predicting the intensification and weakening of the gyres in all three basins.

2.11 Barotropic instability

Many of the "mean" flows in the atmosphere and ocean, like the Jet and Gulf Streams, are not steady at all. Instead, they meander and generate eddies (storms). The reason is that these flows are *unstable*. If the flow is perturbed slightly, for instance by a slight change in heating or wind forcing, the perturbation will grow, extracting energy from the mean flow. These perturbations then develop into mature storms, both in the atmosphere and ocean.

We'll first study instability in the barotropic context. In this we ignore forcing and dissipation, and focus exclusively on the interaction between the mean flow and the perturbations. A constant mean flow, like we used when deriving the dispersion relation for free Rossby waves, is stable. But a mean flow which is *sheared* can be unstable. To illustrate this, we examine a mean flow which varies in y. We will see that wave solutions exist in this case too, but that they can grow in time.

The barotropic vorticity equation with a flat bottom and no forcing or bottom drag is:

$$\frac{d_g}{dt}(\zeta + \beta y) = 0 \tag{231}$$

We again linearize the equation assuming a zonal flow, but now this can vary in y, i.e. $U = U(y)$. Significantly, the mean flow now has an associated vorticity:

$$\bar{\zeta} = -\frac{\partial}{\partial y}U \tag{232}$$

So the PV equation is now:

$$\frac{d_g}{dt}(\zeta' - \frac{\partial}{\partial y}U + \beta y) = 0 \tag{233}$$

The mean flow again is time independent, so its vorticity doesn't change in time either. As such, the mean vorticity *alters the geostrophic contours*. In particular, we have:

$$q_s = \beta y - \frac{\partial}{\partial y}U \tag{234}$$

This implies the mean flow will affect the way Rossby waves propagate in the system.

The linearized version of the vorticity equation is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\zeta' + v'\frac{\partial}{\partial y}q_s = 0 \tag{235}$$

Written in terms of the streamfunction, this is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\psi + (\frac{\partial}{\partial y}q_s)\frac{\partial\psi}{\partial x} = 0 \tag{236}$$

Now because the mean flow varies in y, we have to be careful about our choice of wave solutions. We can in any case assume a sinusoidal dependence in x and t. The form we will use is:

$$\psi = Re\{\hat{\psi}(y)\, e^{ik(x-ct)}\} \tag{237}$$

As we know, the amplitude can be complex, i.e.:

$$\hat{\psi} = \hat{\psi}_r + i\hat{\psi}_i$$

But now the phase speed, c, *also* can be complex. If you assume the phase speed is purely real, the problem turns out to be inconsisten. So we can write:

$$c = c_r + ic_i \tag{238}$$

This is an important change. With a complex c, we have:

$$e^{ik(x-ct)} = e^{ik(x-(c_r+ic_i)\,t)} = e^{ik(x-c_rt)+kc_it} \tag{239}$$

The argument of the exponential has both real and imaginary parts. The real part determines how the phases change, as before. But the imaginary part can change the amplitude of the wave. In particular, if $c_i > 0$, the wave amplitude will *grow exponentially in time*. If this happens, we say the flow is *barotropically unstable*. Then the wave solution grows in time, eventually becoming as strong as the background flow itself.

If we substitute the wave solution into (236), we get:

$$(-ikc + ikU)(-k^2\hat{\psi} + \frac{\partial^2}{\partial y^2}\hat{\psi}) + ik\hat{\psi}\frac{\partial}{\partial y}q_s = 0 \quad (240)$$

Canceling the ik yields:

$$(U - c)\,(\frac{\partial^2}{\partial y^2}\hat{\psi} - k^2\hat{\psi}) + \hat{\psi}\frac{\partial}{\partial y}q_s = 0 \quad (241)$$

This is known as the "Rayleigh equation". The solution of this determines which waves are unstable. However, because U and q_s are functions of y, this is generally not easy to solve.

One alternative is to solve (241) numerically. If you know $U(y)$, you could put that into the equation and crank out a solution. If the solution has growing waves, you know the mean flow is unstable. But then say you wish to examine a slightly different flow. Then you would have to start again, and solve the equation all over. What would be nice is if we could figure out a way to determine if the flow is unstable without actually solving (241). It turns out this is possible.

2.11.1 Rayleigh-Kuo criterion

We do this as follows. First we divide (241) by $U - c$:

$$(\frac{\partial^2}{\partial y^2}\hat{\psi} - k^2\hat{\psi}) + \frac{\hat{\psi}}{U - c}\frac{\partial}{\partial y}q_s = 0 \quad (242)$$

This assumes that $U \neq c$ anywhere in the flow.[4] Then we multiply by the complex conjugate of the streamfunction:

$$\hat{\psi}^* = \hat{\psi}_r - i\hat{\psi}_i$$

[4]When $U = c$ at some point, the flow is said to have a *critical layer*. Then the analysis is more involved than that here.

This yields:

$$(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_i) + i(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r) - k^2|\hat{\psi}|^2 + \frac{|\hat{\psi}|^2}{U-c}\frac{\partial}{\partial y}q_s = 0 \tag{243}$$

The denominator in the last term is complex. We write it in a more convenient form this way:

$$\frac{1}{U-c} = \frac{1}{U-c_r-ic_i} = \frac{U-c_r+ic_i}{|U-c|^2}$$

Now the denominator is purely real. So we have:

$$(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_i) + i(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r) - k^2|\hat{\psi}|^2$$

$$+(U-c_r+ic_i)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s = 0 \tag{244}$$

This equation has both real and imaginary parts, and each must separately equal zero.

Consider the imaginary part of (244):

$$(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r) + c_i \frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s = 0 \tag{245}$$

Let's integrate this in y, over a region from $y = [0, L]$:

$$\int_0^L (\hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r - \hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i)\, dy = c_i \int_0^L \frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s\, dy \tag{246}$$

We can rewrite the first terms by noting:

$$\hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r - \hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i = \frac{\partial}{\partial y}(\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i) - \frac{\partial}{\partial y}\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r + \frac{\partial}{\partial y}\hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i$$

$$= \frac{\partial}{\partial y}(\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i) \tag{247}$$

Substituting this into the LHS of (246), we get:

$$\int_0^L \frac{\partial}{\partial y}(\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i)\, dy = (\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i)\,|_0^L \tag{248}$$

Now, to evaluate this, we need the boundary conditions on ψ at $y = 0$ and $y = L$. Let's imagine the flow is confined to a channel. Then the normal flow vanishes at the northern and southern walls. This implies that the streamfunction is constant on those walls, and we can take the constant to be zero. Thus:

$$\hat{\psi}(0) = \hat{\psi}(L) = 0$$

Then (248) vanishes.

In fact, we obtain the same result if we simply pick $y = 0$ and $y = L$ to be latitudes where the perturbation vanishes (i.e. far away from the mean flow). Either way, the equation for the imaginary part reduces to:

$$c_i \int_0^L \frac{|\hat{\psi}|^2}{|U - c|^2} \frac{\partial}{\partial y} q_s \, dy = 0 \tag{249}$$

In order for this to be true, either c_i or the integral must be zero. If $c_i = 0$, the wave amplitude is not growing and the wave is stable. For unstable waves, $c_i > 0$. Then the integral must vanish to satisfy the equation. The squared terms in the integrand are always greater than zero, so a necessary condition for instability is that:

$$\frac{\partial}{\partial y} q_s = 0 \tag{250}$$

Thus *the meridional gradient of the background PV must change sign somewhere in the domain*. This is the *Rayleigh-Kuo criterion*. Under the β-plane approximation, we have:

$$\frac{\partial}{\partial y} q_s \equiv \beta - \frac{\partial^2}{\partial y^2} U \tag{251}$$

Thus instability requires $\beta = \frac{\partial^2}{\partial y^2} U$ somewhere in the domain.

Think about what this means. If $U = 0$, then $q_s = \beta y$. Then we have Rossby waves, all of which propagate westward. With a background flow, the waves need not propagate westward. If $\beta - \frac{\partial^2}{\partial y^2}U = 0$ somewhere, the mean PV gradient vanishes and the Rossby waves are *stationary*. So the wave holds its position in the mean flow, extracting energy from it. In this way, the wave grows in time.

The Rayleigh-Kuo criterion is a *necessary condition* for instability. So instability requires that this condition be met. But it is not a *sufficient condition*—it doesn't guarantee that a jet will be unstable. However, the opposite case is a sufficient condition; if the gradient does *not* change sign, the jet must be stable.

As noted, the Rayleigh-Kuo condition is useful because we don't actually need to solve for the unstable waves to see if the jet is unstable. Such a solution is often very involved.

We can derive another stability criterion, following Fjørtoft (1950), by taking the real part of (244). The result is similar to the Rayleigh-Kuo criterion, but a little more specific. Some flows which are unstable by the Rayleigh criterion may be stable by Fjørtoft's. However this is fairly rare. Details are given in Appendix E.

2.11.2 Examples

Let's consider some examples of barotropically unstable flows. Consider a westerly jet with a Gaussian profile (Kuo, 1949):

$$U = U_0\, exp[-(\frac{y - y_0}{L})^2] \tag{252}$$

Shown in the two right panels of Fig. (26) is $\beta - \frac{\partial^2}{\partial y^2}U$ for two jet amplitudes, U_0. We take $\beta = L = 1$, for simplicity. With $U_0 = 0.04$, the PV gradient is positive everywhere, so the jet is stable. With $U_0 = 0.1$, the PV gradient changes sign both to the north and south of the jet maximum. So

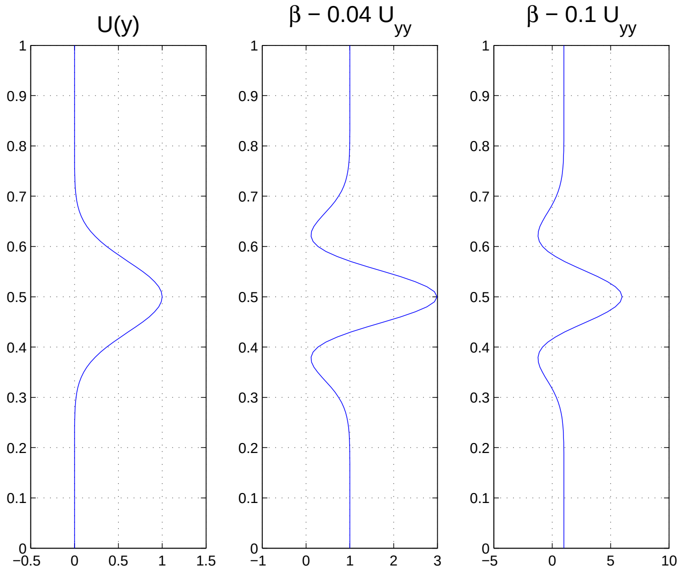


Figure 26: A westerly Gaussian jet (left panel). The middle and right panels show $\beta - \frac{\partial^2}{\partial y^2}u$ for the jet with amplitudes of 0.04 and 0.1, respectively. Only the latter satisfies Rayleigh's criterion for instability.

this jet *may* be unstable.

Now consider an *easterly* jet (Fig. 27), with $U_0 < 0$. With both amplitudes, $\beta - \frac{\partial^2}{\partial y^2}U$ is negative at the centers of the jets. So the jet is unstable with both amplitudes. This is a general result: easterly jets are more unstable than westerly jets.

An example of an evolving barotropic instability is shown in Fig. (28). This derives from a numerical simulation of a jet with a Gaussian profile of relative vorticity. So:

$$\zeta = -\frac{\partial}{\partial y}U = Ae^{-y^2/L^2} \tag{253}$$

In this simulation, $\beta = 0$, so the PV gradient is:

$$\frac{\partial}{\partial y}q_s = -\frac{\partial^2}{\partial y^2}U = -\frac{2y}{L^2}Ae^{-y^2/L^2} \tag{254}$$

This is zero at $y = 0$ and so satisfies Rayleigh's criterion. We see in the simulation that the jet is unstable, wrapping up into vortices. These have

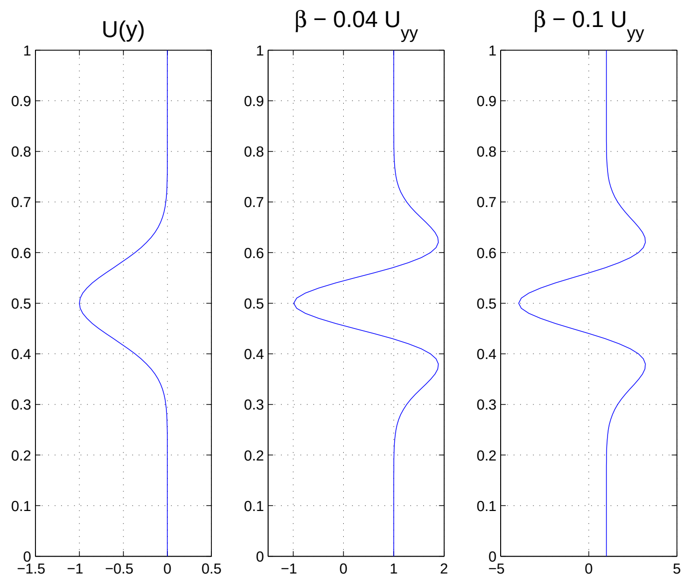


Figure 27: An easterly Gaussian jet (left panel). The middle and right panels show $\beta - \frac{\partial^2}{\partial y^2}u$ for the jet, with amplitudes of 0.04 and 0.1. Note that both satisfy Rayleigh's criterion for instability.

positive vorticity, like the jet itself.

An example of barotropic instability in the atmosphere is seen in Fig. (29). This shows three infrared satellite images of water vapor above the US. Note in particular the dark band which stretches over the western US in into Canada. This is a filament of air, near the tropopause. We see that the filament is rolling up into vortices, much like in the numerical simulation in (28).

Barotropic instability also occurs in the ocean. Consider the following example, from the southern Indian and Atlantic Oceans (Figs. 30-32). Shown in (30) is a Stommel-like solution for the region. Africa is represented by a barrier attached to the northern wall, and the island to its east represents Madagascar. The wind stress curl is indicated in the right panel; this is negative in the north, positive in the middle and negative in the south.

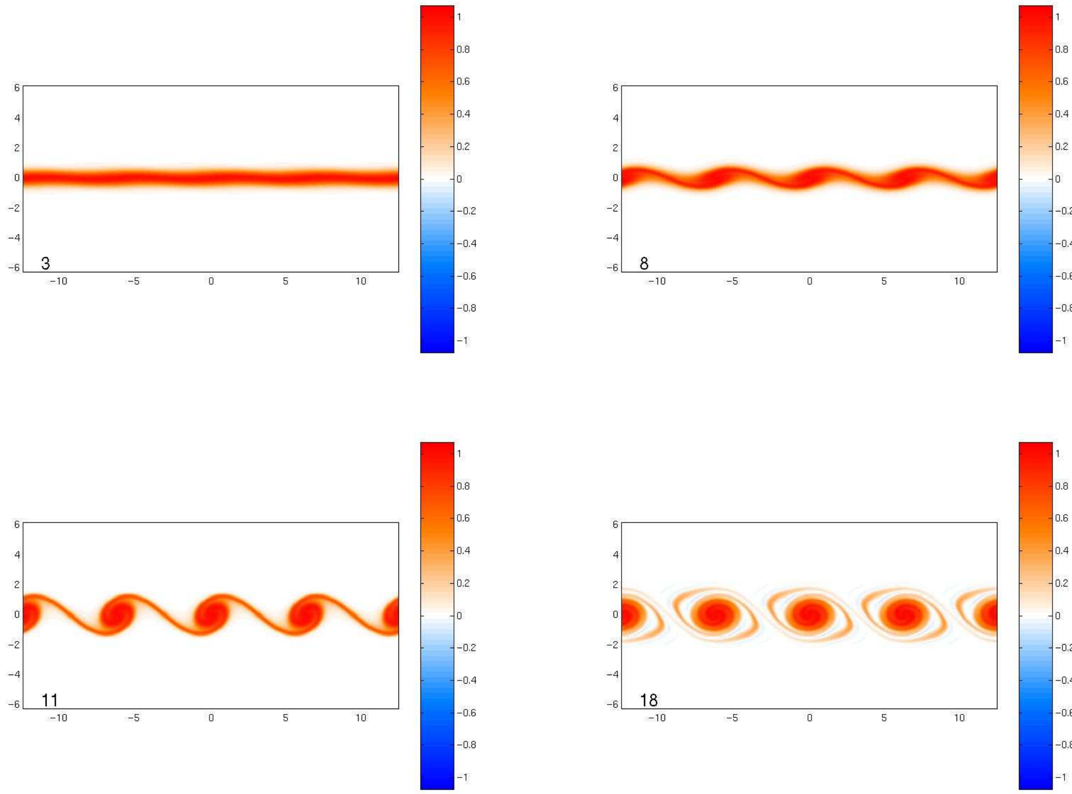


Figure 28: Barotropic instability of a jet with a Gaussian profile in relative vorticity. Courtesy of G. Hakim, Univ. of Washington.

In the southern part of the domain, the flow is eastward. This represents the Antarctic Circumpolar Current (the largest ocean current in the world). In the “Indian ocean”, the flow is to the west, towards Madagascar. This corresponds to the South Equatorial Current, which impinges on Madagascar. There are western boundary currents to the east of Africa and Madagascar. The boundary currents east of Madagascar flow westward toward Africa in two jets, to the north and south of the Island. Similarly, the western boundary current leaves South Africa to flow west and join the flow in the South Atlantic.

Shown in Fig. (31) is the PV gradient for this solution, in the region near South Africa and Madagascar. Clearly the gradient is dominated by

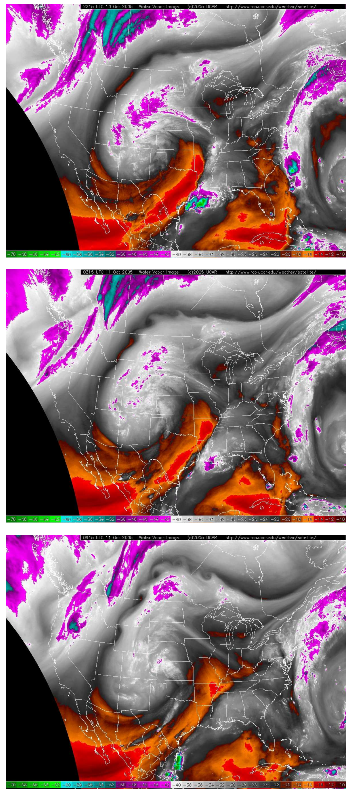


Figure 29: Barotropic instability of filaments on the tropopause, observed from water vapor infrared satellite imagery. The images were taken on the 11th of October, 2005, at 22:45 pm, 3:15 am and 9:45 am, respectively. Courtesy of G. Hakim, Univ. of Washington.

the separated jets. Moreover, the gradient changes sign several times in each of the jets. So we would expect the jets might be unstable, by the Rayleigh-Kuo criterion.

A snapshot from a numerical solution of the barotropic flow is shown in Fig. (32). In this simulation, the mean observed winds were used to drive the ocean, which was allowed to spin-up to a statistically steady state. The figure shows a snapshot of the sea surface height, after the model has spun up. We see that all three of the eastward jets have become unstable and are generating eddies (of both signs). The eddies drift westward, linking up with the boundary currents to their west.

Barotropic instability occurs when the lateral shear in a current is too

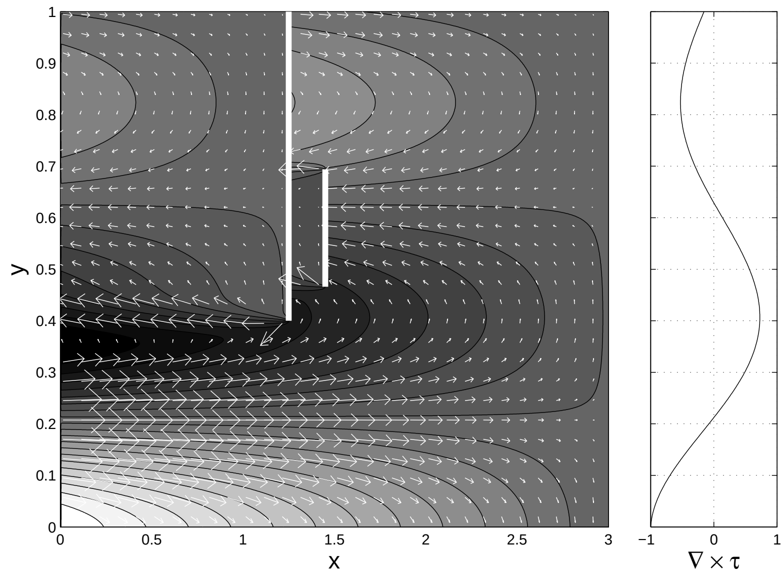


Figure 30: A Stommel-like solution for the Indian Ocean. The curl of the wind stress is indicated in the right panel. From LaCasce and Isachsen (2007).

large. The unstable waves extract energy from the mean flow, reducing the shear by mixing momentum laterally. However, in the atmosphere *baroclinic instability* is more important, in terms of storm formation. Under baroclinic instability, the waves act to reduce the *vertical shear* of the mean flow. In order to study that, we have to take account of density changes.

Exercise 2.9: Barotropic instability. We have a region with $0 \leq x < 1$ and $-1 \leq y < 1$. Consider the following velocity profiles:

a) $U = 1 - y^2$

b) $U = exp(-y^2)$

c) $U = sin(\pi y)$

d) $U = \frac{1}{6}y^3 + \frac{5}{6}y$

Which profiles are unstable by the Rayleigh-Kuo criterion if $\beta = 0$? How large must β be to stabilize *all* the profiles? Note that the terms here

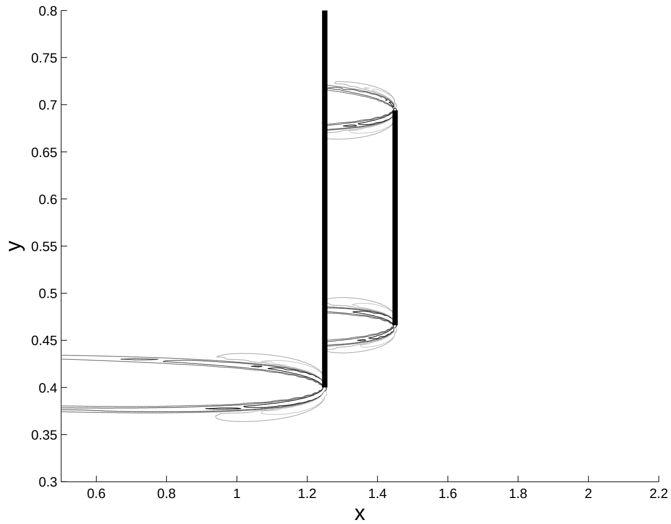


Figure 31: The PV gradient for the solution in Fig. (30). The gradient changes sign rapidly in the three jet regions. From LaCasce and Isachsen (2007).

have been non-dimensionalized, so that β can be any number (e.g. an integer).

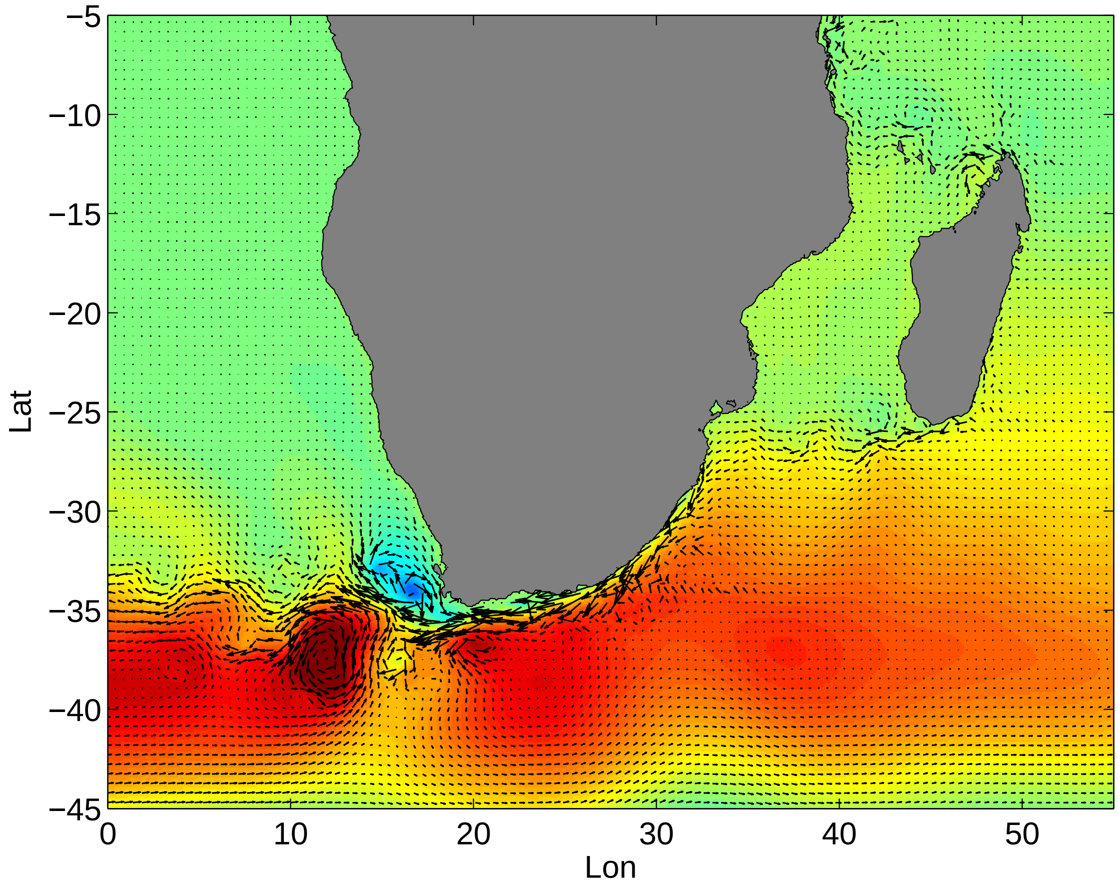


Figure 32: The sea surface height from a barotropic numerical simulation of the southern Indian and Atlantic Oceans. From LaCasce and Isachsen (2007).

3 Baroclinic flows

We will now examine what happens with vertical shear. In this case the winds at higher levels need not be parallel to or of equal strength with those at lower levels. Baroclinic flows are inherently more three dimensional than barotropic ones. Nevertheless, we will see that we get the same type of solutions with baroclinic flows as with barotropic ones. We have baroclinic Rossby waves and baroclinic instability. These phenomena involve some modifications though, as seen hereafter.

Consider the vorticity equation (123):

$$(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x})(\nabla^2 \psi + f) = f_0 \frac{\partial}{\partial z} w \tag{255}$$

When we derived this, we made no demands about the vertical structure of the flows. Thus this equation works equally well with baroclinic flows as barotropic ones. The equation has two unknowns, ψ and w. For barotropic flows, we eliminate w by integrating over the depth of the fluid. Then the vertical velocity only enters at the upper and lower boundaries.

With baroclinic flows however it is not so simple to dispose of w. We require a second equation which also has ψ and w in it.

3.1 Density Equation

For this, we use the equation for the fluid density (temperature). In the atmosphere, we have the thermodynamic equation (36):

$$c_p \frac{d(ln\theta)}{dt} = \frac{J}{T} \tag{256}$$

With zero heating, $J = 0$, this implies:

$$\frac{d\theta}{dt} = 0 \tag{257}$$

i.e. that the potential temperature is conserved. This equation can be rewritten in terms of ψ and ω and then combined with the pressure coordinate version of the vorticity equation (Appendix F).

To illustrate this, we'll do the derivation in z-coordinates. The corresponding thermodynamic equation for the ocean is:

$$\frac{d\rho}{dt} = \frac{\partial}{\partial t}\rho + \vec{\ }\cdot\nabla\rho = 0 \tag{258}$$

Here the velocity here is the full velocity, not just the geostrophic one.

Now, we have seen that the hydrostatic approximation is an excellent approximation for synoptic scale flows. This implies that we can decompose the pressure into static and moving parts:

$$p = p_0(z) + p'(x, y, z, t) = -\rho_0 g z + p'(x, y, z, t)$$

Now ρ_0 is allowed to vary with height, but only height. So we can write:

$$\rho = \rho_0(z) + \rho'(x, y, z, t) \tag{259}$$

Only the perturbation fields are important for horizontal motion. We assume too, as always, that the dynamic parts are much smaller:

$$|\rho'| \ll \rho_0, \quad |p'| \ll p_0 \tag{260}$$

Moreover, the perturbation terms are also linked by the hydrostatic relation, as shown in sec. (1.7). So:

$$\frac{\partial}{\partial z}p' = -\rho' g \tag{261}$$

Using the static and dynamic densities, along with the geostrophic horizontal velocities, in the simplied density equation yields:

$$\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)\rho' + w\,\frac{\partial}{\partial z}\rho_0 = 0 \tag{262}$$

Note we neglect the term involving the vertical advection of the perturbation density, as this is smaller than the advection of background density. Using the hydrostatic balance, we have:

$$(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}) \frac{\partial p'}{\partial z} - gw \frac{\partial}{\partial z}\rho_0 = 0 \tag{263}$$

after multiplying through by $-g$. Lastly, we can substitute in the geostrophic streamfunction defined in (120). Then we obtain:

$$(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}) \frac{\partial \psi}{\partial z} + \frac{N^2}{f_0} w = 0 \tag{264}$$

This is the quasi-geostrophic density equation. Here N^2 is the *Brunt-Vaisala frequency*:

$$N^2 = -\frac{g}{\rho_c}\frac{d\rho_0}{dz} \tag{265}$$

The Brunt-Vaisala frequency is a measure of the stratification in z-coordinates. It reflect the frequency of oscillation of parcels in a stably stratified fluid which are displaced up or down (see problem 3.1).

Consider what the density equation means. If there is vertical motion in the presence of background stratification, the perturbation density will change. For example, if the background density decreases going up (as it must for a stably stratified fluid), a rising parcel has:

$$w \frac{\partial}{\partial z}\rho_0 < 0$$

This implies that the pertubation density must increase in time. So as the parcel rises, it becomes heavier relative to the background density.

There is an interesting parallel here. The vorticity equation implies that *meridional* motion changes the parcels *vorticity*. Here we see that *vertical* motion affects its *density*. The two effects are intimately linked when you have baroclinic instability (sec. 3.7).

Equation (264) gives us a second equation involving ψ and w. Combined with the vorticity equation (123), we now have a complete system.

Problem 3.1: The Brunt-Vaisala Frequency.

Consider a fluid parcel which is displaced from its initial vertical position, z_0, a distance δz. Assume we have a mean background stratification for which:

$$\frac{\partial}{\partial z}p = -\rho_0 g$$

Substitute this into the vertical momentum equation to find:

$$\frac{dw}{dt} = g(\frac{\rho_0 - \rho}{\rho})$$

Estimate ρ_0 at $z_0 + \delta z$ by Taylor-expanding about z_0. Assume the parcel conserves its density from z_0. Then use the vertical momentum equation to show that:

$$\frac{d^2(\delta z)}{dt^2} = -N^2 \delta z$$

and define N^2. This is known as the Brunt-Vaisala frequency. What happens if $N^2 > 0$? What if it is negative?

3.2 QG Potential vorticity

We now have two equations with two unknowns. It is straightforward to combine them to produce a single equation with only one unknown. We eliminate w from (123) and (264). First we multiply (264) by f_0^2/N^2 and take the derivative with respect to z:

$$\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial}{\partial t}\frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial z}[\vec{u}_g \cdot \nabla(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})] = -f_0\frac{\partial}{\partial z}w \qquad (266)$$

Now the second term can be expanded thus:

$$(\frac{\partial}{\partial z}\vec{u}_g) \cdot \nabla(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}) + \vec{u}_g \cdot \nabla(\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}))$$

The first term vanishes. You can see this by writing the velocity in terms of the streamfunction:

$$\frac{f_0^2}{N^2}[-\frac{\partial}{\partial z}(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial z}(\frac{\partial \psi}{\partial x})\frac{\partial}{\partial y}(\frac{\partial \psi}{\partial z})] = 0 \qquad (267)$$

The physical reason for this is that the the geostrophic velocity is parallel to the pressure; thus the dot product between $(\frac{\partial}{\partial z}\vec{u}_g)$ and the gradient of $\frac{\partial}{\partial z}\psi$ must be zero. So (266) reduces to:

$$(\frac{\partial}{\partial t} + \vec{u}_g \cdot \nabla)\,[\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})] = -f_0\frac{\partial}{\partial z}w$$

If we combine (266) with (123), we get:

$$(\frac{\partial}{\partial t} + \vec{u}_g \cdot \nabla)\,[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}) + \beta y] = 0 \qquad (268)$$

This is the *quasi-geostrophic potential vorticity* (QGPV) equation. It has only one unknown, ψ. The equation implies that the potential vorticity:

$$q = \nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}) + \beta y \qquad (269)$$

is *conserved following a parcel moving with the geostrophic flow*. This is a powerful constraint. The flow evolves in such a way that q is only redistributed, not changed.

The first term in the QGPV is the QG relative vorticity and the third term is the planetary vorticity, as noted before. The second term is new; this is the *stretching vorticity*. This is related to vertical gradients in the density.

The QGPV equation can be used to model synoptic scale flows. If one were to codes this up, you would solve for the flow in several steps. First, the QGPV equation is advanced in time, to obtain the PV at the next time step. Then the PV is *inverted* to obtain the streamfunction. From this, we can obtain the velocities and then advance the QGPV equation again. However, the inversion step is often non-trivial. Doing this requires *boundary conditions*. We consider these next.

3.3 Boundary conditions

Notice the QGPV equation (268) doesn't contain any Ekman or topographic terms. This is because the PV equation pertains to the interior. In the barotropic case, we introduced those terms by integrating between the lower and upper boundaries. But here, we must treat the boundary conditions separately.

We obtain these by evaluating the density equation (264) at the boundaries. We can rewrite the relation slightly this way:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial\psi}{\partial z} = -w \tag{270}$$

As discussed in section (2.2), the vertical velocity at the boundary can come from either pumping from an Ekman layer or flow over topography. Thus for the lower boundary, we have:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial\psi}{\partial z}|_{zb} = -u_g \cdot \nabla h - \frac{\delta}{2}\nabla^2\psi \tag{271}$$

where the velocities and streamfunction are evaluated at the bottom boundary, which we take to be at $z = z_b$.

The upper boundary condition is similar. For the ocean, with the ocean surface at $z = z_u$, we have:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial \psi}{\partial z}\,|_{zu} = -\frac{1}{\rho_c f_0}\nabla \times \vec{\tau}_w \qquad (272)$$

The upper boundary condition for the atmosphere depends on the application. If we are considering the entire atmosphere, we could demand that the amplitude of the motion decay as $z \to \infty$, or that the energy flux is directed upwards. However, we will primarily be interested in motion in the troposphere. Then we can treat the tropopause as a surface, either rigid or freely moving. If it is a rigid surface, we would have simply:

$$\frac{1}{N^2}\frac{d_g}{dt}\frac{\partial \psi}{\partial z^*}\,|_{zu} = 0 \qquad (273)$$

at $z = z_u$. A free surface is only slightly more complicated, but the rigid upper surface will surfice for what follows.

3.4 Baroclinic Rossby waves

We now look at some specific solutions. We will begin with seeing how stratification alters the Rossby wave solutions.

First we linearize the PV equation (268) assuming a constant background flow:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\,[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})] + \beta\frac{\partial}{\partial x}\psi = 0 \qquad (274)$$

We assume moreover that the domain lies between two rigid, flat surfaces. With the ocean in mind, we'll take the boundaries at $z = 0$ and $z = -D$ (the result would be the same with positive z). We will also neglect Ekman layers on those surfaces. So the linearized boundary condition on each surface is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial \psi}{\partial z} = 0 \tag{275}$$

This implies that the density (or temperature) doesn't change on parcels advected by the mean flow along the boundary. Thus the density is constant on the boundaries, and we take the constant to be zero, so that:

$$\frac{\partial \psi}{\partial z} = 0 \tag{276}$$

The coefficients in the PV equation do not vary with time or in (x, y). But the Brunt-Vaisala frequency, N, can vary in z. So an appropriate choice of wave solution would be:

$$\psi = Re\{\hat{\psi}(z)e^{i(kx+ly-\omega t)}\} \tag{277}$$

Substituting this into the PV equation, we get:

$$(-i\omega + ikU)[-(k^2 + l^2)\hat{\psi} + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \hat{\psi}}{\partial z})] + i\beta k\hat{\psi} = 0 \tag{278}$$

or:

$$\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \hat{\psi}}{\partial z}) + \lambda^2\hat{\psi} = 0 \tag{279}$$

where:

$$\lambda^2 \equiv -k^2 - l^2 + \frac{\beta k}{Uk - \omega} \tag{280}$$

Equation (279) determines the vertical structure, $\hat{\psi}(z)$, of the Rossby waves. With the boundary conditions (276), this constitutes an *eigenvalue* or "Sturm-Liouville" problem. Only specific values of λ will be permitted. In order to find the dispersion relation for the waves, we must first solve for the vertical structure.

3.4.1 Baroclinic modes with constant stratification

To illustrate, consider the simplest case, with $N^2 = const$. Then we have:

$$\frac{\partial^2}{\partial z^2}\hat{\psi} + \frac{N^2\lambda^2}{f_0^2}\hat{\psi} = 0 \tag{281}$$

This has a general solution:

$$\hat{\psi} = Acos(\frac{N\lambda z}{f_0}) + Bsin(\frac{N\lambda z}{f_0}) \tag{282}$$

In order to satisfy $\frac{\partial}{\partial z}\hat{\psi} = 0$ on the upper boundary (at $z = 0$), we require that $B = 0$. But in addition, it must work on the lower boundary, at $z = -D$. So either $A = 0$ (so that we have no wave at all) or:

$$sin(\frac{N\lambda D}{f_0}) = 0 \tag{283}$$

For this to be true:

$$\frac{N\lambda D}{f_0} = n\pi \tag{284}$$

where $n = 0, 1, 2...$ is an integer. In other words, only specific combinations of of the parameters will work. Solving for λ, we get:

$$\lambda^2 = \frac{n^2\pi^2 f_0^2}{N^2 D^2} = \frac{n^2}{L_D^2} \tag{285}$$

Here,

$$L_D = \frac{ND}{\pi f_0}$$

is the baroclinic *deformation radius*. Combining this with the definition of λ^2, we get:

$$\frac{n^2}{L_D^2} \equiv -k^2 - l^2 + \frac{\beta k}{Uk - \omega} \tag{286}$$

Solving for ω, we obtain:

$$\omega \equiv \omega_n = Uk - \frac{\beta k}{k^2 + l^2 + n^2/L_D^2} \tag{287}$$

This is the *dispersion relation for baroclinic Rossby waves*. In fact, we have an infinite number of relations, one for each value of n. And for each n, we have a different vertical struture. The wave structure corresponding to each is given by:

$$\psi = Acos(kx + ly - \omega_n t)\, cos(\frac{n\pi z}{D}) \tag{288}$$

These are the baroclinic Rossby waves.

Consider first the case with $n = 0$. Then the dispersion relation is:

$$\omega_0 = Uk - \frac{\beta k}{k^2 + l^2} \tag{289}$$

This is just the dispersion relation for the barotropic Rossby wave obtained earlier (sec. 2.4). The wave solution with $n = 0$ is

$$\psi_0 = Acos(kx + ly - \omega_n t) \tag{290}$$

This doesn't vary in the vertical, exactly like the barotropic case we considered before. So the *barotropic mode* exists, even though there is stratification. All the properties that we derived before apply to this wave as well.

With $n = 1$, the streamfunction is:

$$\psi_1 = Acos(kx + ly - \omega_n t)cos(\frac{\pi z}{D}) \tag{291}$$

This is the *first baroclinic mode*. The streamfunction (and thus the velocities) change sign in the vertical. Thus if the velocity is eastward near the upper boundary, it is westward near the bottom. There is also a "zero-crossing" at $z = -D/2$, where the velocities vanish. The waves have an associated density perturbation as well:

$$\rho_1 \propto \frac{\partial}{\partial z}\psi_1 = -\frac{n\pi}{D}Acos(kx + ly - \omega_n t)sin(\frac{\pi z}{D}) \tag{292}$$

So the density perturbation is largest at the mid-depth, where the horizontal velocities vanish. In the ocean, first mode baroclinic Rossby waves cause large deviations in the *thermocline*, which is the subsurface maximum in the density gradient.

We have assumed the surface and bottom are flat, and our solution has no density perturbations on those surfaces. However, if we had allowed the upper surface to move, we would have found that the first baroclinic mode has an associated surface deflection. Moreover, this deflection is of the opposite in sign to the density perturbation at mid-depth. If the density contours are pressed down at mid-depth, the surface rises. This means one can observe baroclinic Rossby waves by satellite.

The dispersion relation for the first mode is:

$$\omega_1 = Uk - \frac{\beta k}{k^2 + l^2 + 1/L_D^2} \tag{293}$$

The corresponding zonal phase speed is:

$$c_1 = \frac{\omega_1}{k} = U - \frac{\beta}{k^2 + l^2 + 1/L_D^2} \tag{294}$$

So the first mode wave also propagates westward relative to the mean flow. But the phase speed is *slower* than that of the barotropic Rossby wave. However, if the wavelength is much smaller than the deformation radius (so that $k^2 + l^2 \gg 1/L_d^2$), then:

$$c_1 \approx U - \frac{\beta}{k^2 + l^2} \tag{295}$$

So small scale baroclinic waves have a phase speed like that of a barotropic wave of the same size.

If on the other hand the wave is much larger than the deformation radius, then:

$$c_1 \approx U - \beta L_D^2 = U - \frac{\beta N^2 D^2}{\pi^2 f_0^2} \tag{296}$$

This means the large waves are *non-dispersive*, because the phase speed is independent of the wavenumber. This phase speed, known as the "long wave speed", is a strong function of latitude, varying inversely with the square of the Coriolis parameter. Where f_0 is small—at low latitudes—the long baroclinic waves move faster.

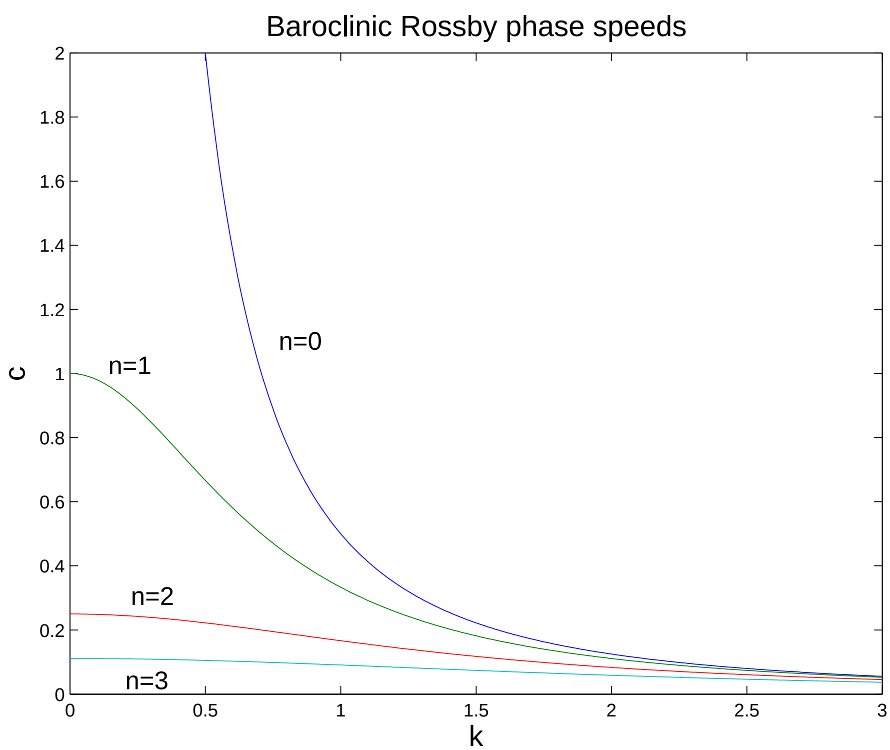


Figure 33: Rossby phase speeds as a function of wavenumber for the first four modes.

The phase speeds from the first four modes are plotted as a function of wavenumber in Fig. (33). Here we plot the function:

$$c_n = \frac{1}{2k^2 + n^2} \tag{297}$$

(note that the actual c is the negative of this). We have set $\beta = L_D = 1$ and $k = l$ and assumed the mean flow is zero. The barotropic mode ($n = 0$) has a phase speed which increases without bound as the wavenumber goes to zero. This is actually a consequence of having a rigid lid at the surface; if we had a free (moving) surface, the wave would have a finite phase speed at $k = 0$. The first baroclinic mode ($n = 1$) has a constant phase speed at low k, equal to $c = 1$. This is the long wave speed with $L_D = 1$. The second and third baroclinic modes ($n = 2, 3$) also have long wave speeds, but these are four and nine times smaller than the first baroclinic long wave speed.

Problem 3.2: Normal modes

We solved for the baroclinic modes assuming the the upper and lower boundaries were flat surfaces, with $w = 0$. As a result, the waves have non-zero flow at the bottom. But if the lower boundary is *rough*, a better condition is to assume that the horizontal velocity vanishes, i.e. $u = v = 0$.

Find the modes with this boundary condition. Compare the solutions to those with a flat bottom. What happens to the barotropic mode? The derivation is slightly simpler if you have the bottom at $z = 0$ and the surface at $z = D$.

Problem 3.3: Baroclinic Rossby waves

a) What is the phase velocity for a long first baroclinic Rossby wave in the ocean at 10N? Assume that $N = 0.01\ sec^{-1}$ and that the ocean depth is 5 km.

b) What about at 30N?

c) What is the group velocity for long first baroclinic Rossby waves?

d) What do you think would happen to a long wave if it encountered a western wall?

3.4.2 Baroclinic modes with exponential stratification

In the preceding section, we assumed a constant Brunt-Vaisala frequency, N. This implies the density has linear profile in the vertical. In reality, the oceanic density varies strongly with z. In many locations, the Brunt-Vaisala frequency exhibits a nearly exponential dependence on depth, with larger values near the surface and smaller ones at depth.

An exponential profile can also be solved analytically. Assume:

$$N^2 = N_0^2 e^{\alpha z} \tag{298}$$

Substituting (298) into (279) yields:

$$\frac{d^2\hat{\psi}}{dz^2} - \alpha\frac{d\hat{\psi}}{dz} + \frac{N_0^2\lambda^2}{f_0^2}e^{\alpha z}\hat{\psi} = 0 \tag{299}$$

Making the substitution $\zeta = e^{\alpha z/2}$, we obtain:

$$\zeta^2\frac{d^2\hat{\psi}}{d\zeta^2} - \zeta\frac{d\hat{\psi}}{d\zeta} + \frac{4N_0^2\lambda^2}{\alpha^2 f_0^2}\zeta^2\hat{\psi} = 0 \tag{300}$$

This is a Bessel-type equation. The solution which satisfies the upper boundary condition (at $z = 0$) is:

$$\hat{\psi} = Ae^{\alpha z/2}[Y_0(2\gamma)J_1(2\gamma e^{\alpha z/2}) - J_0(2\gamma)Y_1(2\gamma e^{\alpha z/2})] \tag{301}$$

where $\gamma = N_0\lambda/(\alpha f_0)$. If we then impose the bottom boundary condition, we get:

$$J_0(2\gamma)Y_0(2\gamma e^{-\alpha H/2}) - Y_0(2\gamma)J_0(2\gamma e^{-\alpha H/2}) = 0 \tag{302}$$

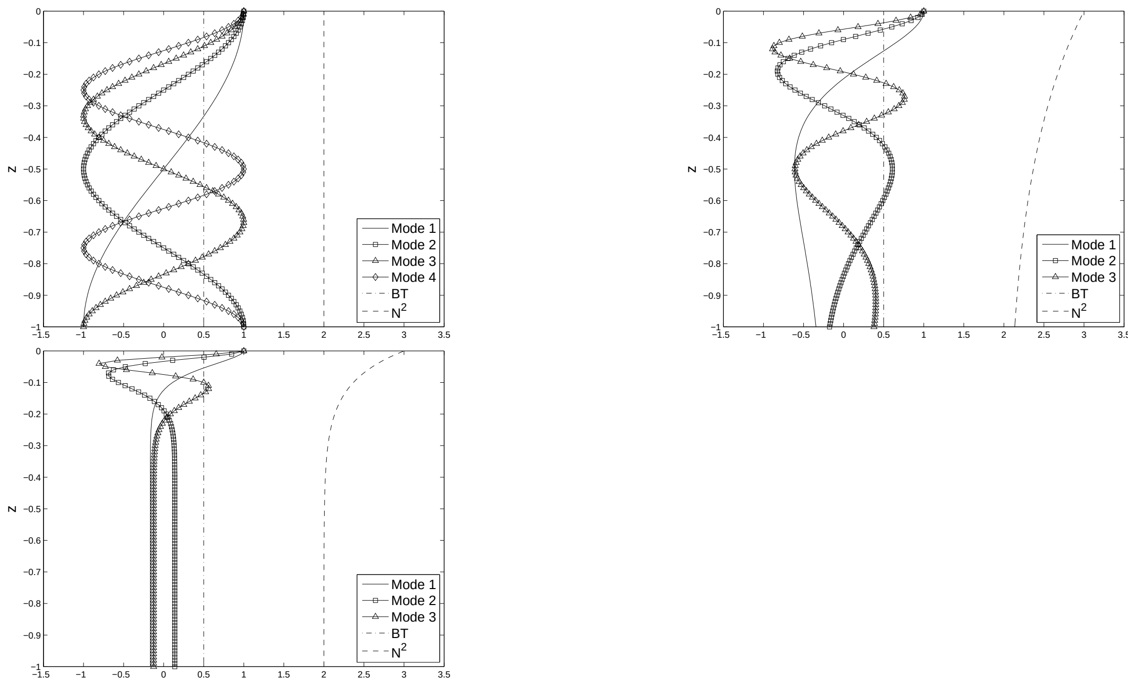


Figure 34: The baroclinic modes with N=const. (upper left panel) and with exponential N. In the upper right panel, $\alpha^{-1} = H/2$, and in the lower left, $\alpha^{-1} = H/10$. In all cases, $H = 1$. From LaCasce (2012).

Equation (302), a *transcendental equation*, admits only certain discrete values, γ_n. In other words, γ_n is quantized, just as it was with constant stratification. Once γ_n is found, the wave frequencies can be determined from the dispersion relation as before. Equation (302) is more difficult to solve than with constant stratification, but it's possible to do this numerically. Notice though that $\gamma = 0$ is also a solution of (302)—so there is also a barotropic mode in this case as well.

Some examples of the wave vertical structure, $\hat{\psi}(z)$, are shown in Fig. (34). In the upper left panel are the cosine modes, with constant N^2. In the upper right panel are the modes with exponential stratification, for the case where α^{-1}, the e-folding depth of the stratification, is equal to half the total depth. In the lower right panel are the modes with the e-folding depth equal to 1/10th the water depth. In all cases, there is a depth-independent barotropic mode plus an infinite set of baroclinic modes. And in all cases, the first baroclinic mode has one zero crossing, the second mode has two, and so forth. But unlike the cosine modes, the exponential modes have their largest amplitudes near the surface. So the Rossby wave velocities and density perturbations are likewise surface-intensified.

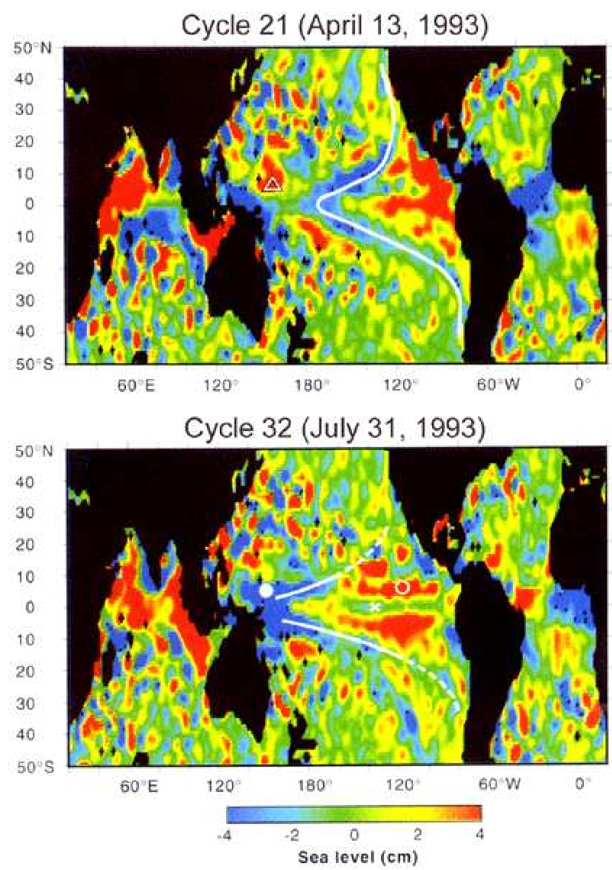


Figure 35: Sea surface height anomalies at two successive times. Westward phase propagation is clear at low latitudes, with the largest speeds occurring near the equator. From Chelton and Schlax (1996).

3.4.3 Observations of Baroclinic Rossby waves

As noted, baroclinic Rossby waves can be seen by satellite. Satellite *altimeters* measure the sea surface height elevation, and because Rossby waves also have a surface signature, then can be observed. Shown in Fig. (35) are two sea surface height fields from 1993. There are large scale anomalies in the surface elevation, and these migrate westward in time. The speed of propagation moreover increases towards the equator, which is evident from a bending of the leading wave front (indicated by the white contours).

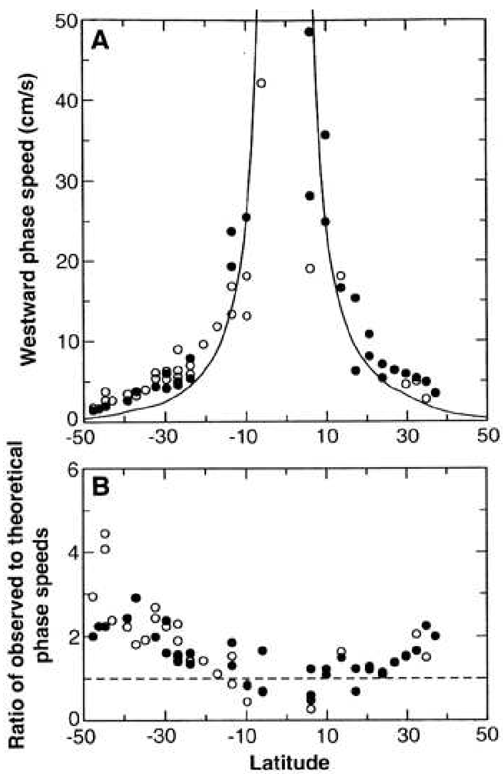


Figure 36: Westward phase speeds deduced from the motion of sea surface height anomalies, compared with the value predicted by the long wave phase speed given in (296). The lower panel shows the ratio of observed to predicted phase speed. Note the observed speeds are roughly twice as fast at high latitudes. From Chelton and Schlax (1996).

One can use satellite date like this to deduce the phase speed. Sections of sea surface height at fixed latitudes are used to construct Hovmuller diagrams (sec. 2.4.4), and then the phase speed is determined from the tilt of the phase lines. This was done by Chelton and Schlax (1996), from the

Hovmuller diagrams shown in Fig. (17); the resulting phase speeds are plotted against latitude in Fig. (36). The observations are plotted over a curve showing the long wave speed for the first baroclinic mode.

There is reasonable agreement at most latitudes. The agreement is very good below about 20 degrees of latitude; at higher latitudes there is a systematic discrepancy, with the observed waves moving perhaps twice as fast as predicted. There are a number of theories which have tried to explain this.[5] For our purposes though, we see that the simple theory does surprisingly well at predicting the observed sea surface height propagation.

There are, in addition, the higher baroclinic modes (with $n > 1$). These waves are even slower than the first baroclinic mode and have more structure in the vertical. The second baroclinic mode thus has two zero-crossings and the third baroclinic mode has three.

Note that the eigenfunctions obtained from the Sturm-Liouville problem form a *complete basis*. That means that we can express an arbitrary function in terms of them, if that function is continuous. So oceanic currents can be decomposed into vertical modes. An early attempt to do this was made by Kundu et al. (1974) using observations off the Oregon coast. Wunsch (1997) studied currents using a large collection of current meters. He found that the variability projects largely onto the barotropic and first baroclinic modes. So these two modes are probably the most important for time-varying motion.

3.5 Mountain waves

In sec. (2.8), we saw how a mean wind blowing over mountains could excite standing Rossby waves. Now we will consider what happens in the

[5] See for example LaCasce and Pedlosky (2004) and Isachsen et al. (2007).

baroclinic case.

We consider the potential vorticity equation (268), without forcing:

$$\frac{d_g}{dt}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}) + \beta y] = 0 \qquad (303)$$

As before, we consider the flow driven by a mean zonal wind:

$$U\frac{\partial}{\partial x}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z})] + \beta\frac{\partial}{\partial x}\psi = 0 \qquad (304)$$

The mean flow is constant, i.e. there is no vertical or lateral shear (we take up a vertically sheared flow later on). As before, we ignore the time dependence; we are looking for stationary, wave-like solutions. Again we will assume that the stratification parameter, N^2, is constant, for simplicity.

With a constant N^2, all the coefficients in the vorticity equation are constant. That means we can use a solution which is wave-like in all directions:

$$\psi = \hat{\psi}e^{ikx+ily+imz} \qquad (305)$$

Substituting this into (304) yields:

$$ikU[-(k^2+l^2) - m^2\frac{f_0^2}{N^2}] + ik\beta\hat{\psi} = 0 \qquad (306)$$

Rearranging, we get:

$$m = \pm\frac{N}{f_0}(\frac{\beta}{U} - k^2 - l^2)^{1/2} \qquad (307)$$

The character of the solution depends on the term in the square root in (307). If this is *positive*, then m is real and we have wave-like solutions. But if the argument is *negative*, then m will be imaginary and the vertical dependence will be *exponential*. If we rule out those solutions which grow with height—recall that the source for the waves is the mountains, at the ground—then the exponential solutions are decaying upward.

But if the argument is positive, then m is real and the solution is wave-like in z. This means the waves can effectively propagate upward to infinity, leaving the troposphere and entering the stratosphere and beyond. Then the waves generated at the surface can alter the circulation higher up in the atmosphere.

In order for the argument to be positive, we require:

$$\frac{\beta}{U} > k^2 + l^2 \qquad (308)$$

This implies that the mean flow, U, must be *positive*, or eastward. Rewriting the relation, we have:

$$0 < U < \frac{\beta}{k^2 + l^2} \equiv U_s \qquad (309)$$

So while U must be positive, neither can it be too strong. It must, in particular, be less than U_s, the speed at which the barotropic Rossby wave is stationary (sec. 2.4.3).

Why is the mean flow limited by speed of the barotropic wave? As we saw in the previous section, the barotropic mode is the *fastest* of all the Rossby modes. So upward propagating waves are possible only when the mean speed is slow enough so that one of the baroclinic Rossby modes is *stationary*.

Notice that we have not said anything about the lower boundary, where the waves are forced. In fact, the form of the mountains determines the structure of the stationary waves. But the general condition above applies to all types of mountain. If the mean flow is westerly and not too strong,

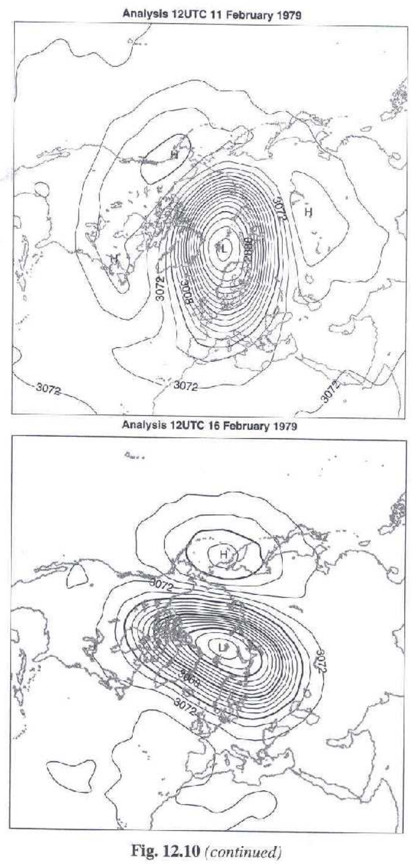


Figure 37: The geopotential height at 10 hPa on February 11 and 16, 1979. The polar vortex is being perturbed by a disturbance over the Pacific. From Holton, *An Introduction to Dynamic Meteorology*.

the waves generated over the mountains can extend upward indefinitely.

Upward propagating Rossby waves are important in the stratosphere, and can greatly disturb the flow there. They can even change the usual equator-to-pole temperature difference, a *stratospheric warming* event.

Consider Figs. (37) and (38). In the first panel of Fig. (37), we see the *polar vortex* over the Arctic. This is a region of persistent low pressure (with a correspondingly low tropopause height). In the second panel, a high pressure is developing over the North Pacific. This high intensifies, eventually causing the polar vortex has split in two, making a mode

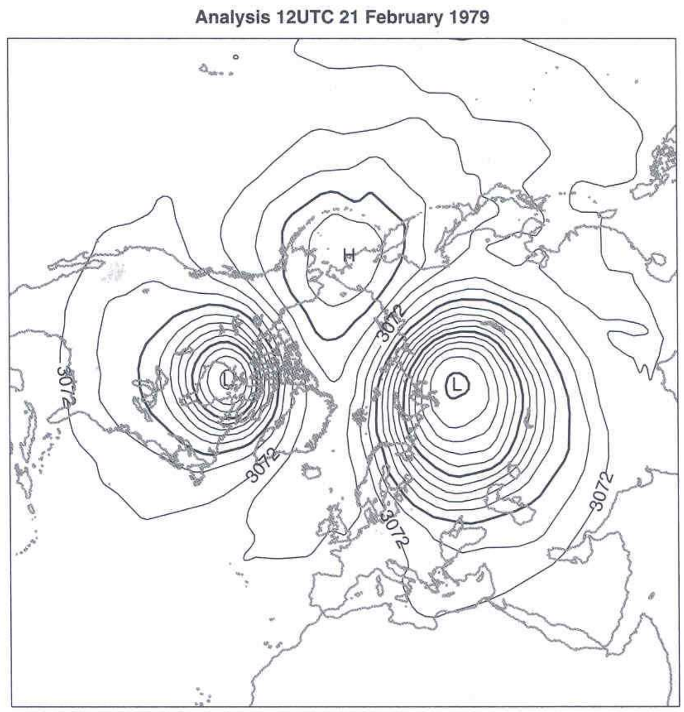


Fig. 12.10 10 hPa geopotential height analyses for February 11, 16, and 21, 1979 at 12UTC showing breakdown of the polar vortex associated with a wave number 2 sudden stratospheric warming. Contour interval: 16 dam. Analysis from ERA-40 reanalysis courtesy of the European Centre for Medium-Range Weather Forecasts (ECMWF).

Figure 38: The geopotential height at 10 hPa on February 21, 1979 (following Fig. 37). The polar vortex has split in two, appearing now as a mode 2 Rossby wave. From Holton, *An Introduction to Dynamic Meteorology*.

2 planetary wave (Fig. 38). The wave has a corresponding temperature perturbation, and in regions the air actually warms moving from south to north.

Stratospheric warming events occur only in the wintertime. Charney and Drazin (1961) used the above theory to explain which this happens. In the wintertime, the winds are westerly ($U > 0$), so that upward propagation is possible. But in the summertime, the stratospheric winds are *easterly* ($U < 0$), preventing upward propagation. So Rossby waves only alter the stratospheric circulation in the wintertime.

Problem 3.4: Mountain waves

Suppose that a stationary linear Rossby wave is forced by flow over sinusoidal topography with height $h(x) = h_0\, cos(kx)$. Show that the lower boundary condition on the streamfunction can be expressed as:

$$\frac{\partial}{\partial z}\psi = -\frac{hN^2}{f_0} \tag{310}$$

Using this, and an appropriate upper boundary condition, solve for $\psi(x, z)$. What is the position of the crests relative to the mountain tops?

3.6 Topographic waves

In an earlier problem, we found that a sloping bottom can support Rossby waves, just like the β-effect. The waves propagate with shallow water to their right (or "west", when facing "north" up the slope). Topographic waves exist with stratification too, and it is useful to examine their structure.

We'll use the potential vorticity equation, linearized with zero mean flow ($U = 0$) and on the f-plane ($\beta = 0$). We'll also assume that the Brunt-Vaisala frequency, N, is constant. Then we have:

$$\frac{\partial}{\partial t}(\nabla^2\psi + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\psi) = 0 \tag{311}$$

Thus the potential vorticity in the interior of the fluid *does not change in time*; it is simply constant. We can take this constant to be zero.

For the bottom boundary condition, we will assume a linear topographic slope. This can be in any direction, but we will say the depth is decreasing toward the north:

$$D = D_0 - \alpha y \tag{312}$$

so that $h = \alpha y$. In fact, this is a general choice because with f=const., the system is rotationally invariant (why?). With this topography, the bottom boundary condition (271) becomes:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial \psi}{\partial z^*}|_{zb} = -u_g \cdot \nabla h \quad \rightarrow \quad \frac{d_g}{dt}\frac{\partial}{\partial z}\psi + \frac{N^2}{f_0}\alpha v = 0 \tag{313}$$

Let's assume further that the bottom is at $z = 0$. We won't worry about the upper boundary, as the waves will be trapped near the lower one.

To see that, assume a solution which is wave-like in x and y:

$$\psi = Re\{\hat{\psi}(z)e^{ikx+ily-i\omega t}\} \tag{314}$$

Under the condition that the PV is zero, we have:

$$(-k^2 - l^2)\hat{\psi} + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\hat{\psi} = 0 \tag{315}$$

or

$$\frac{\partial^2}{\partial z^2}\hat{\psi} - \frac{N^2\kappa^2}{f_0^2}\hat{\psi} = 0 \tag{316}$$

where $\kappa = (k^2 + l^2)^{1/2}$ is again the total wavenumber. This equation only has exponential solutions. The one that decays going up from the bottom boundary has:

$$\hat{\psi}(z) = Ae^{-N\kappa z/|f_0|} \tag{317}$$

This is the vertical structure of the topographic waves. It implies the waves have a vertical e-folding scale of:

$$H \propto \frac{|f_0|}{N\kappa} = \frac{|f_0|\lambda}{2\pi N}$$

if λ is the wavelength of the wave. Thus the vertical scale of the wave *depends on its horizontal scale*. Larger waves extend further into the interior. Note too that we have a *continuum* of waves, not a discrete set like we did with the baroclinic modes (sec. 3.4).

Notice that we would have obtained the same result with the mountain waves in the previous section. If we take (307) and set $\beta = 0$, we get:

$$m = \pm\frac{N}{f_0}(-k^2 - l^2)^{1/2} = \pm\frac{iN\kappa}{f_0} \tag{318}$$

So with $\beta = 0$, we obtain *only* exponential solutions in the vertical. The wave-like solutions require an interior PV gradient.

Now we can apply the bottom boundary condition. We linearize (313) with zero mean flow and write v in terms of the streamfunction:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\psi + \frac{N^2\alpha}{f_0}\frac{\partial\psi}{\partial x} = 0 \tag{319}$$

Substituting in the wave expression for ψ, we get:

$$-\frac{\omega N\kappa}{|f_0|}A - \frac{N^2\alpha k}{f_0}A = 0 \tag{320}$$

so that:

$$\omega = -\frac{N\alpha k}{\kappa}sgn(f_0) \tag{321}$$

where $sgn(f_0)$ is +1 if $f > 0$ and -1 if $f < 0$.

This is the dispersion relation for stratified topographic waves. The phase speed in the x-direction (along the isobaths, the lines of constant depth) is:

$$c_x = -\frac{N\alpha}{\kappa} sgn(f_0) \tag{322}$$

This then is "westward" in the Northern Hemisphere, i.e. with the shallow water on the right. As with planetary waves, the fastest waves are the largest ones (with small κ). These are also the waves the penetrate the highest into the water column. Thus the waves which are closest to barotropic are the fastest.

Topographic waves are often observed in the ocean, particularly over the continental slope. Observations suggest that disturbances originating at the equator propagate north (with shallow water on the right) past California towards Canada. At the same time, waves also propagate south (with the shallow water on the left) past Peru.

Problem 3.5: Topographic waves

Say we are in a region where there is a steep topographic slope rising to the east, as off the west coast of Norway. The bottom decreases by 1 km over a distance of about 20 km. Say there is a southward flow of 10 cm/sec over the slope (which is constant with depth). Several fishermen have seen topographic waves which span the entire slope. But they disagree about which way they are propagating—north or south. Solve the problem for them, given that $N \approx 10 f_0$ and that we are at 60N.

3.7 Baroclinic instability

Now we return to instability. As discussed before, solar heating of the earth's surface causes a temperature gradient, with a warmer equator and colder poles. This north-south temperature gradient is accompanied by a vertically sheared flow in the east-west direction. The flow is weak near the surface and increases moving upward in the troposphere.

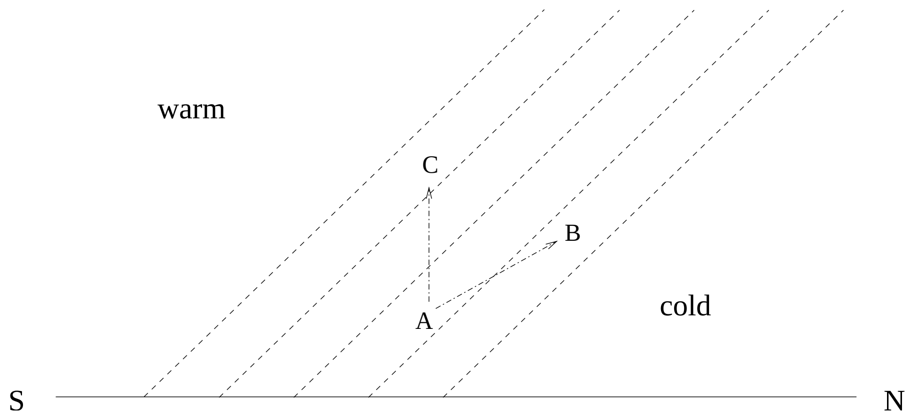


Figure 39: Slantwise convection. The slanted isotherms are accompanied by a thermal wind shear. The parcel A is colder, and thus heavier, than parcel C, implying static stability. But A is lighter than B. So A and B can be interchanged, releasing potential energy.

3.7.1 Basic mechanism

The isotherms look (crudely) as sketched in Fig. (39). The temperature decreases to the north, and also increases going up. Thus the parcel A is colder (and heavier) than parcel C, which is directly above it. The air is stably stratified, because exchanging A and C would *increase* the potential energy.

However, because the isotherms tilt, there is a parcel B which is above A and heavier. So A and B can be exchanged, *releasing* potential energy. This is often referred to as "slantwise" convection, and it is the basis for baroclinic instability. Baroclinic instability simultaneously *reduces the vertical shear* while *decreasing the north-south temperature gradient*. In effect, it causes the temperature contours to slump back to a more horizontal configuration, which reduces the thermal wind shear while decreasing the meridional temperature difference.

Baroclinic instability is extremely important. For one, it allows us to live at high latitudes—without it, the poles would be much colder than the equator.

3.7.2 Charney-Stern criterion

We can derive conditions for baroclinic instability, just as we did to obtain the Rayleigh-Kuo criterion for barotropic instability. We begin, as always, with the PV equation (268):

$$\frac{d_g}{dt}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}) + \beta y] = 0 \tag{323}$$

We linearize this about a mean flow, U, which varies in *both* the y and z-directions. Doing this is the same thing if we had writen the streamfunction as:

$$\psi = \Psi(y, z) + \psi'(x, y, z, t) \tag{324}$$

where the primed streamfunction is much smaller than the mean streamfunction. The mean streamfunction has an associated zonal flow:

$$U(y, z) = -\frac{\partial}{\partial y}\Psi \tag{325}$$

Note it has no meridional flow (V) because Ψ is independent of x. Using this, we see the mean PV is:

$$\frac{\partial^2}{\partial y^2}\Psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\Psi}{\partial z}) + \beta y \tag{326}$$

So the full linearized PV equation is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z})] + (\frac{\partial}{\partial y}q_s)\frac{\partial}{\partial x}\psi = 0 \tag{327}$$

where:

$$\frac{\partial}{\partial y}q_s = \beta - \frac{\partial^2}{\partial y^2}U - \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}) \tag{328}$$

We saw the first two terms before, in the barotropic case. The third term however is new. It comes about because the mean velocity (and hence the mean streamfunction) varies in z.

In addition, we need the boundary conditions. We will assume flat boundaries and no Ekman layers, to make this simple. Thus we use (273), linearized about the mean flow:

$$\begin{aligned}\frac{d_g}{dt}\frac{\partial\psi}{\partial z} &= (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial\psi}{\partial z} + v\frac{\partial}{\partial y}\frac{\partial\Psi}{\partial z} \\ &= (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial\psi}{\partial z} - v\frac{\partial U}{\partial z} = 0\end{aligned} \tag{329}$$

We'll assume that we have boundaries at the ground, at $z = 0$, and an upper level, $z = D$. The latter could be the tropopause. Alternatively, we could have no upper boundary at all, as with the mountain waves. But we will use an upper boundary in the Eady model in the next section, so it's useful to include that now.

Because U is potentially a function of both y and z, we can only assume a wave structure in (x, t). So we use a Fourier solution with the following form:

$$\psi = \hat{\psi}(y, z)e^{ik(x-ct)} \tag{330}$$

Substituting into the PV equation (327), we get:

$$(U - c)[-k^2\hat{\psi} + \frac{\partial^2}{\partial y^2}\hat{\psi} + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}}{\partial z})] + (\frac{\partial}{\partial y}q_s)\hat{\psi} = 0 \tag{331}$$

after canceling the factor of k. Similarly, the boundary conditions are:

$$(U-c)\frac{\partial}{\partial z}\hat{\psi}-(\frac{\partial}{\partial z}U)\hat{\psi}=0 \tag{332}$$

We now do as we did in sec. (2.11.1): we divide (331) by $U-c$ and then multiply by the complex conjugate of $\hat{\psi}$:

$$\hat{\psi}^*[\frac{\partial^2}{\partial y^2}\hat{\psi}+\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}}{\partial z})]-k^2|\hat{\psi}|^2+\frac{1}{U-c}(\frac{\partial}{\partial y}q_s)|\hat{\psi}|^2=0 \tag{333}$$

We then separate real and imaginary parts. The imaginary part of the equation is:

$$\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_i-\hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_r+\hat{\psi}_r\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}_i}{\partial z})-\hat{\psi}_i\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}_r}{\partial z})$$
$$+\frac{c_i}{|U-c|^2}(\frac{\partial}{\partial y}q_s)|\hat{\psi}|^2=0 \tag{334}$$

We have again used:

$$\frac{1}{U-c}=\frac{1}{U-c_r-ic_i}=\frac{U-c_r+ic_i}{|U-c|^2}$$

As we did previously, we use a channel domain and demand that $\hat{\psi}=0$ at the north and south walls, at $y=0$ and $y=L$. We integrate the PV equation in y and then invoke integration by parts. Doing this yields, for the first two terms on the LHS:

$$\int_0^L(\hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_r-\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_i)\,dy=\hat{\psi}_i\frac{\partial}{\partial y}\hat{\psi}_r|_0^L-\int_0^L\frac{\partial}{\partial y}\hat{\psi}_i\frac{\partial}{\partial y}\hat{\psi}_r\,dy$$
$$-\hat{\psi}_r\frac{\partial}{\partial y}\hat{\psi}_i|_0^L+\int_0^L\frac{\partial}{\partial y}\hat{\psi}_r\frac{\partial}{\partial y}\hat{\psi}_i\,dy=0 \tag{335}$$

We can similarly integrate the PV equation in the vertical, from $z = 0$ to $z = D$, and again integrate by parts. This leaves:

$$\hat{\psi}_r \frac{f_0^2}{N^2} \frac{\partial \hat{\psi}_i}{\partial z} |_0^D - \hat{\psi}_i \frac{f_0^2}{N^2} \frac{\partial \hat{\psi}_r}{\partial z} |_0^D \tag{336}$$

(because the leftover integrals are the same and cancel each other). We then evaluate these two terms using the boundary condition. We rewrite that as:

$$\frac{\partial}{\partial z} \hat{\psi} = (\frac{\partial}{\partial z} U) \frac{\hat{\psi}}{U - c} \tag{337}$$

The real part of this is:

$$\frac{\partial}{\partial z} \hat{\psi}_r = (\frac{\partial}{\partial z} U)[\frac{(U - c_r)\hat{\psi}_r}{|U - c|^2} - \frac{c_i \hat{\psi}_i}{|U - c|^2}] \tag{338}$$

and the imaginary part is:

$$\frac{\partial}{\partial z} \hat{\psi}_i = (\frac{\partial}{\partial z} U)[\frac{(U - c_r)\hat{\psi}_i}{|U - c|^2} + \frac{c_i \hat{\psi}_r}{|U - c|^2}] \tag{339}$$

If we substitute these into (336), we get:

$$\frac{f_0^2}{N^2}(\frac{\partial}{\partial z} U) \frac{c_i \hat{\psi}_i^2}{(U - c_r)^2 + c_i^2} |_0^D + \frac{f_0^2}{N^2}(\frac{\partial}{\partial z} U) \frac{c_i \hat{\psi}_r^2}{(U - c_r)^2 + c_i^2} |_0^D =$$

$$\frac{f_0^2}{N^2}(\frac{\partial}{\partial z} U) \frac{c_i \hat{|\psi|}^2}{(U - c_r)^2 + c_i^2} |_0^D \tag{340}$$

So the doubly-integrated (336) reduces to:

$$c_i [\int_0^L \int_0^D \frac{|\hat{\psi}|^2}{|U - c|^2} (\frac{\partial}{\partial y} q_s) \, dz \, dy + \int_0^L \frac{f_0^2}{N^2} \frac{\hat{|\psi|}^2}{|U - c|^2} (\frac{\partial}{\partial z} U) \ |_0^D \, dy \,] = 0 \tag{341}$$

This is the *Charney-Stern criterion* for instability. In order to have instability, $c_i > 0$ and that requires that the term in brackets vanish.

Note that the first term is identical to the one we got for the Rayleigh-Kuo criterion (249). In that case we had:

$$\frac{\partial}{\partial y}q_s = \beta - \frac{\partial^2}{\partial y^2}U \tag{342}$$

For instability, we required that $\frac{\partial}{\partial y}q_s$ had to be zero somewhere in the domain.

The baroclinic condition is similar, except that now the background PV is given by (328), so:

$$\frac{\partial}{\partial y}q_s = \beta - \frac{\partial^2}{\partial y^2}U - \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}) = 0$$

So now the vertical shear can also cause the PV gradient to vanish.

In addition, the boundary contributions also come into play. In fact we have *four* possibilities:

- $\frac{\partial}{\partial y}q_s$ vanishes in the interior, with $\frac{\partial}{\partial z}U = 0$ on the boundaries
- $\frac{\partial}{\partial z}U$ at the upper boundary has the opposite sign as $\frac{\partial}{\partial y}q_s$
- $\frac{\partial}{\partial z}U$ at the lower boundary has the same sign as $\frac{\partial}{\partial y}q_s$
- $\frac{\partial}{\partial z}U$ has the same sign on the boundaries, with $\frac{\partial}{\partial y}q_s = 0$ in the interior

The first condition is the Rayleigh-Kuo criterion. This is the only condition in the baroclinic case too if the vertical shear vanishes at the boundaries. Note that from the thermal wind balance:

$$\frac{\partial}{\partial z}U \propto \frac{\partial}{\partial y}T$$

So having zero vertical shear at the boundaries implies the temperature is *constant* on them. So the boundaries are important if there is a temperature gradient on them.

The fourth condition applies when the PV (and hence the gradient) is zero in the interior. Then the two boundaries can interact to produce instability. This is Eady's (1949) model of baroclinic instability, which we consider in the next section.

In the atmosphere, the mean relative vorticity is generally smaller than the β-effect. So the interior gradient is positive (and approximately equal to β). Then the main effect is for the lower boundary to cancel the interior term. This is what happens in Charney's (1947) model of baroclinic instability.

It is also possible to construct a model with zero shear at the boundaries and where the gradient of the interior PV vanishes because of the vertical gradient. This is what happens in Phillip's (1954) model of instability. His model has two fluid layers, with the flow in each layer being barotropic. Thus the shear at the upper and lower boundaries is zero. But because there are two layers, the PV in each layer can be different. If the PV in the layers is of opposite sign, then they can potentially sum to zero. Then Philip's model is unstable.

As with the Rayleigh-Kuo criterion, the Charney-Stern criteria represent a necessary condition for instability but not a sufficient one. So satisfying one of the conditions above indicates instability *may* occur. Note that only one needs to be satisfied. But if none of the conditions are satisfied, the flow is stable.

Problem 3.6: Instability and the Charney-Stern relation

Consider a region with $-1 \leq y < 1$ and $0 \leq z \leq D$. We have the following velocity profiles:

a) $U = Acos(\frac{\pi z}{D})$
b) $U = Az + B$
c) $U = z(1 - y^2)$

Which profiles are stable or unstable if $\beta = 0$ and $N^2 = const.$? What if $\beta \neq 0$?

(Note the terms have been non-dimensionalized, so β can be any number, e.g. 1, 3.423, .5, etc.).

3.8 The Eady model

The simplest model of baroclinic instability with continuous stratification is that of Eady (1949). This came out two years after Charney's (1947) model, which also has continuous stratification *and* the β-effect—something not included in the Eady model. But the Eady model is comparatively simple, and illustrates the major aspects.

The configuration for the Eady model is shown in Fig. (40). We will make the following assumptions:

- A constant Coriolis parameter ($\beta = 0$)
- Uniform stratification ($N^2 = const.$)
- The mean velocity has a constant shear, so $U = \Lambda z$

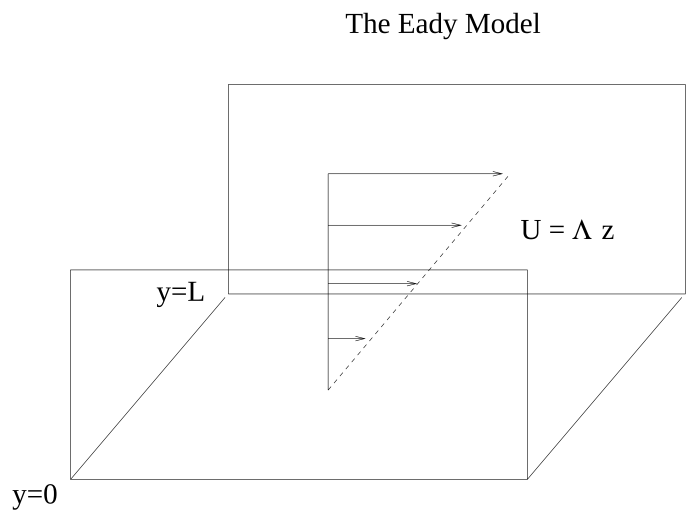


Figure 40: The configuration for the Eady model.

- The motion occurs between two rigid plates, at $z = 0$ and $z = D$
- The motion occurs in a channel, with $v = 0$ on the walls at $y = 0, L$

The uniform stratification assumption is reasonable for the troposphere but less so for the ocean (where the stratification is greater near the surface, as we have seen). The rigid plate assumption is also unrealistic, but simplifies the boundary conditions.

From the Charney-Stern criteria, we see that the model can be unstable because the vertical shear is the same on the two boundaries. The interior PV on the other hand is zero, so this cannot contribute to the instability. We will see that the interior in the Eady model is basically passive. It is the interaction between temperature anomalies on the boundaries which are important.

We will use a wave solution with the following form:

$$\psi = \hat{\psi}(z) sin(\frac{n\pi y}{L}) e^{ik(x-ct)}$$

The sin term satisfies the boundary conditions on the channel walls because:

$$v = \frac{\partial}{\partial x}\psi = 0 \quad \rightarrow \quad ik\hat{\psi} = 0 \tag{343}$$

which implies that $\hat{\psi} = 0$. The sin term vanishes at $y = 0$ and $y = L$.

The linearized PV equation for the Eady model is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\psi + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\psi) = 0 \tag{344}$$

Because there is no β term, the PV is constant on air parcels advected by the mean flow. Inserting the wave solution in yields:

$$(U - c)[(-(k^2 + \frac{n^2\pi^2}{L^2})\hat{\psi} + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\hat{\psi}] = 0 \tag{345}$$

So either the phase speed equals the mean velocity or the PV itself is zero. The former case defines what is known as a *critical layer*; we won't be concerned with that at the moment. So we assume instead the PV is zero. This implies:

$$\frac{\partial^2}{\partial z^2}\hat{\psi} = \alpha^2\hat{\psi} \tag{346}$$

where

$$\alpha \equiv \frac{N\kappa}{f_0}$$

and where $\kappa = (k^2 + (n\pi/L)^2)^{1/2}$ is the total horizontal wavenumber. This is exactly the same as in the topographic wave problem in (3.6). Equation (346) determines the vertical structure of the waves.

First, let's consider what happens when the vertical scale factor, α, is large. This is the case when the waves are short, because κ is then large.

In this case the solutions to (346) are exponentials which decay away from the boundaries:

$$\hat{\psi} = Ae^{-\alpha z}, \quad \hat{\psi} = Be^{\alpha(z-D)} \tag{347}$$

near $z = 0$ and $z = D$, respectively. The waves are thus trapped on each boundary and have a vertical structure like topographic waves.

To see how the waves behave, we use the boundary condition. This is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x}\frac{dU}{dz} = 0 \tag{348}$$

(see eq. (329)). Inserting the wave solution and the mean shear, this is simply:

$$(\Lambda z - c)\frac{\partial \psi}{\partial z} - \Lambda \hat{\psi} = 0 \tag{349}$$

after cancelling the factor of ik. At $z = 0$, this is:

$$(\alpha c - \Lambda)A = 0 \tag{350}$$

after inserting the vertical dependence at the lower boundary. At $z = D$, we have:

$$[\alpha(\Lambda D - c) - \Lambda]B = 0 \tag{351}$$

To have non-trivial solutions, A and B are non-zero. So we require:

$$c = \frac{\Lambda}{\alpha}, \quad c = \Lambda D - \frac{\Lambda}{\alpha} \tag{352}$$

at $z = 0, D$ respectively.

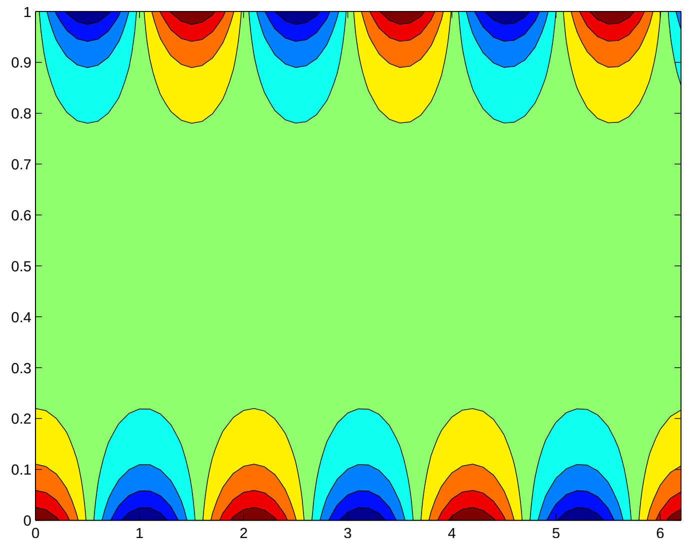


Figure 41: The Eady streamfunction in the limit of large α.

First we notice that the phase speeds are *real*—so there is no instability. The waves are simply propagating on each boundary. In the limit that α is large (the decay from the boundaries is rapid), these are:

$$c \approx 0, \quad c \approx \Lambda D \tag{353}$$

So the phase speeds are equal to the mean velocities on the boundaries. Thus the waves are just swept along by the background flow.

If α is not so large, the boundary waves propagate at speeds different than the mean flow.

The solution is shown in Fig. (41). We have two waves, each advected by the mean flow at its respective boundary and each decaying exponentially away from the boundary. These waves are *independent* because they decay so rapidly with height; they do not interact with each other.

Now let's look at the case where α is not so large, so that the waves extend further into the interior. Then we would write for the wave solution:

$$\hat{\psi} = Ae^{\alpha z} + Be^{-\alpha z} \tag{354}$$

This applies over the whole interior, including both boundaries. Plugging into the boundary equation (349) we get, at $z = 0$:

$$(-c\alpha - \Lambda)A + (\alpha c - \Lambda)B = 0 \tag{355}$$

while at the upper boundary, at $z = D$, we get:

$$(\alpha(\Lambda D - c) - \Lambda)e^{\alpha D}A + (-\alpha(\Lambda D - c) - \Lambda)e^{-\alpha D}B = 0 \tag{356}$$

We can rewrite these equations in matrix form as follows:

$$\begin{pmatrix} c\alpha + \Lambda & -c\alpha + \Lambda \\ (-\alpha c + \Lambda(\alpha D - 1))e^{\alpha D} & (\alpha c - \Lambda(\alpha D + 1))e^{-\alpha D} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{357}$$

Note we multiplied the first equation through by -1. Because this system is homogeneous, solutions exist *only* if the determinant of the coefficients vanishes. Multiplying this out, we get:

$$c^2\alpha^2(-e^{\alpha D} + e^{-\alpha D}) + c\alpha(\Lambda - \Lambda\alpha D - \Lambda)e^{-\alpha D} + c\alpha(\Lambda\alpha D - \Lambda + \Lambda)e^{\alpha D} -$$

$$\Lambda^2(\alpha D + 1)e^{-\alpha D} - \Lambda^2(\alpha D - 1)e^{\alpha D} = 0 \tag{358}$$

or:

$$-2c^2\alpha^2 sinh(\alpha D) + 2c\alpha^2\Lambda D sinh(\alpha D) - 2\Lambda^2\alpha D cosh(\alpha D)$$

$$+2\Lambda^2 sinh(\alpha D) = 0 \tag{359}$$

Dividing through by $-2\alpha^2 sinh(\alpha D)$:

$$c^2 - \Lambda Dc + \frac{\Lambda^2 D}{\alpha} coth(\alpha D) - \frac{\Lambda^2}{\alpha^2} = 0 \tag{360}$$

This quadratic equation has the solutions:

$$c = \frac{\Lambda D}{2} \pm \frac{\Lambda D}{2}[1 - \frac{4}{\alpha D} coth(\alpha D) + \frac{4}{\alpha^2 D^2}]^{1/2} \tag{361}$$

We can rewrite the part in the square root using the identity:

$$cothx = \frac{1}{2}[tanh\frac{x}{2} + coth\frac{x}{2}]$$

Then, pulling in a factor of $\alpha D/2$, the solution is:

$$\begin{aligned} c &= \frac{\Lambda D}{2} \pm \frac{\Lambda}{\alpha}[\frac{\alpha^2 D^2}{4} - \frac{\alpha D}{2} coth(\frac{\alpha D}{2}) - \frac{\alpha D}{2} tanh(\frac{\alpha D}{2}) + 1]^{1/2} \\ &= \frac{\Lambda D}{2} \pm \frac{\Lambda}{\alpha}[(\frac{\alpha D}{2} - coth[\frac{\alpha D}{2}])(\frac{\alpha D}{2} - tanh[\frac{\alpha D}{2}])]^{1/2} \end{aligned} \tag{362}$$

Now for all x, $x > tanh(x)$; so the second factor in the root is always positive. Thus if:

$$\frac{\alpha D}{2} > coth[\frac{\alpha D}{2}] \tag{363}$$

the term inside the root is positive. Then we have two phase speeds, both of which are real. This occurs when α is large. In particular, if $\alpha \gg (2/D)coth(\alpha D/2)$, these phase speeds are:

$$c = 0, \quad \Lambda D \tag{364}$$

So we recover the trapped-wave solutions that we derived first.

If, on the other hand:

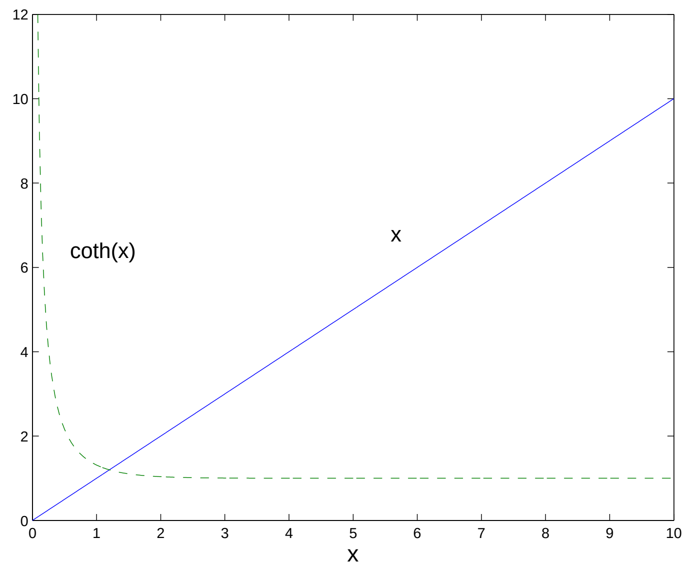


Figure 42: x and $coth(x)$.

$$\frac{\alpha D}{2} < coth[\frac{\alpha D}{2}] \tag{365}$$

the term inside the root of (362) is negative. In Fig. (42), we plot x and $coth(x)$. You can see that x is less for small values of x. Thus the condition for instability is met when α is small. Since we have:

$$\alpha = \frac{N}{f_0}(k^2 + \frac{n^2\pi^2}{L^2})^{1/2}$$

this occurs when the wavenumbers, k and n, are small. Thus large waves are more unstable.

When this condition is met, we can write the phase speed as:

$$c = \frac{\Lambda D}{2} \pm ic_i \tag{366}$$

where

$$c_i = \frac{\Lambda}{\alpha}[(coth[\frac{\alpha D}{2}] - \frac{\alpha D}{2})(\frac{\alpha D}{2} - tanh[\frac{\alpha D}{2}])]^{1/2}$$

Putting this into the wave expression, we have that:

$$\psi \propto e^{ik(x-ct)} = e^{ik(x-\Lambda Dt/2) \mp kc_i t} \tag{367}$$

Thus at each wavenumber there is a growing wave and a decaying wave. The growth rate is equal to kc_i.

The real part of the phase speed is:

$$c_r = \frac{\Lambda D}{2} \tag{368}$$

This is how fast the wave is propagating. We see that the speed is equal to the mean flow speed at the midpoint in the vertical. So it is moving slower than the mean flow speed at the upper boundary and faster than that at the lower boundary. We call the midpoint, where the speeds are equal, the *steering level*.

The growth rate is just kc_i. This is plotted in Fig. (43) for the $n = 1$ mode in the y-direction. We use the following parameters:

$$N = 0.01\, sec^{-1}, \quad f_0 = 10^{-4}\, sec^{-1}, \quad \Lambda = 0.005\, sec^{-1},$$
$$D = 10^4\, m, \quad L = 2 \times 10^6\, m$$

This shear parameter yields a velocity of 50 m/sec at the tropopause height (10 km), similar to the peak velocity in the Jet Stream. For these values, the Eady model yields complex phase speeds, indicating the troposphere is baroclinically unstable.

The growth rate increases from zero as k increases, reaches a maximum value and then goes to zero. For k larger than a critical value, the waves are stable. Thus there is a *short wave cut-off* for the instability. The shorter

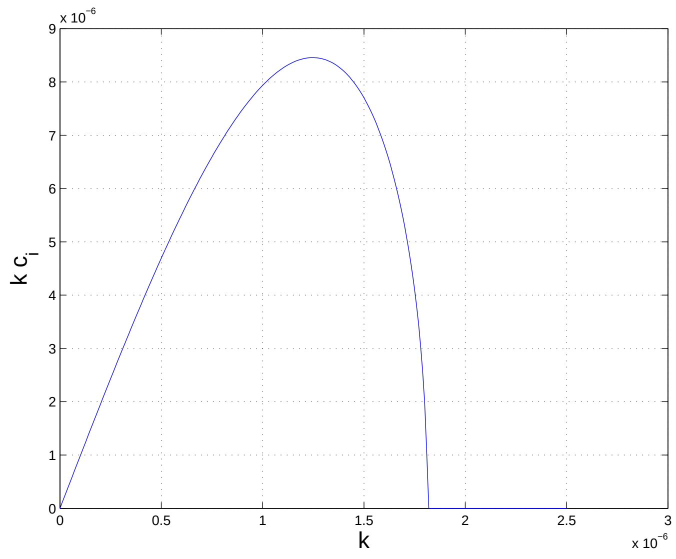


Figure 43: The Eady growth rate as a function of the wavenumber, k.

the waves are, the more trapped they are at the boundaries and thus less able to interact with each other.

The growth rate is a maximum at $k = 1.25 \times 10^{-6}\, m$, corresponding to a wavelength of $2\pi/k = 5027$ km. The wave with this size will grow faster than any other. If we begin with a random collection of waves, this one will dominate the field after a period of time.

The distance from a trough to a crest is one-fourth of a wavelength, or roughly 1250 km for this wave. So this is the scale we'd expect for storms. The maximum value of kc_i is 8.46 $\times 10^{-6}$ sec^{-1}, or equivalently $1/1.4\ day^{-1}$. Thus the growth time for the instability is on the order of a day. So both the length and time scales in the Eady model are consistent with observations of storm development in the troposphere.

Using values typical of oceanic conditions:

$$N = 0.0005\, sec^{-1}, \quad f_0 = 10^{-4}\, sec^{-1}, \quad \Lambda = 0.0001\, sec^{-1},$$

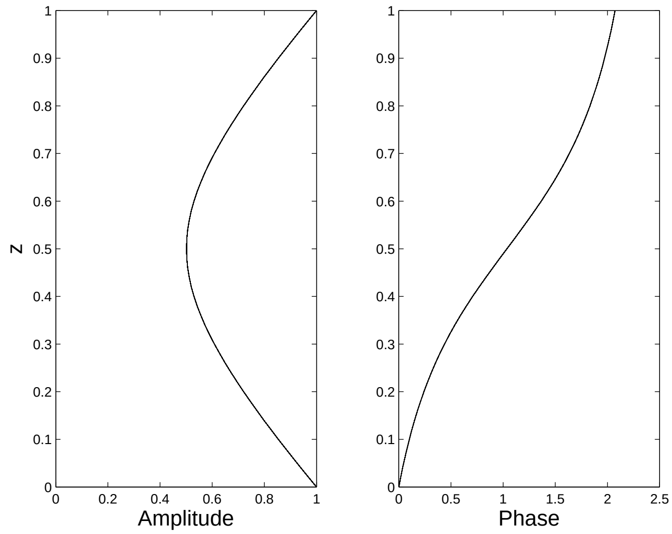


Figure 44: The amplitude (left) and phase (right) of the Eady streamfunction vs. height.

$$D = 5 \times 10^3 \, m, \quad L = 2 \times 10^6 \, m$$

we get a maximum wavelength of about 100 km, or a quarter wavelength of $25 \, km$. Because the deformation radius is so much less in the ocean, the "storms" are correspondingly smaller. The growth times are also roughly ten times longer than in the troposphere. But these values should be taken as very approximate, because N in the ocean varies greatly between the surface and bottom.

Let's see what the unstable waves look like. To plot them, we rewrite the solution slightly. From the condition at the lower boundary, we have:

$$(c\alpha + \Lambda)A + (-c\alpha + \Lambda)B = 0$$

So the wave solution can be written:

$$\psi = A[e^{\alpha z} + \frac{c\alpha + \Lambda}{c\alpha - \Lambda} e^{-\alpha z}] sin(\frac{n\pi y}{L}) e^{ik(x-ct)}$$

Rearranging slightly, we get:

$$\psi = A[cosh(\alpha z) - \frac{\Lambda}{c\alpha} sinh(\alpha z)] sin(\frac{n\pi y}{L}) e^{ik(x-ct)} \tag{369}$$

We have absorbed the αc into the unknown A. Because c is complex, the second term in the brackets will affect the phase of the wave. To take this into account, we rewrite the streamfunction thus:

$$\psi = A\Phi(z) sin(\frac{n\pi y}{L}) cos[k(x - c_r t) + \gamma(z)] e^{kc_i t} \tag{370}$$

where

$$\Phi(z) = [(cosh(\alpha z) - \frac{c_r \Lambda}{|c|^2 \alpha} sinh(\alpha z))^2 + (\frac{c_i \Lambda}{|c|^2 \alpha} sinh(\alpha z))^2]^{1/2}$$

is the magnitude of the amplitude and

$$\gamma = tan^{-1}[\frac{c_i \Lambda sinh(\alpha z)}{|c|^2 \alpha cosh(\alpha z) - c_r \Lambda sinh(\alpha z)}]$$

is its phase. These are plotted in Fig. (44). The amplitude is greatest near the boundaries. But it is not negligible in the interior, falling to only about 0.5 at the mid-level. Rather than two separate waves, we have one which spans the depth of the fluid. Also, the phase changes with height. So the streamlines *tilt* in the vertical.

We see this in Fig. (45), which shows the streamfunction, temperature, meridional and vertical velocity for the most unstable wave. The streamfunction extends between the upper and lower boundaries, and the streamlines tilt to the west going upward. This means the wave is tilted *against* the mean shear. You get the impression the wave is working against the mean flow, trying to reduce its shear (which it is). The meridional velocity (third panel) is similar, albeit shifted by 90 degrees. The temperature on the other hand tilts toward the east with height, and so is offset from the meridional velocity.

We can also derive the vertical velocity for the Eady wave. Inverting the linearized temperature equation, we have:

$$w = -\frac{f_0}{N^2}(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x})\frac{\partial \psi}{\partial z} + \frac{f_0}{N^2}\Lambda \frac{\partial \psi}{\partial x} \tag{371}$$

This is shown in the bottom panel for the most unstable wave. There is generally downward motion when the flow is toward the south and upward motion when toward the north. This fits exactly with our expectations for slantwise convection, illustrated in Fig. (39). Fluid parcels which are higher up and to the north are being exchanged with parcels lower down to the south. So the Eady model captures most of the important elements of baroclinic instability.

However, the Eady model lacks an interior PV gradient (it has no β-effect). Though this greatly simplifies the derivation, the atmosphere possesses such gradients, and it is reasonable to ask how they alter the instability. Interior gradients are considered in both the the Charney (1947) and Phillips (1954) models. Details are given by Pedlosky (1987) and by Vallis (2006).

Problem 3.7: Eady waves

a) Consider a mean flow $U = -Bz$ over a flat surface at $z = 0$ with no Ekman layer and no upper surface. Assume that $\beta = 0$ and that $N = const.$. Find the phase speed of a perturbation wave on the lower surface.

b) Consider a mean flow with $U = Bz^2$. What is the phase speed of the wave at $z = 0$ now? Assume that $\beta = Bf_0^2/N^2$, so that there still is no

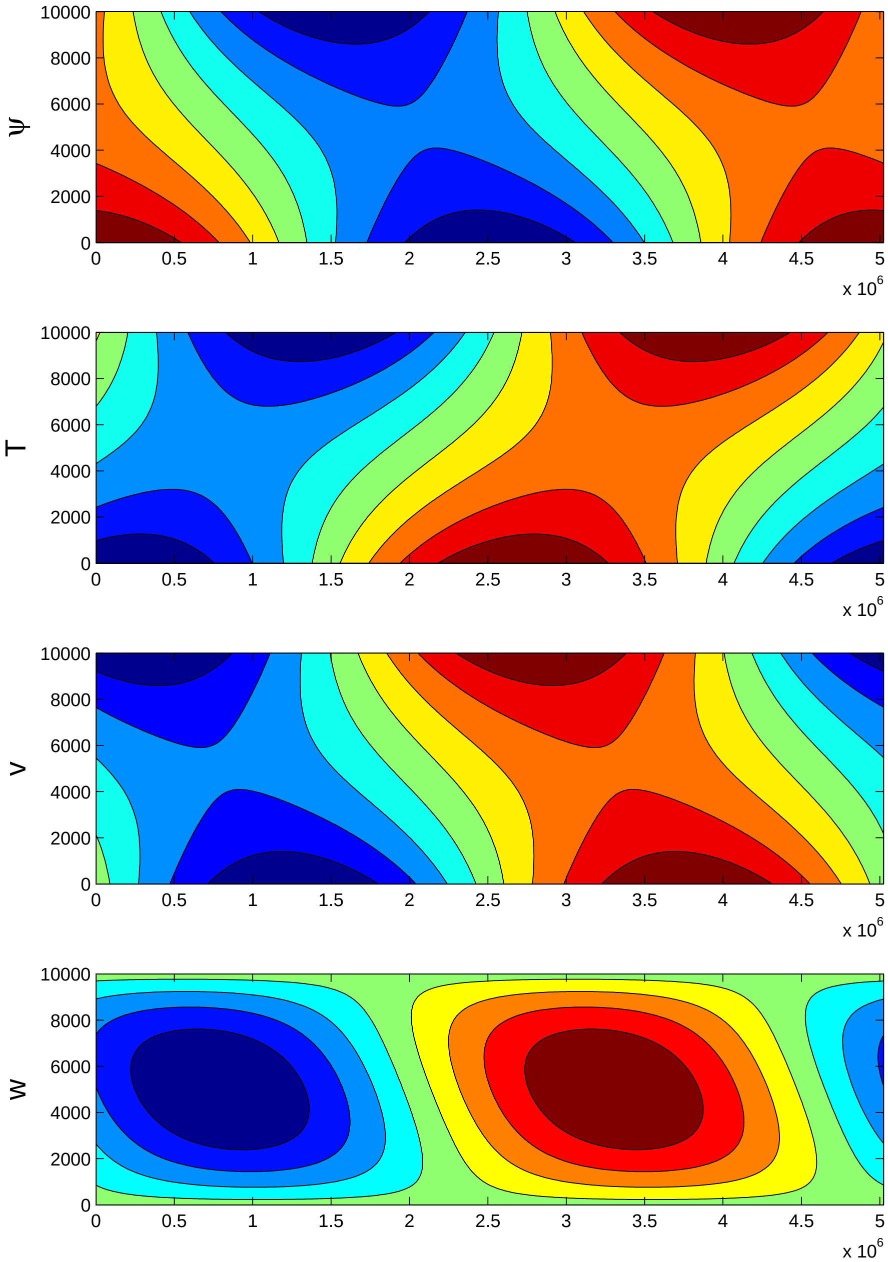


Figure 45: The streamfunction (upper), temperature (second), meridional velocity (third) and vertical velocity for the most unstable wave in the Eady problem.

PV gradient in the interior. What is the mean temperature gradient on the surface?

c) Now imagine a sloping bottom with zero mean flow. How is the slope oriented and how steep is it so that the topographic waves are propagating at the same speed as the waves in (a) and (b)?

Problem 3.8: Eady heat fluxes

Eady waves can flux heat. To see this, we calculate the correlation between the northward velocity and the temperature:

$$\overline{vT} \propto \overline{\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial z}} \equiv \frac{1}{L}\int_0^L \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial z} dx$$

where L is the wavelength of the wave. Calculate this for the Eady wave and show that it is positive; this implies that the Eady waves transport warm air northward. You will also find that the heat flux is *independent of height*.

- Hint: use the form of the streamfunction given in (370).
- Hint:

$$\int_0^L sin(k(x-ct))\, cos(k(x-ct))\; dx = 0$$

- Hint:

$$\frac{d}{dz} tan^{-1}\frac{y}{x} = \frac{x^2}{x^2+y^2}\left(\frac{xdy/dz - ydx/dz}{x^2}\right) = \frac{xdy/dz - ydx/dz}{x^2+y^2}$$

- Hint: The final result will be proportional to c_i. Note that c_i is *positive* for a growing wave.

Problem 3.9: Eady momentum fluxes

Unstable waves can flux momentum. The zonal *momentum flux* is defined as:

$$\overline{uv} \propto -\overline{\frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x}} \equiv -\frac{1}{L}\int_0^L \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} dx$$

Calculate this for the Eady model. Why do you think you get the answer you do?

4 Appendices

4.1 Appendix A: Kelvin's theorem

The vorticity equation can be derived in an elegant way. This is based on the *circulation*, which is the integral of the vorticity over a closed area:

$$\Gamma \equiv \iint \vec{\zeta} \cdot \hat{n}\, dA \tag{372}$$

where $\hat{n}$ is the normal vector to the area. From Stoke's theorem, the circulation is equivalent to the integral of the velocity around the circumference:

$$\Gamma = \iint (\nabla \times \vec{u}) \cdot \hat{n}\, dA = \oint \vec{u} \cdot \vec{dl} \tag{373}$$

Thus we can derive an equation for the circulation if we integrate the momentum equations around a closed circuit. For this, we will use the momentum equations in vector form. The derivation is somewhat easier if we work with the fixed frame velocity:

$$\frac{d}{dt}\vec{u}_F = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{374}$$

If we integrate around a closed area, we get:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{g} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl} \tag{375}$$

The gravity term vanishes because it can be written in terms of a potential (the geopotential):

$$\vec{g} = -g\hat{k} = \frac{\partial}{\partial z}(-gz) \equiv \nabla \Phi \tag{376}$$

and because the closed integral of a potential vanishes:

$$\oint \nabla\Phi \cdot \vec{dl} = \oint d\Phi = 0 \tag{377}$$

So:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl} \tag{378}$$

Now the circulation, Γ_F, has two components:

$$\Gamma_F = \oint \vec{u}_F \cdot \vec{dl} = \iint \nabla \times \vec{u}_F \cdot \hat{n}\, dA = \iint (\vec{\zeta} + 2\vec{\Omega}) \cdot \hat{n}\, dA \tag{379}$$

As noted above, the most important components of the vorticity are in the vertical. So a natural choice is to take an area which is in the horizontal, with $\hat{n} = \hat{k}$. Then:

$$\Gamma_F = \iint (\zeta + f)\, dA \tag{380}$$

Putting this back in the circulation equation, we get:

$$\frac{d}{dt} \iint (\zeta + f)\, dA = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl} \tag{381}$$

Now, the first term on the RHS of (381) is zero under the Boussinesq approximation because:

$$\oint \frac{dp}{\rho} = \frac{1}{\rho_c} \oint dp = 0$$

It is also zero if we use pressure coordinates because:

$$\oint \frac{dp}{\rho}|_z \rightarrow \oint d\Phi|_p = 0$$

Thus, in both cases, we have:

$$\frac{d}{dt}\Gamma_a = \oint \vec{F} \cdot \vec{dl} \tag{382}$$

So the absolute circulation can only change under the action of friction. If $\vec{F} = 0$, the absolute circulation is conserved on the parcel. This is Kelvin's theorem.

4.2 Appendix B: Solution in the Ekman layer

Ekman's solution requires that we we *parametrize* the stress in the boundary layer. To do this, we make a typical assumption that the stress is proportional to the velocity shear:

$$\frac{\vec{\tau}}{\rho_c} = A_z \frac{\partial}{\partial z}\vec{u} \tag{383}$$

where A_z, is a *mixing coefficient*. Thus the stress acts down the gradient of the velocity. If the vertical shear is large, the stress is large and vice versa. Generally, A_z varies with height, and often in a non-trivial way, but in such cases it can be difficult to find analytical solutions.

So we assume that A_z is constant. This follows Ekman's (1905) original formulation, and the solutions is now referred to as an *Ekman* boundary layer. We assume the flow is purely geostrophic in the fluid interior, above the boundary layer, with velocities (u_g, v_g). The boundary layer's role then is to bring the velocities to rest at the lower boundary. With these stresses, we can solve for the ageostrophic velocities in the layer (the details are given in Appendix B). Integrating the velocities with height, one finds:

$$U = -\frac{\delta_e}{2}(u_g + v_g), \quad V = \frac{\delta_e}{2}(u_g - v_g)$$

where (u_g, v_g) are the velocities in the interior. In the solutions, the depth of the Ekman layer, δ, is determined by the mixing coefficient, A_z. This is:

$$\delta = \sqrt{\frac{2A_z}{f_0}} \tag{384}$$

So we have:

$$w(\delta) = \frac{\delta}{2}(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial x}) + \frac{\delta}{2}(-\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial y})$$

$$= \frac{\delta}{2}(-\frac{\partial v_g}{\partial y} + \frac{\partial v_g}{\partial x}) + \frac{\delta}{2}(-\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial y}) = \frac{\delta}{2}(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y})$$

The solution for velocities in the Ekman layer is as follows. Substituting the parametrized stresses (383) into the boundary layer equations (84-85) yields:

$$-f_0 v_a = -A_z \frac{\partial^2}{\partial z^2} u_a \tag{385}$$

$$f_0 u_a = -A_z \frac{\partial^2}{\partial z^2} v_a \tag{386}$$

Note that the geostrophic velocity was assumed to be independent of height, so it doesn't contribute to the RHS. If we define a variable χ thus:

$$\chi \equiv u_a + i v_a \tag{387}$$

we can combine the two equations into one:

$$\frac{\partial^2}{\partial z^2}\chi = i\frac{f_0}{A_z}\chi \tag{388}$$

The general solution to this is:

$$\chi = A\, exp(\frac{z}{\delta_E})\, exp(i\frac{z}{\delta_E}) + B\, exp(-\frac{z}{\delta_E})\, exp(-i\frac{z}{\delta_E}) \tag{389}$$

where:

$$\delta_E = \sqrt{\frac{2A_z}{f_0}} \tag{390}$$

This is the Ekman depth. So the depth of the Ekman layer is determined by the mixing coefficient and by the Coriolis parameter.

To proceed, we need boundary conditions. The solutions should decay moving upward, into the interior of the fluid, as the boundary layer solutions should be confined to the boundary layer. Thus we can set:

$$A = 0$$

From the definition of χ, we have:

$$\begin{aligned} u_a = Re\{\chi\} = Re\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E}) \\ + Im\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E}) \end{aligned} \tag{391}$$

and

$$\begin{aligned} v_a = Im\{\chi\} = -Re\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E}) \\ + Im\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E}) \end{aligned} \tag{392}$$

Thus there are two unknowns. To determine these, we evaluate the velocities at $z = 0$. To satisfy the no-slip condition, we require:

$$u_a = -u_g, \quad v_a = -v_g \quad at \;\; z = 0$$

Then the total velocity will vanish. So we must have:

$$Re\{B\} = -u_g$$

and

$$Im\{B\} = -v_g$$

Now we must integrate the velocities to obtain the transports. Strictly speaking, the integrals are over the depth of the layer. But as the ageostrophic velocities decay with height, we can just as well integrate them to infinity. So, we have:

$$\begin{aligned} U_a = -u_g \int_0^\infty exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})\, dz - v_g \int_0^\infty exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})\, dz \\ = -\frac{\delta}{2}(u_g + v_g) \end{aligned} \tag{393}$$

(using a standard table of integrals). Likewise:

$$\begin{aligned} V_a = u_g \int_0^\infty exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})\, dz - v_g \int_0^\infty exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})\, dz \\ = \frac{\delta}{2}(u_g - v_g) \end{aligned} \tag{394}$$

Integrating the velocities with height, we obtain:

$$U = -\frac{\delta_e}{2}(u_g + v_g), \quad V = \frac{\delta_e}{2}(u_g - v_g)$$

where (u_g, v_g) are the velocities in the interior. In the solutions, the depth of the Ekman layer, δ, is determined by the mixing coefficient, A_z. This is:

$$\delta = \sqrt{\frac{2A_z}{f_0}} \tag{395}$$

So we have:

$$w(\delta) = \frac{\delta}{2}(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial x}) + \frac{\delta}{2}(-\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial y})$$
$$= \frac{\delta}{2}(-\frac{\partial v_g}{\partial y} + \frac{\partial v_g}{\partial x}) + \frac{\delta}{2}(-\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial y}) = \frac{\delta}{2}(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y})$$
$$= \frac{\delta}{2}\nabla \times \vec{u}_g = \frac{\delta}{2}\zeta_g \tag{396}$$

4.3 Appendix C: Rossby wave energetics

Another way to derive the group velocity is via the energy equation for the waves. For this, we first need the energy equation for the wave. As the wave is barotropic, it has only kinetic energy. This is:

$$E = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}[(-\frac{\partial \psi}{\partial y})^2 + (\frac{\partial \psi}{\partial x})^2] = \frac{1}{2}|\nabla\psi|^2$$

To derive an energy equation, we multiply the wave equation (131) by ψ. The result, after some rearranging, is:

$$\frac{\partial}{\partial t}(\frac{1}{2}|\nabla\psi|^2) + \nabla \cdot [-\psi\nabla\frac{\partial}{\partial t}\psi - \hat{i}\beta\frac{1}{2}\psi^2] = 0 \tag{397}$$

We can also write this as:

$$\frac{\partial}{\partial t}E + \nabla \cdot \vec{S} = 0 \tag{398}$$

So the kinetic energy changes in response to the divergence of an energy *flux*, given by:

$$\vec{S} \equiv -\psi\nabla\frac{\partial}{\partial t}\psi - \hat{i}\beta\frac{1}{2}\psi^2$$

The energy equation is thus like the continuity equation, as the density also changes in response to a divergence in the velocity. Here the kinetic energy changes if there is a divergence in $\vec{S}$.

Let's apply this to the wave. We have

$$E = \frac{k^2 + l^2}{2} A^2 sin^2(kx + ly - \omega t) \tag{399}$$

So the energy varies sinusoidally in time. Let's average this over one wave period:

$$< E > \equiv \int_0^{2\pi/\omega} E\, dt = \frac{1}{4}\,(k^2 + l^2)\, A^2 \tag{400}$$

The flux, $\vec{S}$, on the other hand is:

$$\vec{S} = -(k\hat{i} + l\hat{j})\,\omega\, A^2 cos^2(kx + ly - \omega t) - \hat{i}\beta\,\frac{A^2}{2} cos^2(kx + ly - \omega t) \tag{401}$$

which has a time average:

$$< S > = \frac{A^2}{2}[-\omega(k\hat{i} + l\hat{j}) - \frac{\beta}{2}\hat{i}] = \frac{A^2}{4}[\beta\frac{k^2 - l^2}{k^2 + l^2}\,\hat{i} + \frac{2\beta kl}{k^2 + l^2}\,\hat{j}] \tag{402}$$

Rewriting this slightly:

$$< S > = [\beta\frac{k^2 - l^2}{(k^2 + l^2)^2}\,\hat{i} + \frac{2\beta kl}{(k^2 + l^2)^2}\,\hat{j}]\; E \equiv \vec{c}_g < E > \tag{403}$$

So the mean flux is the product of the mean energy and the group velocity, $\vec{c}_g$. It is straightforward to show that the latter is the same as:

$$c_g = \frac{\partial \omega}{\partial k}\,\hat{i} + \frac{\partial \omega}{\partial l}\,\hat{j} \tag{404}$$

Since c_g only depends on the wavenumbers, we can write:

$$\frac{\partial}{\partial t} < E > + \vec{c}_g \cdot \nabla < E > = 0 \tag{405}$$

We could write this in Lagrangian form then:

$$\frac{d_c}{dt} < E >= 0 \tag{406}$$

where:

$$\frac{d_c}{dt} = \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla \tag{407}$$

In words, this means that the energy is conserved when moving at the group velocity. The group velocity then is the relevant velocity to consider when talking about the energy of the wave.

4.4 Appendix D: Munk's model of the Gulf Stream

(Coming soon).

4.5 Appendix E: Fjørtoft's criterion

This is an alternate condition for barotropic instability, derived by Fjørtoft (1950). This follows from taking the real part of (244):

$$(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_i) - k^2|\hat{\psi}|^2 + (U - c_r)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s = 0 \tag{408}$$

If we again integrate in y and rearrange, we get:

$$\int_0^L (U - c_r)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s = \\ -\int_0^L (\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_i)dy + \int_0^L k^2|\hat{\psi}|^2 dy \tag{409}$$

We can use integration by parts again, on the first term on the RHS. For instance,

$$\int_0^L \hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_r \, dy = \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_r|_0^L - \int_0^L (\frac{\partial}{\partial y}\hat{\psi}_r)^2 \, dy \tag{410}$$

The first term on the RHS vanishes because of the boundary condition. So (408) can be written:

$$\int_0^L (U - c_r)\frac{|\hat{\psi}|^2}{|U - c|^2}\frac{\partial}{\partial y}q_s \, dy = \int_0^L (\frac{\partial}{\partial y}\hat{\psi}_r)^2 + (\frac{\partial}{\partial y}\hat{\psi}_i)^2 + k^2|\hat{\psi}|^2 \, dy \tag{411}$$

The RHS is always *positive*. Now from Rayleigh's criterion, we know that:

$$\int_0^L \frac{|\hat{\psi}|^2}{|U - c|^2}\frac{\partial}{\partial y}q_s \, dy = 0 \tag{412}$$

So we conclude that:

$$\int_0^L (U - c_r)\frac{|\hat{\psi}|^2}{|U - c|^2}\frac{\partial}{\partial y}q_s > 0 \tag{413}$$

We don't know what c_r is, but the condition states essentially that this integral must be positive for *any* real constant, c_r.

To test this, we can just pick a value for c_r. The usual procedure is to pick some value of the velocity, U; call that U_s. A frequent choice is to use the value of U at the point where $\frac{\partial}{\partial y}q_s$ vanishes; Then we must have that:

$$(U - U_s)\frac{\partial}{\partial y}q_s > 0 \tag{414}$$

somewhere in the domain. If this fails, the flow is stable.

Fjørtoft's criterion is also a necessary condition for instability. It represents an additional constraint to Rayleigh's criterion. Sometimes a flow will satisfy the Rayleigh criterion but not Fjørtoft's—then the flow is stable. Interestingly, it's possible to show that Fjørtoft's criterion requires the flow have a relative vorticity maximum somewhere in the domain interior, not just on the boundaries.

4.6 Appendix F: QGPV in pressure coordinates

The PV equation in pressure coordinates is very similar to that in z-coordinates. First off, the vorticity equation is given by:

$$\frac{d_H}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{415}$$

Using the incompressibility condition (66), we rewrite this as:

$$\frac{d_H}{dt}(\zeta + f) = (\zeta + f)\frac{\partial \omega}{\partial p} \tag{416}$$

The quasi-geostrophic version of this is:

$$\frac{d_g}{dt}(\zeta + f) = f_0 \frac{\partial \omega}{\partial p} \tag{417}$$

where $\zeta = \nabla^2 \Phi / f_0$.

To eliminate ω, we use the potential temperature equation (36). For simplicity we assume no heating, so the equation is simply:

$$\frac{d\theta}{dt} = 0 \tag{418}$$

We assume:

$$\theta_{tot}(x, y, p, t) = \theta_0(p) + \theta(x, y, p, t) \ , \quad |\theta| \ll |\theta_0|$$

where θ_{tot} is the full temperature, θ_0 is the “static” temperature and θ is the “dynamic” temperature. Substituting these in, we get:

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial}{\partial p}\theta_0 = 0 \tag{419}$$

We neglect the term $\omega\partial\theta/\partial p$ because it is much less than the term with θ_0.

The geopotential is also dominated by a static component:

$$\Phi_{tot} = \Phi_0(p) + \Phi(x, y, p, t)\,, \quad |\Phi| \ll |\Phi_0| \tag{420}$$

Then the hydrostatic relation (67) yields:

$$\frac{d\Phi_{tot}}{dp} = \frac{d\Phi_0}{dp} + \frac{d\Phi}{dp} = -\frac{RT_0}{p} - \frac{RT'}{p} \tag{421}$$

and where:

$$T_{tot} = T_0(p) + T(x, y, p, t)\,, \quad |T| \ll |T_0| \tag{422}$$

Equating the static and dynamic parts, we find:

$$\frac{d\Phi}{dp} = -\frac{RT'}{p} \tag{423}$$

Now we need to rewrite the hydrostatic relation in terms of the potential temperature. From the definition of potential temperature, we have:

$$\theta = T\,(\frac{p_s}{p})^{R/c_p}, \quad \theta_0 = T_0\,(\frac{p_s}{p})^{R/c_p}$$

where again we have equated the dynamic and static parts. Thus:

$$\frac{\theta}{\theta_0} = \frac{T}{T_0} \tag{424}$$

So:

$$\frac{1}{T_0}\frac{d\Phi}{dp} = -\frac{RT}{pT_0} = -\frac{R\theta}{p\theta_0} \tag{425}$$

So, dividing equation (419) by θ_0, we get:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,\frac{\theta'}{\theta_0} + \omega\,\frac{\partial}{\partial p}ln\theta_0 = 0 \tag{426}$$

Finally, using (425) and approximating the horizontal velocities by their geostrophic values, we obtain the QG temperature equation:

$$(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y})\,\frac{\partial\Phi}{\partial p} + \sigma\omega = 0 \tag{427}$$

The parameter:

$$\sigma(p) = -\frac{RT_0}{p}\frac{\partial}{\partial p}ln(\theta_0)$$

reflects the static stratification and is proportional to the Brunt-Vaisala frequency (sec. 3.1). We can write this entirely in terms of Φ and ω:

$$(\frac{\partial}{\partial t} - \frac{1}{f_0}\frac{\partial}{\partial y}\Phi\frac{\partial}{\partial x} + \frac{1}{f_0}\frac{\partial}{\partial x}\Phi\frac{\partial}{\partial y})\,\frac{\partial\Phi}{\partial p} + \omega\sigma = 0 \tag{428}$$

As in sec. (3.2), we can combine the vorticity equation (417) and the temperature equation (428) to yield a PV equation. In pressure coordinates, this is:

$$(\frac{\partial}{\partial t} - \frac{1}{f_0}\frac{\partial}{\partial y}\Phi\frac{\partial}{\partial x} + \frac{1}{f_0}\frac{\partial}{\partial x}\Phi\frac{\partial}{\partial y})\,[\frac{1}{f_0}\nabla^2\Phi + \frac{\partial}{\partial p}(\frac{f_0^2}{\sigma}\frac{\partial\psi}{\partial p}) + \beta y] = 0 \tag{429}$$