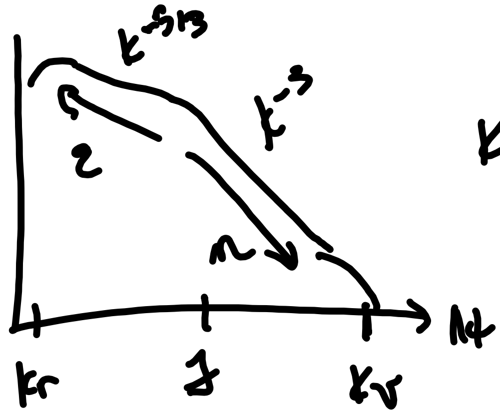
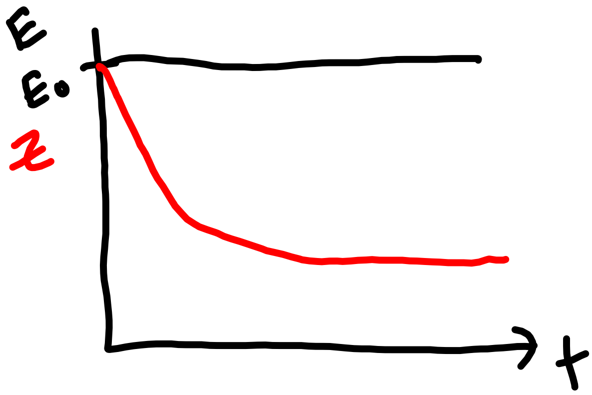
2D turbulence

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla\zeta = F + \nu\nabla^2\zeta - r\zeta$$



Kraichnan 1967



$$\zeta \quad \frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla\zeta = 0$$

$$\frac{\partial}{\partial t}\zeta^2/2 + \vec{u}\cdot\nabla\,\zeta^2/2 = 0$$

$$\iint \frac{\partial}{\partial t}\underline{\zeta^2/2} + \nabla\cdot\cancel{(\vec{u}\,\zeta^2/2)} = 0 \quad dA$$

$$\frac{\partial}{\partial t}\iint \frac{\zeta^2}{2}\,dA = 0$$

β-effect
$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla\zeta + \beta v = 0$$

$$\iint \frac{\partial}{\partial t}\zeta^2/2 + \vec{u}\cdot\nabla\,\zeta^2/2 + \beta v\zeta = 0 \quad dA$$

$$\frac{\partial}{\partial t}\iint \frac{\zeta^2}{2} + 0 + \iint \beta\psi_x(\psi_{xx}+\psi_{yy}) = 0$$

$$\beta\int \psi_x\psi_{xx} = \int \frac{\partial}{\partial x}\psi_x^2/2\,dx = \frac{\psi_x^2}{2}\Big|_0^L = 0$$

$$\iint \psi_x\,\psi_{yy}\,dx\,dy = 0 \ ?$$

$$\iint \zeta \frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta) + \vec{u}\cdot\nabla h = 0$$

$$\frac{d}{dt}\iint \zeta^2 + 0 + \iint -\psi_y h_x + \psi_x h_y = 0$$

topography = $\neq 0$
source enstrophy

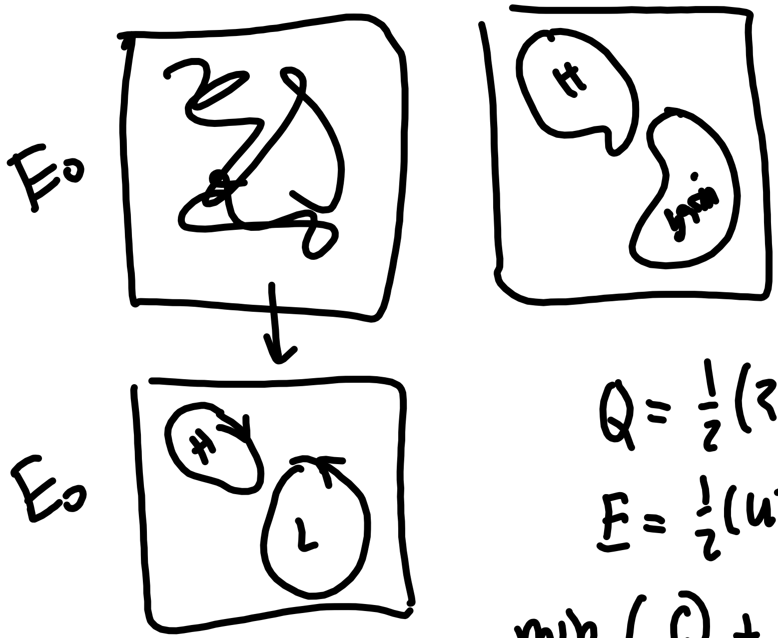
$$(\zeta+h)\left[\frac{\partial}{\partial t}(\zeta+h) + \vec{u}\cdot\nabla(\zeta+h)\right] = 0$$

$$\frac{d}{dt}\frac{1}{2}(\zeta+h)^2 + \vec{u}\,\nabla\frac{1}{2}(\zeta+h)^2 = 0$$

$$\frac{d}{dt}\iint \frac{1}{2}(\zeta+h)^2\, dA = 0$$

relative topographic

total

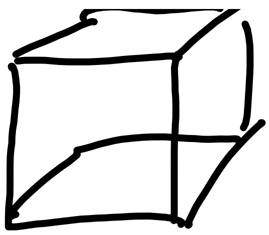


$$Q = \frac{1}{2}(\zeta + h)^2$$

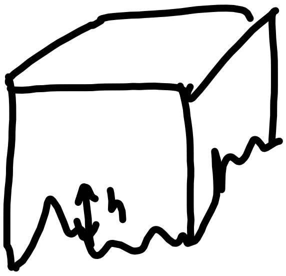
$$\underline{E} = \frac{1}{2}(u^2 + v^2)$$

$$\min\left(Q + \mu(\underline{E} - \underline{E}_0)\right)$$

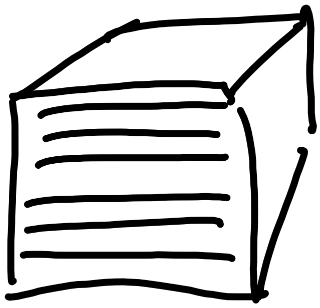
varistional calculus



$\rightarrow f = f_0 + \beta y$



$\rightarrow \xi + h(x,y)$



$\rho \neq \rho_0 = \rho(z)$

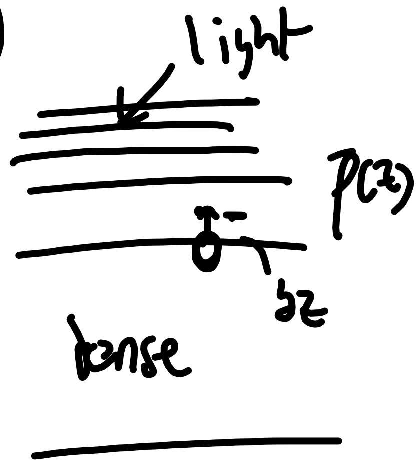
stratification

$$\frac{\partial}{\partial t} q + \vec{u}\cdot\nabla q = 0$$

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)$$

$\nabla^2\psi \downarrow$ relative vorticity

$\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) \downarrow$ stretching vorticity



$$N^2 = -\frac{g}{\rho_c}\frac{d\bar{\rho}}{dz}$$

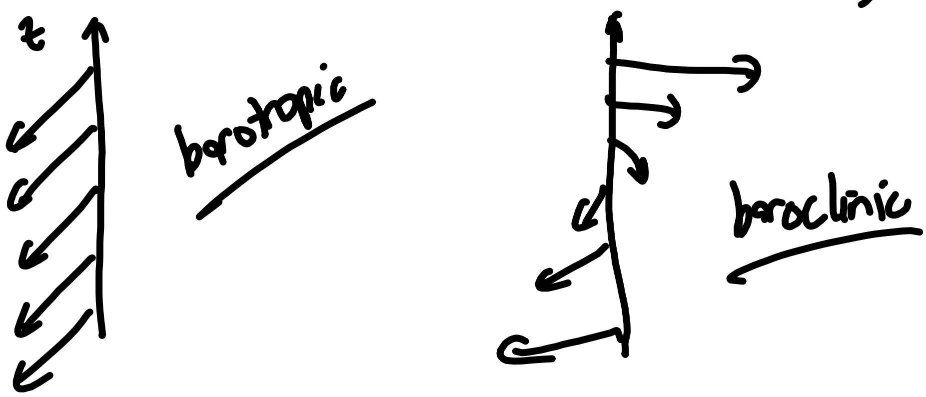
$$\frac{d^2}{dt^2}\delta z + N^2\delta z = 0$$

$N^2 > 0$ $\quad \delta z \sim A\cos(Nt)$

$N^2 < 0$ $\quad \delta z \sim A e^{Nt}$

convection

$$\psi(x, y, t) \longrightarrow \psi(x, y, z, t)$$



Charney (1971)

"Geostrophic turbulence"

q. $\frac{\partial q}{\partial t} + \vec{u}\cdot\nabla q = 0$

$$\frac{\partial}{\partial t}\frac{q^2}{2} + \vec{u}\cdot\nabla\frac{q^2}{2} = 0$$

$$\iiint \frac{\partial}{\partial t}\frac{q^2}{2} + \cancel{\nabla\cdot\left(\vec{u}\frac{q^2}{2}\right)} = 0 \quad dx\,dy\,dz$$

$$\frac{d}{dt}\iint \frac{1}{2}q^2\,dA = 0$$ potential enstrophy (Charney?)

$$\iiint \psi \frac{\partial}{\partial t}\left(\nabla^2\psi + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \vec{u}\cdot\nabla q = 0$$

$$\int \psi \frac{\partial}{\partial t}\psi_{xx}\, dx = \cancel{\psi \frac{\partial}{\partial t}\psi_x \Big|_0^L} - \int \psi_x \frac{\partial}{\partial t}\psi_x\, dx$$

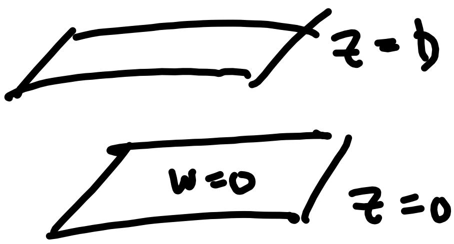
$$= -\int \frac{\partial}{\partial t} \psi_x^2/2\, dx = -\int \frac{\partial}{\partial t}\frac{v^2}{2}$$

$$\int \psi \frac{\partial}{\partial t}\psi_{yy}\, dy \rightarrow -\frac{\partial}{\partial t}\int \frac{\psi_y^2}{2} = -\frac{\partial}{\partial t}\int \frac{u^2}{2}$$

$$-\frac{\partial}{\partial t}\iiint \frac{u^2}{2} + \frac{v^2}{2} + \dots$$

kinetic energy

$$\int_0^D \psi \frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{f^2}{N}\frac{\partial\psi}{\partial z}\right) dz$$



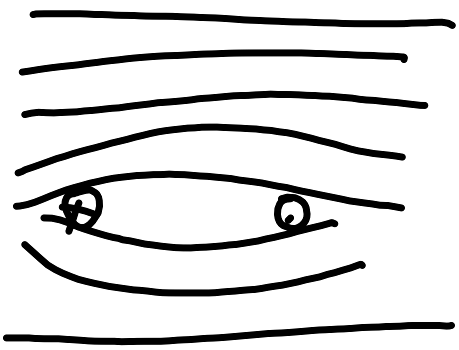
$$w=0 \rightarrow \frac{\partial\psi}{\partial z} = 0$$

$$= \cancel{\psi \frac{\partial}{\partial t}\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}}\Big|_0^D - \int \psi_z \frac{\partial}{\partial t}\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\, dz$$

$$= -\int \frac{f^2}{N^2}\frac{\partial}{\partial t}\frac{\psi_z^2}{2}\, dz$$

$$\frac{d}{dt}\iiint \underbrace{\frac{1}{2}\left(\psi_y^2+\psi_x^2\right)}_{\text{kinetic}} + \underbrace{\frac{f^2}{N^2}\frac{1}{2}\psi_z^2}_{\text{potential}} \; dx\,dy\,dz = 0$$

$$\iiint \psi\,\vec{u}\cdot\nabla q \rightarrow \iint \nabla\cdot\left(\vec{u}\,\psi q\right) \rightarrow 0$$



two conserved quantities

$$\frac{q^2}{2} = \left(\nabla^2\psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)^2/2$$

$$E = \frac{1}{2}\left(\psi_x^2 + \psi_y^2 + \frac{f^2}{N^2}\psi_z^2\right)$$

Batchelor (1953) Homogeneous Turbulence

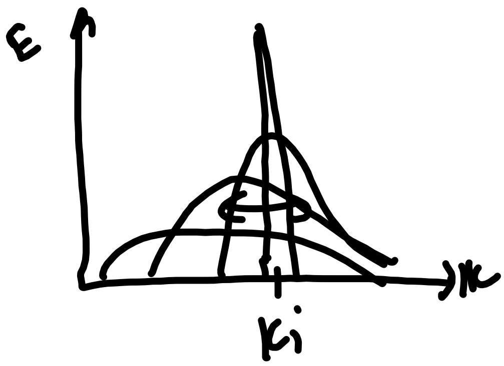
2D turbulence

$$E = \frac{1}{2}\left(\psi_x^2 + \psi_y^2\right)$$

$$Z = \frac{1}{2}\left(\nabla^2\psi\right)^2 \qquad \psi \sim e^{ikx+ily}\hat{\psi}$$

$$E = \sum_{k,l}\frac{1}{2}(k^2+l^2)|\hat{\psi}|^2 = \int \frac{1}{2}|\mathbf{k}|^2|\hat{\psi}|^2 d\mathbf{k}$$

$$Z = \sum_{k,l}\frac{1}{2}(k^2+l^2)^2|\hat{\psi}|^2 = \int \frac{1}{2}|\mathbf{k}|^4|\hat{\psi}|^2 d\mathbf{k}$$



$$\frac{d}{dt}\int (k-k_i)^2 E\, dk > 0$$

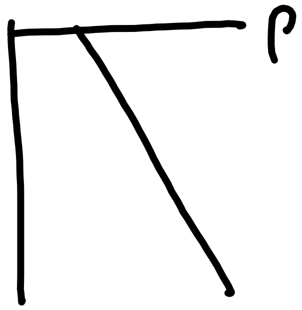
(variance)

$$\cancel{\frac{d}{dt}\int k^2 E} - 2k_i \frac{d}{dt}\int k E + \cancel{k_i^2 \frac{d}{dt}\int E} > 0$$

$$-2k_i \frac{d}{dt}\int k E > 0$$

$$\longrightarrow \int k E\, dk < 0$$

<u>mean</u> ∴ inverse cascade



$N = \text{const.} \quad \frac{\partial \rho}{\partial z} = \text{const.} \quad \rho \sim z$

$$\frac{f^2}{N^2}\frac{\partial^2}{\partial z^2} \to \frac{\partial^2}{\partial z_*^2}$$

$z^* = \frac{Nz}{f}$ "stretched coordinate"

$$\frac{1}{2}\left(\nabla^2\psi + \frac{f^2}{N^2}\psi_{zz}\right)^2 \to \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_*^2} \to \nabla_3^2$$

$$\frac{1}{2}\left(\nabla_3^2\psi\right)^2 \quad \left[\frac{1}{2}\left(\nabla^2\psi\right)^2\right]$$

$$\frac{1}{2}\left(\psi_x^2 + \psi_y^2 + \frac{f^2}{N^2}\psi_z^2\right)$$

$$\rightarrow \frac{1}{2}\left(\psi_x^2 + \psi_y^2 + \psi_{z_*}^2\right)$$

$$\rightarrow \frac{1}{2}|\nabla_3 \psi|^2 \qquad \left[\frac{1}{2}|\nabla_2 \psi|^2\right]$$

$$\psi = \hat{\psi}\, e^{ikx + ily} \cos\left(\frac{n\pi z}{D}\right)$$

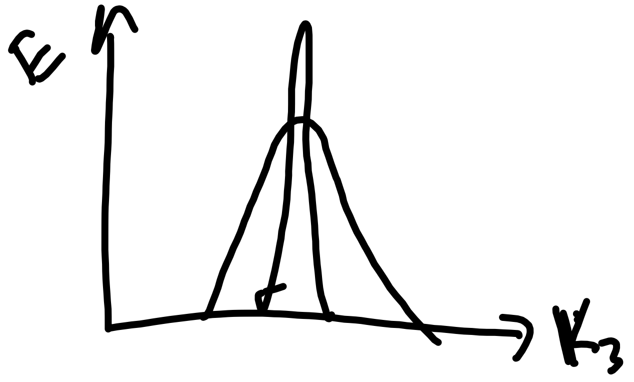
$$\nabla_3^2 \psi = \left[-k^2\hat{\psi} - l^2\hat{\psi} - \frac{n^2\pi^2}{D^2}\hat{\psi}\right] e^{ikx+ily}\cos\left(\frac{n\pi z}{D}\right)$$

$$= -K_3^2\hat{\psi} \qquad K_3^2 = k^2 + l^2 + \frac{n^2\pi^2}{D^2}$$

$$E = \sum_{k,l,n} \frac{1}{2} |K_3^2| \hat{\psi}|^2 = \frac{1}{2}\int |K_3^2| \hat{\psi}|^2$$

$$Q = \sum_{k,l,n} \frac{1}{2} |K_3^4| \hat{\psi}|^2 = \frac{1}{2}\int |K_3^4| \hat{\psi}|^2$$

$$= \frac{1}{2}\int K_3^2 E$$



isotropic

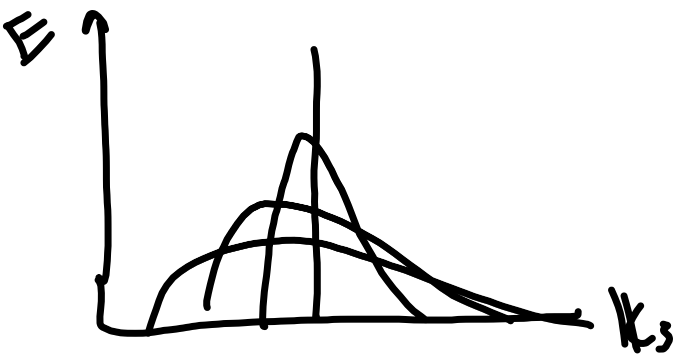
$x \leftrightarrow y \leftrightarrow z$ *

$$\frac{d}{dt}\int (K_3 - K_{3i})^2 E \, dK_3 > 0$$

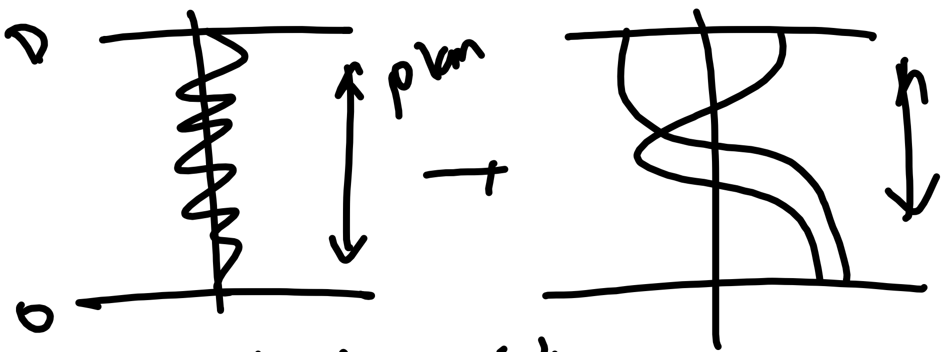
$$\frac{d}{dt}\cancel{\int K_3^2 E} - 2K_{3i}\frac{d}{dt}\int K_3 E + K_{3i}^2 \cancel{\frac{d}{dt}\int E} > 0$$

$$\frac{d}{dt}\int K_3 E < 0$$

mean decreasing



smaller k, l
smaller $\underline{n}$

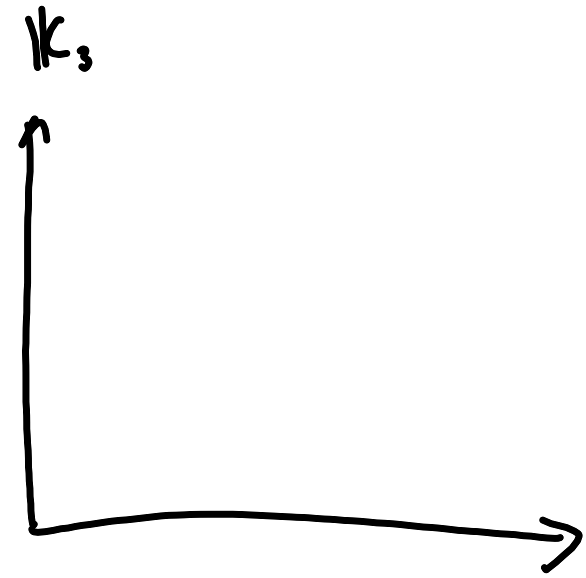


"barotropization"

$$z_* = \frac{Nz}{f}$$

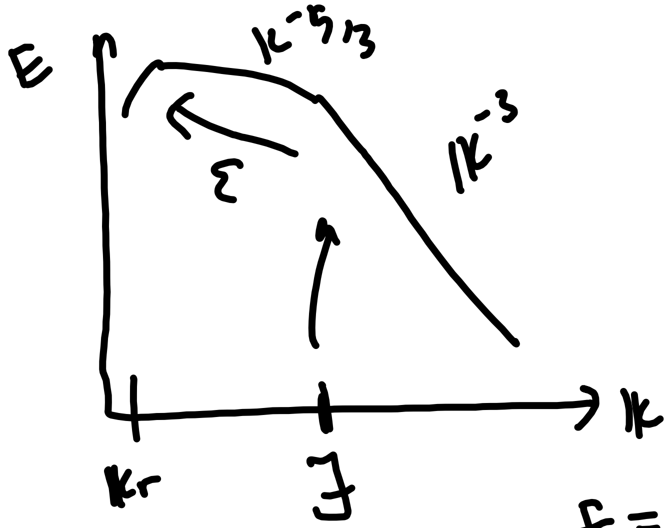
probably applies to smaller scales (isotropy)

$$E \;,\; Q$$
$$\downarrow \qquad \downarrow$$
$$\varepsilon \qquad m$$
$$\frac{m^2}{s^3} \qquad \frac{1}{s^3}$$



$$Q = \Big(\underset{\downarrow\; 1/s}{\nabla^2 \psi} + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial \psi}{\partial z}\Big)^2 \rightarrow \frac{1}{s^2}$$

$$1/s \qquad \frac{d}{dt} Q \sim \frac{1}{s^3}$$



$$E = \int E(k)\, dk$$

$$\frac{m^2}{s^2} \quad \frac{1}{m}$$

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

$$\frac{m^3}{s^2} \quad \left(\frac{m^2}{s^3}\right)^{2/3} \left(\frac{1}{m}\right)^{-5/3}$$

$$E(k) \sim \eta^{2/3} k^{-3}$$

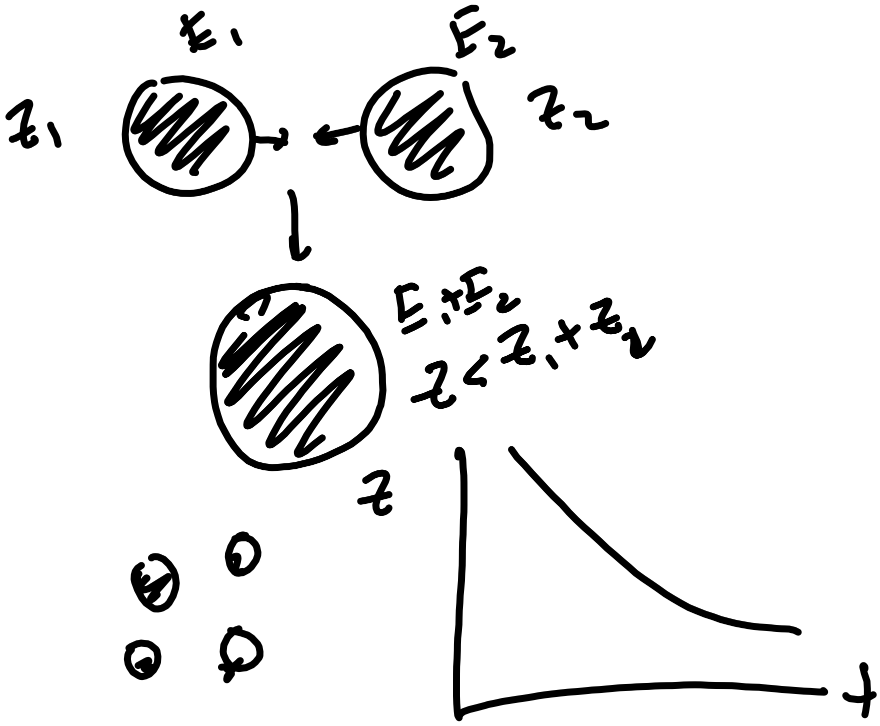
$$\frac{m^3}{s^2} \qquad \left(\frac{1}{s^3}\right)^{2/3}$$

Charney $z \rightarrow z^* = \frac{Nz}{f}$

Batchelor, Kraichnan

2D works @ synoptic scales

observed ?


E_1
E_2
Z_1
Z_2
$E_1 + E_2$
$Z < Z_1 + Z_2$
Z
t

$$q = \nabla^2 \psi + \frac{f^2}{N^2}\psi_{zz}$$

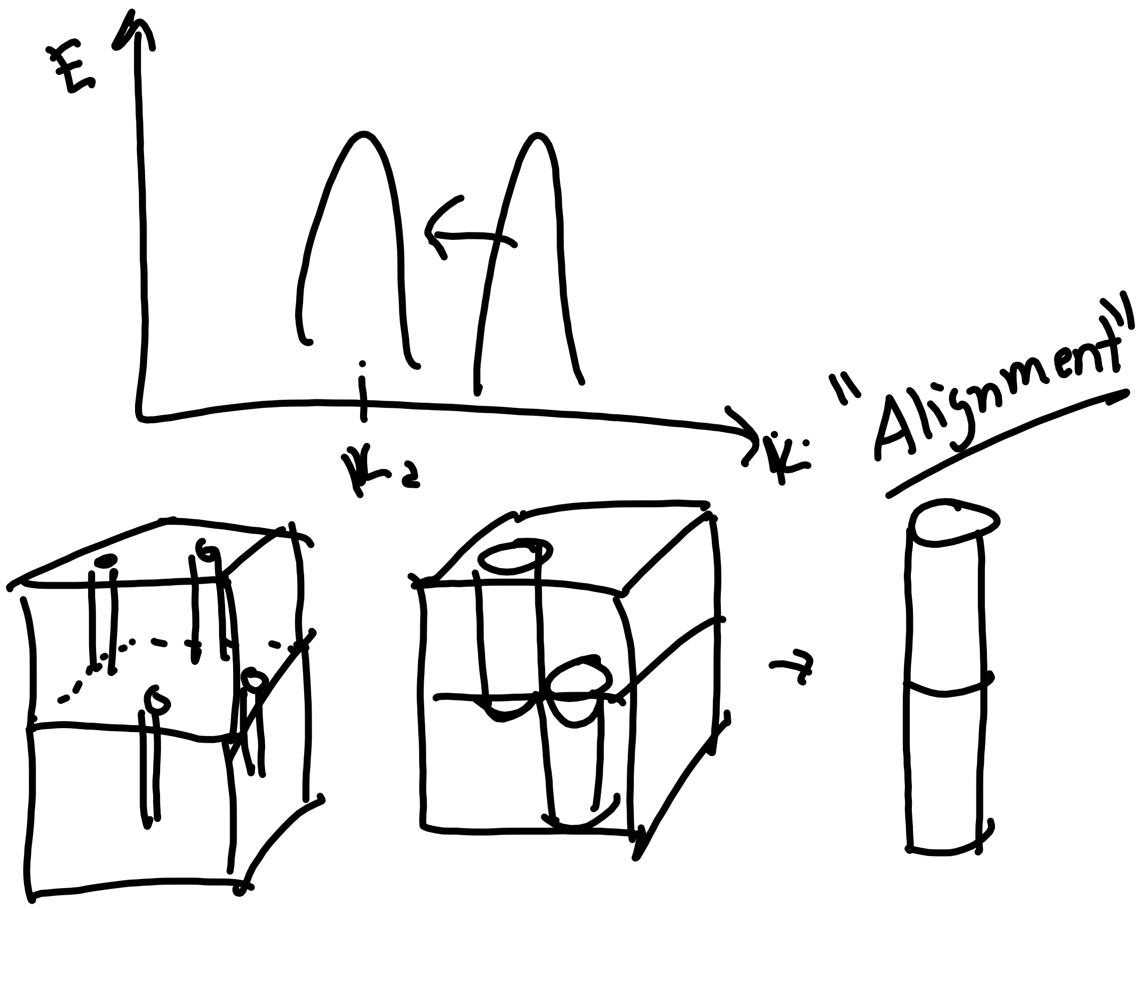
$$\frac{\psi}{L^2} \qquad \frac{f^2\psi}{N^2H^2}$$

$$1 \qquad \frac{f^2L^2}{N^2H^2} = \frac{L^2}{L_d^2}$$

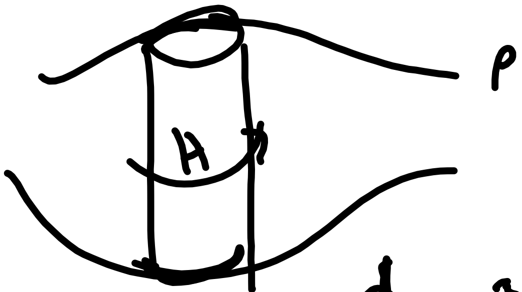
$L_d = \frac{NH}{f}$ deformation radius

$L > L_d$ stretching vorticity

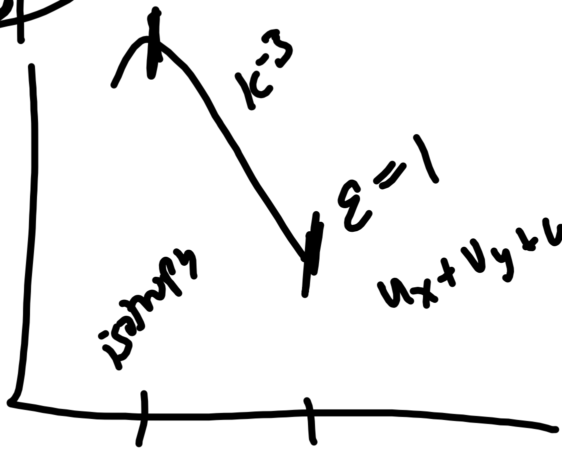
$L < L_d$ relative vorticity


E
k₂
k
"Alignment"

$u_g, v_g, w \qquad |w| << u_g, v_g$



$$u = \frac{-1}{\rho f}\frac{\partial p}{\partial y}$$



$\varepsilon = 1$

$u_x + v_y + w_z = 0$