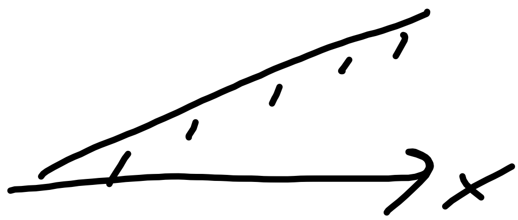
$$\frac{\partial \zeta}{\partial t} + \vec{u} \cdot \nabla(\zeta + h) = 0, \quad h = \alpha x$$



$$\frac{\partial}{\partial t} \nabla^2 \psi - \frac{\partial \psi}{\partial y} \overbrace{\frac{\partial h}{\partial x}}^{=\alpha} = 0$$

$$\psi = \hat{\psi} e^{i(kx + ly - \omega t)}$$

$$\kappa^2 = k^2 + l^2$$

$$\omega = \frac{\alpha l}{\kappa^2}$$

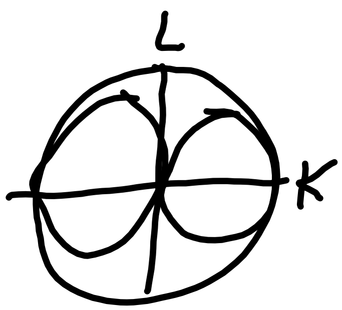
$$k = \kappa \cos(\theta)$$
$$l = \kappa \sin(\theta)$$

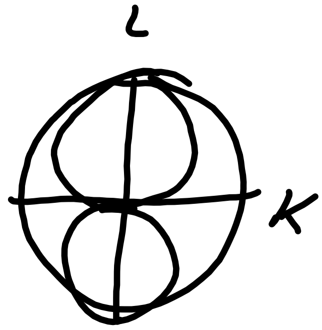
$$c_y = \frac{\alpha}{\kappa^2}$$

$$T_\alpha = T_R$$

$$\varepsilon^{-\frac{1}{3}} \kappa_R^{-\frac{2}{3}} = \frac{\kappa_R}{\alpha \sin\theta}$$

$$\kappa_R = \varepsilon^{-\frac{1}{5}} \alpha^{\frac{3}{5}} \sin(\theta)^{\frac{3}{5}}$$





β plane

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta+\beta y) = 0$$

topography

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta+h) = 0$$

Bretherton + Haidvogel 1975

Calculus of variations

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta+f) = \mathcal{F} + \mathcal{D}$$

$f_0+\beta y$ topography?

Shallow water

$\rho = \rho_0$

conserve PV $\quad q = \dfrac{\zeta + f}{H}$

$$\frac{\partial}{\partial t} q + \vec{u} \cdot q = 0 \qquad H = \text{const}$$

$$\frac{\partial}{\partial t} \zeta + \vec{u} \cdot \nabla(\zeta + f) = 0$$

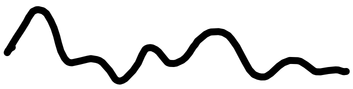
$$\frac{\partial}{\partial t}\left(\frac{\zeta + f}{H}\right) + \vec{u} \cdot \nabla\left(\frac{\zeta + f}{H}\right) = 0 \qquad H \neq \text{const.}$$

QG → quasigeostrophic approximation

$f \approx f_0 + \beta y \quad |\beta y| \ll f_0$

$u, v \approx u_g, v_g \quad \zeta/f_0 \sim \frac{U}{fL} = R_0 \ll 1$

$H = H_0 - h \quad |h| \ll H_0$



$$q = \frac{\zeta + f_0 + \beta y}{H_0 - \tilde{h}} = \frac{f_0}{H_0}\,\frac{(1 + \zeta/f_0 + \beta y/f_0)}{(1 - \tilde{h}/H_0)}$$

$$q \approx \frac{f_0}{H_0}\left(1 + \frac{\zeta}{f_0} + \frac{\beta y}{f_0}\right)\left(1 + \frac{\tilde{h}}{H_0}\right)$$

$$q \approx \frac{f_0}{H_0}\left(1 + \frac{\zeta}{f_0} + \frac{\beta y}{f_0} + \frac{\tilde{h}}{H_0} + \text{smaller}\right)$$

$$q = \frac{1}{H_0}\left(f_0 + \zeta + \beta y + \frac{f_0}{H_0}\tilde{h}\right)$$

$$\frac{\partial}{\partial t} q + \vec{u} \cdot \nabla q = 0$$

$$\frac{1}{H_0}\left(\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla\left(\zeta + \beta y + \frac{f_0 \tilde{h}}{H_0}\right)\right) = 0$$

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla\left(\zeta + \beta y + \frac{f_0}{H_0}\tilde{h}\right) = 0$$

$$h = \frac{f_0}{H_0}\tilde{h}$$

$\beta = 0 \quad \frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta + h) = 0$

conserved quantities

2D turbulence (Fjørtoft 1953)

$\longrightarrow$ Energy, enstrophy

$\iint \frac{1}{2}(u^2+v^2) \quad \iint \frac{1}{2}\zeta^2$

Energy

$\iint \psi \quad \frac{\partial}{\partial t}\zeta + u\cdot\nabla(\zeta+h) = 0 \quad dA$

$\iint \psi \frac{\partial}{\partial t}(\psi_{xx}+\psi_{yy})\,dx\,dy$

$\cancel{\psi \frac{\partial}{\partial t}\psi_x \Big|_0^L} - \int \psi_x \frac{\partial}{\partial t}\psi_x\,dx$

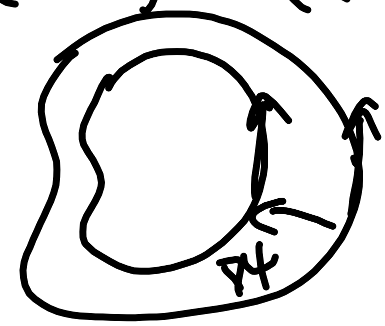
$-\int \frac{1}{2}\frac{\partial}{\partial t}\psi_x^2\,dx = -\int \frac{1}{2}v^2\,dx$

$\iint \psi \frac{\partial}{\partial t}\zeta\,dx\,dy = -\iint \frac{1}{2}(u^2+v^2)\,dx\,dy = -\frac{dE}{dt}$

$$\iint \psi \vec{u} \cdot \nabla (\zeta + h) \, dx \, dy$$

$$\nabla \cdot (\vec{u} \psi (\zeta + h)) = \psi \vec{u} \cdot \nabla (\zeta + h)$$

$$+ \cancel{\psi (\zeta + h) (\nabla \cdot \vec{u})} + \cancel{(\zeta + h) \vec{u} \cdot \nabla \psi}$$



$$\iint \nabla \cdot (\vec{u} \psi (\zeta + h)) \, dA =$$

$$\oint \psi (\zeta + h) \vec{u} \cdot \hat{n} \, dl = 0$$

$$\frac{dE}{dt} = 0$$

$$\zeta \left(\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta + h) = 0 \right)$$

$$\iint \frac{\partial}{\partial t}\frac{\zeta^2}{2}\, dA = Z$$

$$\frac{dZ}{dt} + \iint \zeta \vec{u}\cdot\nabla(\zeta + h)\, dA = 0$$

$$\iint \zeta \vec{u}\cdot\nabla\zeta + \iint \zeta \vec{u}\cdot\nabla h$$

$$\iint \vec{u}\cdot\nabla\frac{\zeta^2}{2} = \iint \nabla\cdot\left(\vec{u}\frac{\zeta^2}{2}\right) dA$$

$$= \oint \frac{\zeta^2}{2}\vec{u}\cdot\hat{n}\, dl = 0$$

$$\iint \zeta \vec{u}\cdot\nabla h\, dA \neq 0$$

topography $\rightarrow$ source of enstophy!

$$(\zeta+h)\quad \frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta+h) = 0$$

$$\frac{\partial}{\partial t}\frac{1}{2}(\zeta+h)^2 = Q$$

$$(\zeta+h)\vec{u}\cdot\nabla(\zeta+f) = \vec{u}\cdot\nabla\frac{1}{2}(\zeta+h)^2$$

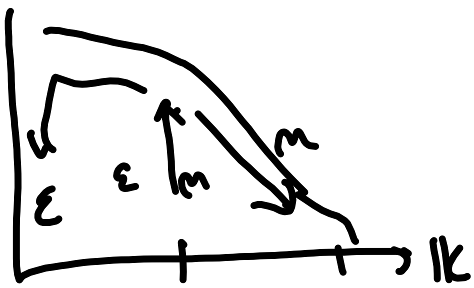
$$\iint \nabla\cdot\left(\vec{u}\,\frac{1}{2}(\zeta+h)^2\right) dA = 0$$

$$\frac{dE}{dt} = 0 \qquad \frac{dQ}{dt} = 0$$

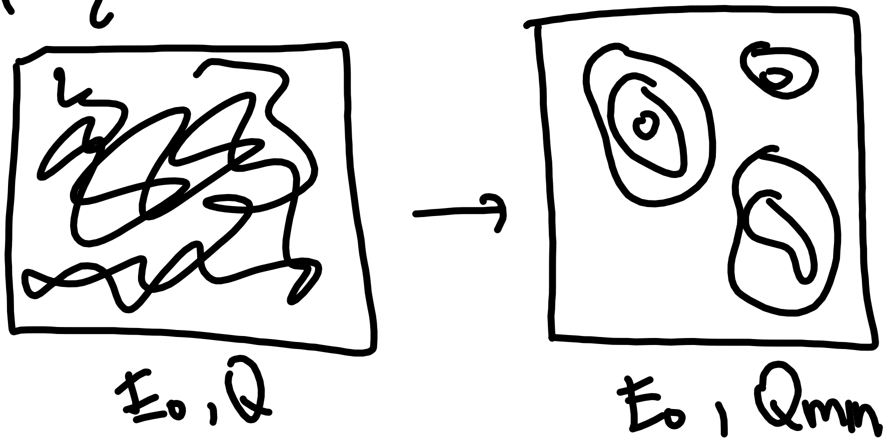
Q = potential enstrophy

2D E, Z conserved $r=0$

E conserved
Z dissipated



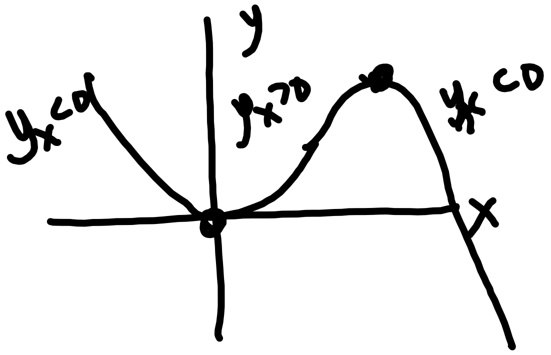
E conserved
$Q = \frac{1}{2}(\zeta+h)^2$ dissipated



$\frac{1}{2}(u^2+v^2) \rightarrow$ same

$\frac{1}{2}(\zeta+h)^2 \rightarrow$ small $\quad \zeta \rightarrow -h$

Calculus of variations (Lagrange Euler)



$\frac{dy}{dx}=0$

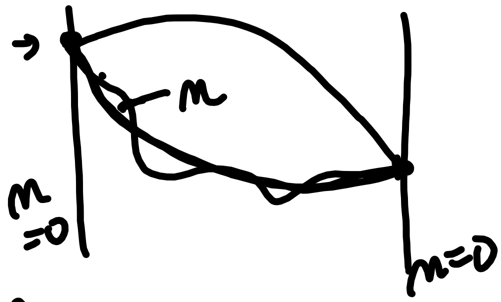
$2x=0 \rightarrow x=0$

$\frac{d^2y}{dx^2}>0$ min $\quad \frac{d^2y}{dx^2}<0$ max

$x^2 \rightarrow \frac{d^2}{dx^2}y = 2$

$$Q = \frac{1}{2}(\zeta + h)^2$$

$$\mathcal{L} = \iint \frac{1}{2}(\zeta + h)^2 \, dx\, dy$$



$$\delta \mathcal{L} = 0$$

$f(x) + m(x)$ (Bcs)

$$\delta \mathcal{L} = \iint (\zeta + h)\, \delta(\zeta + h)\, dA = 0$$

$$= \iint (\zeta + h)\, \delta\zeta \, dA$$

$$= \iint (\nabla^2\psi + h)\, \delta\nabla^2\psi \, dA$$

$$\iint (\nabla^2\psi + h)\,\delta\psi_{xx}\,dx$$

$$\cancel{(\nabla^2\psi+h)\,\delta\psi_x}\Big|_0^L - \iint \frac{\partial}{\partial x}(\nabla^2\psi+h)\,\delta\psi_x\,dx$$

$$\cancel{\frac{\partial}{\partial x}(\nabla^2\psi+h)\,\delta\psi}\Big|_0^L + \iint \frac{\partial^2}{\partial x^2}(\nabla^2\psi+h)\,\delta\psi\,dx\,dy$$

$$\delta J = \iint \nabla^2(\nabla^2\psi+h)\,\delta\psi\,dA = 0$$

$$\nabla^2\psi + h = 0 \rightarrow \underline{\zeta = -h}$$

"Lagrange multiplier"

$$Q + \mu(E - E_0)$$

$$\mathcal{L}(\psi, \mu) = \iint Q + \mu(E - E_0) \, dA$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0 \rightarrow \iint (E - E_0) \, dA = 0 \qquad E = E_0$$

$$\delta \mathcal{L} = \iint \delta Q + \mu \, \delta E \, dA = 0 \qquad E = \frac{1}{2} |\nabla \psi|^2$$

$$\iint (\zeta + h) \delta \zeta + \mu \nabla \psi \cdot \delta \nabla \psi \qquad \delta E = \nabla \psi \cdot \delta \nabla \psi$$

$$\iint (\zeta + h)\,\delta\zeta = \iint (\nabla^2\psi + h)\,\delta\nabla^2\psi$$

$$- \iint \frac{\partial}{\partial x}(\nabla^2\psi + h)\,\delta\psi_x - \iint \frac{\partial}{\partial y}(\ )\,\delta\psi_y$$

$$\iint \frac{\partial^2}{\partial x^2}(\nabla^2\psi + h)\,\delta\psi + \iint \frac{\partial^2}{\partial y^2}(\nabla^2\psi + h)\,\delta\psi$$

$$\iint \nabla^2(\nabla^2\psi + h)\,\delta\psi$$

$$\iint \mu\,\nabla\psi \cdot \delta\nabla\psi\,dA$$

$$\iint \mu\,\psi_x\,\delta\psi_x + \iint \mu\,\psi_y \cdot \delta\psi_y$$

$$- \iint \mu\,\psi_{xx}\,\delta\psi - \iint \mu\,\psi_{yy}\,\delta\psi$$

$$= - \iint \mu\,\nabla^2\psi\,\delta\psi$$

$$\delta \mathcal{L} = \iint \nabla^2 (\nabla^2 \psi + h - \mu \psi)\, \delta \psi = 0$$

$$\nabla^2 \psi - \mu \psi = -h$$ Euler-Lagrange Eqn.

$$\psi = \hat{\psi} e^{ikx + ily} \qquad h = \hat{h} e^{ikx + ily}$$

$$(-k^2 - l^2)\hat{\psi} \cancel{e^{ikx+ily}} - \mu \hat{\psi} \cancel{e^{ikx+ily}} = -\hat{h} \cancel{e^{ikx+ily}}$$

$$(k^2 + l^2 + \mu)\hat{\psi} = \hat{h}$$

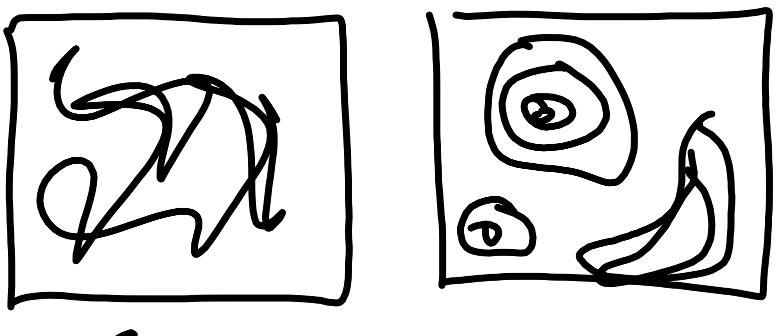
$$\hat{\psi} = \frac{\hat{h}}{k^2 + l^2 + \mu} = \frac{\hat{h}}{|\mathbf{k}|^2 + \mu}$$

$$E = \frac{1}{2}|\nabla\psi|^2 = \frac{1}{2}(\psi_x^2 + \psi_y^2)$$

$$= \frac{1}{2}\left(k^2|\hat{\psi}|^2 + l^2|\hat{\psi}|^2\right) = \frac{1}{2}|\mathbb{k}^2||\hat{\psi}|^2$$

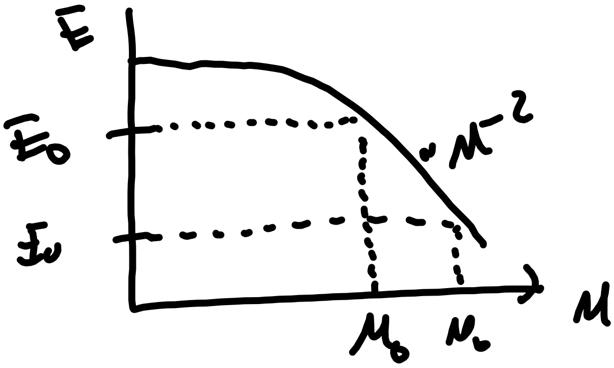
$$E = \sum_{k,l} \frac{1}{2}\mathbb{k}^2 \frac{|\hat{h}|^2}{(\mathbb{k}^2+\mu)^2}$$

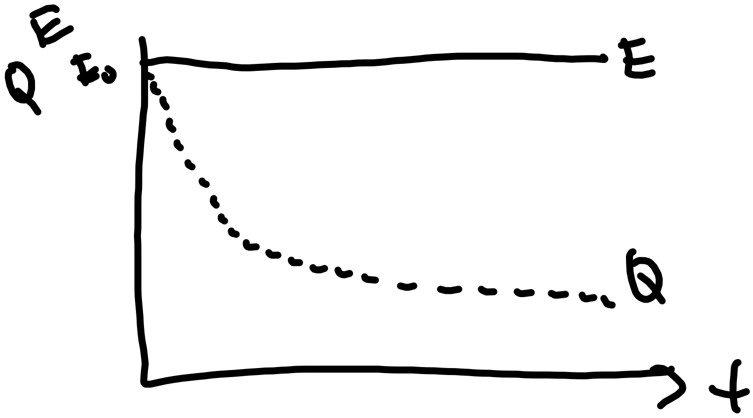
ψ



h

E_0


E
E0
E0
M-2
M
M0
M0


Q
E
E0
E
Q
t