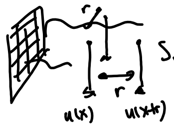


$S_n = |u(x+r) - u(x)|^n$

$n = 2$

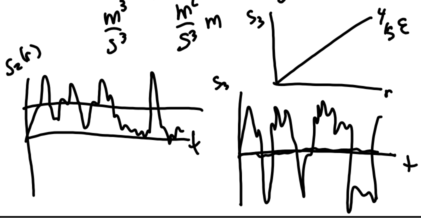
$S_n \sim \varepsilon^{2/3} r^{2/3}$?

$\frac{m^2}{s^2}$ $\left(\frac{m^2}{s^3}\right)^{2/3} m^{2/3}$ "$\frac{2}{3}$ law"

$\frac{m^{4/3}}{s^2} m^{2/3}$

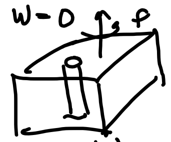
$S_3 = -\frac{4}{5} \varepsilon r$ "$\frac{4}{5}$ law"

$\frac{m^3}{s^3}$ $\frac{m^2}{s^3} m$



2 dimensions

$\frac{\partial}{\partial z} = 0$



$f = 2\Omega \sin\theta$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_c}\frac{\partial p}{\partial x} + \mathcal{F}^x + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_c}\frac{\partial p}{\partial y} + \mathcal{F}^y + \nu\nabla^2 v$$

$(u, v, w) \to (u, v, 0)$

$\nabla \times \vec{u} = \vec{\omega} \Rightarrow \left(0, 0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$\downarrow$

ζ

$\begin{vmatrix} \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix}$

$(\cancel{w_y - v_z}, \cancel{u_z - w_x}, v_x - u_y)$

$\vec{\omega} = \zeta \hat{k}$

$$\frac{\partial \zeta}{\partial t} + \vec{u} \cdot \nabla \zeta + f\cancel{(u_x + v_y)} = \nabla \times \mathcal{F} + \nu \nabla^2 \zeta$$

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \nabla \vec{\omega} + \cancel{\omega(\nabla \cdot u)} - \cancel{\vec{\omega}_a \cdot \nabla u} = \nabla \times \mathcal{F} + \nu \nabla^2 \vec{\omega}$$

$\vec{\omega}_a = (\zeta + f)\hat{k} \cdot \nabla \vec{u} = 0$

$(u, v, 0)$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$

ζ $\frac{\partial \zeta}{\partial t} + \vec{u} \cdot \nabla \zeta = \nabla \times \mathcal{F} + \nu \nabla^2 \zeta$

$\frac{\partial}{\partial t}\frac{\zeta^2}{2} + \vec{u} \cdot \nabla \frac{\zeta^2}{2} = \zeta(\nabla \times \vec{\mathcal{F}}) + \nu \zeta \nabla^2 \zeta$

$\frac{\zeta^2}{2}$ enstophy

$\mathcal{F} \to 0$ $\nu \to 0$

$\frac{\partial}{\partial t}\frac{\zeta^2}{2} + \vec{u} \cdot \nabla \frac{\zeta^2}{2} = 0$

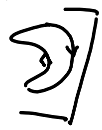
$\iint \frac{\partial}{\partial t}\frac{\zeta^2}{2} + \nabla \cdot \left(\vec{u}\frac{\zeta^2}{2}\right) = 0 \quad dA$

$\frac{d}{dt} Z = 0$ $Z = \iint \frac{1}{2}\zeta^2 \, dA$

$E = \iint \frac{1}{2}(u^2 + v^2) \, dA$

$u_x + v_y = 0 \longrightarrow u = -\frac{\partial \psi}{\partial y}$ $v = \frac{\partial \psi}{\partial x}$

$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$



$\frac{\partial}{\partial t} \zeta + \vec{u} \cdot \nabla \zeta = 0$

ψ $\frac{\partial}{\partial t} \nabla^2 \psi + \vec{u} \cdot \nabla(\nabla^2 \psi) = 0$

$E = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}(\psi_y^2 + \psi_x^2)$

$\iint \psi \frac{\partial}{\partial t} \nabla^2 \psi \, dA = \iint \psi \dot{\psi}_{xx} \, dx\, dy + \ldots \psi_{yy}$

$= \cancel{\psi \dot{\psi}_x \Big|_0^L} - \iint \psi_x \dot{\psi}_x \, dx\, dy$

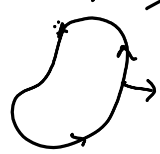
$= -\frac{1}{2}\iint \frac{\partial}{\partial t} \psi_x^2 \, dx\, dy - \frac{1}{2}\iint \frac{\partial}{\partial t} \psi_y^2 \, dx\, dy$

$-\frac{d}{dt} E$

$\iint \psi \vec{u} \cdot \nabla(\nabla^2 \psi) \, dA$

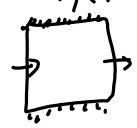
$\nabla \cdot (\vec{u} \psi \nabla^2 \psi) = \psi \vec{u} \cdot \nabla(\nabla^2 \psi) + \cancel{\psi \nabla^2 \psi (\nabla \cdot u)} + \cancel{\nabla^2 \psi \, \vec{u} \cdot \nabla \psi}$

$u\psi_x + v\psi_y$



$-\psi_y \psi_x + \psi_x \psi_y = 0$

$\iint \nabla \cdot (\vec{u} \psi \nabla^2 \psi) \, dA = \oint \psi \nabla^2 \psi \, \vec{u} \cdot n \, dl$

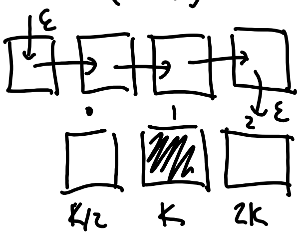


$= 0$

$$\frac{d}{dt} E = 0 \qquad \frac{d}{dt} Z = 0$$

$$E = \iint \frac{1}{2}(u^2+v^2) \qquad Z = \iint \frac{1}{2} \zeta^2$$

Fjørtoft (1953)



$$J = 0$$
$$v = 0$$

$$E_0 + E_1 + E_2 = E_{10}$$
$$Z_0 + Z_1 + Z_2 = Z_{20}$$

$$E = \frac{1}{2}(u^2+v^2) = \frac{1}{2}(\psi_y^2 + \psi_x^2)$$

$$\psi = \sum_{k,l} \hat{\psi} e^{ikx+ily}$$

$$\frac{1}{2}\left(l^2 \hat{\psi}^2 + k^2 \hat{\psi}^2\right) = \sum \frac{1}{2} |K^2| \hat{\psi}|^2 \qquad K^2 = k^2 + l^2$$

$$Z = \frac{\zeta^2}{2} = \frac{1}{2}(\nabla^2 \psi)^2 = \frac{1}{2}\left[(-k^2 - l^2)\hat{\psi}\right]^2 = \sum \frac{1}{2} K^4 |\hat{\psi}|^2$$

$$Z = K^2 E$$

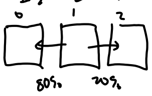
$$Z_0 + Z_1 + Z_2 \rightarrow K_0^2 E_0 + K_1^2 E_1 + K_2^2 E_2$$

$$u^2 + v^2 = \psi_x^2 + \psi_y^2$$
$$= \sum_{k,l} K^2 |\hat{\psi}|^2$$

$$\iint u^2 + v^2 = \sum K^2 |\hat{\psi}|^2$$ Parseval's theorem

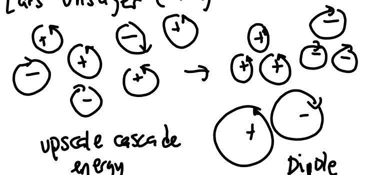
$$E_0 + E_1 + E_2 = E_{10}$$
$$\left(\frac{k}{2}\right)^2 E_0 + k^2 E_1 + (2k)^2 E_2 = k^2 E_{10}$$
$$\frac{1}{4} E_0 + E_1 + 4E_2 = E_{10}$$

$$E_0 + E_2 = (E_{10} - E_1) = \delta E_1$$
$$-4\times \quad \frac{1}{4} E_0 + 4E_2 = (E_{10} - E_1) = \delta E_1$$
$$E_2 - 16E_2 = \delta E_1 - 4\delta E_1 = 3\delta E_1$$
$$-15 E_2 = -3\delta E_1$$
$$E_2 = +\frac{1}{5}\delta E_1 \rightarrow E_0 = \frac{4}{5}\delta E$$



energy going upscale!

Lars Onsager (1949)

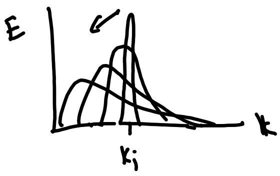


upscale cascade energy

enstrophy? $K^2 E$

$$E_2 = \frac{1}{5}\delta E_1 \qquad 4K^2 E_2 = \frac{4}{5} K^2 \delta E_1$$
$$Z_2 = \frac{4}{5}\delta Z_1$$
$$E_0 = \frac{4}{5}\delta E_1$$
$$\frac{1}{4} K^2 E_0 = \frac{1}{5} K^2 \delta E_1 \rightarrow Z_0 = \frac{1}{5}\delta Z_1$$

Batchelor 1953 Homogeneous Turbulence



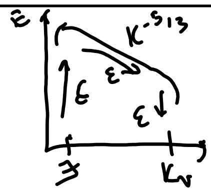
$$\frac{d}{dt}\int (k - k_i)^2 E \, dk > 0$$

mean $\int k E \, dk = k_i$

$$\frac{d}{dt}\int (k^2 - 2kk_i + k_i^2) E \, dk > 0$$

$$\cancel{\frac{d}{dt}\int k^2 E} - 2k_i \frac{d}{dt}\int kE + \cancel{k_i^2 \frac{d}{dt}\int E} > 0$$

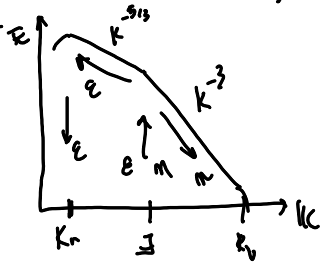
$$\frac{d}{dt}\int k E \, dk < 0$$



3D

2D?

Kraichnan (1967)



$$\varepsilon \sim \frac{dE}{dt} \sim \frac{m^2}{s^3}$$

$$\eta \sim \frac{dZ}{dt} \sim \frac{1}{s^3}$$

$$\zeta = v_x - u_y \sim \frac{1}{s}$$

$$Z \sim \zeta^2 \sim \frac{1}{s^2}$$

$E(k) \sim \varepsilon^{2/3} k^{-5/3}$

$\frac{m^3}{s^2} \quad \left(\frac{m^2}{s^3}\right)^{2/3} \left(\frac{1}{m}\right)^{-5/3}$

$E(k) \sim \eta^{2/3} k^{-3}$

$\frac{m^3}{s^2} \quad \left(\frac{1}{s^3}\right)^{2/3} \left(\frac{1}{m}\right)^{-3}$

m^3

$K_r \quad T_\nu = T_\eta \longrightarrow m^{-1/3}$

$L^2/\nu = \nu^{-1} k^{-2}$

$T_\eta \sim \eta^{-1/3} \qquad \nu^{-1} k^{-2} = \eta^{-1/3}$

$s \quad \left(\frac{1}{s}\right)^3 \qquad \frac{\eta^{1/3}}{\nu} = k^2$

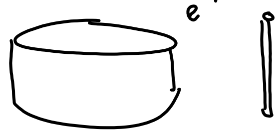
$k_r = \frac{\eta^{1/6}}{\nu^{1/2}}$ dissipation rate $= \eta$

Ekman (1905)

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla\zeta = F + \nu\nabla^2\zeta - r\zeta$$

$\frac{\partial}{\partial t}\zeta = -r\zeta \qquad \zeta = \zeta_0 e^{-rt}$

e^{-rt}



$r \sim \frac{1}{\text{2-3 days}}$

K_r? $T_\eta = T_r$

$T_\eta = \varepsilon^{-2/3} k^{-4/3} \qquad T_r = \frac{1}{r}$

$s \quad \frac{m^2}{s^3} \quad \frac{1}{m} \qquad \varepsilon^{-2/3} k^{-4/3} = r^{-1}$

$\frac{m^{-4/3}}{s^2} m^{4/3} \qquad \frac{r^{3/4}}{\varepsilon^{1/2}} = K_r$