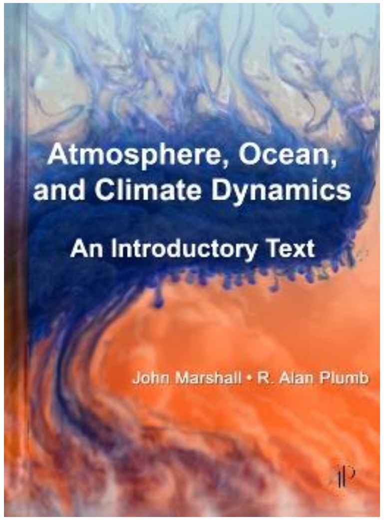
GEF 1100 – Klimasystemet

Chapter 6: The equations of fluid motion



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Who am I?

Prof. Dr. Kirstin Krüger

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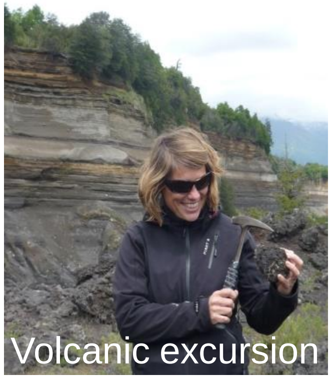
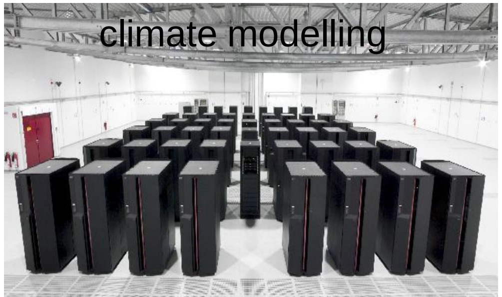
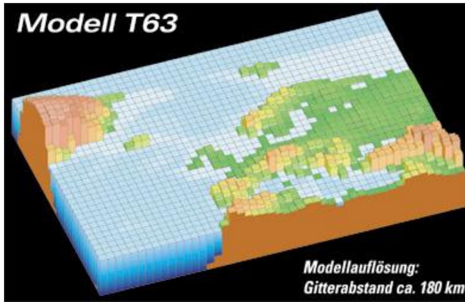


Research interests:

- **Stratosphere-ocean interactions**
- **Dynamics of the stratosphere-troposphere**
- **Stratospheric ozone and ozone depleting substances**
- **Influence of volcanic eruptions on climate/ earth system**



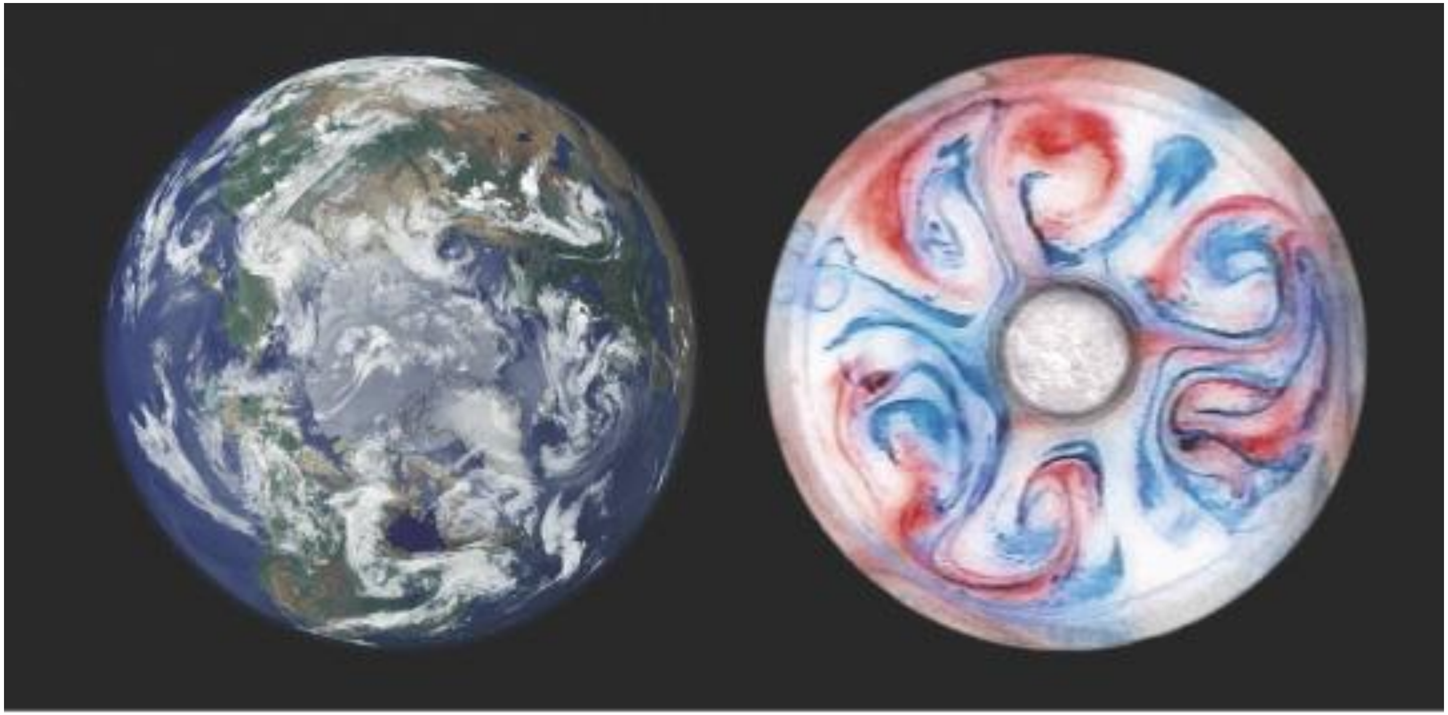


Lecture Outline – Ch. 6

Ch. 6 - The equations of fluid motions

**Addition, not in book.*

Motivation



Marshall and Plumb (2008)

- Derivation of the general equation of motion
- Different horizontal wind forms
- General circulation of atmosphere/ ocean

Equations of fluid motions

- **Hydrodynamics* focuses on moving liquids and gases.**
- **Hydrodynamics** and **thermodynamics** are fundamental for meteorology and oceanography.
- **Equations of fluid motions** for the **atmosphere and ocean** are constituted by macroscopic conservation laws for substances, impulses and internal energy (hydrodynamics and thermodynamics).

  The characteristics of very small volume elements are looked at (e.g. overall mass, average speed).

Derivation of hydrodynamics is based on the conservation of momentum, mass and energy.

**hydro (greek): water*

State variables for atmosphere and ocean

$\boldsymbol{u}$	Wind $\boldsymbol{u}$ = (u, v, w)	in m/s
T	Temperature	in ° C or K
p	Pressure	in N/m^2 = 1 Pa (1 hPa = 10^2 Pa = 1 mbar)

Specifically for the **atmosphere**:

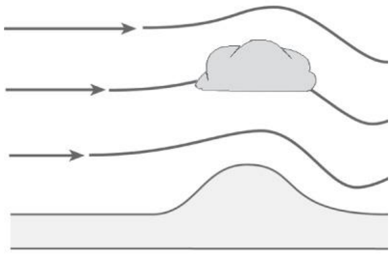
q	Specific humidity	in kg water vapour/kg moist air

Specifically for the **ocean**:

S	Salinity	in kg salt/kg seawater (often the salt content is expressed in practical salinity unit: psu = kg salt/1,000 kg seawater).

Differentiation following the motion - **Euler – Lagrange derivatives**

Example: wind blows over hill, cloud forms at the ridge of the lee wave; steady state assumption eq. cloud does not change in time.



Eulerian derivative (after Euler; 1707-1783):

$$\left(\frac{\partial C}{\partial t}\right)_{\text{fixed point in space}} = 0,$$

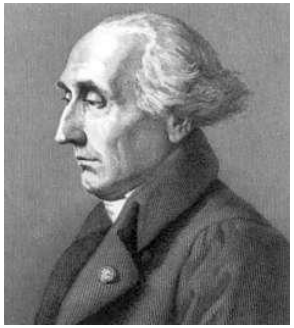
Leonhard Euler (CH)

Lagrangian derivative (after Lagrange; 1736-1813):

$$\left(\frac{\partial C}{\partial t}\right)_{\text{fixed particle}} \neq 0,$$

C = C(x,y,z,t): e.g. cloud amount

$\frac{\partial}{\partial t}$: partial derivatives; other variables are kept fixed during the differentiation



Joseph-Louis Lagrange (I)

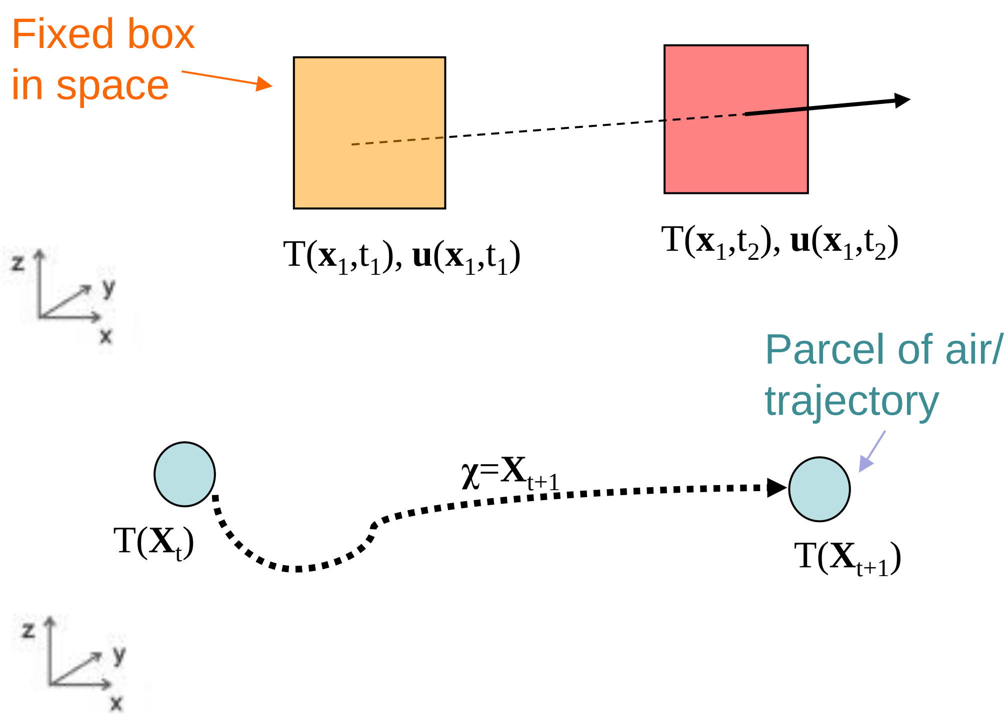
Differentiation following the motion -
Euler and Lagrange perspectives

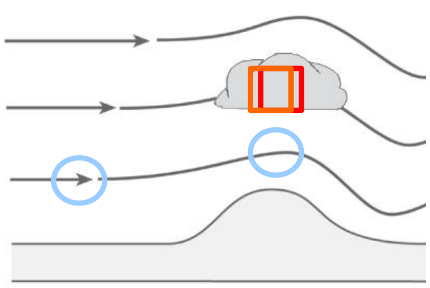
Euler perspective: The control volume is anchored stationary within the coordinate system, e.g. a cube through which the air or the seawater flows through.

Lagrange perspective: The control volume is an air or seawater parcel that always consists of the same particles and which moves along with the wind or the current.

Euler

Lagrange





Euler and Lagrangian differentiation

Think about examples in your everyday life for Euler's and Lagrange's differentiation. Discuss in groups of 2 and give one example.

Differentiation following the motion mathematically

For arbitrary small variations:

$$\delta C = \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial x}\delta x + \frac{\partial C}{\partial y}\delta y + \frac{\partial C}{\partial z}\delta z$$

δ: Greek small delta, infinitesimal small size

Insert $\delta x = u\delta t$, $\delta y = v\delta t$, $\delta z = w\delta t$

$$(\delta C)_{fixed\ particle} = \left(\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z}\right)\delta t$$

Dividing by δt and in limit of small variations:

$$\left(\frac{\partial C}{\partial t}\right)_{fixed\ particle} = \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{DC}{Dt}$$

$\frac{D}{Dt}$:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}$$

Advection term of a quantity: "$\boldsymbol{u}\cdot\boldsymbol{\nabla}$ <…> "

Differentiation following the motion mathematically

Lagrangian (or material, substantial or *total)* ***derivative***
(after Lagrange; 1736-1813):

$$\left(\frac{D}{Dt}\right)_{\text{fixed particle}} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z}$$

$$\equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} \qquad \text{(6-1)}$$

$\boldsymbol{u} = (u,v,w)$ velocity vector
$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ gradient operator, called "nabla"

"Time rate of change of some characteristic of a particular element of fluid, which in general is ***changing its position****."*

Eulerian derivative (after Euler; 1707-1783):

$$\left(\frac{\partial}{\partial t}\right)_{\text{fixed point}}$$

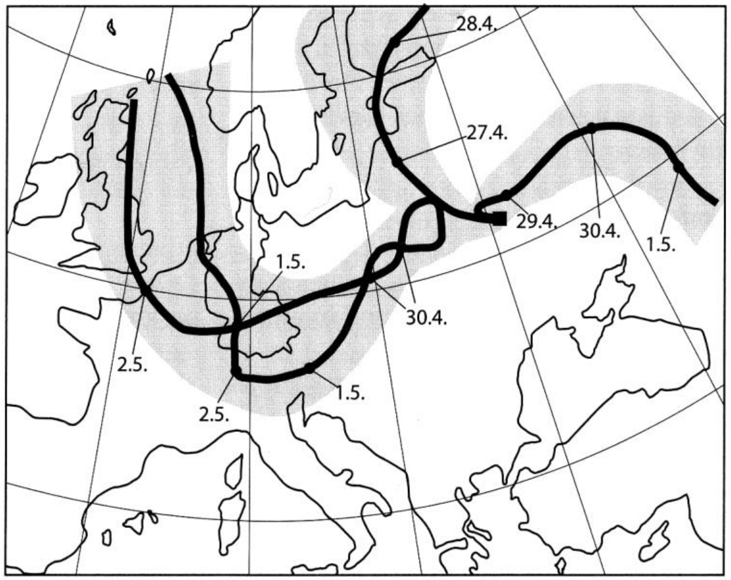
"Time rate of change of some characteristic at a ***fixed point in space*** *but with constantly changing fluid element because the fluid is moving."*

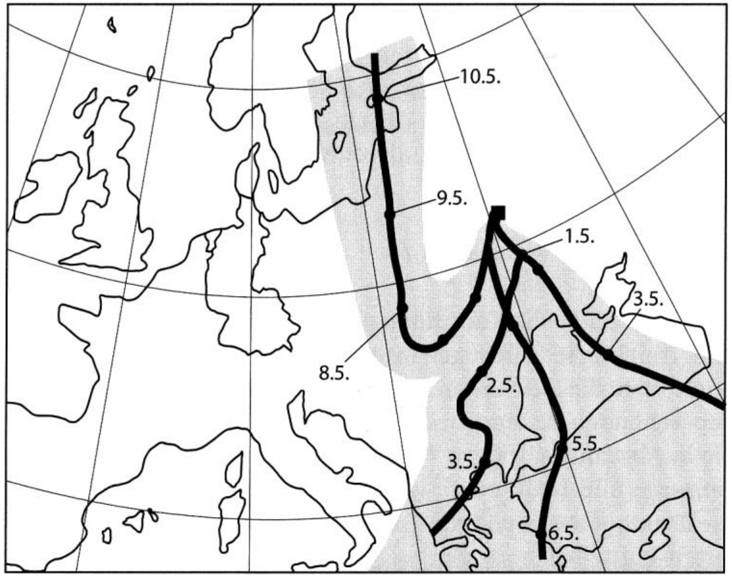
Differentiation following the motion - Lagrangian perspective

Forward trajectories started @850 hPa:

- top: between 26.04.-29.04.1986

- bottom: between 30.04.-05.05.1986



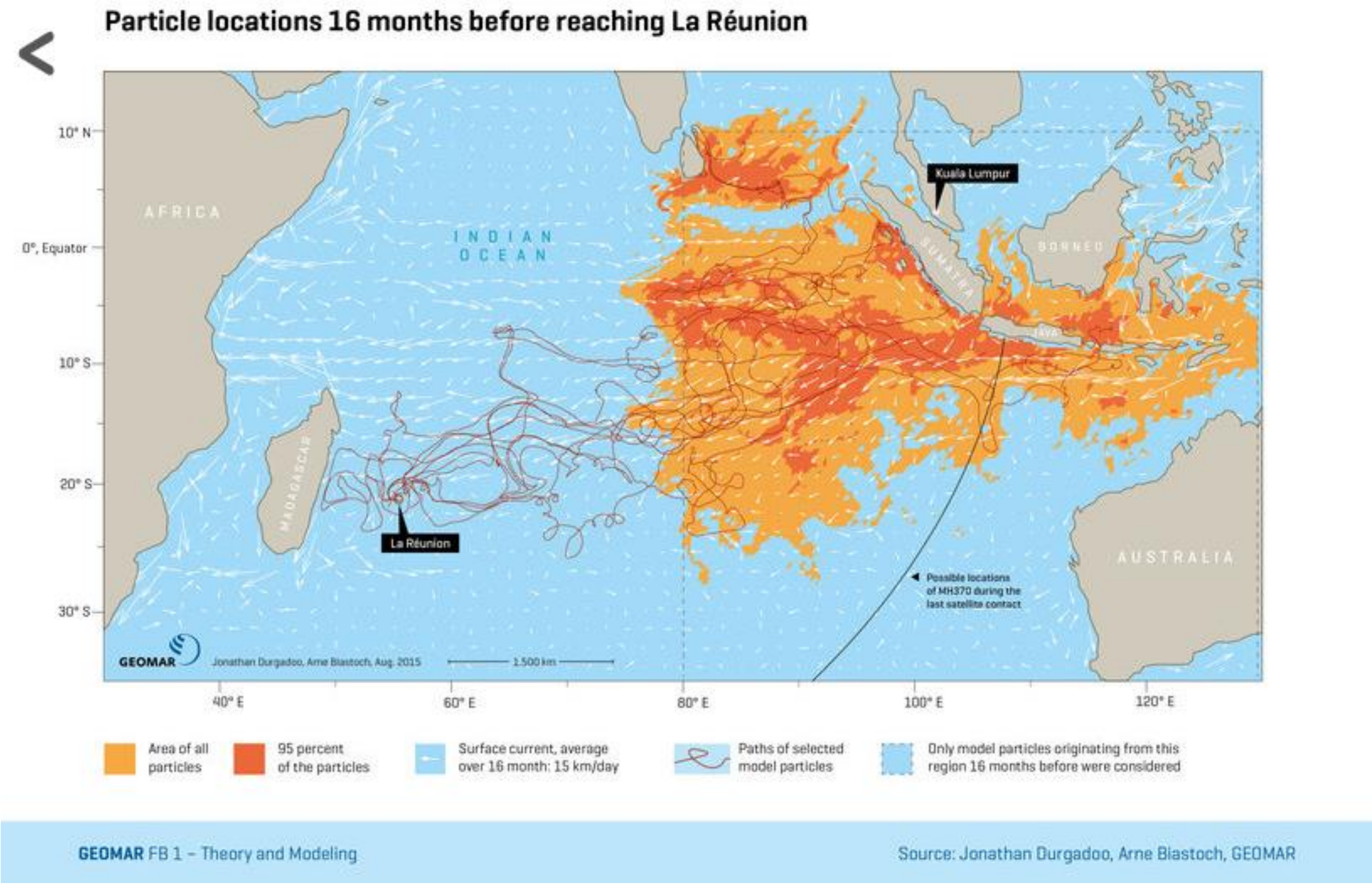


Example: nuclear reactor accident Chernobyl on 26.04.1986, 01:30 am

Trajectories (=flight path) are used for the prediction of air movement.

Kraus (2004)

Where is the missing Malaysian Airline MH370 (8 March 2014)?



GEOMAR press release from 1. September 2015

2.2 Basics - Mathematical add ons

- Nomenclature
- Vector operations
- Nabla operator

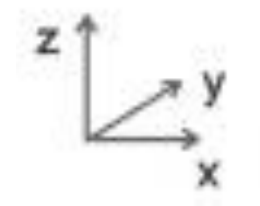
Nomenclature summary

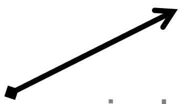
- δ : Greek small delta, infinitesimal small size,
  e.g. *δV:= small air volume*.
- $\frac{\partial}{\partial x}$: partial derivative of a quantity with respect to
  a coordinate, e.g. *x*
- $\frac{d}{dt}$*: total differential, derivative of a quantity with respect to all dependent coordinates (x,z,y)*
- $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ *: Lagrangian (or total) derivative*

Vector operations

- Multiplication with a scalar *a* -

$$a\vec{v} = \vec{v}a = av_i = v_i a = a\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} av_x \\ av_y \\ av_z \end{pmatrix}$$





wind vector

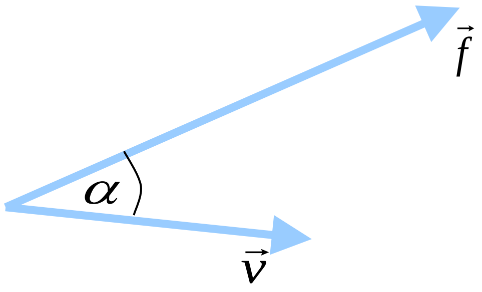
- **With a scalar multiplication the vector stays a vector.**
- Each element of a vector is multiplied individually with a scalar *a*.
- The vector extends (or shortens) itself by the factor *a.*
- Convention: *With the multiplication scalar – vector we* ***don't*** *use a point (like with scalar – scalar).*

Vector operations
- Scalar product -

$$\vec{v}\cdot\vec{f}=\vec{f}\cdot\vec{v}=\begin{pmatrix}v_x\\v_y\\v_z\end{pmatrix}\cdot\begin{pmatrix}f_x\\f_y\\f_z\end{pmatrix}=v_x f_x+v_y f_y+v_z f_z=f_i v_i\equiv\sum_i f_i v_i=f_i v_i$$

- **The scalar product of two vectors is a scalar.**
- It is multiplied component-wise, then added.
- Convention: *The scalar product is marked by a multiplication sign (·).*
- **It is at a maximum with parallel vectors and disappears when the vectors are perpendicular to each other.**

$$\vec{v}\cdot\vec{f}=|\vec{v}|\,|\vec{f}|\cos\alpha$$

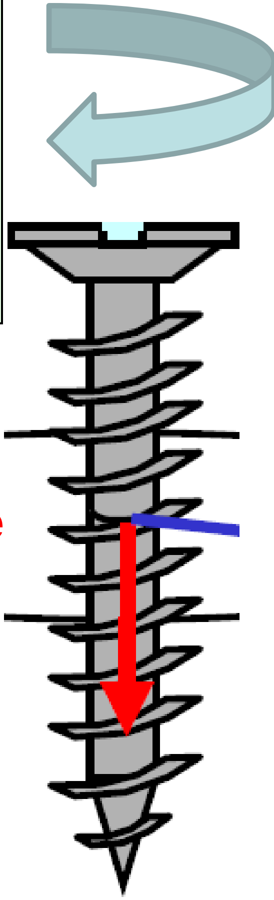


Vector operations – vector product

$$\vec{c} = \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \vec{i}(a_y b_z - a_z b_y) + \vec{j}(a_z b_x - a_x b_z) + \vec{k}(a_x b_y - a_y b_x)$$

$$= \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

$$|\vec{c}| = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\alpha \quad , \vec{c} \perp \vec{a} \wedge \vec{c} \perp \vec{b}$$



- **The vector product of two vectors is again a vector.**
- Convention: The vector product or cross product is marked by an “x”.
- **If $\vec{a}$ is turned to $\vec{b}$ on the fastest route possible, the vector (from the vector product) points in the direction in which a right-hand screw would move (right hand rule).**
- With parallel vectors, it disappears, and it reaches its maximum when the two vectors are at right angles to each other!

Nabla operator – spatial gradient

$$\text{Nabla - Operator } \vec{\nabla}:$$

$$\text{with } \vec{\nabla} = \nabla = \partial_i \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

- **Nabla is a (vector) operator that works to the right.**
- **It only results in a value if it stands left of an arithmetic expression.**
- **If it stands to the right of an arithmetic expression, it keeps its operator function (and "waits" for application).**

Graphically:

- Pooling of the spatial gradients towards the spatial coordinate axes → vector
- Gradient points towards the largest increase of the quantity.
- Amount is the magnitude of the derivative pointing towards the largest increase.

Nabla operator

Product with a Scalar, vector

$$\vec{\nabla} T \equiv grad\ T = \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} T$$

Scalar product “·”, scalar

$$\vec{\nabla} \cdot \vec{u} \equiv div\ \vec{u} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Vector product “x”, vector

$$\vec{\nabla} \times \vec{u} \equiv rot\ \vec{u} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$$

Unit vectors:
i: x direction
j: y direction
k: z direction

Nabla operator – algorithms:

Note: Nabla is a (vector) operator, i.e. the sequence must not be changed here!

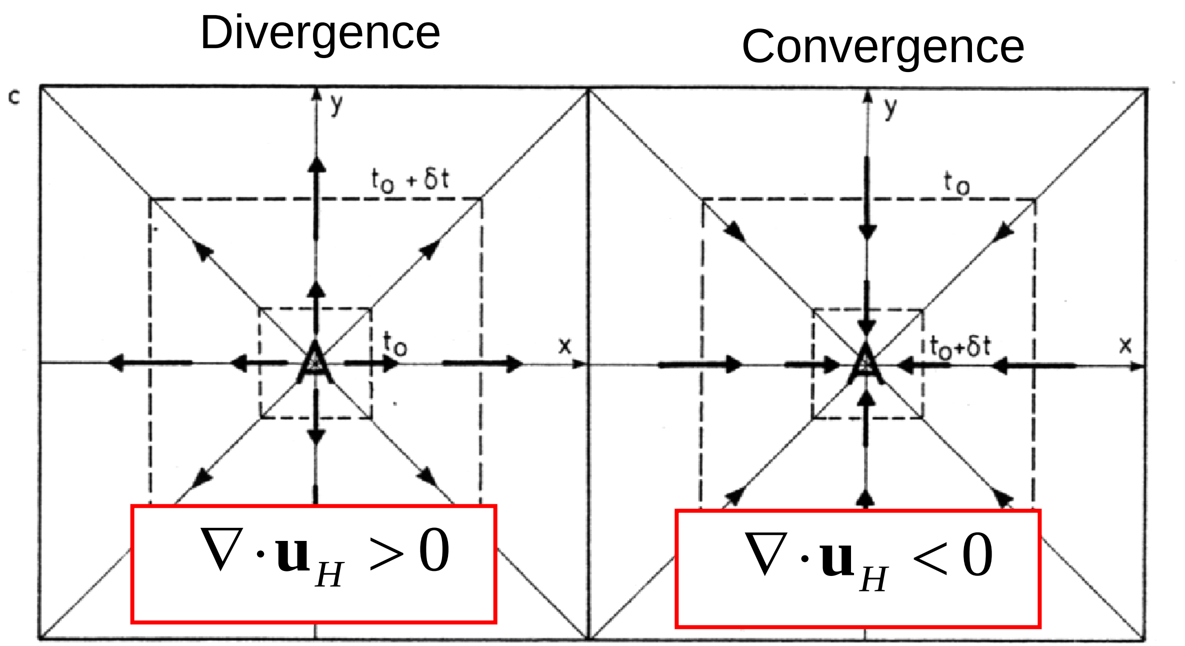
$$\begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} T = \vec{\nabla} T \neq T\vec{\nabla} = \begin{pmatrix} T\,\frac{\partial}{\partial x} \\ T\,\frac{\partial}{\partial y} \\ T\,\frac{\partial}{\partial z} \end{pmatrix}$$

$$\vec{\nabla} \cdot \vec{u} \equiv div\,\vec{u} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\neq$$

$$\vec{u} \cdot \vec{\nabla} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z}$$

Characteristics of horizontal wind



$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

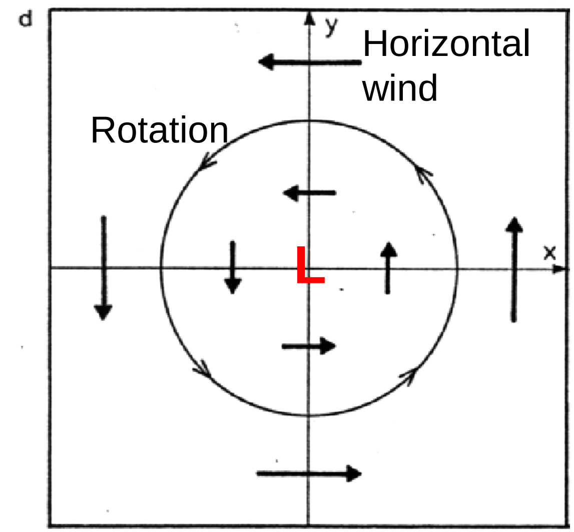
Characteristics of horizontal wind

Rotation and vorticity*

(*comes from the word “vortex”)

Rotation of horizontal flow:

$$\boldsymbol{rot}\ \boldsymbol{u} = \nabla \times \boldsymbol{u}$$



Rotation of a vector field

- Vector product of the Nabla operator with a vector -

$$\vec{\nabla} \times \vec{u} \equiv rot\,\vec{u} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$$

Definition vorticity

The rotation of a flow field is constituted by

$$rot\,\boldsymbol{u} \equiv \nabla \times \boldsymbol{u}$$

The vorticity is a scalar!

The vorticity ζ is defined as a vertical component of the velocity fields's rotation.

ζ : Greek „Zeta“

$$\zeta = \boldsymbol{k} \cdot \nabla \times \boldsymbol{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity

Relative vorticity ζ:

Rotation relative to the earth.

$$\boxed{\zeta = \mathbf{k} \cdot \nabla \times \mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$$

Absolute vorticity η:

Sum of relative vorticity and planetary vorticity.

$$\boxed{\eta = \zeta + f}$$

ζ : Zeta
η : Eta
f : Coriolis parameter

Rotation in an absolute coordinate system oriented towards a fixed star.

Planetary vorticity f:

Contribution of earth rotation.

$$\boxed{f = 2\, \Omega\, sin\varphi}$$

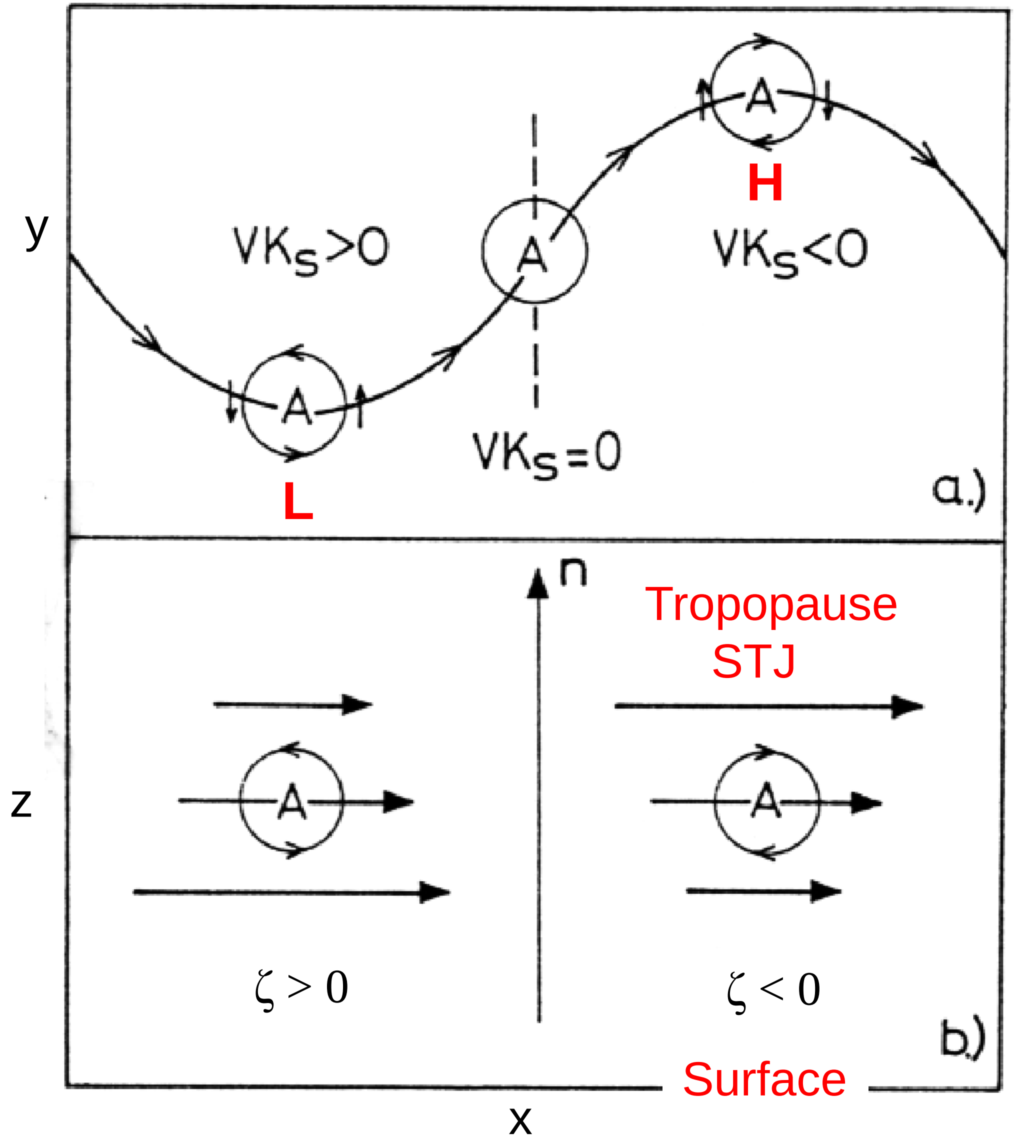
f: Coriolis parameter, φ: latitude, Ω: angular velocity
$\Omega = 2\pi/24h$ is the angular velocity of the earth rotation.

Application examples: e.g. vorticity advection in synoptic; strength of a vortex (e.g. stratospheric polar vortex); stratosphere: vorticity correlates with ozone.

Vorticity (NH)

Curvature vorticity (VK_s)

Shear vorticity



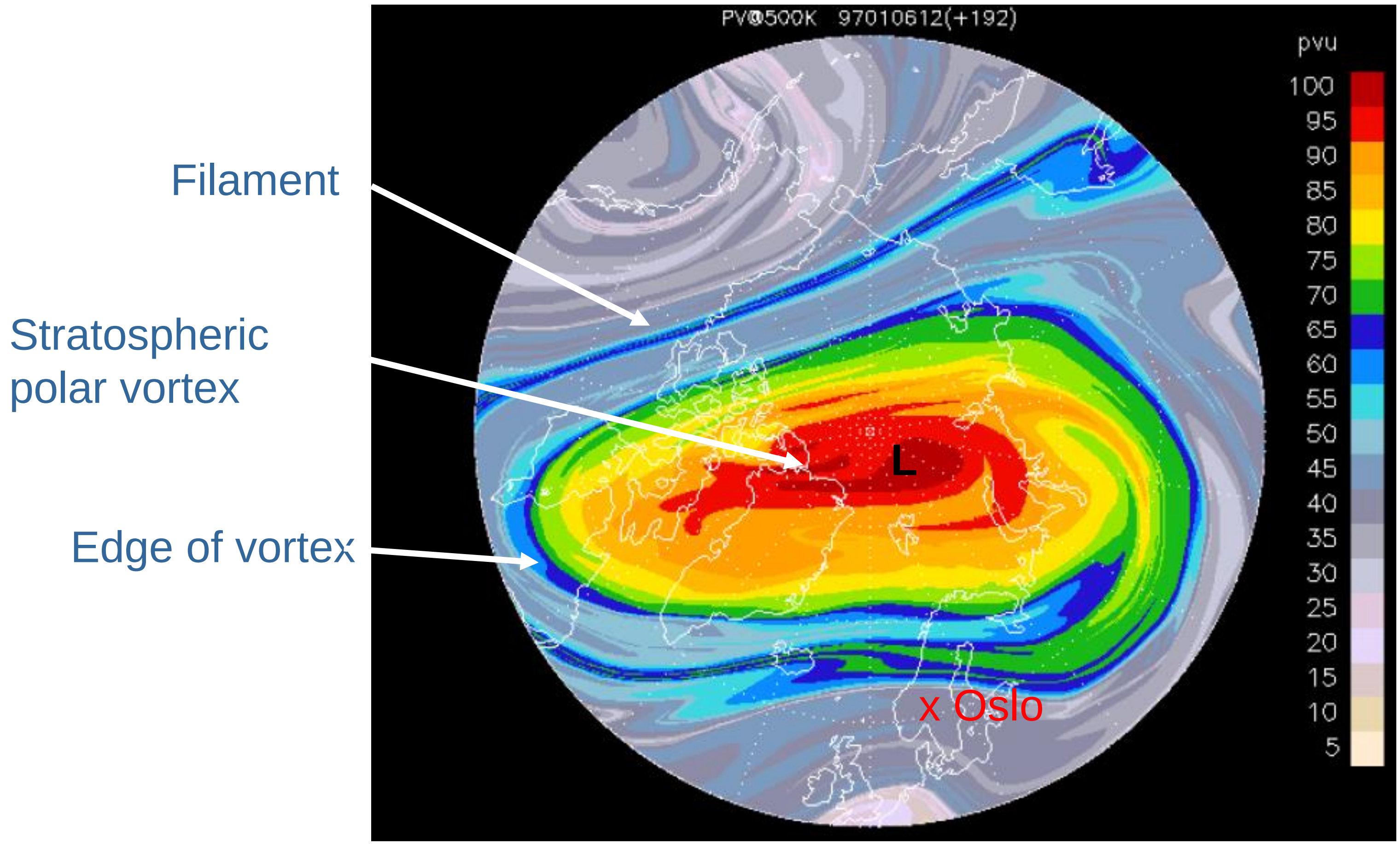
Examples

STJ: Sub-tropical Jet

Note: In general, the horizontal wind field contains both the curvature as well as the shear forces (varying strengths of winds on the verticals).

NH stratospheric polar vortex

Potential vorticity unit (PVU) at ~20 km altitude



- High vorticity (PV) in the centre of the polar vortex (low pressure centre).
- Greatest PV gradient at the edge of the polar vortex (→ wind maximum).

Lecture Outline – Ch. 6

Ch. 6 - The equations of fluid motions

1. Motivation
2. Basics
   - 2.1 Lagrange - Euler
   - 2.2* Mathematical add ons, Vorticity
3. Equation of motion for a nonrotating fluid
   - 3.1 Forces
   - 3.2 Equations of motion
   - 3.3 Hydrostatic balance
4. Conversation of mass
5. Thermodynamic equation
6. Equation of motion for a rotating fluid
   - 6.1 Forces
   - 6.2 Equations of motion
7. Take home messages

**Addition, not in book.*

3. Equations of motion - nonrotating fluid

I. Laws of motion for a fluid parcel in x-,y-,z-directions (applying Newton's 1. and 2. law)

II. Conservation of mass

III. Law of thermodynamics (including motion)

→5 equations for the evolution of the fluid (5 unknowns: ($\boldsymbol{u}$, p, T))

Momentum conservation

The conservation of momentum is represented by Newton's first law ("Lex prima"):

(momentum = mass · velocity)

momentum = const. *at absence of forces*

Momentum sentence ("Lex secunda"):

temporal change in momentum = Force

(mass · acceleration = Force)

$$M\,\frac{d\boldsymbol{u}}{dt} = \boldsymbol{F}$$

t: time; M: mass; ***u****: velocity vector* ***F****: Force vector*

fluid parcel - cube

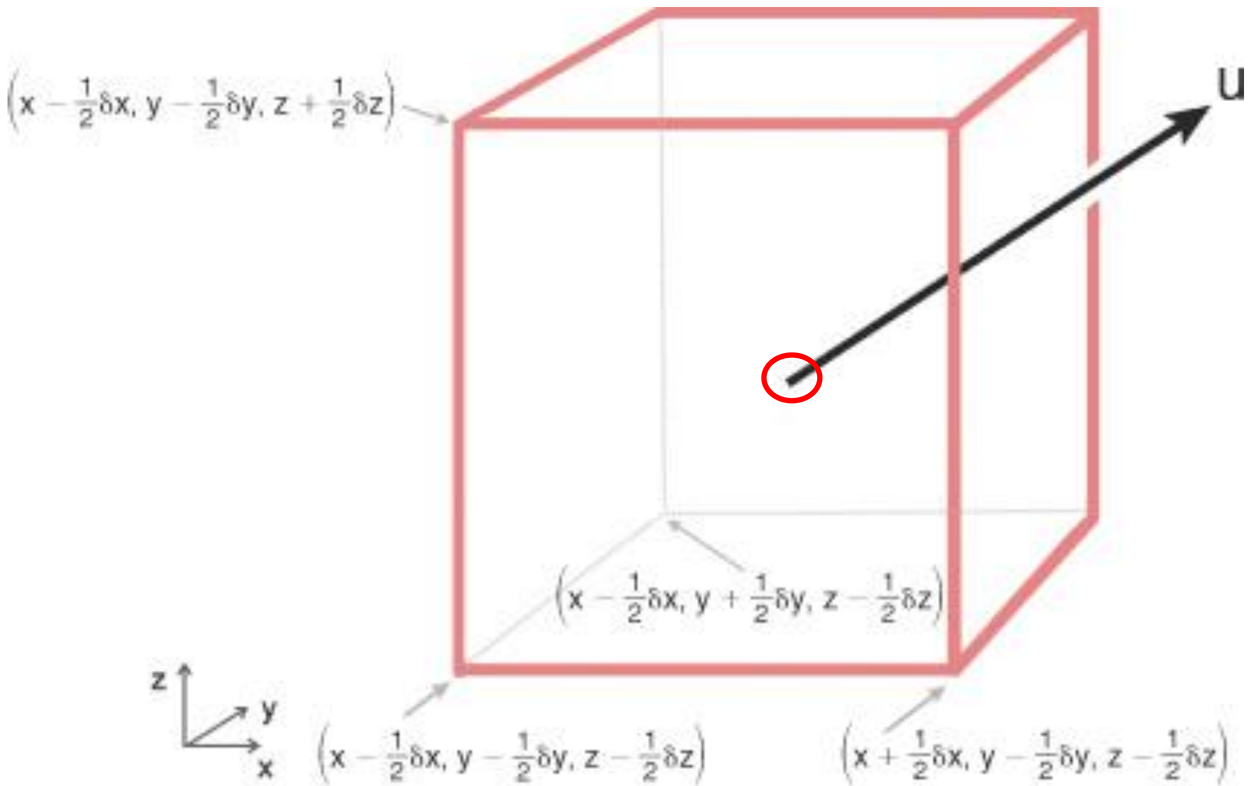


Figure 6.2: An elementary fluid parcel, conveniently chosen to be a cube of sides $\delta x, , \delta y, , \delta z$, centered on $(x, , y, , z)$. The parcel is moving with velocity u.

Marshall and Plumb (2008)

Forces on a fluid parcel

$$\rho\, \delta x\, \delta y\, \delta z\, \frac{D\boldsymbol{u}}{Dt} = \mathbf{F} \quad \text{(Eq. 6-2)}$$

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y} + w\frac{\partial \boldsymbol{u}}{\partial z}$$
$$= \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}$$

ρ: density
$(\delta x, \delta y, \delta z)$: fluid parcel with infinitesimal dimensions
δM: mass of the parcel; $\delta M = \rho\, \delta x\, \delta y\, \delta z$
$\boldsymbol{u}$: parcel velocity
$\mathbf{F}$: net force

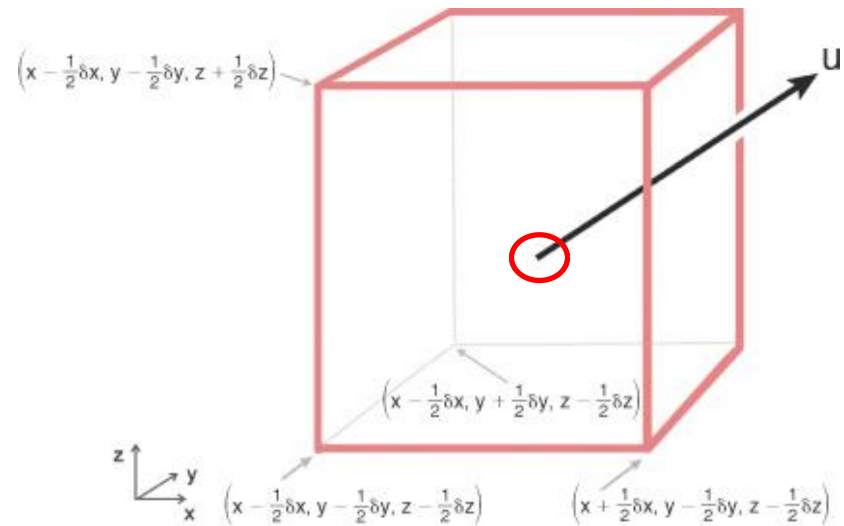


Figure 6.2: An elementary fluid parcel, conveniently chosen to be a cube of sides $\delta x, , \delta y, , \delta z$, centered on $(x, , y, , z)$. The parcel is moving with velocity u.

Marshall and Plumb (2008)

Gravity

The gravitational force, $g\,\delta M$, is directed downward:

$$\mathbf{F}_{\text{gravity}} = -g\,\rho\,\hat{\mathbf{z}}\,\delta x\,\delta y\,\delta z \qquad \text{(Eq. 6-3)}$$

$\mathbf{F}_{\text{gravity}}$: Gravitational force
M: mass
g: gravity acceleration, g ≈ const.
ρ: density
$\hat{\mathbf{z}}$: unit vector in upward direction

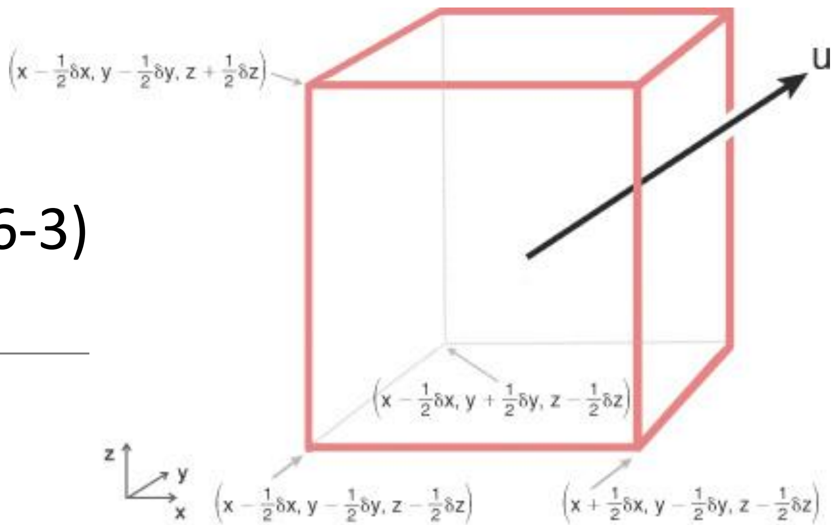


Figure 6.2: An elementary fluid parcel, conveniently chosen to be a cube of sides δx, , δy, , δz, centered on $(x, , y, , z)$. The parcel is moving with velocity u.

Marshall and Plumb (2008)

Pressure gradient force

x-component of the pressure force is:

$$F(A) = p \cdot A = p\,(x - \frac{\delta x}{2}, y, z)\,\delta y\,\delta z$$

$$F(B) = p \cdot B = -p\,(x + \frac{\delta x}{2}, y, z)\,\delta y\,\delta z$$

$$F_x = F(A) + F(B)$$

Apply Taylor expansion (A.2.1) at midpoint of parcel and neglect small terms (see book):

$$F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

Apply for all sides (y and z).

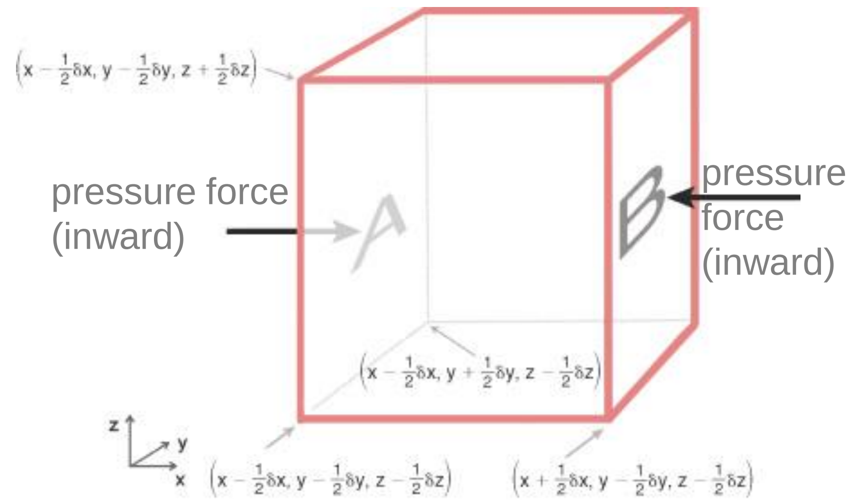


Figure 6.3: Pressure gradient forces acting on the fluid parcel. The pressure of the surrounding fluid applies a force to the right on face A and to the left on face B.

Marshall and Plumb (2008)

p: pressure

Pressure gradient force

$$\mathbf{F}_{\text{pressure}} = \left(F_x, F_y, F_z\right)$$

$$= -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) \delta x\, \delta y\, \delta z$$

$$= -\nabla p\, \delta x\, \delta y\, \delta z \qquad \text{(Eq. 6-4)}$$

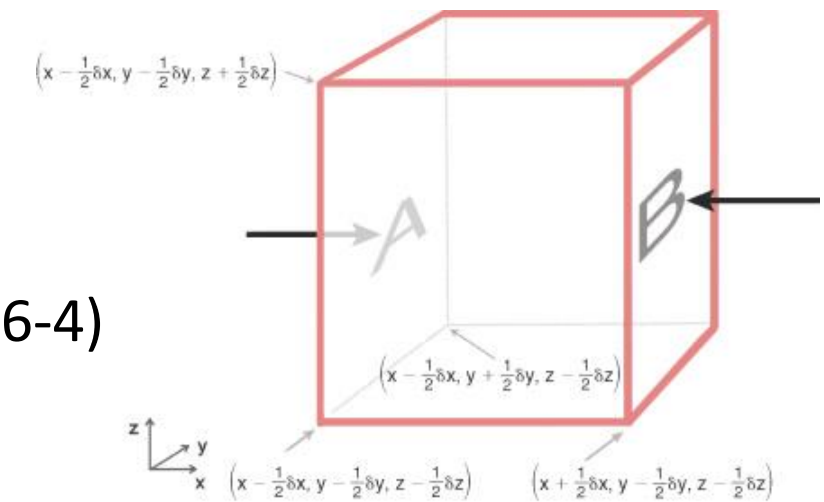


Figure 6.3: Pressure gradient forces acting on the fluid parcel. The pressure of the surrounding fluid applies a force to the right on face A and to the left on face B.

Marshall and Plumb (2008)

p: pressure

Friction force

- Friction force operates on the earth's surface,
- the greater the surface roughness, the higher the friction force,
- the greater the wind velocity, the higher the friction force,
- friction force takes effect in vertical distances of ~ 100 m up to 1,000 m (→ atmospheric boundary layer Chapter 7).

$$\mathbf{F}_{\text{friction}} = \rho\, \mathcal{F}\, \delta x\, \delta y\, \delta z \qquad \text{(Eq. 6-5)}$$

$\mathcal{F}$: frictional force *per unit mass (see book chapters 7.4.2 and 10.1)*
ρ: density

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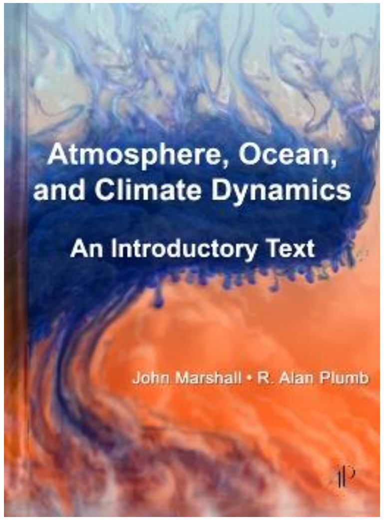
Chapter 6: The equations of fluid motion



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Lecture Outline – Ch. 6

Ch. 6 - The equations of fluid motions

1. Motivation

2. Basics
   - 2.1 Lagrange - Euler
   - 2.2* Mathematical add ons, Vorticity

3. Equation of motion for a nonrotating fluid
   - 3.1 Forces
   - 3.2 Equations of motion
   - 3.3 Hydrostatic balance

4. Conversation of mass

5. Thermodynamic equation

6. Equation of motion for a rotating fluid
   - 6.1 Forces
   - 6.2 Equations of motion

7. Take home messages

**Addition, not in book.*

Equations of motion

- All three forces together in Eq. 6-2 gives:

$$\rho\, \delta x\, \delta y\, \delta z \frac{D\mathbf{u}}{Dt} = \mathbf{F}_{pressure} + \mathbf{F}_{gravity} + \mathbf{F}_{friction}$$

- Re-arranging leads to **equation of motion for fluid parcels**:

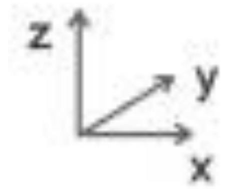
$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \mathrm{g}\hat{\mathbf{z}} = \mathcal{F}$$

(Equation 6-6)

ρ: density
$\mathbf{u}$: velocity vector
$\mathbf{F}$: Force vector
$\mathcal{F}$: frictional force *per unit mass*
p: pressure
g: gravity acceleration
$\hat{\mathbf{z}}$: unit vector in z-direction

Equations of motion – component form

(Eq. 6-6) in Cartesian coordinates:



- $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \mathcal{F}_x \qquad \text{(6-7a)}$$
- $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \mathcal{F}_y \qquad \text{(6–7b)}$$
- $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \mathcal{F}_z \qquad \text{(6-7c)}$$

---

u: zonal velocity
v:meridional velocity
w: vertical velocity
ρ: density
$\mathcal{F}$: frictional force *per unit mass*
p: pressure
g: gravity acceleration

Hydrostatic balance

If friction $\mathcal{F}_z$ and vertical acceleration $\frac{Dw}{Dt}$ are negligible, we derive from the vertical eq. of motion (6-7c) the **hydrostatic balance** *(Ch.3, Eq.3-3):*

$$\frac{\partial p}{\partial z} = -\rho g \quad \text{(Eq. 6-8)}$$

Balance between vertical pressure gradient and gravitational force!

"Pressure decreases with height in proportion to the weight of the overlying atmosphere."

Note: This approximation holds for large-scale atmospheric and oceanic circulation with weak vertical motions.

$\frac{D}{Dt}$: *Lagrangian derivative* $\quad \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$

Conservation of mass
– continuity equation

Conserved quantity:

A quantity of which the derivative equals zero.

Conservation of mass for liquids is also called **continuity equation**.

Conservation of mass

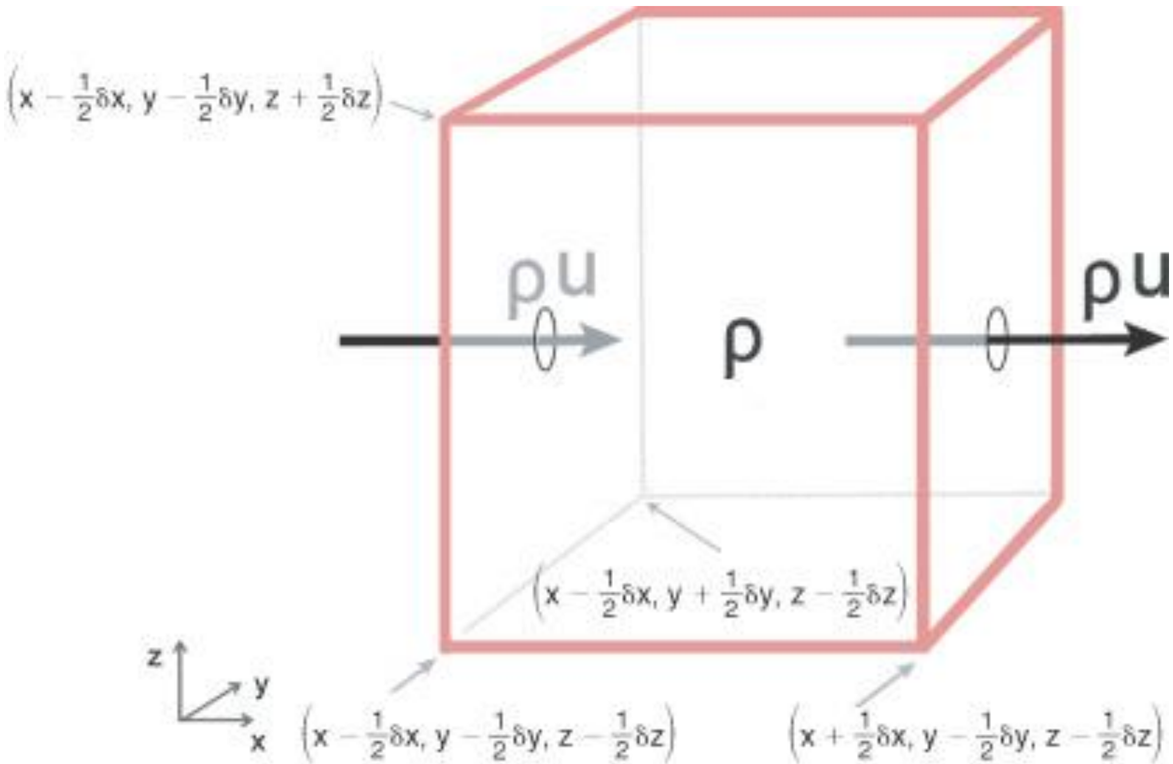


Figure 6.4: The mass of fluid contained in the fixed volume, $\rho \delta x\, \delta y\, \delta z$, can be changed by fluxes of mass out of and into the volume, as marked by the arrows.

Marshall and Plumb (2008)

Conservation of mass -1

Mass continuity requires:

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z$$

= (net mass flux into the volume)

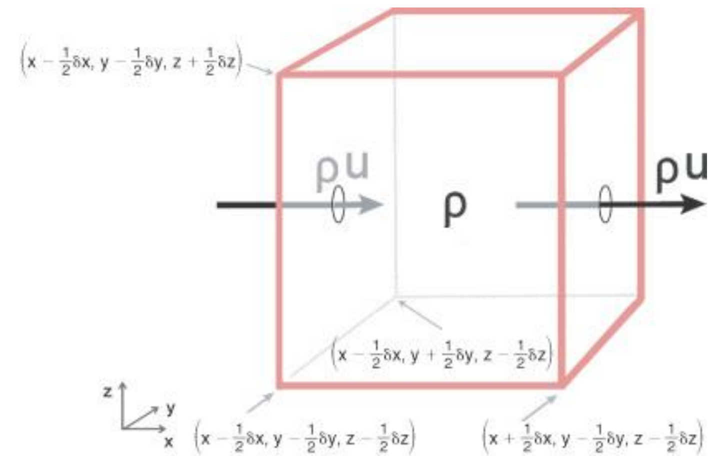


Figure 6.4: The mass of fluid contained in the fixed volume, $\rho\delta x\,\delta y\,\delta z$, can be changed by fluxes of mass out of and into the volume, as marked by the arrows.

- mass flux in x-direction per unit time ***into*** the volume

$$[\rho u]\left(x - \frac{1}{2}\,\delta x, y, z\right)\delta y\,\delta z$$

- mass flux in x-direction per unit time ***out of*** the volume

$$[\rho u]\left(x + \frac{1}{2}\,\delta x, y, z\right)\delta y\,\delta z$$

Marshall and Plumb (2008)

Conservation of mass -2

- net mass flux in x-direction *into* the volume is then (employing Taylor expansion A.2.2):
  $-\frac{\partial}{\partial x}(\rho u)\,\delta x\,\delta y\,\delta z\,,$

- net mass flux in y- and z- direction accordingly (see book).

- Net mass flux into the volume:

  $$-\nabla\cdot(\rho\boldsymbol{u})\,\delta x\,\delta y\,\delta z\,,$$

- substituting into mass continuity:

  $$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = -\nabla\cdot(\rho\boldsymbol{u})\,\delta x\,\delta y\,\delta z$$

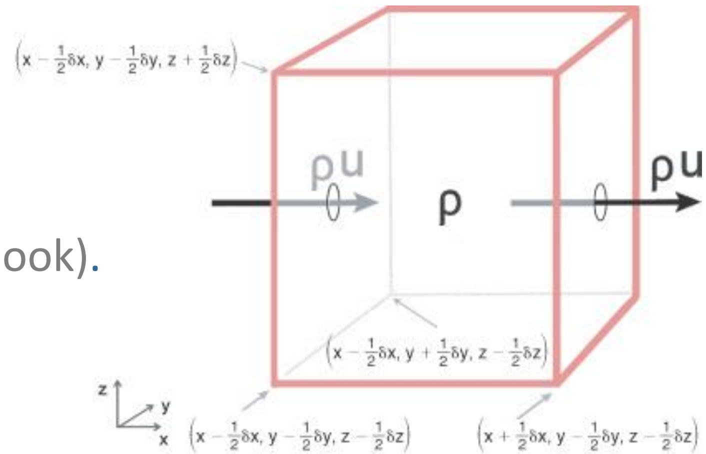


Figure 6.4: The mass of fluid contained in the fixed volume, $\rho\delta x\,\delta y\,\delta z$, can be changed by fluxes of mass out of and into the volume, as marked by the arrows.

Marshall and Plumb (2008)

Conservation of mass -3

- leads to equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \qquad \text{(Eq. 6-9),}$$

which has the form of conservation law:

$$\frac{\partial\, Concentration}{\partial t} + \nabla \cdot (flux) = 0.$$

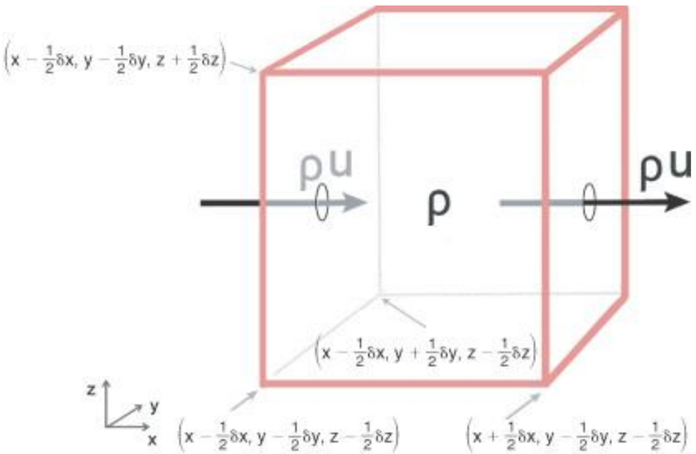


Figure 6.4: The mass of fluid contained in the fixed volume, $\rho\delta x\, \delta y\, \delta z$, can be changed by fluxes of mass out of and into the volume, as marked by the arrows.

Marshall and Plumb (2008)

- Using D/Dt (Eq. 6-1) and $\nabla \cdot (\rho \boldsymbol{u}) = \rho \nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \rho$ (see A.2.2) we can rewrite:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0 \qquad \text{(Eq. 6-10)}$$

$\frac{D}{Dt}$: *Lagrangian derivative* $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$

Approximations for continuity equation: incompressible - compressible flows

- Incompressible flow (e.g. ocean):

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(Eq.6-11)}$$

- Compressible flow (e.g. air, ρ varies) expressed in pressure coordinate p applying hydrostatic assumption:

$$\nabla_p \cdot \boldsymbol{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial p} = 0 \qquad \text{(Eq.6-12)}$$

Thermodynamic equation

- Temperature evolution can be derived from first law of thermodynamics (Ch.4, 4-2):

$$\frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} \qquad \text{(Eq. 6-13)}$$

- If the heating rate is zero ($\frac{DQ}{Dt} = 0$, Ch. 4.3.1) then:

$$\frac{DT}{Dt} = \frac{1}{\rho c_p} \frac{Dp}{Dt}$$

The temperature of a parcel will decrease/ increase in ascent/ descent with decreasing/increasing pressure.

→ Introduction of potential temperature θ (Eq. 4-17)

$$\theta = T \left(\frac{p_0}{p}\right)^{\kappa}$$

Q: heat
$\frac{DQ}{Dt}$: diabatic heating rate per unit mass
c_p : specific heat at constant pressure
T: temperature
ρ: density
p: pressure; p_0= 1000 hPa
$\kappa = R/c_p$
R: gas constant for dry air

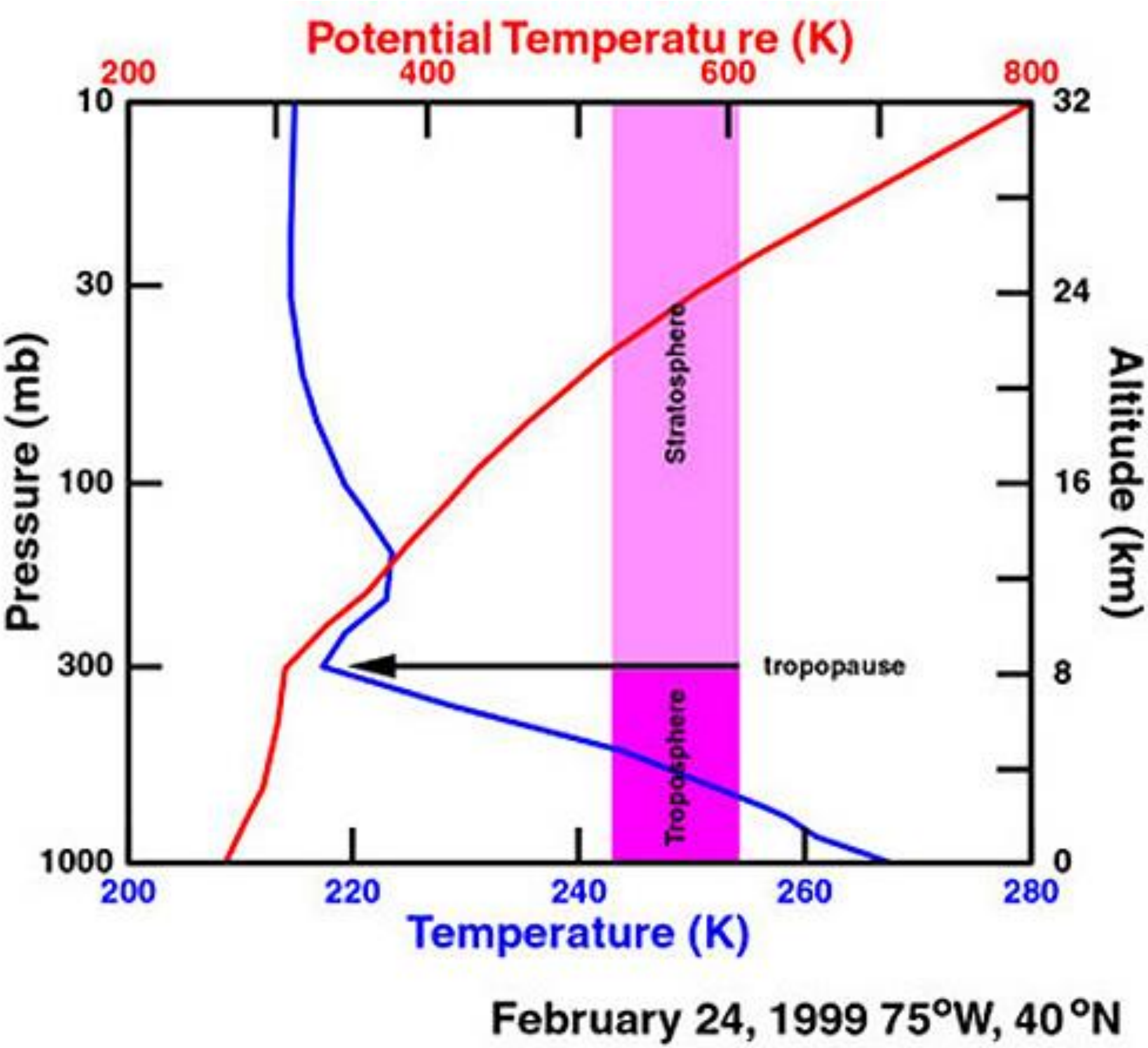
Thermodynamic equation - potential temperature

- Eq. 6-13 for potential temperature θ *(see book for details)*:

$$\frac{D\theta}{Dt} = \left(\frac{p}{p_0}\right)^{-\kappa} \frac{\frac{DQ}{Dt}}{c_p} \qquad \text{(Eq. 6-14)}$$

Note: For adiabatic motions $(\delta Q = 0)$ θ is conserved.

Q: heat
$\frac{DQ}{Dt}$: diabatic heating rate per unit mass
c_p : specific heat at constant pressure
T: temperature


Potential Temperatu re (K)
200
400
600
800
10
30
100
300
1000
Pressure (mb)
32
24
16
8
0
Altitude (km)
Stratosphere
tropopause
Troposphere
200
220
240
260
280
Temperature (K)
February 24, 1999 75°W, 40°N

Summary

→5 equations for the evolution of the fluid (5 unknowns)

Ia.) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \mathcal{F}_x$ (6-7a)

Ib.) $\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \mathcal{F}_y$ (6–7b)

Ic.) $\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \mathcal{F}_z$ (6-7c)

II.) $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$ (6-9 or 6-11/6-12)

III.) $\frac{DQ}{Dt} = c_p\frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$ (6-13 or 6-14)

Restrictions:
- application to average motion is often incorrect i.e. turbulence,
- fixed coordinate system.

Take home message I

- Eulerian – Lagrangian derivations/ perspectives.
- Equations of motion on a *non-rotating fluid*: *Pressure gradient force, gravitational force and friction force act.* $\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathcal{F}$
- 5 equations for the evolution of the fluid with 5 unknowns (u,v,w,p,T): *equations of motion (3), conservation of mass (1), thermodynamic equation (1)*

Lecture Outline – Ch. 6

Ch. 6 - The equations of fluid motions

Addition, not in book.

Forces on rotating sphere

Fictitious forces:

- occur in revolving/accelerating system
  (e.g. car in curve)
- occur on the earth's surface
  (Fixed system on earth's surface forms an accelerated system, because a circular motion is performed once a day. Movement of earth around the sun compared to earth's rotation is insignificant.)

→ **Coriolis force**

→ **Centrifugal force**

Coordinate systems in meteorology

- Cartesian coordinates (x,y,z)
- Spherical polar coordinates (λ,φ,z)
- Height coordinates:

  geometrical altitude (z),

  pressure (p) and

  potential temperature (θ)

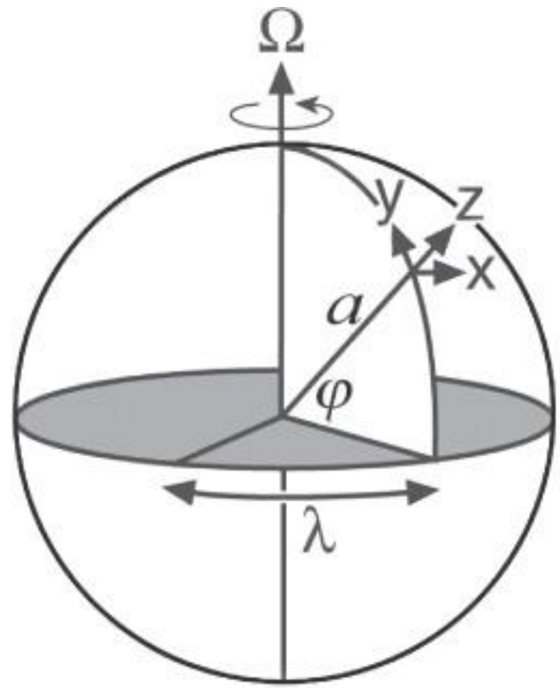


Figure 6.19: At latitude φ and longitude λ, we define a local coordinate system such that the three coordinates in the (x, , y, , z) directions point (eastward, northward, upward): $dx = a \cos \varphi d\lambda$; $dy = ad\varphi$; $dz = dz$, where a is the radius of the Earth. The velocity is $\mathbf{u} = (u, , v, , w)$ in the directions (x, , y, , z). See also Appendix A.2.3.

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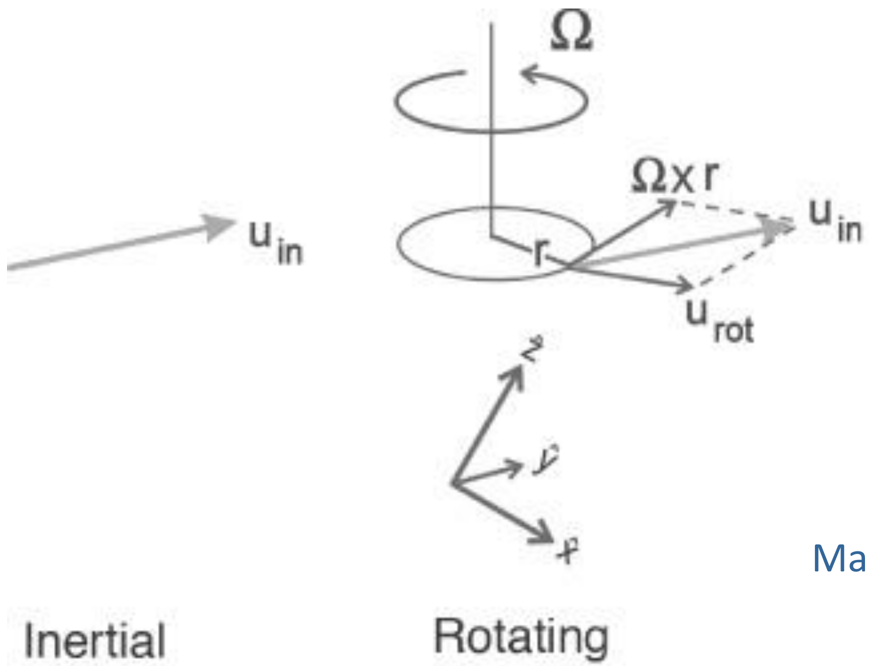
a : radius; Ω (Greek: omega): rotation vector

spherical polar coordinates: φ: $latitude$, λ: $longitude$, z: $radial\ distance$

Inertial - rotating frames



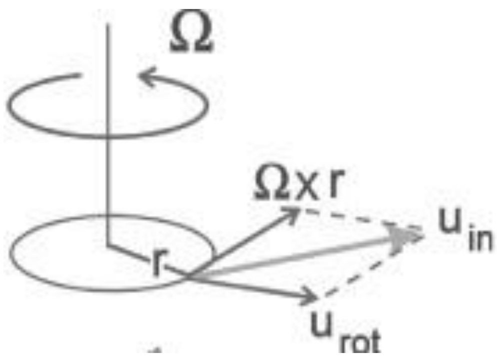
Marshall and Plumb (2008)

Figure 6.9: On the left is the velocity vector of a particle $\mathbf{u}_{in}$ in the inertial frame. On the right is the view from the rotating frame. The particle has velocity $\mathbf{u}_{rot}$ in the rotating frame. The relation between $\mathbf{u}_{in}$ and $\mathbf{u}_{rot}$ is $\mathbf{u}_{in} = \mathbf{u}_{rot} + \Omega \times \mathbf{r}$, where $\Omega \times \mathbf{r}$ is the velocity of a particle fixed (not moving) in the rotating frame at position vector $\mathbf{r}$. The relationship between the rate of change of any vector A in the rotating frame and the change of A as seen in the inertial frame is given by: $(DA/Dt)_{in} = (DA/Dt)_{rot} + \Omega \times A$.

$\boldsymbol{r}$: position vector; Ω (Greek: omega): rotation vector

Transformation into rotating coordinates

Consider figure 6.9:



- $\boldsymbol{u}_{in} = \boldsymbol{u}_{rot} + \Omega \times \boldsymbol{r}$ (Eq. 6-24)

We set $\boldsymbol{A} = \boldsymbol{r}$. The transformation rule for D/Dt acting on a vector is given by:

- $\left(\frac{D\boldsymbol{A}}{Dt}\right)_{in} = \left(\frac{D\boldsymbol{A}}{Dt}\right)_{rot} + \Omega \times \boldsymbol{A}$ (Eq. 6-26)

Then we set $\boldsymbol{A} \rightarrow \boldsymbol{u}_{in}$ in Eq. 6-26 using 6-24, we derive:

$$\left(\frac{D\boldsymbol{u}_{in}}{Dt}\right)_{in} = \left[\left(\frac{D}{Dt}\right)_{rot} + \Omega \times\right] (\boldsymbol{u}_{rot} + \Omega \times \boldsymbol{r})$$

$$= \left(\frac{D\boldsymbol{u}_{rot}}{Dt}\right)_{rot} + 2\Omega \times \boldsymbol{u}_{rot} + \Omega \times \Omega \times \boldsymbol{r} \qquad \text{(Eq. 6-27)}$$

$\boldsymbol{r}$: position vector
$\boldsymbol{A}$: any vector
Ω: rotation vector

$$\left(\frac{D\boldsymbol{r}}{Dt}\right)_{rot} = \boldsymbol{u}_{rot}$$

Rotating equations of motion

Substituting $\left(\frac{D\boldsymbol{u}_{in}}{Dt}\right)_{in}$ from Eq. 6-27 into Eq. 6-6 (inertial frame equation of motion) we derive in rotating frame, dropping subscript “rot“:

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathcal{F} \qquad \text{(Eq. 6-6)}$$

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \underbrace{-\,2\Omega \times \boldsymbol{u} - \Omega \times \Omega \times \boldsymbol{r}}_{?} + \mathcal{F} \qquad \text{(Eq. 6-28)}$$

$\boldsymbol{u}$: velocity vector, t: time, p: pressure, g: gravity acceleration, $\hat{\boldsymbol{z}}$: unit vector in z−direction
Ω: rotation vector, $\mathcal{F}$: frictional force *per unit mass*

Coriolis acceleration

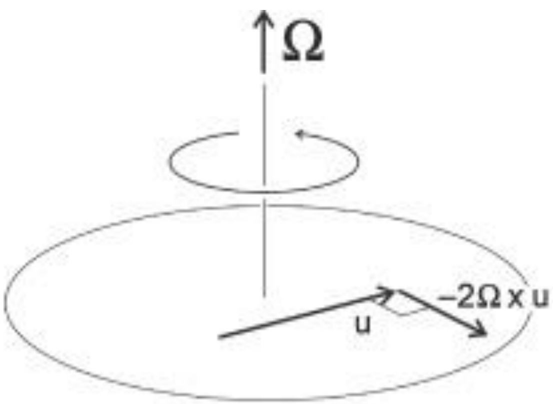


Figure 6.10: A fluid parcel moving with velocity u_{rot} in a rotating frame experiences a Coriolis acceleration, $-2\Omega \times u_{rot}$ directed "to the right" of u_{rot} if, as here, Ω is directed upwards, corresponding to anticlockwise rotation.

Marshall and Plumb (2008)

NH (viewed from above the Northpole):
$\Omega > 0$: rotation anticlockwise

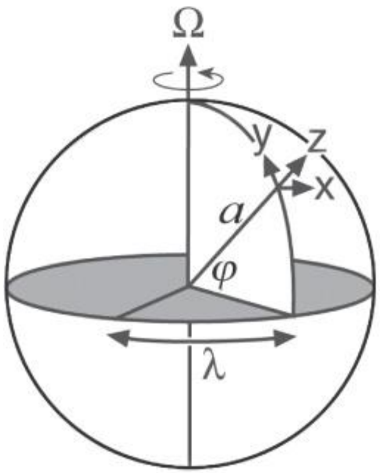
SH (viewed from above the Southpole):
$\Omega < 0$: rotation clockwise

Absence of other forces:

$$\frac{D\mathbf{u}}{Dt} = -2\boldsymbol{\Omega} \times \boldsymbol{u} \qquad \text{Eq. 6-31}$$

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 \\ \Omega_y \\ \Omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ \Omega\, cos\varphi \\ \Omega \sin\varphi \end{pmatrix}$$: Angular velocity

Coriolis force on the sphere – Coriolis parameter



Assuming that 1) Ω_z is negligible as $\Omega u << g$; 2) $w << \mathbf{u}_h$ leads to:

$$-2\boldsymbol{\Omega} \times \boldsymbol{u} \simeq (-2\Omega \sin\varphi\, v,\ 2\Omega \sin\varphi\, u, 0) \qquad \text{(Eq. 6-41)}$$
$$= f\hat{\boldsymbol{z}} \times \boldsymbol{u}$$

Coriolis parameter f:

$$f = 2\Omega \sin\varphi \qquad \text{(Eq. 6-42)}$$

Note that $\Omega \sin\varphi$ is the *vertical component* of the Earths rotation rate; only component that matters.

TABLE 6.1. Values of the Coriolis parameter, $f = 2\Omega\sin\varphi$ (Eq. 6 42), and its meridional gradient, $\beta = df/dy = 2\Omega/a \cos\varphi$ (Eq. 10-10), tabulated as a function of latitude. Here Ω is the rotation rate of the Earth and a is the radius of the Earth.

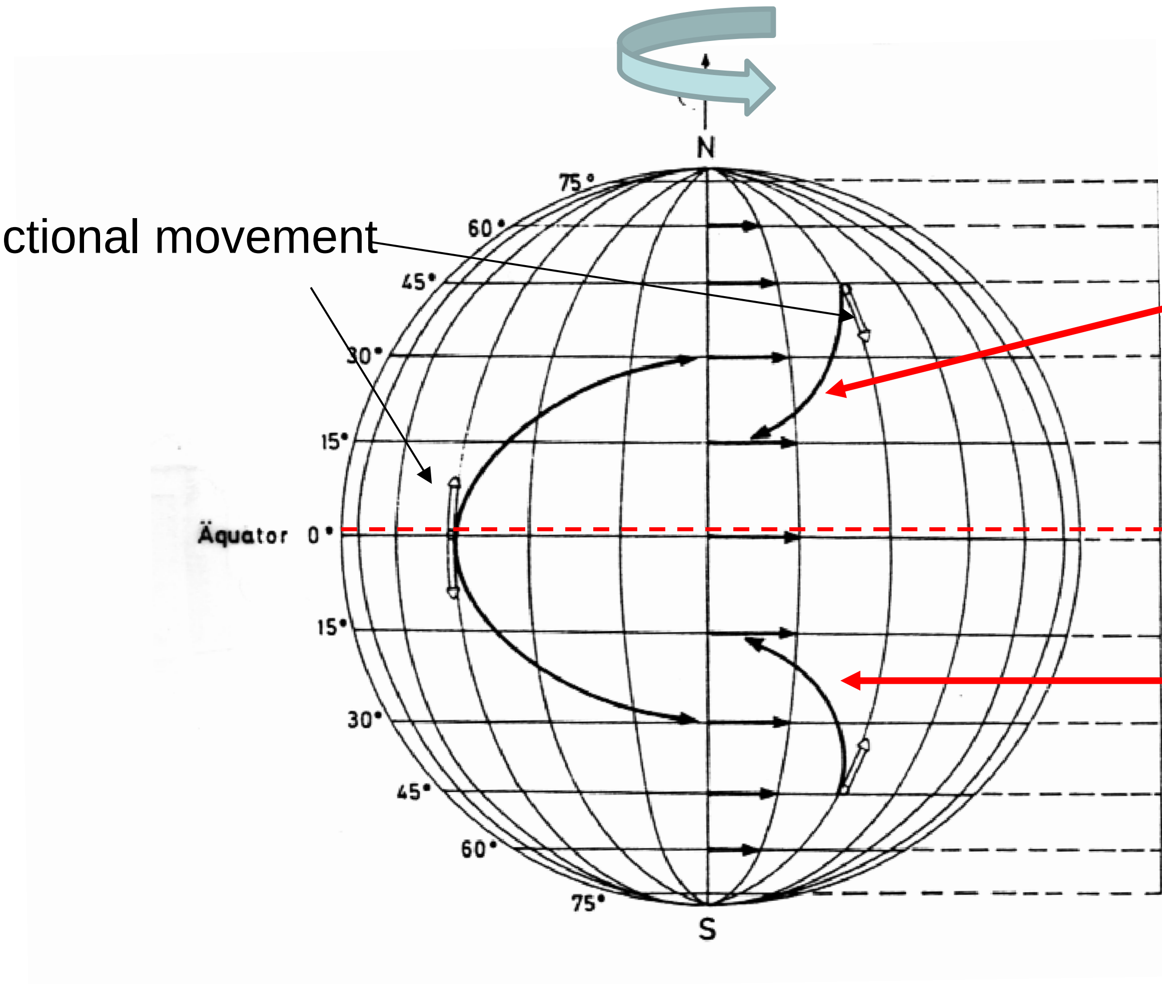
Latitude	$f\left(\times 10^{-4}\,\mathrm{s}^{-1}\right)$	$\beta\left(\times 10^{-11}\,\mathrm{s}^{-1}\,\mathrm{m}^{-1}\right)$
90°	1.46	0
60°	1.26	1.14
45°	1.03	1.61
30°	0.73	1.98
10°	0.25	2.25
0°	0	2.28

Marshall and Plumb (2008)

$\Omega = 7.27 \times 10^{-5}\ \mathrm{s}^{-1}$
$a = 6.37 \times 10^{6}$ m

Coriolis force

Directional movement



Schematic picture for Coriolis force

In direction of movement, in

NH:
deflection of air particles to the right

SH:
deflection of air particles to the left

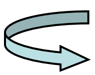
takes place.

Rotation: Cyclonic – Anticyclonic

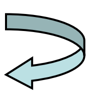
We call a rotating movement “cyclonic”, if it follows the same rotational direction as the earth.

As $f > 0$, in the NH, the rotary movement is:

$\zeta > 0$: cyclonic, motion in anti-clockwise direction, low pressure



$\zeta < 0$: anticyclonic, motion in clockwise direction > high pressure



What is the rotary motion like in the SH?

In the SH, because of $f < 0$, it is exactly the opposite.

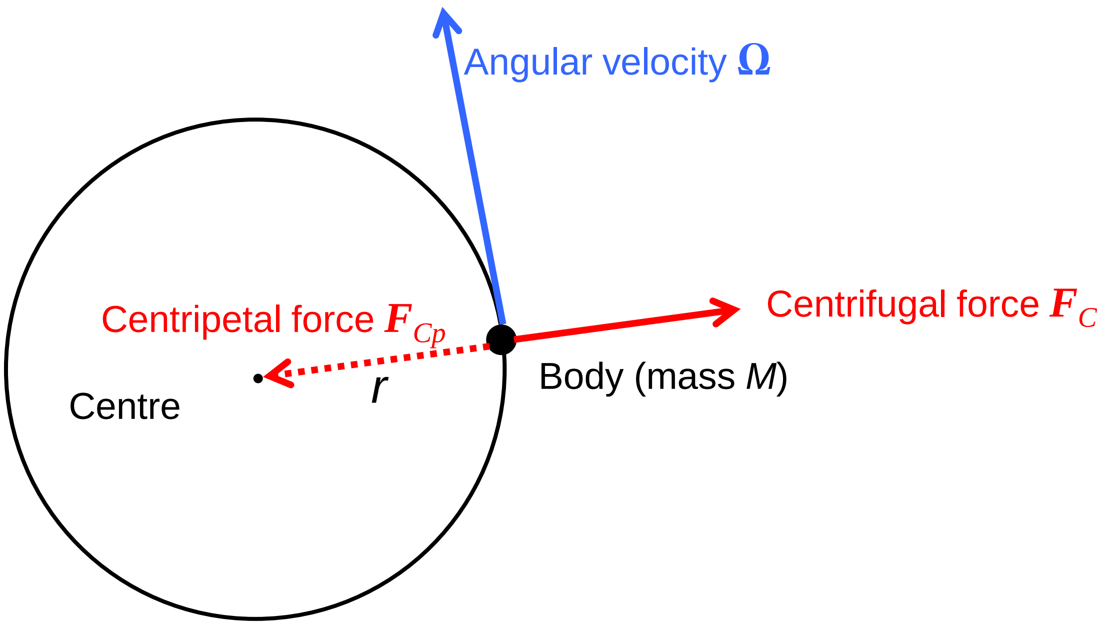
ζ : vorticity
f : Coriolis parameter

Centrifugal force – Centripetal force

- **Directed radially outward.**
- If no other forces are present the particle would accelerate outwards.
- Fictitious force, balanced by Centripetal force

$$\boldsymbol{F}_C = -M\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

$$\boldsymbol{F}_{Cp} = -\,\boldsymbol{F}_C$$



Rotating equations of motion

Substituting $\left(\frac{D\boldsymbol{u}_{in}}{Dt}\right)_{in}$ from Eq. 6-27 into Eq. 6-6 (inertial frame equation of motion) we derive in rotating frame, dropping subscript "rot":

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathcal{F} \qquad \text{(Eq. 6-6)}$$

$$\frac{D\mathbf{u}}{Dt} + \underbrace{\frac{1}{\rho}\nabla p}_{\text{Pressure gradient accel.}} + \underbrace{g\hat{\mathbf{z}}}_{\text{Gravi-tational accel.}} = \underbrace{-\,2\Omega \times \boldsymbol{u}}_{\text{Coriolis acceleration}} \underbrace{-\,\Omega \times \Omega \times \boldsymbol{r}}_{\text{Centrifugal acceleration}} + \underbrace{\mathcal{F}}_{\text{Friction acceleration}} \qquad \text{(Eq. 6-28)}$$

$\boldsymbol{u}$: velocity vector, *t:* time, *p*: pressure, g: gravity acceleration,$\hat{\mathbf{z}}$: unit vector in z−direction
Ω: rotation vector, $\mathcal{F}$: frictional force *per unit mass*

Centrifugal acceleration – modified gravitational potential

Combine the gradient of Centrifugal potential $-\Omega \times \Omega \times \boldsymbol{r} = \nabla\left(\frac{\Omega^2 \mathrm{r}^2}{2}\right)$ and Gravitational potential $\mathrm{g}\hat{\mathbf{z}} = \nabla(\mathrm{gz})$ in Eq. 6-28*:*

$$\frac{D\mathbf{u}}{Dt} + \underbrace{\frac{1}{\rho}\nabla p}_{\text{Pressure gradient accel.}} + \underbrace{\nabla\Phi}_{\text{Gravitational +Centrifugal accelerations}} = \underbrace{-\,2\Omega \times \boldsymbol{u}}_{\text{Coriolis acceleration}} + \underbrace{\mathcal{F}}_{\text{Friction acceleration}} \qquad \text{Eq. 6–29}$$

where: $\Phi = \mathrm{g}z - \frac{\Omega^2 \mathrm{r}^2}{2}$ Eq. 6–30

Φ(Greek "Phi") is a modified (by centrifugal accelerations) gravitational potential "measured" in the rotating frame.

…on the sphere

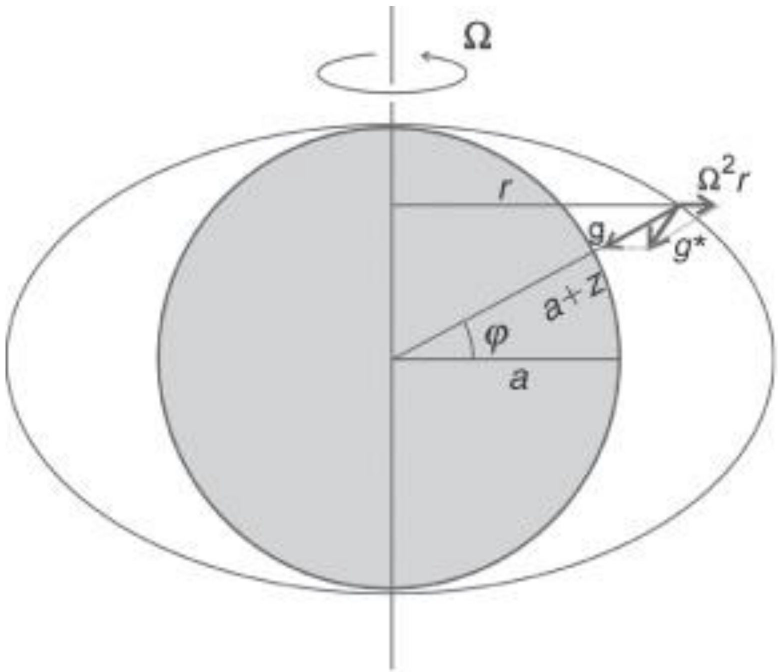


Figure 6.18: The centrifugal vector $\Omega \times \Omega \times r$ has magnitude $\Omega^2 r$, directed outward normal to the rotation axis. Gravity, g, points radially inward to the center of the Earth. Over geological time the surface of the Earth adjusts to make itself an equipotential surface—close to a reference ellipsoid—which is always perpendicular the the vector sum of $\Omega \times \Omega \times r$ and **g**. This vector sum is "measured" gravity: $\mathbf{g}* = -g\hat{\mathbf{z}} - \Omega \times \Omega \times r$.

"Shallow atmosphere" approx. ($a+z \simeq a$):

$r=(a+z)\cos\varphi \simeq a \cos\varphi$, hence Eq. 6-30 becomes:

$$\Phi = gz - \frac{\Omega^2 a^2 \cos^2\varphi}{2}$$

Definition of geopotential surfaces:

$$z^* = z + \frac{\Omega^2 a^2 \cos^2\varphi}{2g} \quad \text{(Eq. 6-40)}$$

$\Omega = 7.27 \times 10^{-5}\ s^{-1}$

$a = 6.37 \times 10^{6}$ m

→At the equator z^* *is* $\left(\frac{\Omega^2 a^2}{2g} \approx\right)$ 11 km, higher than at the Pole (z^*=z).

Marshall and Plumb (2008)

Equations of motion

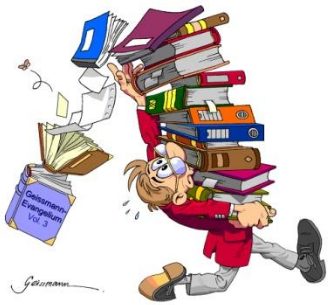
$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi + f\hat{\mathbf{z}} \times \boldsymbol{u} = \mathcal{F} \quad \text{(Eq. 6–43)}$$

- Fluid in a thin spherical shell on a rotating sphere, applying hydrostatic balance for vertical component and neglecting $\mathcal{F}_z$ compared with gravity:

$$\frac{D\mathrm{u}}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = \mathcal{F}_x \quad \text{(Eq. 6–44)}$$

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = \mathcal{F}_y$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + \mathrm{g} = 0$$



Take home message I+II

- Eulerian – Lagrangian derivations/ perspectives.
- 5 equations for the evolution of the fluid with 5 unknowns (u,v,w,p,T): *equations of motion (3), conservation of mass (1), thermodynamic equation (1)*
- Equations of motion on a *non-rotating fluid*: *Pressure gradient force, gravitational force and friction force act.* $\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathcal{F}$
- Equations of motion on a *rotating fluid*: *Pressure gradient force, modified gravitational potential (gravitational and centrifugal force), Coriolis force and friction force act.* $\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi + f\hat{\mathbf{z}} \times \boldsymbol{u} = \mathcal{F}$