$\vec{D}_n = \vec{D}_{n-1} + \vec{S}$

$\vec{D}_1 = 0 + \vec{S}$

$\vec{D}_2 = \vec{D}_1 + \vec{S}$

$\vec{D}_3 = \vec{D}_2 + \vec{S}$

$\langle |D_n^i|^2 \rangle$

$|\vec{D}_n^i|^2 = |\vec{D}_{n-1}^i|^2 + |\vec{S}|^2 + 2(\vec{D}_{n-1}^i \cdot \vec{S})$

$|\vec{S}|^2 = S^2 \quad (1\,m^2)$

$\langle |\vec{D}_n^m|^2 \rangle = \langle |\vec{D}_{n-1}^m|^2 \rangle + S^2 + \overset{0}{\cancel{\langle 2 \vec{D}_{n-1}^m \cdot \vec{S} \rangle}}$

$\langle |\vec{D}_1^m|^2 \rangle = 0 + S^2$

$\langle |\vec{D}_2^m|^2 \rangle = S^2 + S^2 = 2S^2$

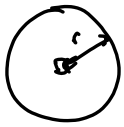
$\langle |\vec{D}_3^m|^2 \rangle = 2S^2 + S^2 = 3S^2$

$\langle |\vec{D}_n^m|^2 \rangle = nS^2 \rightarrow$ variance

---

standard deviation

$\sqrt{|\vec{D}_n^m|^2} = \sqrt{n}\, S$

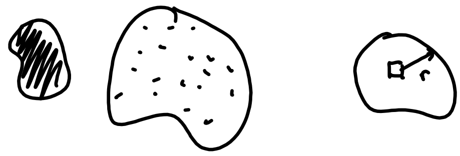


$r \sim S\sqrt{t}$

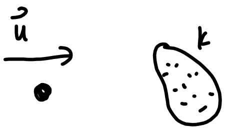
Brownian motion

Random walk $\quad$ std $\sim \sqrt{t}$

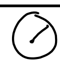
Drunkard's walk



$\frac{\partial C}{\partial t} + \vec{u} \cdot \nabla C = K \nabla^2 C$ (no sources)



$r^2 \frac{\partial C}{\partial t} = K \nabla^2 C = K \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial C}{\partial r}\right)$



$\int_0^\infty r^2 \frac{\partial C}{\partial t} r\, dr = \frac{d}{dt} \int_0^\infty r^2 C r\, dr = \frac{d}{dt} \langle r^2 \rangle$

$C(r,t)$

$\left(\begin{array}{c} \text{variance} \\ p(r,t) \\ \int r^2 p \end{array} \right)$

$\int_0^\infty K r^2 \frac{1}{\cancel{r}} \frac{d}{dr}\left(r \frac{\partial C}{\partial r}\right) \cancel{r}\, dr$

$K \int_0^\infty r^2 \frac{\partial}{\partial r}\left(r \frac{\partial C}{\partial r}\right) dr$

$\cancel{K r^2 \left(r \frac{\partial C}{\partial r}\right)}\Big|_0^\infty - 2K \int_0^\infty r^2 \frac{\partial C}{\partial r}\, dr$

$\cancel{-2K \cdot r^2 C}\Big|_0^\infty + 4K \int_0^\infty r C\, dr$

total $\int_0^\infty C r\, dr = \text{const.}$

$\frac{d}{dt} \int_0^\infty r^2 C r\, dr = 4K \int_0^\infty C r\, dr$

$\frac{d}{dt} \frac{\int r^2 C r\, dr}{\int C r\, dr} = \frac{d}{dt} \langle r^2 \rangle = 4K$

$\langle r^2 \rangle = 4Kt$

---

std dev $\quad \sqrt{\langle r^2 \rangle} = 2K^{1/2} t^{1/2}$

$\sqrt{\langle |D_n^2| \rangle} = S\sqrt{n} \sim S\sqrt{t}$

random walk = diffusion

$\vec{D}_n = \vec{D}_{n-1} + \vec{S} \quad$ Lagrangian

$\frac{\partial C}{\partial t} = K \nabla^2 C \quad$ Eulerian

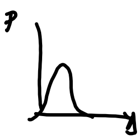
---

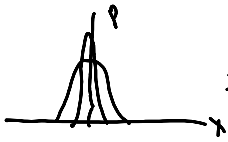
Einstein's approach (1905)

$p(x,t)$

$p(x, t+1) = \int p(x+\Delta, t)\, p(\Delta)\, d\Delta$

previously $\quad p(\Delta) = \delta(\Delta - 1)$





$\int_{-\infty}^{\infty} p(x,t)\, dx = 1$

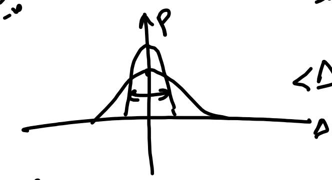
$$p(x, t+1) = p(x,t) + \frac{\partial p}{\partial t} + \dots$$

$$p(x+\Delta, t) = p(x,t) + \frac{\partial p}{\partial x}\Delta + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\Delta^2 + \dots$$

$$p(x,t) + \frac{\partial p}{\partial t} = \int \left[p(x,t) + \frac{\partial p}{\partial x}\Delta + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\Delta^2 \right] \times p(\Delta)\, d\Delta$$

$$\int p(x,t)\, p(\Delta)\, d\Delta = p(x,t) \int_{-\infty}^{\infty} p(\Delta)\, d\Delta = p(x,t)$$

$$\int_{-\infty}^{\infty} \frac{\partial p}{\partial x} \Delta\, p(\Delta)\, d\Delta = \frac{\partial p}{\partial x} \int_{-\infty}^{\infty} \Delta\, p(\Delta)\, d\Delta = 0$$

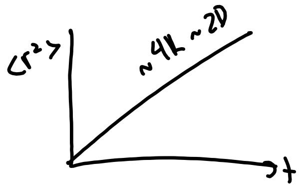


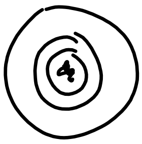
$\langle \Delta \rangle = 0$

$$\int \frac{1}{2}\frac{\partial^2 p}{\partial x^2} \Delta^2 p(\Delta)\, d\Delta = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} \int_{-\infty}^{\infty} \Delta^2 p(\Delta)\, d\Delta = \frac{D}{2}\frac{\partial^2 p}{\partial x^2}$$

$$\cancel{p(x,t)} + \frac{\partial p}{\partial t} = \cancel{p(x,t)} + \cancel{0} + \frac{D}{2}\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial p}{\partial t} = \frac{D}{2}\frac{\partial^2 p}{\partial x^2}$$





1921 G.I Taylor

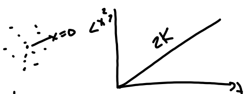
$$\frac{\partial}{\partial t} C = K \nabla^2 C$$

$10^{3}\ m^2/s \neq 10^{-5}\ m^2/s$

calculate diffusivity?

$$\langle X^2 \rangle = 2Kt$$

$$\frac{1}{2}\frac{d}{dt}\langle x^2 \rangle = K$$

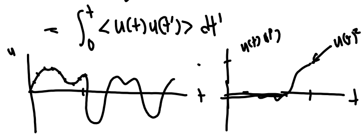


$$K = \frac{1}{2}\frac{d}{dt}\langle x^2 \rangle$$

$$= \langle x u \rangle \qquad x = \int_0^t u(t')\, dt'$$

$$= \left\langle u(t) \int_0^t u(t')\, dt' \right\rangle$$

$$= \int_0^t \langle u(t) u(t') \rangle\, dt'$$



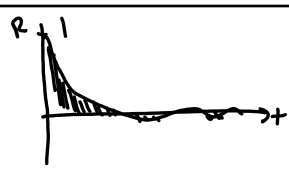
$$v^2 = \langle u^2(t) \rangle = \text{const.}$$

$$K = v^2 T_L \qquad T_L = \frac{1}{v^2}\int_0^t u(t)\, u(t')\, dt'$$

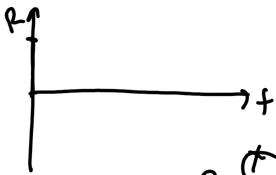
$$R = \frac{\langle u(0) u(t') \rangle}{v^2} \qquad = \frac{1}{v^2}\int_0^t \langle u(0) u(t') \rangle dt \rightarrow \text{time}$$

$T_L \rightarrow$ Lagrangian time

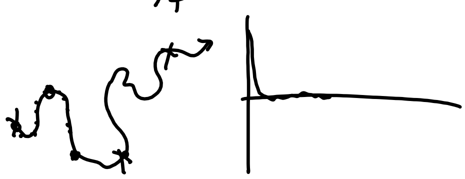
$$K = v^2 T_L$$

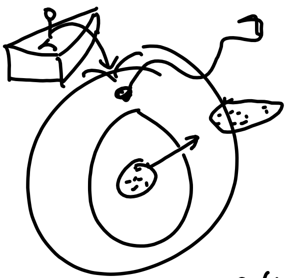


random walk



$$\int R\, dt = T_L$$

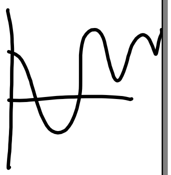




search + rescue

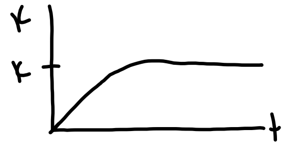
$T_L = 1.0$ days

$R(t) = 1$
$t \to 0$



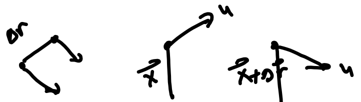
$$K = v^2 \int_0^t R \, dt = v^2 t$$

$$K = \lim_{t \to \infty} v \int_0^t R \, dt = v^2 T_L$$



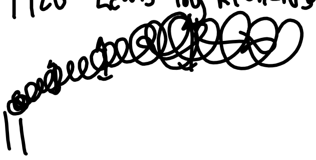
Taylor → Eddy diffusivity

"Relative dispersion"

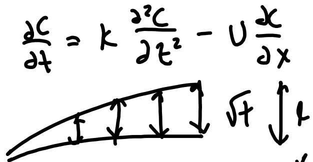


$$S_2 = \langle |\vec{u}(x+r) - u(x)|^2 \rangle$$

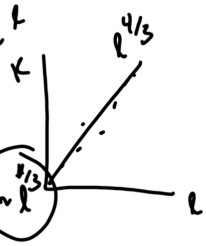
1926 Lewis Fry Richardson



$$\frac{\partial c}{\partial t} = K \frac{\partial^2 c}{\partial z^2} - U \frac{\partial c}{\partial x}$$



$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial c}{\partial z}\right)$$



Richardson's Law $K \sim l^{4/3}$

1941 Obukhov 1952 Batchelor
~ Kolmogorov 1941

δr

$$\langle |u_1 - u_2|^2 \rangle = \langle u_1^2 \rangle + \langle u_2^2 \rangle - 2 \langle (u_1 u_2) \rangle < 2v^2$$

homogeneous

$$K_2 \sim \int_0^t (u_1(t') - u_2(t'))(u_1(0) - u_2(0)) \, dt'$$

$$\int u_1(t') u_1(0) \, dt' + \int u_2(t') u_2(0) \, dt'$$

$$- \int u_1(t') u_2(0) \, dt' - \int u_2(t') u_1(0) \, dt$$

$$K_2 = 2K_1 - 2 \int_{i \neq j} \langle u_i(t') u_j(0) \rangle \, dt'$$

turbulence

$$K_2 \sim \varepsilon^{1/3} r^{4/3}$$ energy cascade

$$\frac{m^2}{s} \quad \left(\frac{m^2}{s^3}\right)^{1/3} m^{4/3}$$

$$\frac{m^{2/3}}{s} \; m^{4/3}$$

$$K \sim \eta^{1/3} r^2$$ enstrophy cascade

$$\frac{m^2}{s} \quad \frac{1}{s^3} \quad m$$

$$K_2 = \frac{1}{2}\frac{d}{dt}\langle \Delta r^2 \rangle = C\eta^{1/3} \langle \Delta r^2 \rangle$$

dispersion

$$\langle \Delta r^2 \rangle = \Delta r_0^2 \, e^{2C\eta^{1/3} t}$$

enstrophy → exponential growth

$$K_2 = \frac{1}{2}\frac{d}{dt}\langle \Delta r^2 \rangle = C\varepsilon^{1/3} \Delta r^{4/3}$$

$$\frac{d\,\Delta r^2}{\Delta r^{4/3}} = 2C\varepsilon^{1/3}$$

$$u = \Delta r^2 \quad u^{2/3}$$

$$\frac{du}{u^{2/3}} \to 3u^{1/3}$$

$$3(\Delta r^2)^{1/3} = 2C\varepsilon^{1/3} t$$

$$\Delta r^2 = (6C\varepsilon)^3 t^3$$

energy → cubic growth