GEF 2220: Dynamics

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Primitive equations

Momentum:

$$\frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w$$

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$$

Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \frac{dq}{dt}$$

Primitive equations

Six equations, six unknowns:

- (u, v, w) — velocities
- p — pressure
- ρ — density
- T — temperature

Primitive equations

Momentum equations $\quad\leftarrow\quad F = ma$

Continuity $\quad\leftrightarrow\quad \rho$

Thermodynamic energy equation $\quad\leftrightarrow\quad T$

Ideal gas law relates ρ, p and T

Prediction

Solve the equations numerically with weather models

Issues:

- Numerical resolution
- Vertical coordinate
- Small scale mixing
- Convection
- Clouds

Goal: forecasting

Dynamics

Solve a simplified set of equations

- Identify dominant balances
- Simplify the equations
- Obtain solutions (analytical, numerical)
- Look for similiarities with observations

Goal: *understanding* the atmosphere

Momentum equations

Take the x-momentum equation:

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial t^2} x = \frac{1}{\rho} \sum_i F_i$$

which is like:

$$a_x = \frac{1}{m} \sum_i F_i$$

Momentum equations

Two types of forces:

    1) Real   2) Apparent

Two ways to write the derivative:

    1) Lagrangian   2) Eulerian

Derivatives

Consider an air parcel, with temperature T

$$T = T(x, y, z, t)$$

The change in temperature, from the chain rule:

$$dT = \frac{\partial}{\partial t} T \, dt + \frac{\partial T}{\partial x} \, dx + \frac{\partial T}{\partial y} \, dy + \frac{\partial T}{\partial z} \, dz$$
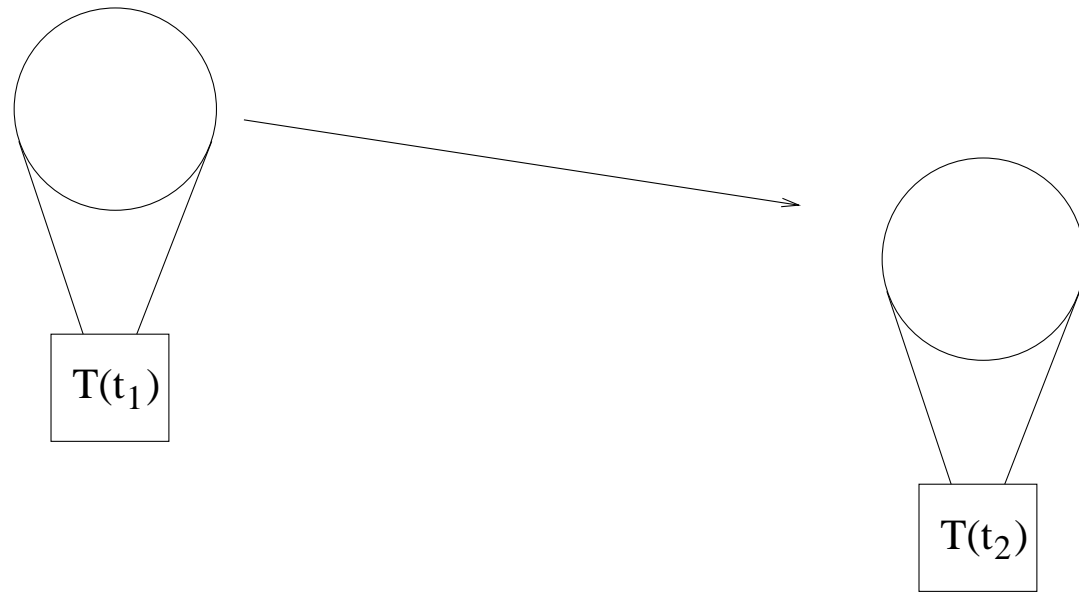
Derivatives

So:

$$\frac{dT}{dt} = \frac{\partial}{\partial t}T + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$= \frac{\partial}{\partial t}T + \vec{u}\cdot\nabla T$$
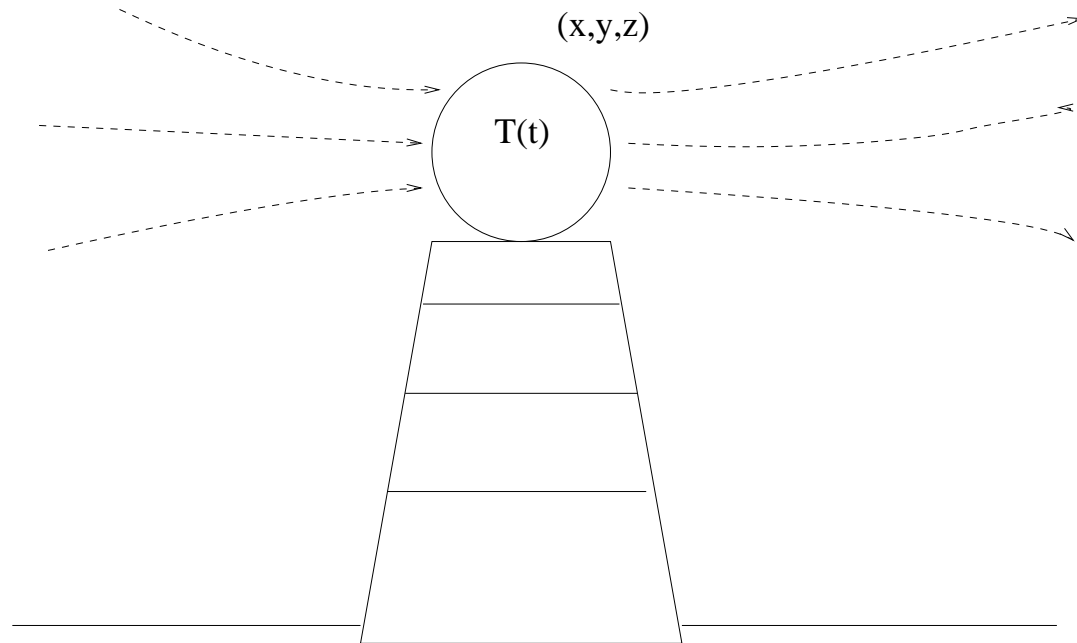
$\frac{d}{dt}$ is the "Lagrangian" derivative

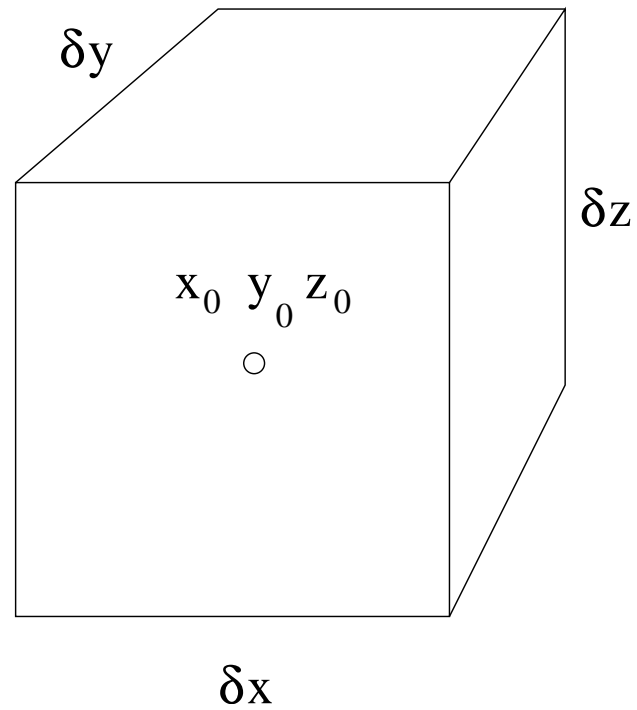$\frac{\partial}{\partial t} + \vec{u}\cdot\nabla$ is the "Eulerian" representation

Lagrangian

$T(t_1)$

$T(t_2)$

Eulerian

(x,y,z)

T(t)

Real forces

- Pressure gradients

- Gravity

- Friction

Pressure gradient

$$\delta V = \delta x \; \delta y \; \delta z$$

Pressure gradient

Use *Taylor series*:

$$G(x_0 + \delta x) = G(x_0) + \frac{\partial G}{\partial x}\,\delta x + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\,\delta x^2 + \dots$$

Pressure on the right side of the box:

$$p = p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2} + \dots$$

Pressure on left side of the box:

$$p = p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2} + \dots$$

Pressure gradient

The force on the right hand side (directed inwards):

$$p = -[p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

On left side:

$$p = [p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

So the net force is:

$$F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

Pressure gradient

The volume weighs:

$$m = \rho \, \delta x \, \delta y \, \delta z$$

So:

$$a_x = \frac{F_x}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

Same derivation for the y and z directions.

Momentum equations

Momentum:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \ldots$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \ldots$$

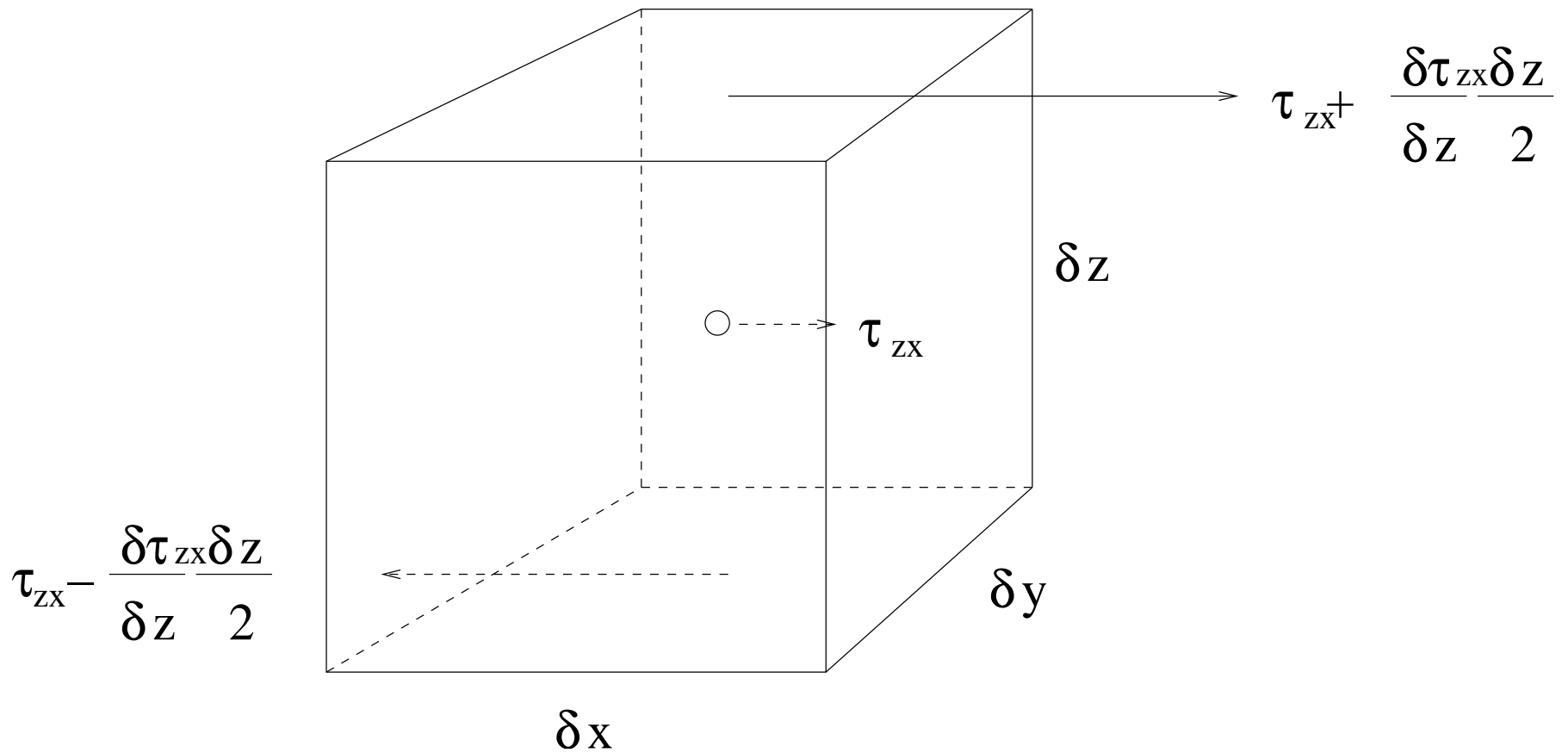$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p + \ldots$$

Gravity

Acts downward (toward the center of the earth):

$$a_z = \frac{F_z}{m} = -g$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

Friction

$$\tau_{zx} + \frac{\delta\tau_{zx}}{\delta z}\frac{\delta z}{2}$$

$$\delta z$$

$$\tau_{zx}$$

$$\tau_{zx} - \frac{\delta\tau_{zx}}{\delta z}\frac{\delta z}{2}$$

$$\delta y$$

$$\delta x$$

Friction

Net viscous force (stress $\times$ area) of the boundaries acting *on the fluid* :

$$(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \frac{\delta z}{2}) \, \delta x \, \delta y - (\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z} \frac{\delta z}{2}) \, \delta x \, \delta y$$

$$= \frac{\partial \tau_{zx}}{\partial z} \, \delta x \, \delta y \, \delta z$$

Divide by the mass of the box:

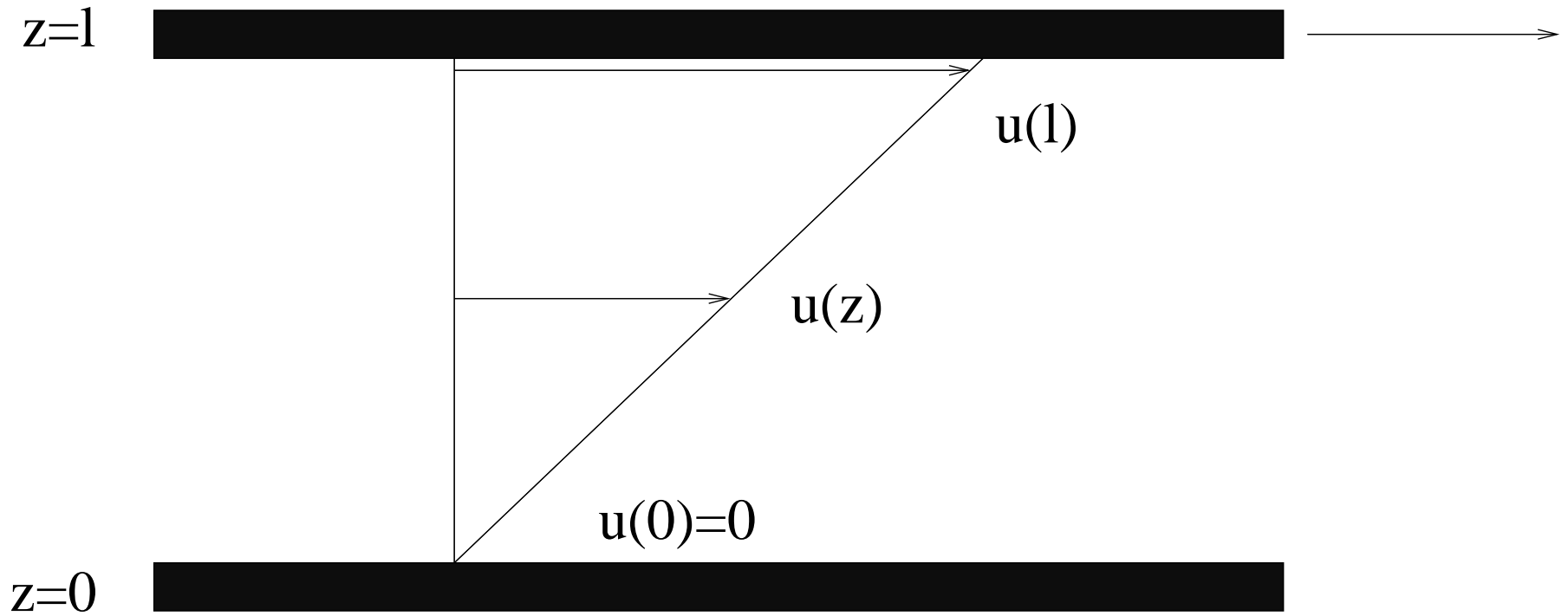$$F_{zx} = \frac{1}{\rho} \frac{\partial \tau_{zx}}{\partial z}$$

Friction

Similar derivations for τ_{zy}, τ_{xx}, ...

Applied to x-direction:

$$\frac{du}{dt} = \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

Friction

z=1

u(1)

u(z)

u(0)=0

z=0

Friction

Experiments show that force required to keep plate moving at a constant velocity is:

$$F_{zx} \propto \frac{u(l)A}{l} = \mu \frac{u(l)A}{l}$$

where μ is the *viscosity coefficient*

Friction

Each fluid layer exerts a stress on layer below it:

$$Fzx = \mu\left(\frac{u(z+\delta z)A}{\delta z} - \frac{u(z)A}{\delta z}\right) \approx \mu A\left(\frac{\delta u}{\delta z}\right)$$

As $\delta z \to 0$:

$$\tau_{zx} = \frac{F_{zx}}{A} = \mu\frac{\partial u}{\partial z}$$

Friction

So:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}(\mu\frac{\partial u}{\partial z})$$

If $\nu = \mu/\rho$ is constant:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \nu\frac{\partial^2}{\partial z^2}u$$

Friction

At sea level:

$$\nu = 1.46 \times 10^{-5} \ m^2/sec$$

Applied to x-direction:

$$\frac{du}{dt} = \nu \frac{\partial^2}{\partial z^2} u$$

• Friction acts to *diffuse momentum*, reducing the vertical shear.

Momenutum equations

With all the friction terms, have:

$$\frac{du}{dt} = \nu \left(\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u + \frac{\partial^2}{\partial z^2} u \right) = \nu \nabla^2 u$$

$$\frac{dv}{dt} = \nu \left(\frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v + \frac{\partial^2}{\partial z^2} v \right) = \nu \nabla^2 v$$

$$\frac{dw}{dt} = \nu \left(\frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w + \frac{\partial^2}{\partial z^2} w \right) = \nu \nabla^2 w$$

Friction

Mixing is due to molecular motion

Creates a *boundary layer*, near the earth's surface

But the molecular layer is only a few centimeters thick

• Molecular mixing *unimportant* for weather-scale motion

Planetary boundary layer

Away from the surface, momentum is mixed by *eddies*

Usually represent the mixing in the same way:

$$F_x = K_m \nabla^2 u$$

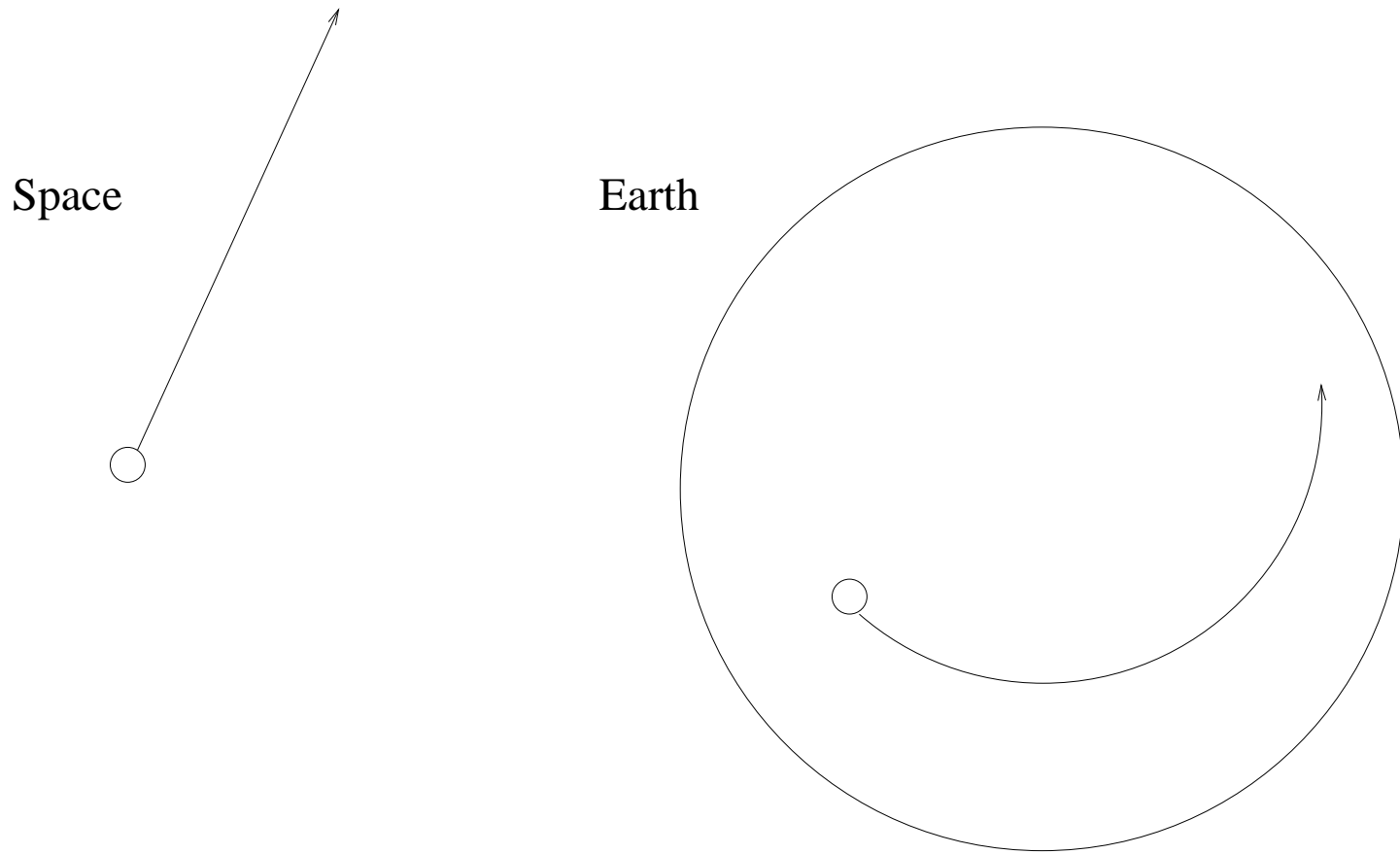where K_m is an *eddy viscosity*

$$K_m \gg \nu$$

Momentum equations

So far:

$$\frac{du}{dt} = \frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$
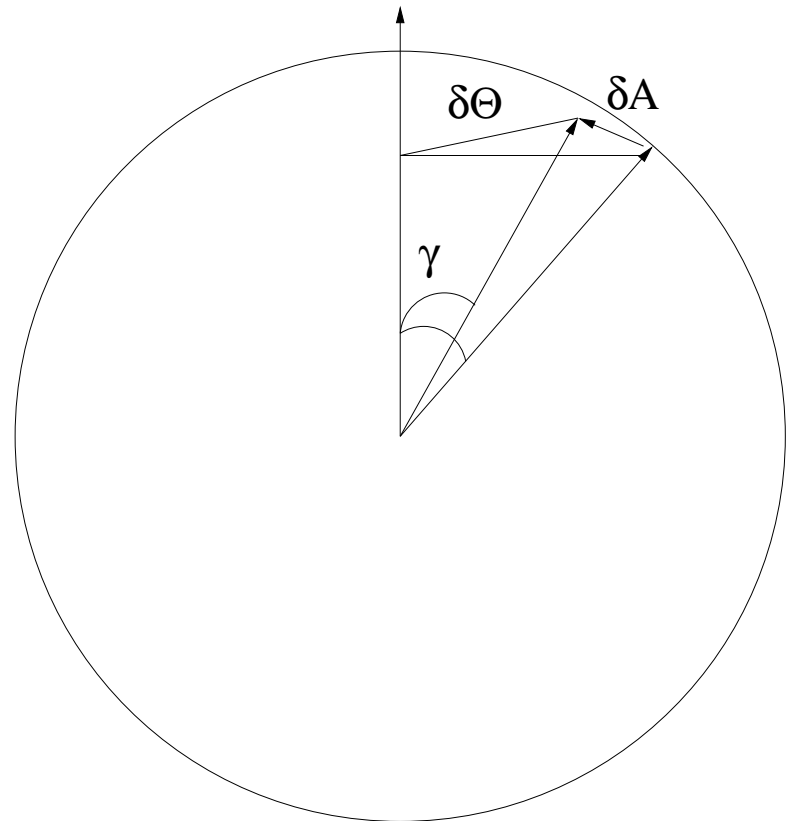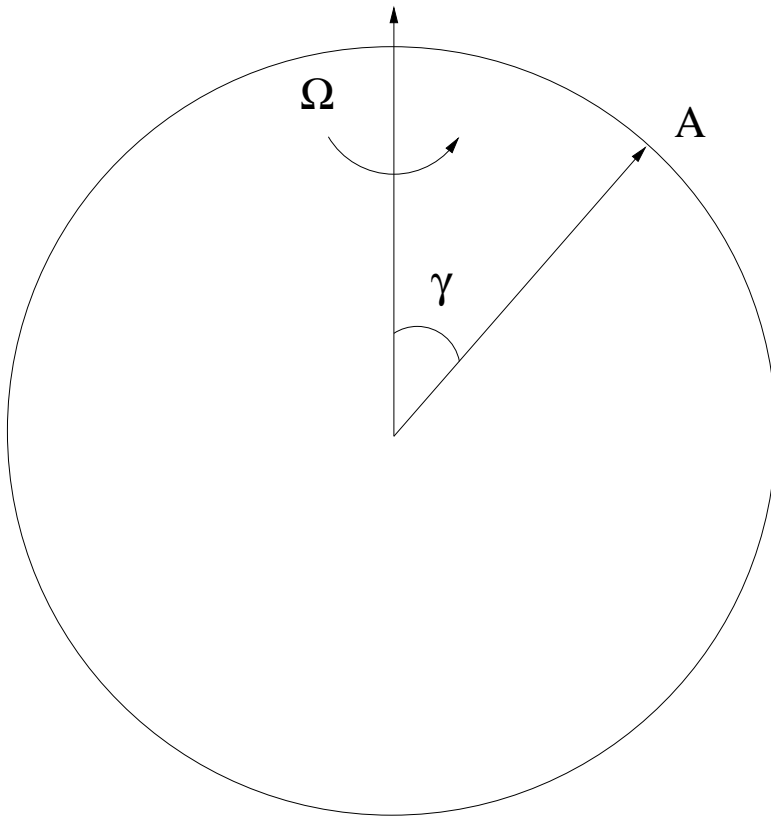
$$\frac{dv}{dt} = \frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\nabla^2 v$$

$$\frac{dw}{dt} = \frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w$$

Apparent forces

Space

Earth

Rotation

Rotation

$$\delta\Theta = \Omega\delta t$$

Assume $\Omega = const.$ (reasonable for the earth)

Change in A is δA, the arc-length:

$$\delta\vec{A} = |\vec{A}|sin(\gamma)\delta\Theta = \Omega|\vec{A}|sin(\gamma)\delta t = \left(\vec{\Omega} \times \vec{A}\right)\delta t$$

Rotation

So:

$$\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A}$$

This is the motion of a *fixed* vector. For a moving vector, we can show:

$$\left(\frac{d\vec{A}}{dt}\right)_F = \left(\frac{d\vec{A}}{dt}\right)_R + \vec{\Omega} \times \vec{A}$$

Rotation

If $\vec{A} = \vec{r}$, the position vector, then:

$$\left(\frac{d\vec{r}}{dt}\right)_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

So the acceleration is:

$$\left(\frac{d\vec{u}_F}{dt}\right)_F = \left(\frac{d\vec{u}_F}{dt}\right)_R + \vec{\Omega} \times \vec{u}_F = \left[\frac{d}{dt}(u_R + \vec{\Omega} \times \vec{r})\right]_R + \vec{\Omega} \times \vec{u}_F$$

$$= \left(\frac{d\vec{u}_R}{dt}\right)_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Rotation

Rearranging:

$$(\frac{d\vec{u}_R}{dt})_R = (\frac{d\vec{u}_F}{dt})_F - 2\vec{\Omega} \times \vec{u}_R - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Two additional terms:

- Coriolis acceleration $\rightarrow$ $-2\vec{\Omega} \times \vec{u}_R$
- Centrifugal acceleration $\rightarrow$ $-\vec{\Omega} \times \vec{\Omega} \times \vec{r}$
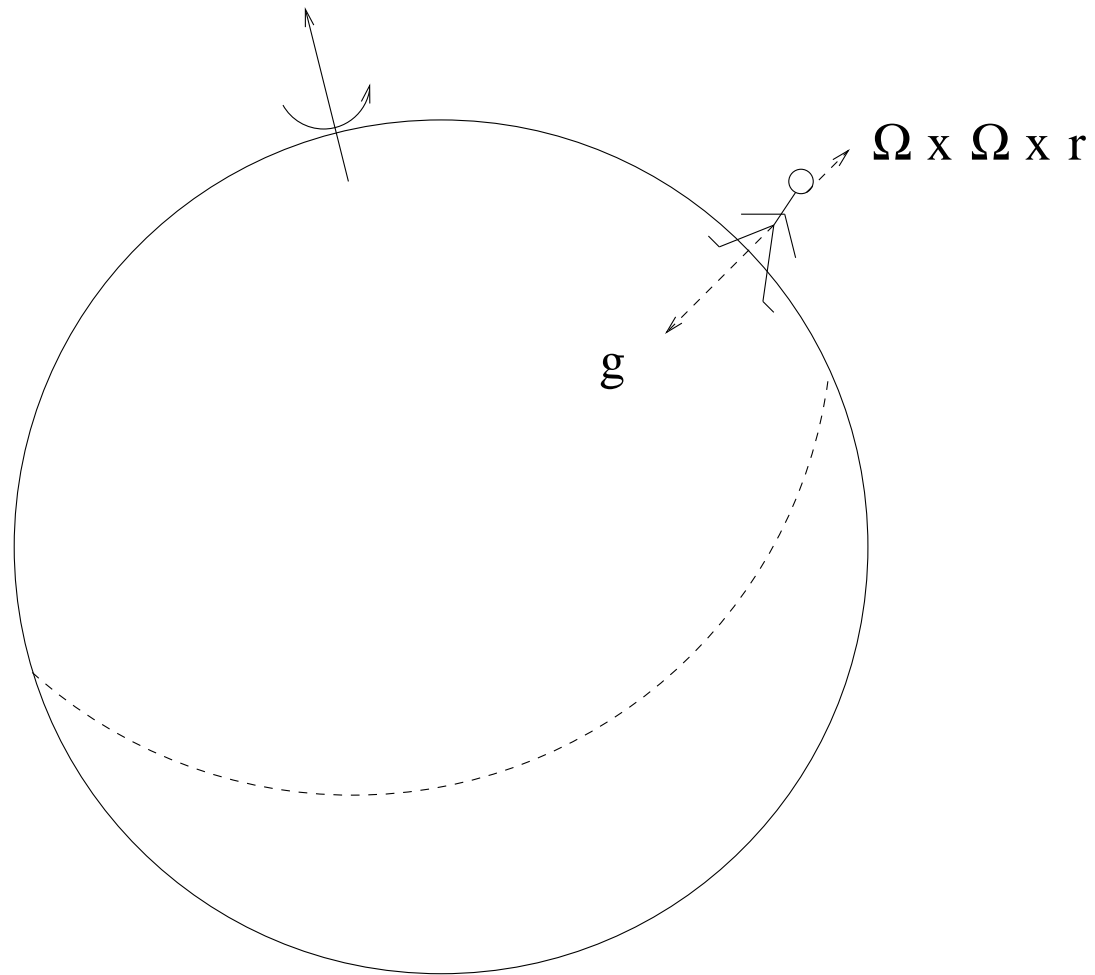
Centrifugal force

Is the negative of centripetal acceleration

Is the outward force we feel on a carousel

If spinning too fast, we would fly off the earth

If not, modifies the effective gravity

Centrifugal force

$\Omega \times \Omega \times r$

g

Centrifugal force

Absorb into gravity:

$$g' = g - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Example: What is the centrifugal force for a parcel at the Equator?

Centrifugal force

At the equator, we have:

$$-\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\Omega \times (\Omega r) = \Omega^2 r$$

with:

$$r_e = 6.378 \times 10^6 \ m$$

and:

$$\Omega = \frac{2\pi}{3600(24)} \ sec$$

Centrifugal force

So:

$$\Omega^2 r_e = 0.034 \ m/sec^2$$

This is much smaller than $g = 9.8 \ m^2/sec$

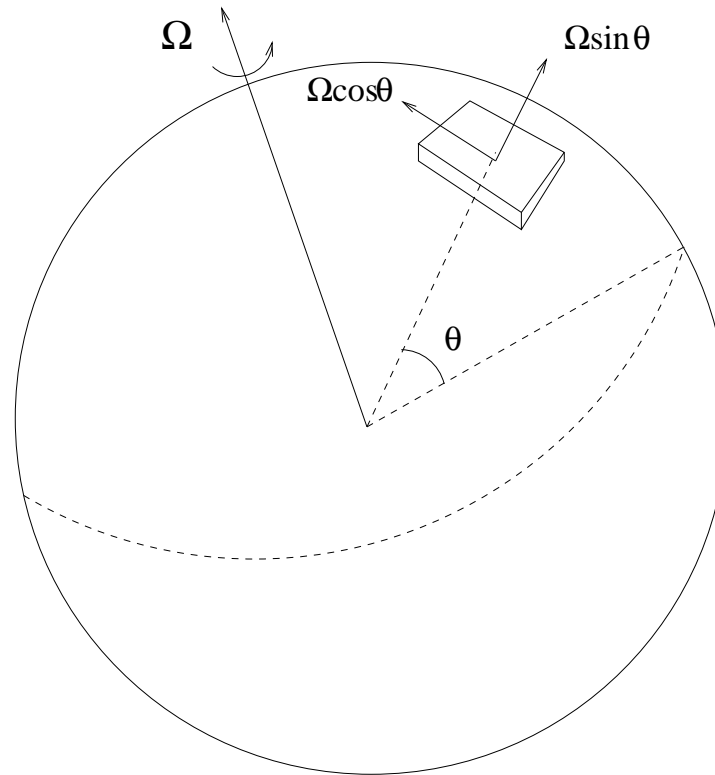• Only a minor change to absorb into g'

Cartesian coordinates

Earth is spherical, so should use spherical coordinates

We will primarily use *Cartesian* coordinates

- Simplifies the math
- The extra terms are usually *unimportant* at weather scales

Cartesian coordinates

Ω

$\Omega\sin\theta$

$\Omega\cos\theta$

θ

Coriolis force

The rotation vector projects onto the local vertical and meridional directions:

$$2\vec{\Omega} = 2\Omega cos\theta \ \hat{j} + 2\Omega sin\theta \ \hat{k} \equiv f_y \ \hat{j} + f_z \ \hat{k}$$

So:

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, v, w)$$

$$= -(f_y w - f_z v, f_z u, -f_y u)$$

Momentum equations

Move Coriolis terms to the LHS:

$$\frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Coriolis force

Example: What is the Coriolis force on a parcel moving eastward at $10 \ m/sec$ at 45 N?

We have:

$$f_y = 2\Omega cos(45) = (7.292 \times 10^{-5})(0.7071)$$

$$= 5.142 \times 10^{-5} \ sec^{-1} = f_z$$

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, 0, 0) = -f_z u \ \hat{j} + f_y u \hat{k}$$

$$= (0, -5.142 \times 10^{-4}, 5.142 \times 10^{-4}) \ m/sec^2$$

Coriolis force

So the force acts to the south and upwards

Upward acceleration is negligible compared to gravity $(g = 9.8\ m/sec^2)$, so has little effect

Horizontal acceleration is to the *south*

- Coriolis acceleration is most important in the horizontal
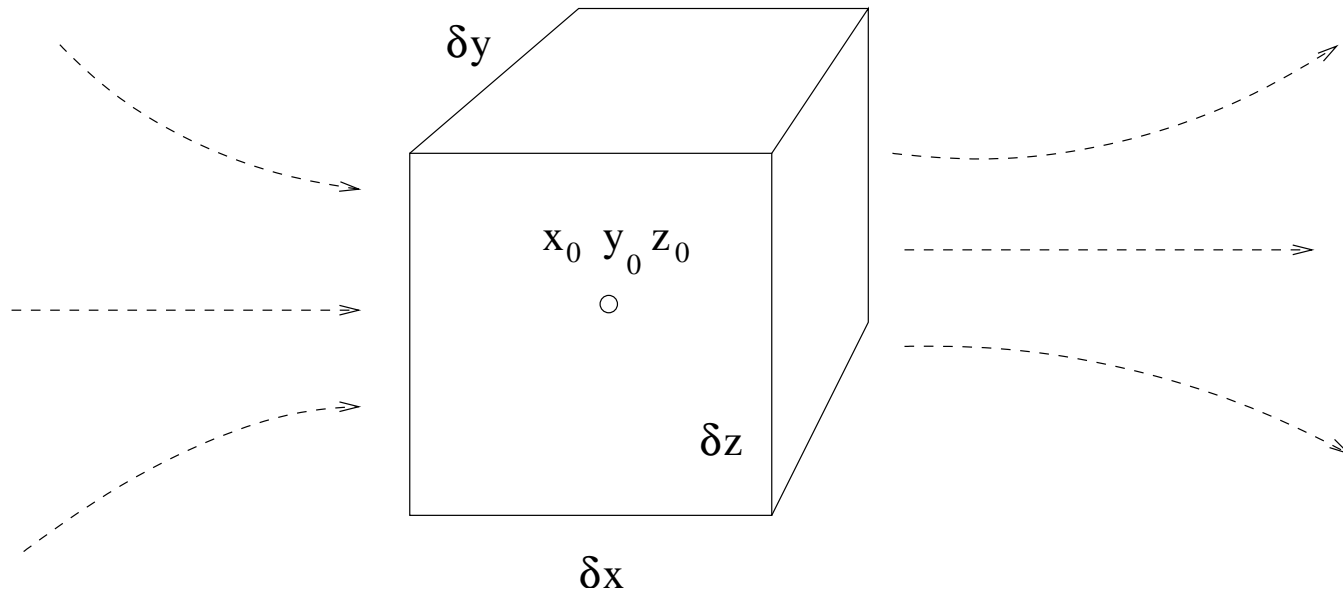- Acts *to the right* in the Northern Hemisphere

Coriolis force

In the Southern hemisphere, $\theta < 0$. Same problem, at 45 S:

$$f_y = 2\Omega cos(-45) = -5.142 \times 10^{-5} \; sec^{-1} = -f_z$$

$$-2\vec{\Omega} \times \vec{u} = (0, +5.142 \times 10^{-4}, -5.142 \times 10^{-4}) \; m/sec^2$$

Acceleration is to the north, to the *left* of the parcel velocity.

Continuity

Continuity

Density flux through the left side:

$$[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Through the right side:

$$[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Continuity

So the net rate of change in mass is:

$$\frac{\partial}{\partial t}(\rho\,\partial x\,\partial y\,\partial z) = [\rho u - \frac{\partial}{\partial x}(\rho u]\frac{\partial x}{2}) \,\partial y\,\partial z$$

$$-[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\partial y\,\partial z = -\frac{\partial}{\partial x}(\rho u)\partial x\,\partial y\,\partial z$$

The volume δV is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u)$$

Continuity

Taking the other sides of the box:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{u})$$

Can rewrite:

$$\nabla \cdot (\rho \vec{u}) = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho .$$

So:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho (\nabla \cdot \vec{u}) = 0$$

Continuity

Can also derive using a Lagrangian box

As the box moves, it conserves it mass. So:

$$\frac{1}{\partial M}\frac{d}{dt}(\partial M) = \frac{1}{\rho \delta V}\frac{d}{dt}(\rho \delta V) = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d\delta V}{dt} = 0$$

Expand the volume term:

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{1}{\delta x}\frac{\partial \delta x}{dt} + \frac{1}{\delta y}\frac{\partial \delta y}{dt} + \frac{1}{\delta z}\frac{\partial \delta z}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Continuity

So:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{u} = 0$$

Same as before

The change in density is proportional to the velocity *divergence*.

If the volume changes, the density changes to keep the mass constant.

Ideal Gas Law

Five of the equations are *prognostic*: they describe the time evolution of fields.

But one "diagnostic" relation.

Relates the density, pressure and temperature

Ideal Gas Law

For dry air:

$$p = \rho R T$$

where

$$R = 287 \ J kg^{-1} K^{-1}$$

Moist air

Law moist air, can write (Chp. 3):

$$p = \rho R T_v$$

where the *virtual temperature* is:

$$T_v \equiv \frac{T}{1 - e/p(1 - \epsilon)}$$

$$\epsilon \equiv \frac{R_d}{R_v} = 0.622$$

Hereafter, we neglect moisture effects

Primitive equations

Continuity:

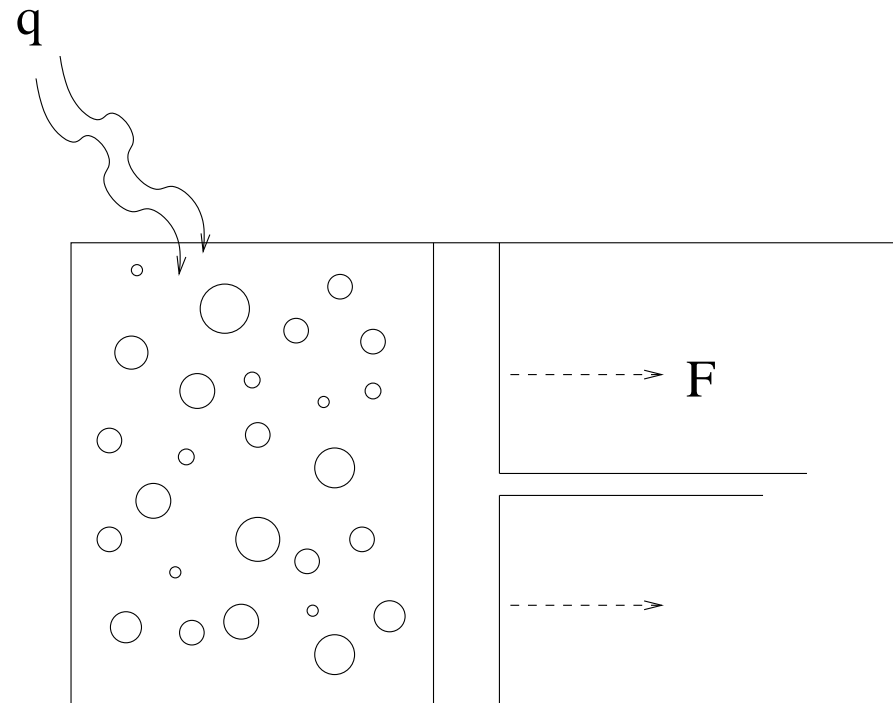$$\frac{\partial}{\partial t}\rho + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$$

Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \frac{dq}{dt}$$

Thermodynamic equation

First law of thermodynamics

Change in internal energy = heat added - work done:

$$dq - dw = de$$

Work is done by expanding against external forces:

$$dw = F dx = p A dx = p dV$$

If $dV > 0$, the volume is doing the work

First law of thermodynamics

If volume has a unit mass, then:

$$\rho V = 1$$

so:

$$dV = d(\frac{1}{\rho}) = d\alpha$$

where α is the *specific volume*. So:

$$dq = p\,d\alpha + de$$

First law of thermodynamics

Add heat to the volume, the temperature rises. The *specific heat* determines how much. At constant volume:

$$c_v \equiv \frac{dq}{dT}\Big|_v$$

This is also the change in internal energy:

$$c_v = \frac{de}{dT}\Big|_v$$

First Law of thermodynamics

Joule's Law: e only depends on temperature for an ideal gas. So even if V changes:

$$c_v = \frac{de}{dT}$$

Result is the First Law:

$$dq = c_v dT + p \, d\alpha$$

First law of thermodynamics

At constant pressure:

$$c_p \equiv \frac{dq}{dT}\big|_p$$

Volume expands keeping p constant. Requires more heat to raise the temperature. Rewrite the first law:

$$dq = c_v dT + d(p\alpha) - \alpha dp$$

First law of thermodynamics

The ideal gas law is:

$$p = \rho RT = \alpha^{-1} RT$$

So:

$$d(p\alpha) = RdT$$

Thus:

$$dq = (c_v + R)dT - \alpha dp$$

First law of thermodynamics

At constant pressure, $dp = 0$, so:

$$\frac{dq}{dT}\Big|_p = c_p = c_v + R$$

So the specific heat at constant pressure is *greater* than at constant volume. For dry air:

$$c_v = 717 J kg^{-1} K^{-1}, \qquad c_p = 1004 J kg^{-1} K^{-1}$$

so:

$$R = 287 \ J kg^{-1} K^{-1}$$

First law of thermodynamics

So First Law can also be written:

$$dq = c_p dT - \alpha dp$$

Obtain the thermodynamic equation by dividing by dt:

$$\frac{dq}{dt} = c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}$$

Basic balances

Not all terms in the momentum equations are equally important for weather systems.

Will simplify the equations by identifying primary balances (throw out as many terms as possible).

Begin with horizontal momentum equations.

Scaling

General technique: *scale* equations using estimates of the various parameters. Take the x-momentum equation, without friction:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \quad fU \qquad \frac{\triangle_H P}{\rho L}$$

Scaling

Now use typical values. Length scales:

$$L \approx 10^6 m, \quad D \approx 10^4 m$$

Horizontal scale is 1000 km, the *synoptic scale*. Large scale motion (not tornadoes).

Velocities:

$$U \approx V \approx 10 m/sec, \quad W \approx 1 cm/sec$$

Winds are *quasi-horizontal*

Scaling

Pressure term:

$$\triangle_H P/\rho \approx 10^3 m^2/sec^2$$

Note typical *horizontal* difference.

Time scale:

$$T = L/U \approx 10^5 sec$$

Called an "advective time scale" (≈ 1 day).

Scaling

Coriolis terms:

$$(f_y, f_z) = 2\Omega(cos\theta, sin\theta)$$

with

$$\Omega = 2\pi(86400)^{-1}sec^{-1}$$

Assume at mid-latitudes:

$$f_y \approx f_z = 2\Omega sin(45) \approx 10^{-4}sec^{-1}$$

Scaling

Plug in:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \qquad fU \qquad \frac{\triangle_H P}{\rho L}$$

$$10^{-4} \qquad 10^{-4} \qquad 10^{-4} \qquad 10^{-5} \qquad 10^{-6} \qquad 10^{-3} \qquad 10^{-3}$$

Geostrophy

Keeping only the 10^{-3} terms:

$$f_z v = \frac{1}{\rho} \frac{\partial}{\partial x} p$$

$$f_z u = -\frac{1}{\rho} \frac{\partial}{\partial y} p$$

These are the *geostrophic* relations.

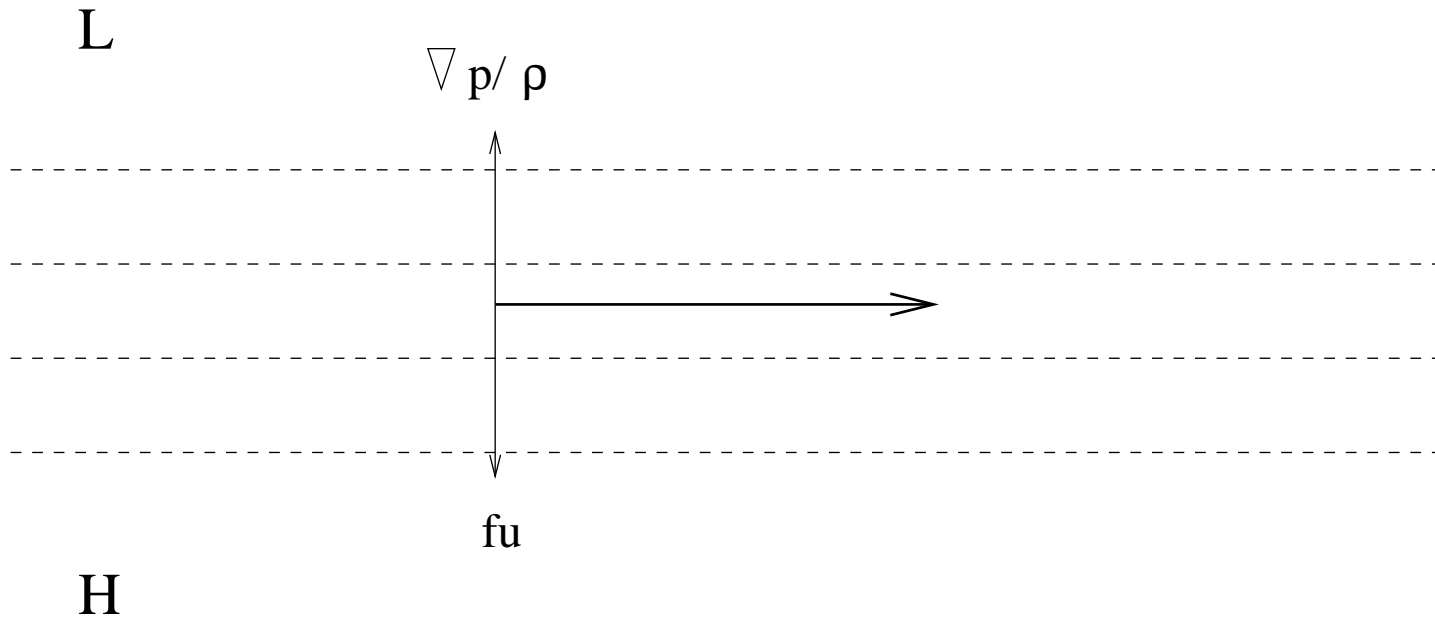Balance between the pressure gradient and Coriolis force.

Geostrophy

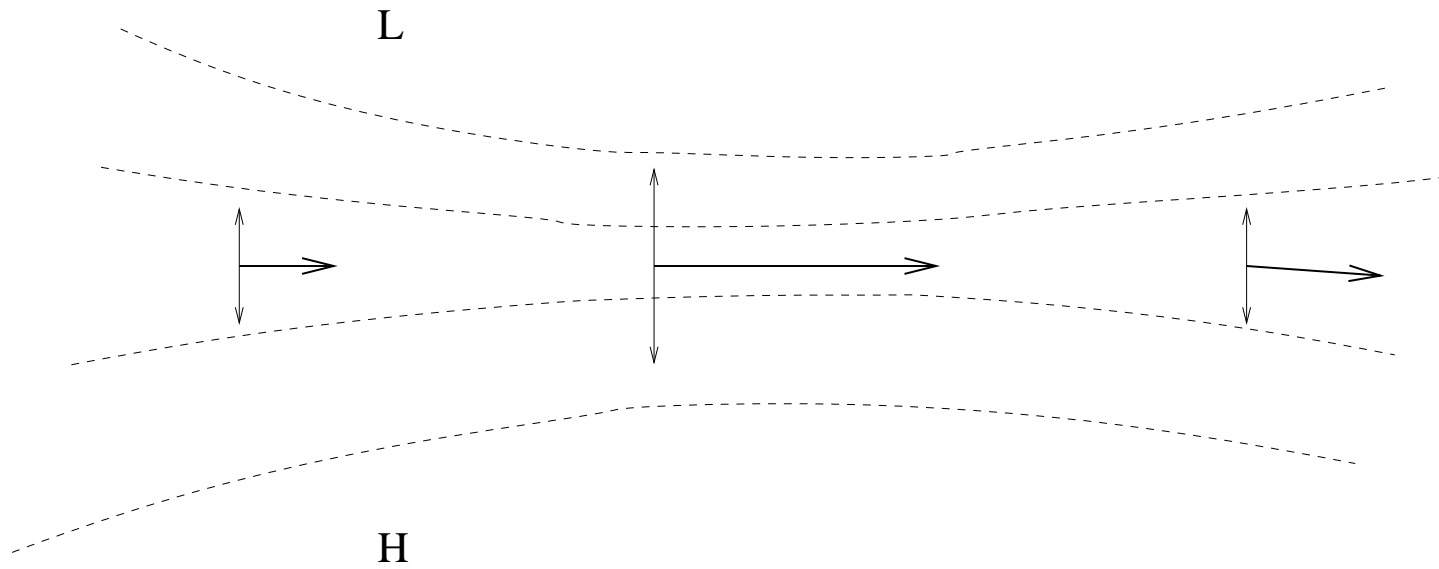Fundamental momentum balance at synoptic scales

- Low pressure to left of the wind in Northern Hemisphere
- Low pressure to *right* in Southern Hemisphere

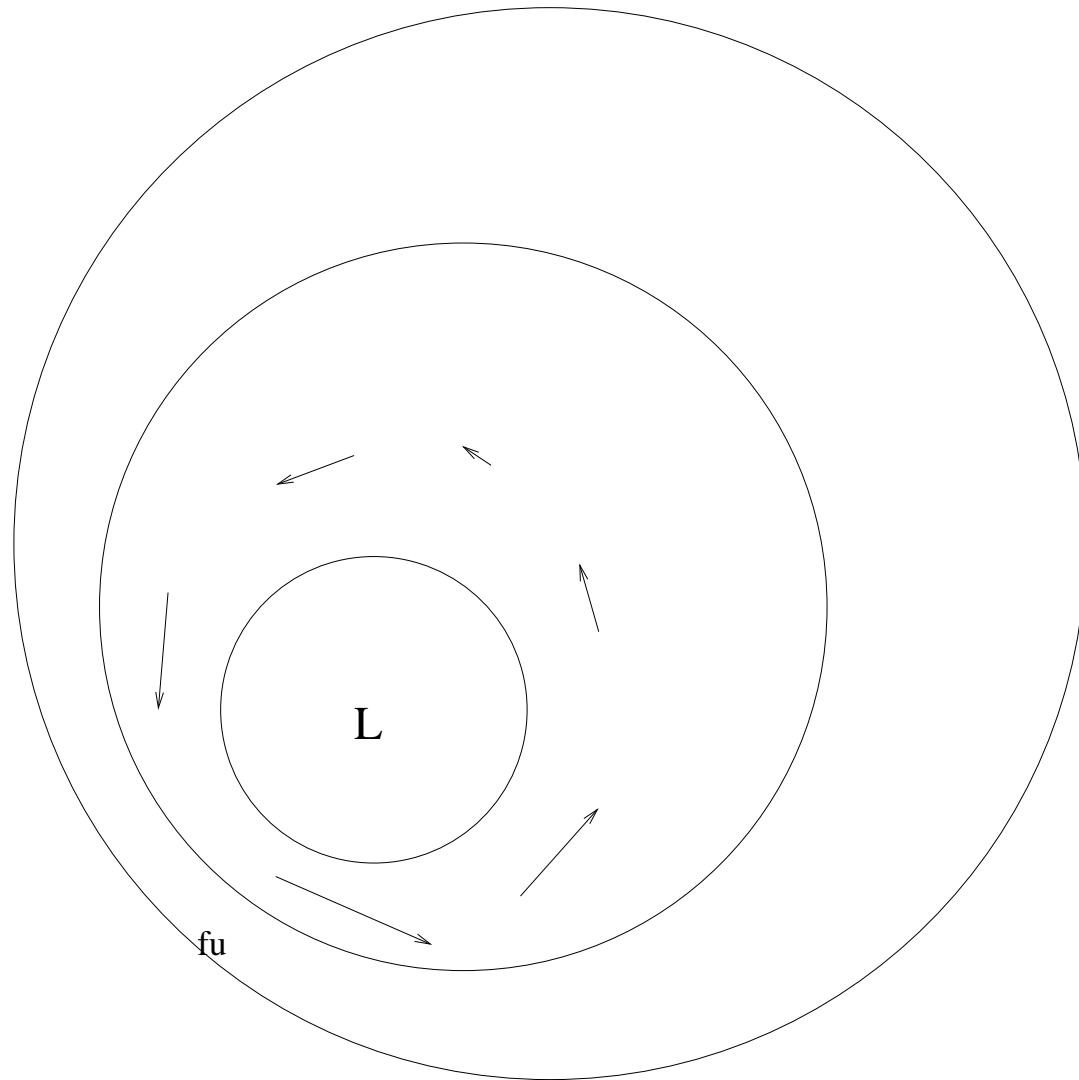But balance fails at equator, because $f_z = 2\Omega sin(0) = 0$

Geostrophy

L

$\nabla p / \rho$

fu

H

Geostrophy

L

H

Geostrophy

L

fu

Geostrophy

Example: What is the pressure gradient required at the earth's surface at 45 N to maintain a geostrophic wind of 30 m/sec?

$$f_z = 2\Omega sin(45) = 1.414*(7.27 \times 10^{-5}) \ sec^{-1} = 1.03 \times 10^{-4} \ sec^{-1}$$

$$\frac{\partial p}{\partial l} = \rho_0 f_z u = (1.2 \ kg/m^3)(1.03 \times 10^{-4} \ sec^{-1})(30 \ m/sec)$$

$$= 3.7 \times 10^{-3} \ N/m^3 = .37 \ kPa/100km$$

Geostrophy

Is a *diagnostic relation*

• Given the pressure, can calculate the horizontal velocities

But geostrophy cannot be used for *prediction*

Approximate horizontal momentum
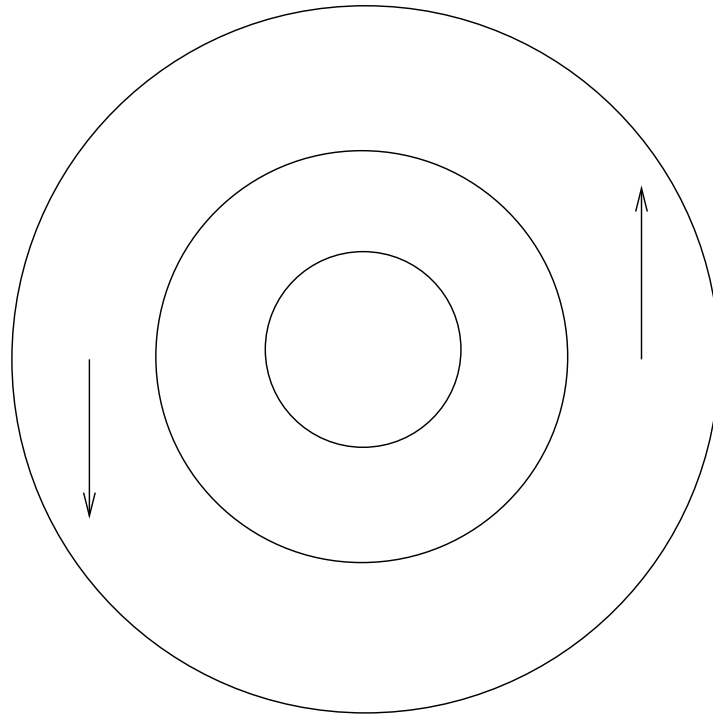
Must also retain the 10^{-4} terms:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

The equations are *quasi-horizontal*: neglect vertical motion

Other momentum balances

Geostrophy most important balance at synoptic scales. But other balances possible. Consider purely circular flow:

Other momentum balances

Use cylindrical coordinates. Acceleration in radial direction is:

$$\frac{\partial}{\partial t} u_r + (u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta}) u_r - \frac{u_\theta^2}{r} - f u_\theta = \frac{d}{dt} u_r - \frac{u_\theta^2}{r} - f u_\theta = -\frac{1}{\rho} \frac{\partial}{\partial r} p$$

u_θ^2 is the *cyclostrophic* term– related to centripetal acceleration.

Other momentum balances

If constant circulation, $\frac{d}{dt}u = 0$. Then:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

Scaling:

$$\frac{U^2}{R} \qquad fU \qquad \frac{\triangle_H P}{\rho R}$$

Other momentum balances

So ratio of the first and second terms is:

$$\frac{U}{fR} \equiv \epsilon$$

This is the *Rossby number*

Other momentum balances

If $\epsilon \ll 1$, recover the geostrophic balance:

$$f u_\theta = \frac{1}{\rho} \frac{\partial}{\partial r} p$$

Other momentum balances

If $\epsilon \gg 1$, the first term dominates. Happens at smaller scales and with stronger winds.

A typical tornado at mid-latitudes has:

$$U \approx 30m/s, \quad f = 10^{-4}sec^{-1}, \quad R \approx 300m$$

so that:

$$\epsilon = 1000$$

Cyclostrophic wind balance

Have:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

or:

$$u_\theta = \pm(\frac{r}{\rho}\frac{\partial}{\partial r}p)^{1/2}$$

- Rotation does not enter.
- Circulation can go *either way*.
- The balance for bathtub vortices.

Inertial oscillations

Third possibility: there is no radial pressure gradient:

$$\frac{u_\theta^2}{r} + f u_\theta = 0$$

then:

$$u_\theta = -fr$$

Rotation is clockwise or *anticyclonic* in the Northern Hemisphere.

Inertial oscillations

The time for a fluid parcel to complete a loop is:

$$\frac{2\pi r}{u_\theta} = \frac{2\pi}{f} = \frac{0.5 \, day}{|sin\theta|} \ ,$$

Called the "inertial period".

Inertial oscillations are seen in the surface layer of the ocean, but are rarer in the atmosphere.

Gradient wind balance

Fourth possibility: all terms are important ($\epsilon = O|1|$).

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r} p$$

Solve using the quadratic formula:

$$u_\theta = -\frac{1}{2} f r \pm \frac{1}{2}\left(f^2 r^2 + \frac{4r}{\rho}\frac{\partial}{\partial r} p\right)^{1/2}$$

$$= -\frac{1}{2} f r \pm \frac{1}{2}\left(f^2 r^2 + 4\, r\, f\, u_g\right)^{1/2}$$

Gradient wind balance

If $u_g < 0$ (anticyclone), require:

$$|u_g| < \frac{fr}{4}$$

If $u_g > 0$ (cyclone), there is *no limit*

Wind gradients are *stronger* in cyclones than in anticyclones

Gradient wind balance

Alternately:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p = f u_g$$

Then:

$$\frac{u_g}{u_\theta} = 1 + \frac{u_\theta}{fr} \approx 1 + \epsilon$$

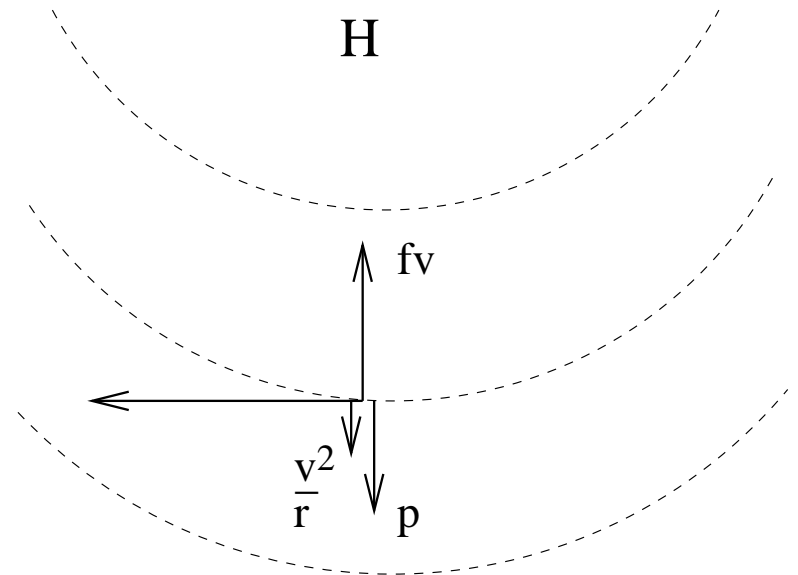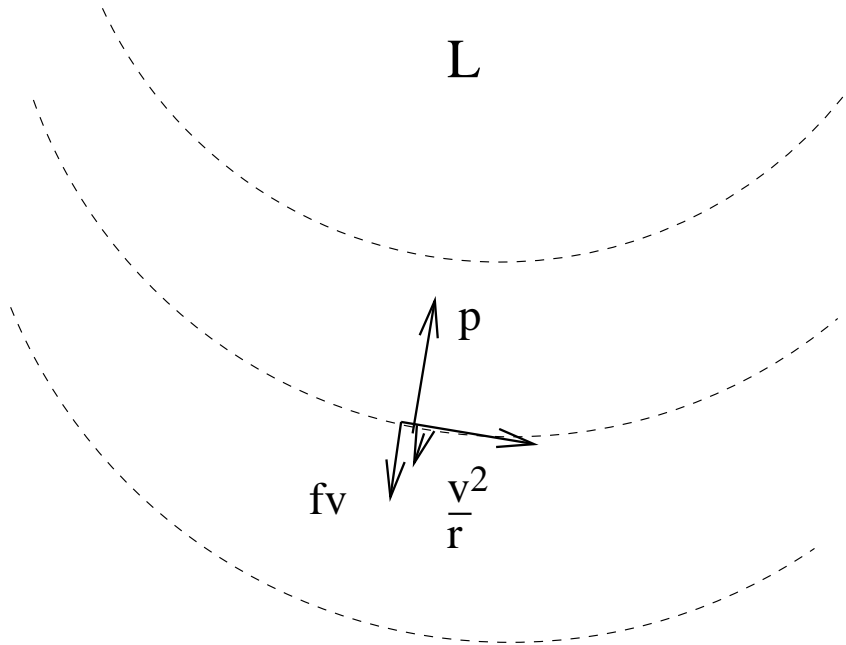If $\epsilon = 0.1$, the gradient wind estimate differs by 10 %.

Gradient wind balance

At low latitudes, where ϵ can be 1-10, the gradient wind estimate is more accurate.

Geostrophy is *symmetric to sign changes*: no difference between cyclones and anticyclones

The gradient wind balance is not symmetric to sign change. Stronger cyclones.

Gradient wind balance

L

p

fv $\dfrac{v^2}{r}$
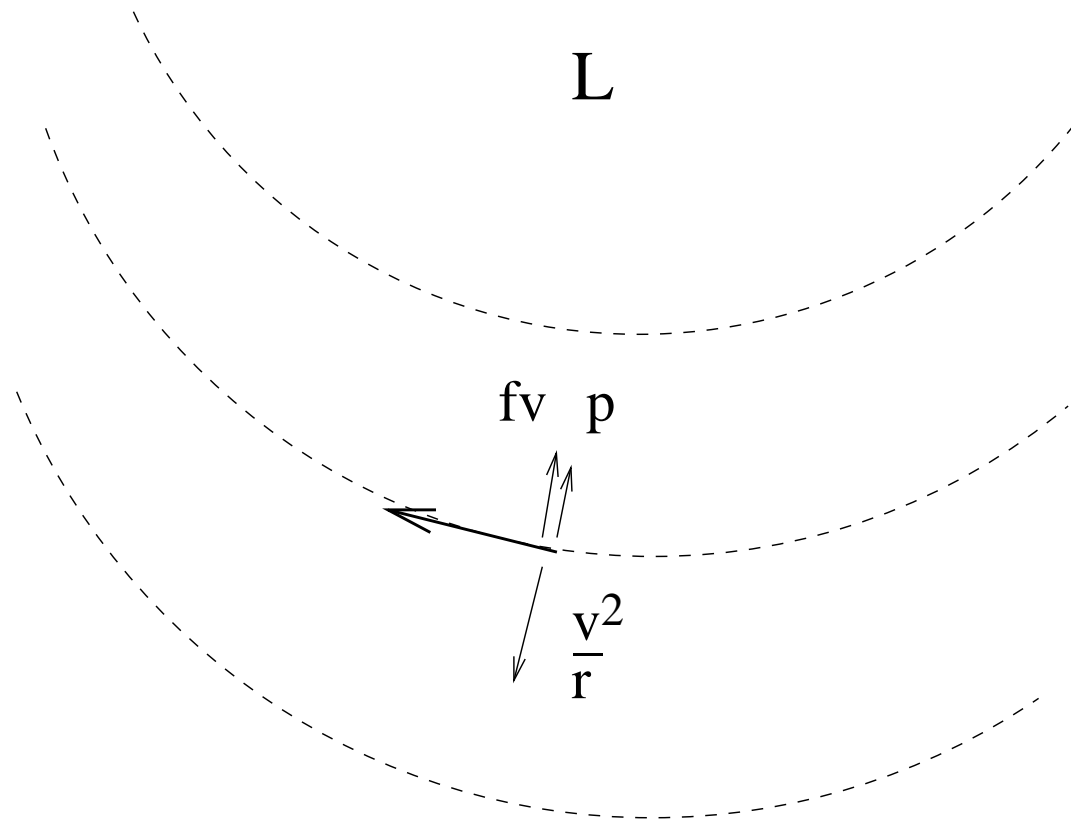
H

fv

$\dfrac{v^2}{r}$ p

Gradient wind balance

Geostrophic motion has Coriolis and pressure gradient forces opposed.

If cyclostrophic term large enough, gradient wind vortices can have the have pressure gradient and Coriolis forces in the same direction.

Gradient wind balance

L

fv   p

$\dfrac{v^2}{r}$

Gradient wind balance

An *anomalous low*: low pressure with clockwise flow

Usually only found near the equator

Hydrostatic balance

Now scale the vertical momentum equation

Must scale:

$$\frac{1}{\rho}\frac{\partial}{\partial z}p$$

Vertical variation of pressure much greater than the horizontal variation:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Hydrostatic balance

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad fU \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \qquad 10 \qquad 10$$

Static atmosphere

Dominant balance between the vertical pressure gradient and gravity

However, same balance *if there no motion at all*!

Set all velocities to zero in the equations of motion:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p = \frac{1}{\rho}\frac{\partial}{\partial y}p = \frac{\partial}{\partial t}\rho = \frac{dT}{dt} = 0$$

Static atmosphere

Only equations left:

$$\frac{\partial}{\partial z} p = -\rho g$$

The *hydrostatic balance*. Also:

$$p = \rho R T$$

Equations describe a non-moving atmosphere

Static atmosphere

Integrate the hydrostatic relation:

$$p(z) = \int_z^\infty \rho g \, dz \ .$$

The pressure at any point is equal to the weight of air above it. Sea level pressure is:

$$p(0) = 101.325 \ kPa \ (1013.25 mb)$$

The average weight per square meter of the entire atmospheric column(!)

Static atmosphere

Say the temperature was constant (*isothermal*):

$$\frac{\partial}{\partial z}p = -\frac{pg}{RT}$$

This implies:

$$ln(p) = -\frac{gz}{RT}$$

Static atmosphere

So that:

$$p = p_0 \, e^{-z/H}$$

where the "scale height" is:

$$H \equiv \frac{RT}{g}$$

Pressure decays exponentially with height

Scaling

Static hydrostatic balance not interesting for weather.
Separate the pressure and density into static and non-static
(moving) components:

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t)$$

$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$$

Assume:

$$|p'| \ll |p_0|, \quad |\rho'| \ll |\rho_0|$$

Scaling

Then:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(p_0 + p') - g$$

$$\approx -\frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}(p_0 + p') - g$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + \left(\frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$\rightarrow$ Neglect $(\rho' p')$

Scaling

Use these terms in the vertical momentum equation

But how to scale?

Vertical variation of the perturbation pressure comparable to the horizontal perturbation:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1}m/sec^2$$

Scaling

Also:

$$|\rho'| \approx 0.001|\rho_0|$$

So:

$$\frac{\rho'}{\rho_0} g \approx 10^{-1} m/sec^2$$

Scaling

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \quad 10^{-1} \quad 10^{-1}$$

Hydrostatic perturbations

Dominant balance still hydrostatic, but with perturbations:

$$\frac{\partial}{\partial z} p' = -\rho' g$$

Vertical accelerations unimportant at synoptic scales.

But lost the vertical velocity! Must calculate w indirectly.

Coriolis parameter

So all terms with f_y are unimportant

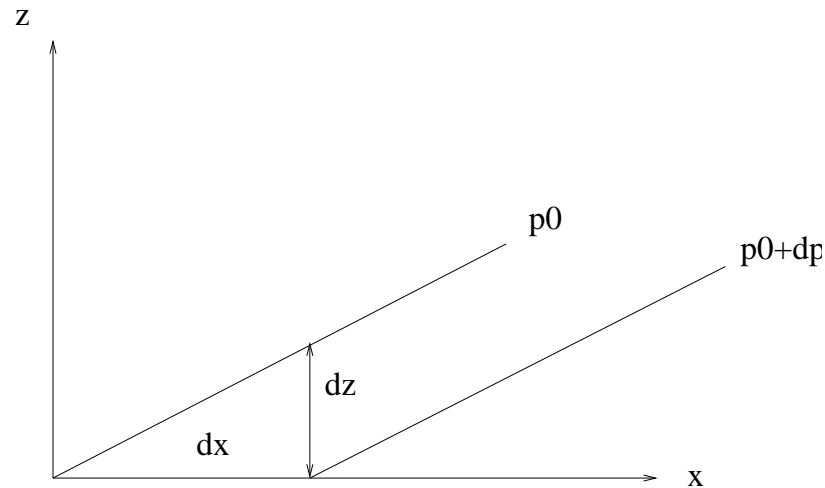From now on, neglect f_y and write f_z simply as f:

$$f \equiv 2\Omega sin(\theta)$$

f_y only important near the equator

Pressure coordinates

Can use the hydrostatic balance to simplify equations

Constant *pressure surfaces* (in two dimensions):

Pressure coordinates

$$dp = \frac{\partial p}{\partial x}\, dx + \frac{\partial p}{\partial z}\, dz = 0$$

Substitute hydrostatic relation:

$$dp = \frac{\partial p}{\partial x}\, dx - \rho g\, dz = 0$$

So:

$$\frac{\partial p}{\partial x}\Big|_z = \rho g \frac{dz}{dx}\Big|_p \equiv \rho\, \frac{\partial \Phi}{\partial x}\Big|_p$$
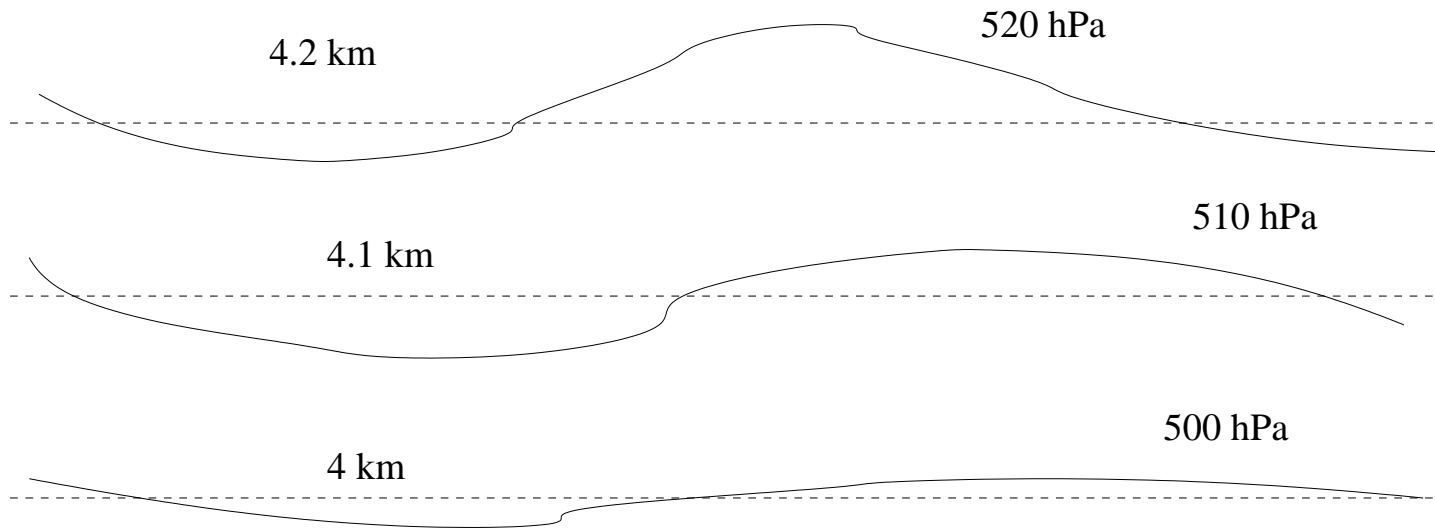
Geopotential

where:

$$\Phi \equiv \int_0^z g\,dz$$

Instead of pressure at a certain height, think:

Height of a certain pressure field

Geopotential

4.2 km

520 hPa

4.1 km

510 hPa

4 km

500 hPa

Geostrophy

Removes density from the momentum equation!

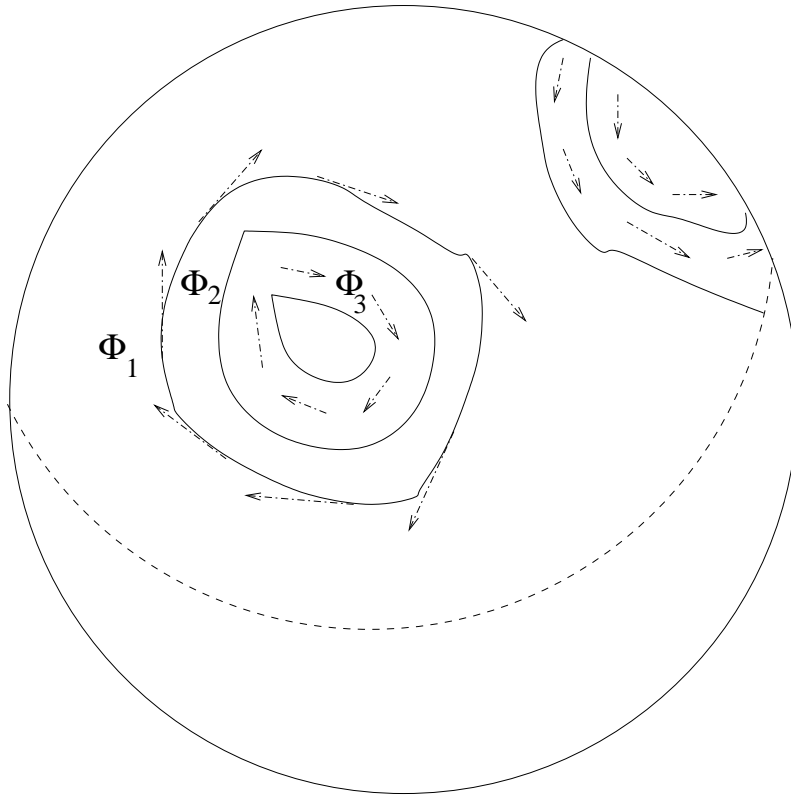$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x}$$

Now the geostrophic balance is:

$$fv = \frac{\partial}{\partial x}\Phi$$

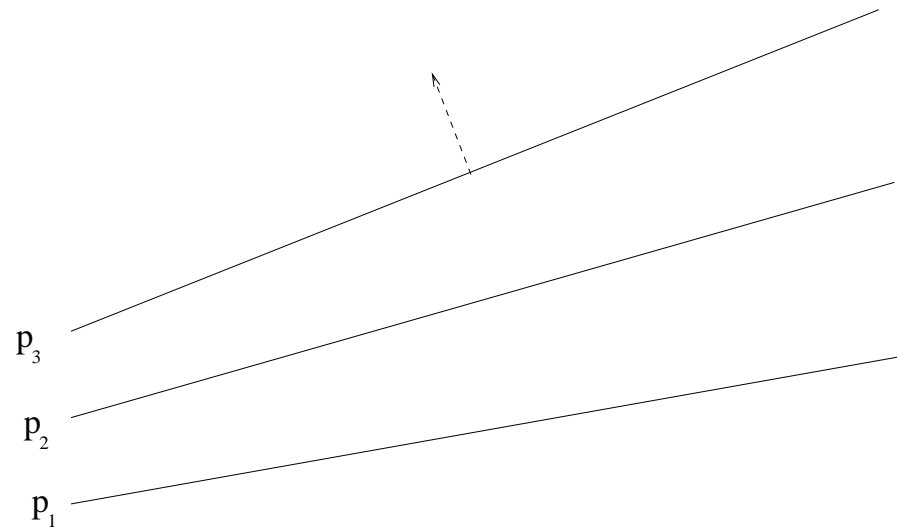$$fu = -\frac{\partial}{\partial y}\Phi$$

Geostrophy

500 hPa

Φ_1 Φ_2 Φ_3

Vertical velocities

Different vertical velocities:

$$w = \frac{dz}{dt} \quad \rightarrow \quad \omega = \frac{dp}{dt}$$

p_3

p_2

p_1

Geopotential

Lagrangian derivative is now:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

Continuity

Lagrangian box:

$$\delta V = \delta x \, \delta y \, \delta z = -\delta x \, \delta y \, \frac{\delta p}{\rho g}$$

with a mass:

$$\rho \delta V = -\delta x \, \delta y \, \delta p / g$$

Continuity

Conservation of mass:

$$\frac{1}{\delta M}\frac{d}{dt}\delta M = \frac{g}{\delta x \delta y \delta p}\frac{d}{dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$$

Rearrange:

$$\frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0$$

Continuity

Let $\delta \to 0$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

The flow is *incompressible* in pressure coordinates

Much simpler to work with

Diagnosing vertical motion

Integrating the continuity equation:

$$\omega = -\int_{p*}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\,dp$$

If the top of the atmosphere, $p* = 0$, so:

$$\omega = -\int_{0}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\,dp$$

Vertical motion

How does ω relate to the actual vertical velocity?

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p + w\frac{\partial}{\partial z}p$$

Using the hydrostatic relation:

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p - \rho g w$$

For geostrophic motion:

$$u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p = -\frac{1}{\rho f}\frac{\partial}{\partial y}p(\frac{\partial}{\partial x}p) + \frac{1}{\rho f}\frac{\partial}{\partial x}p(\frac{\partial}{\partial y}p) = 0$$

Vertical motion

So

$$\omega \approx \frac{\partial}{\partial t}p - \rho g w$$

Also:

$$\frac{\partial}{\partial t}p \approx 10 hPa/day$$

$$\rho g w \approx \left(1.2 kg/m^3\right)\left(9.8 m/sec^2\right)(0.01 m/sec) \approx 100 hPa/day$$
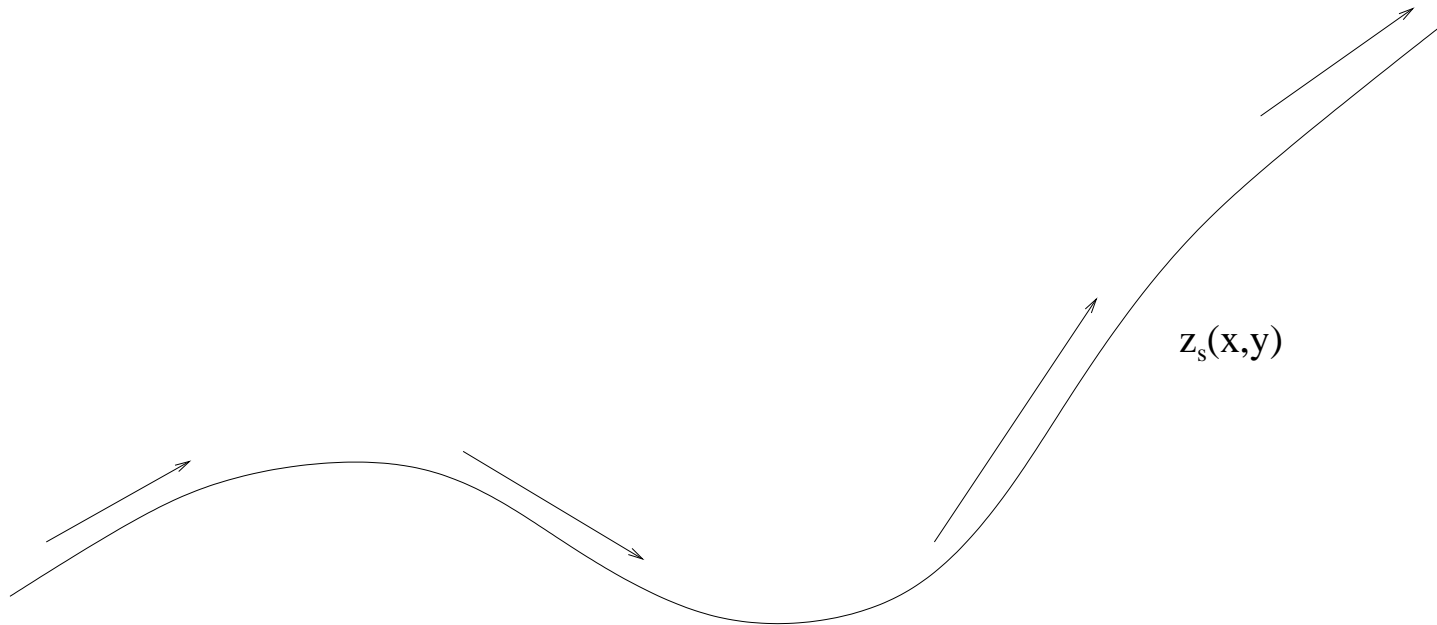
Vertical motion

So:

$$\omega \approx -\rho g w$$

This is accurate within 10 % in the mid-troposphere

In the lowest 1-2 km, topography alters the balances

At the surface:

$$w_s = u \frac{\partial}{\partial x} z_s + v \frac{\partial}{\partial y} z_s$$

Vertical motion

$z_s(x,y)$

Thermal wind

Geostrophy tells us what the velocities are if we know the geopotential on a pressure surface

What about the velocities on other pressure surfaces?

Need to know the velocity *shear*

Shear is determined by the thermal wind relation

Thermal wind

Can use geostrophy to calculate the shear between two pressure surfaces:

$$v_g(p_1) - v_g(p_0) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \equiv \frac{g}{f}\frac{\partial}{\partial x}Z_{10}$$

and:

$$u_g(p_1) - u_g(p_0) = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \equiv -\frac{g}{f}\frac{\partial}{\partial y}Z_{10}$$

Thermal wind

where:

$$Z_{10} = \frac{1}{g}\left(\Phi_1 - \Phi_0\right)$$

is the layer *thickness* between p_0 and p_1.

- Shear proportional to gradients of layer thickness

Thermal wind

Alternate expression.

From the definition of the geopotential:

$$d\Phi = g\,dz = -\frac{dp}{\rho}$$

With the ideal gas law:

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p}$$

Thermal wind

Now take the derivative wrt pressure of the geostrophic relation:

$$\frac{\partial}{\partial p}\left(fv_g = \frac{\partial \Phi}{\partial x}\right)$$

But:

$$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

So:

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

Thermal wind

Or:

$$\frac{\partial v_g}{\partial \ln(p)} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

also:

$$\frac{\partial u_g}{\partial \ln(p)} = \frac{R}{f}\frac{\partial T}{\partial y}$$

- Shear proportional to temperature gradient on p-surface

Thermal wind

If we know the velocity at p_0, can calculate it at p_1

Integrate between two pressure levels:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} \frac{\partial T}{\partial x} \, d\ln(p)$$

$$= -\frac{R}{f} \frac{\partial}{\partial x} \int_{p_0}^{p_1} T \, d\ln(p)$$

Mean temperature

Define the *mean temperature* in the layer bounded by p_0 and p_1:

$$\overline{T} \equiv \frac{\int_{p_0}^{p_1} T \, d(lnp)}{\int_{p_0}^{p_1} d(lnp)} = \frac{\int_{p_0}^{p_1} T \, d(lnp)}{ln(\frac{p_1}{p_0})}$$

Then:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f} \, ln(\frac{p_1}{p_0}) \frac{\partial \overline{T}}{\partial x}$$

Thermal wind

From before:

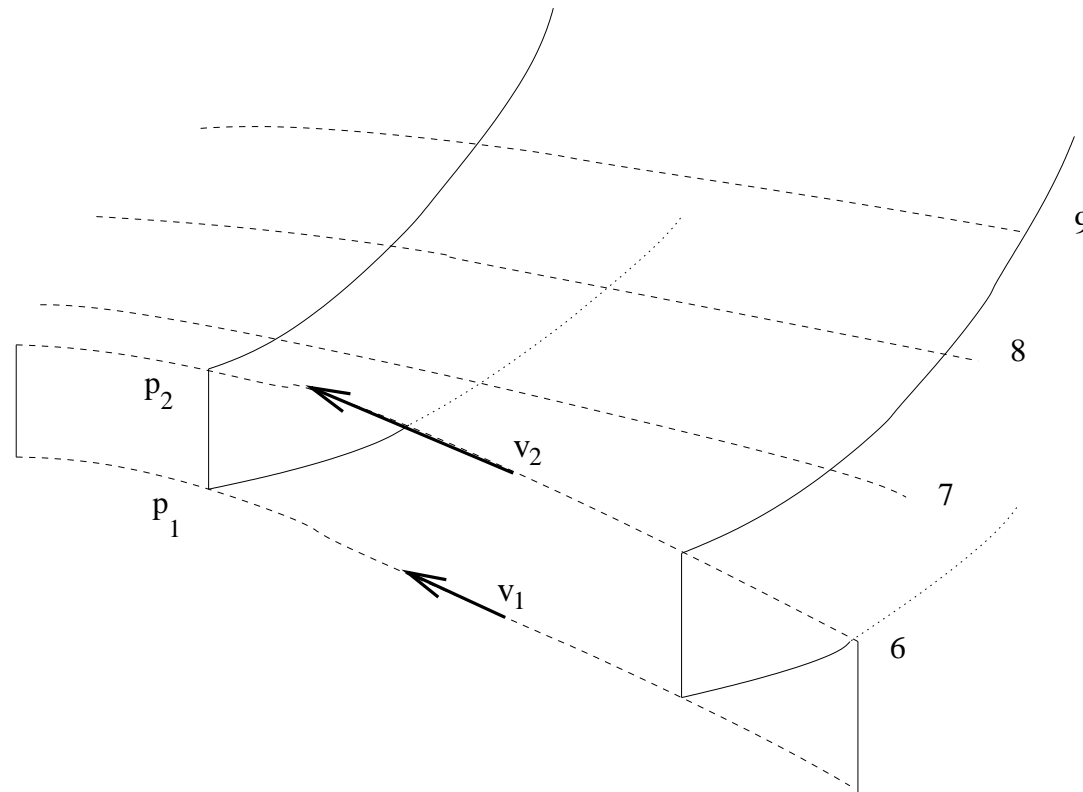$$v_g(p_1) - v_g(p_0) = \frac{g}{f} \frac{\partial}{\partial x} Z_{10}$$

so:

$$Z_{10} = \frac{R}{g} \overline{T} \, ln(\frac{p_1}{p_0})$$

- Layer thickness *proportional to its mean temperature*

Layer thickness

Barotropic atmosphere

What if temperature constant on all pressure surfaces?

Then $\nabla T = 0 \quad \rightarrow \quad$ *no vertical shear*

Belocities have the same direction and magnitude with height

All layers have *equal thickness*

Stacked like pancakes

Equivalent barotropic

If temperature and geopotential contours are parallel, thermal wind is parallel to the wind itself

Wind changes magnitude, but *not direction* with height

Geostrophic wind *increases* with height if

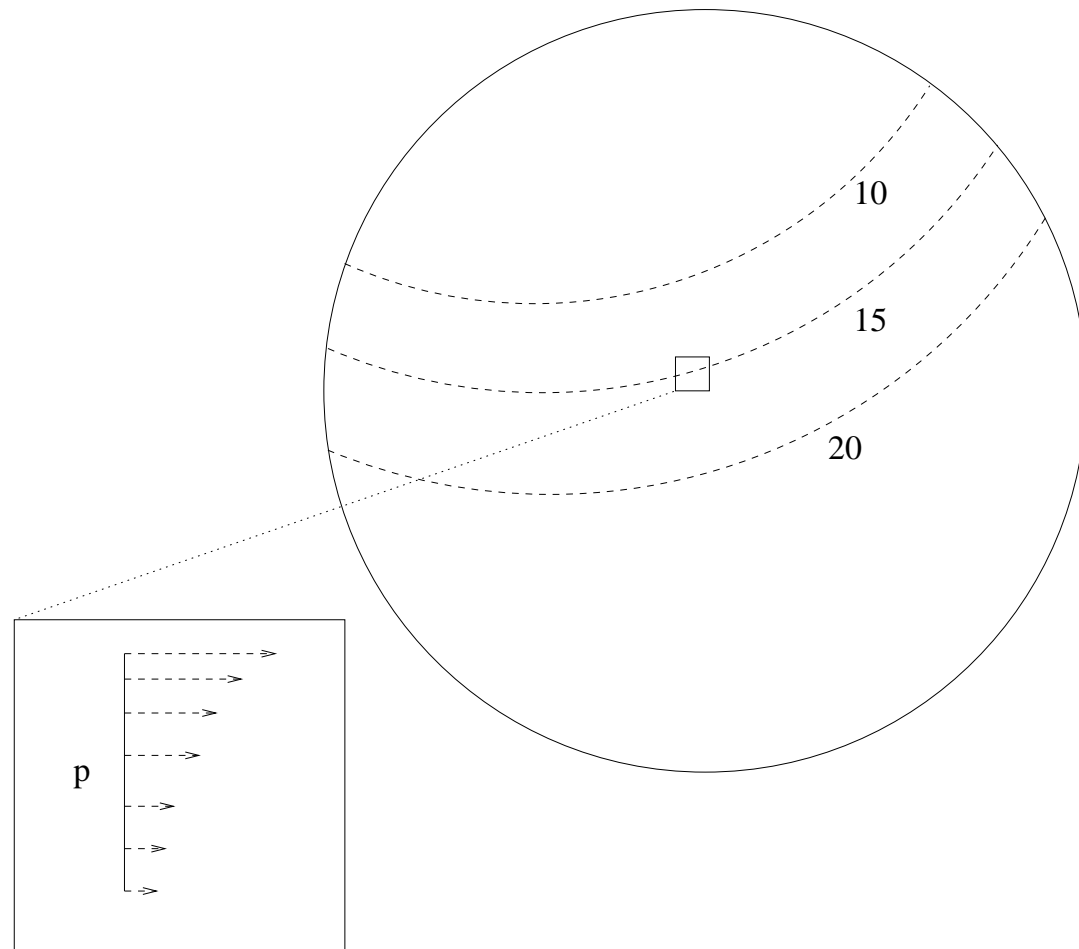- Warm high pressure
- Cold low pressure

Equivalent barotropic

Earth: zonal-average temperature decreases with latitude

Poles are colder than equator

$\rightarrow$ the winds increase with height

Jet stream

10

15

20

p

Thermal wind

Example: At 30N, the zonally-averaged temperature gradient is $0.75\ Kdeg^{-1}$, and the average wind is zero at the earth's surface. What is the mean zonal wind at the level of the jet stream ($250\ hPa$)?

$$u_g(p_1) - u_g(p_0) = u_g(p_1) = \frac{R}{f}\ ln(\frac{p_1}{p_0})\ \frac{\partial \overline{T}}{\partial y}$$

$$u_g(250) = \frac{287}{2\Omega sin(30)}\ ln(\frac{250}{1000})\ (-\frac{0.75}{1.11 \times 10^5\ m}) = 36.8\ m/sec$$
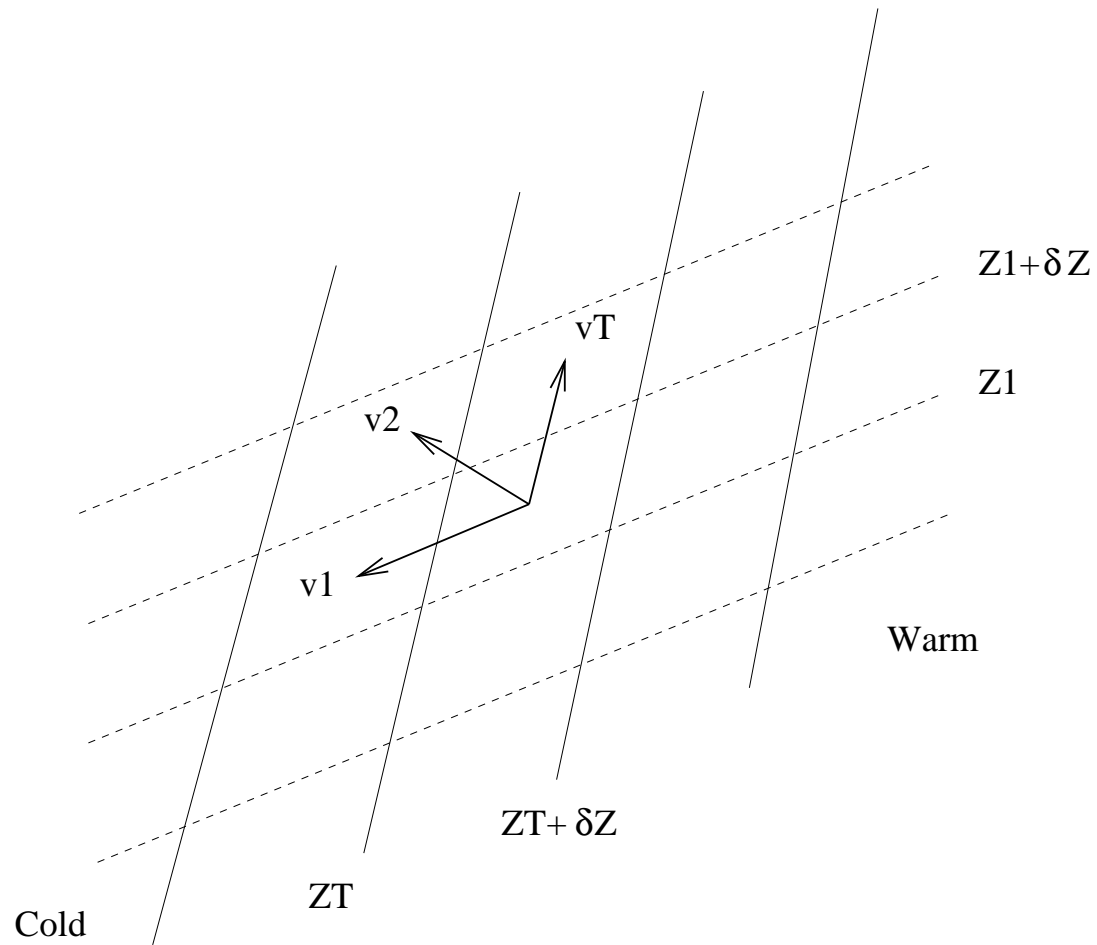
Baroclinic atmosphere

Temperature and geopotential contours usually not parallel

Geostrophic wind has a component normal to the temperature contours (isotherms)

Produces *geostrophic temperature advection*

Winds blow from warm to cold or vice versa

Temperature advection

Temperature advection

Warm advection   →   *veering*

- Anticyclonic (clockwise) rotation with height

Cold advection   →   *backing*

- Cyclonic (counter-clockwise) rotation with height

Summary

Geostrophic wind parallel to geopotential contours

- Wind with high pressure to the right (North Hemisphere)

Thermal wind parallel to thickness (mean temperature) contours

- Wind with high thickness to the right
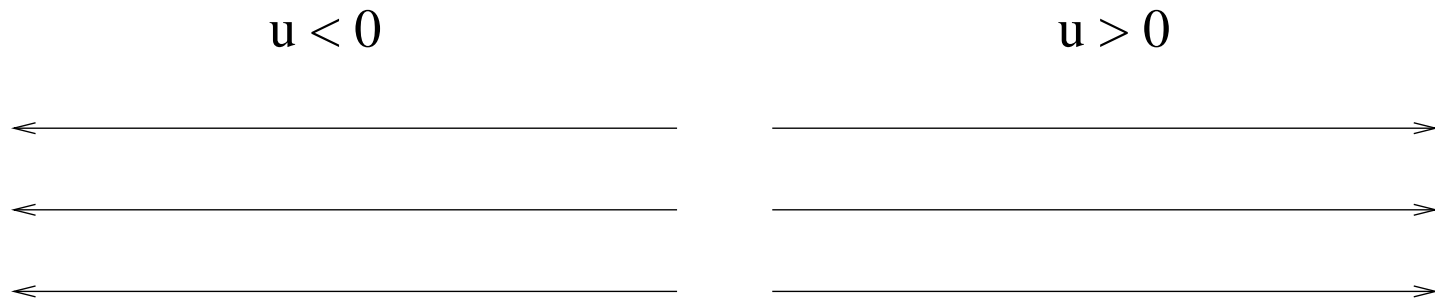
Divergence

Continuity equation:

$$\frac{d\rho}{dt} + \rho \, \nabla \cdot \underline{u} = 0$$

or:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{u} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$$
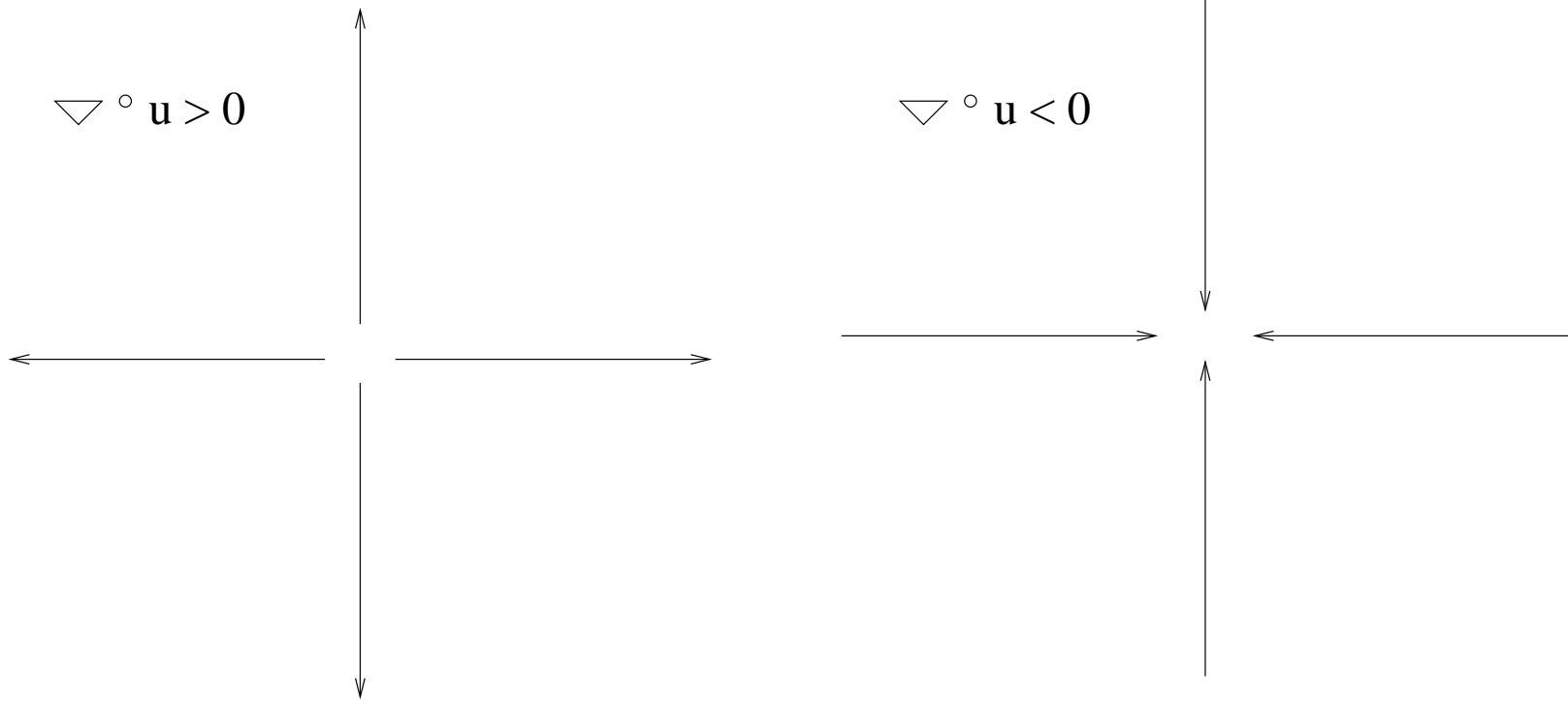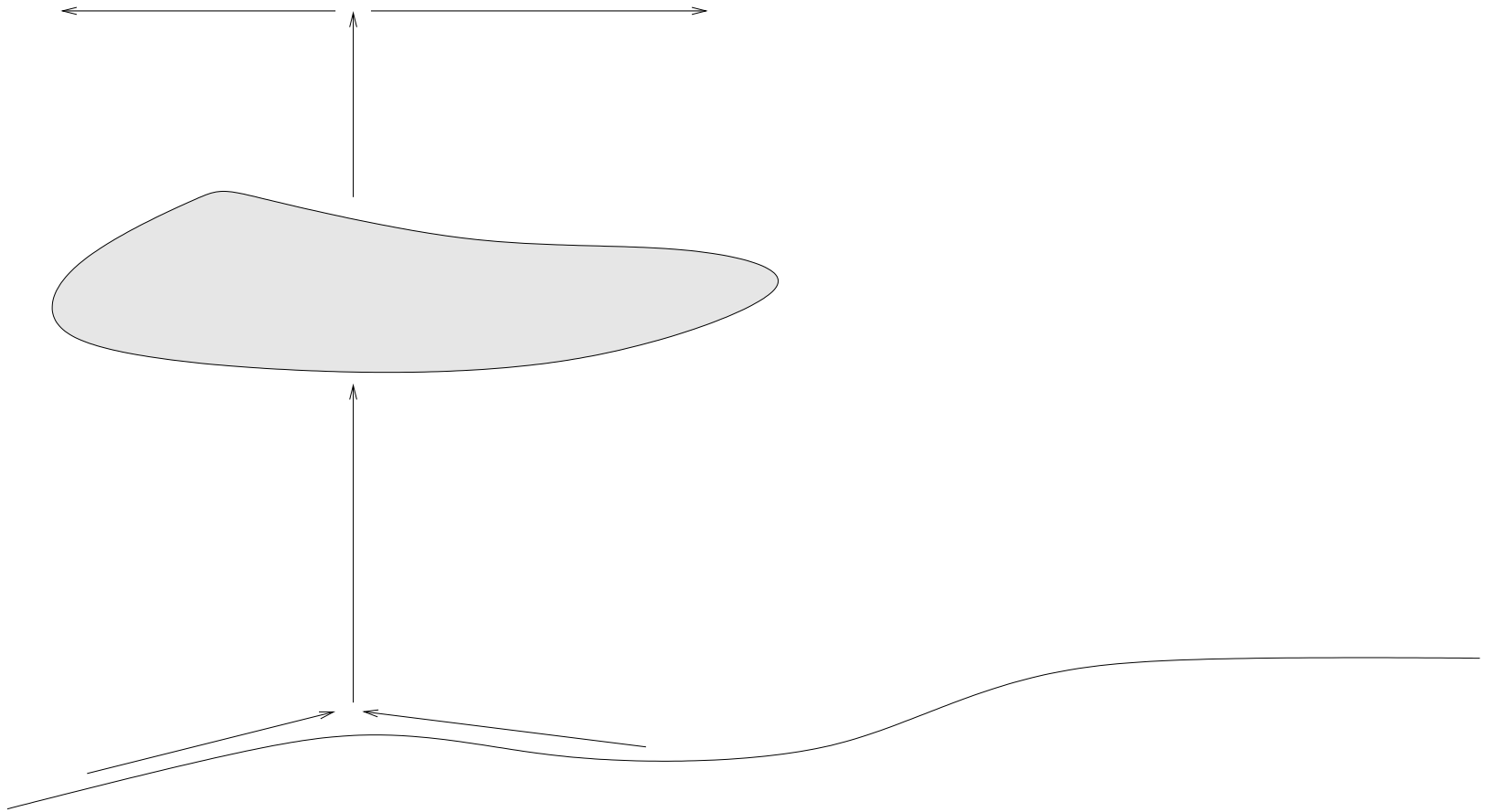
Density changes due to flow *divergence*

Divergence

u < 0

u > 0

$$\frac{\delta u}{\delta x} > 0$$

Divergence

$$\bigtriangledown \circ \, u > 0$$

$$\bigtriangledown \circ \, u < 0$$

Divergence

Example

The divergence in a region is constant and positive:

$$\nabla \cdot \vec{u} = D > 0$$

What happens to the density of an air parcel?

Example

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{u} = -D$$

$$\frac{d\rho}{dt} = -\rho D$$

$$\rho(t) = \rho(0)\, e^{-Dt}$$
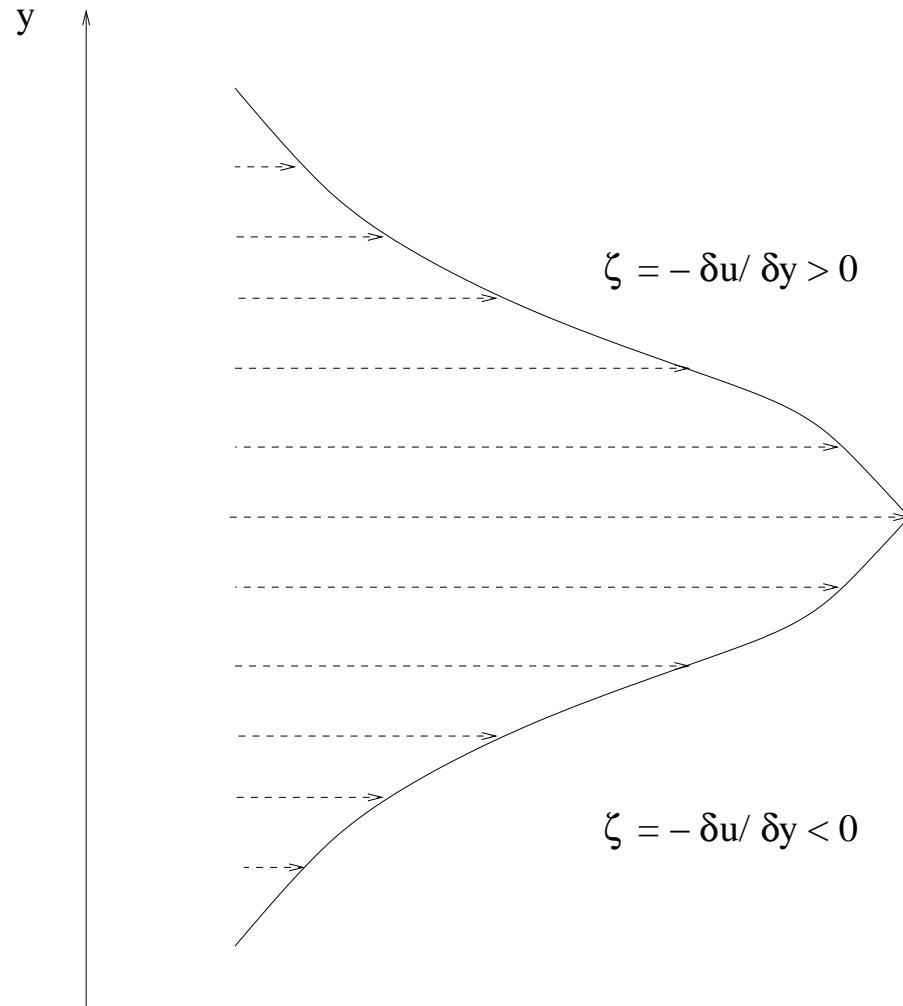
Vorticity

Central quantity in dynamics

$$\vec{\zeta} \equiv \nabla \times \vec{u}$$

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$
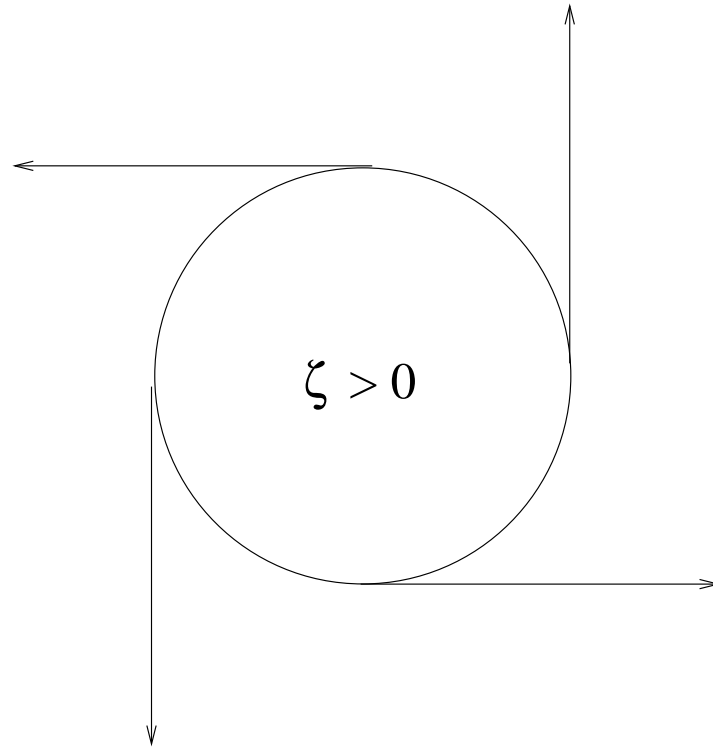
Most important at synoptic scales is vertical component:

$$\vec{\zeta} = \zeta \hat{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity

$$\zeta = -\delta u / \delta y > 0$$

$$\zeta = -\delta u / \delta y < 0$$

y

Vorticity

$$\zeta > 0$$

Example

What is the vorticity of a typical tornado? Assume *solid body rotation*, with a velocity of 100 m/sec, 20 m from the center.

In cylindrical coordinates, the vorticity is:

$$\zeta = \frac{1}{r}\frac{\partial \, rv_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

For solid body rotation, $v_r = 0$ and

$$v_\theta = \omega r$$

Vorticity

Here $\omega =$ const. So:

$$\zeta = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = \frac{1}{r}\frac{\partial \omega r^2}{\partial r} = 2\omega$$

We have $v_\theta = 100$ m/sec at $r = 20$ m:

$$\omega = \frac{v_\theta}{r} = \frac{100}{20} = 5\, rad/sec$$

So:

$$\zeta = 10\, rad/sec$$

2-D Shears

Divergence:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Vorticity:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

2-D Shears

Shear:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

Deformation:

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Absolute vorticity

Now add rotation. The velocity in the fixed frame is:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

So:

$$\vec{\zeta_a} = \nabla \times (\vec{u} + \vec{\Omega} \times \vec{r}) = \vec{\zeta} + 2\vec{\Omega}$$

(as in solid body rotation example)

Absolute vorticity

Two components:

- $\nabla \times \vec{u}$ — the *relative vorticity*
- 2Ω — the *planetary vorticity*

Vertical component dominates:

$$\zeta_a \cdot \hat{k} = (\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u) + 2\Omega sin(\theta) \equiv \zeta + f$$

(ζ now refers to vertical relative vorticity)

Absolute vorticity

Scaling:

$$\zeta \propto \frac{U}{L}$$

So:

$$\frac{|\zeta|}{f} \approx \frac{U}{fL} = \epsilon$$

The Rossby number

Absolute vorticity

- $\epsilon \ll 1$

Geostrophic velocities

Planetary vorticity dominates

Mid-latitude weather system

Absolute vorticity

- $\epsilon \gg 1$

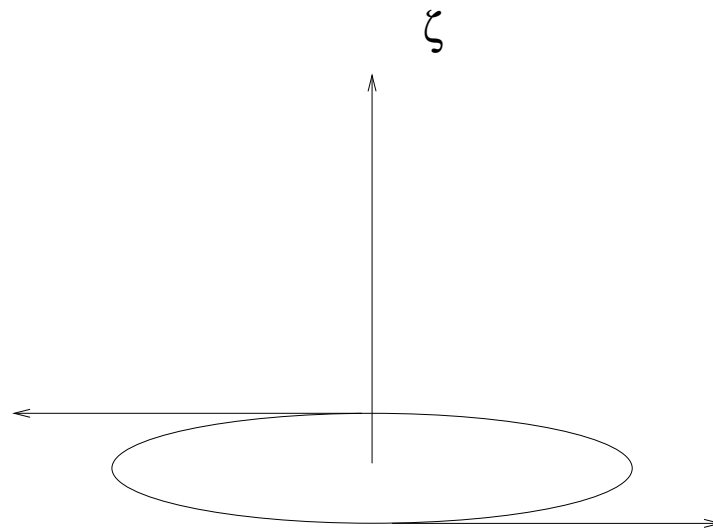 Cyclostrophic velocities

  Relative vorticity dominates

  Tornado

Relative circulation

Integrate the relative vorticity over an area of fluid:

$$\Gamma \equiv \int\int \vec{\zeta} \cdot \hat{n}\, dA = \int\int (\nabla \times \vec{u}) \cdot \hat{n}\, dA = \oint \vec{u} \cdot \vec{dl}$$

ζ

Circulation

Derive equation for the circulation by integrating the momentum equations around a closed curve.

First write momentum equations in vector form:

$$\frac{d}{dt}\vec{u} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F}$$

Integrate around a closed area:

$$\frac{d}{dt}\Gamma + \oint (2\vec{\Omega} \times \vec{u}) \cdot \vec{dl} = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{g} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl}$$

Circulation

Gravity vanishes because can write as a potential:

$$\vec{g} = -g\hat{k} = \frac{\partial}{\partial z}(-gz) \equiv \nabla \Phi_g$$

and the closed integral of a potential vanishes:

$$\oint \nabla \Phi_g \cdot \vec{dl} = \oint d\Phi_g = 0$$

Absolute circulation

So:

$$\frac{d}{dt}\Gamma_a = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl}$$

where

$$\Gamma_a \equiv \Gamma + \oint (2\vec{\Omega} \times \vec{u}) \cdot \vec{dl}$$

Absolute circulation

It is possible to show that:

$$\oint \left(2\vec{\Omega} \times \vec{u} \right) \cdot \vec{dl} = \int \int 2\vec{\Omega} \cdot \hat{n} \, dA$$

So:

$$\frac{d}{dt} \oint \left(2\vec{\Omega} \times \vec{u} \right) \cdot \vec{dl} \quad \propto \quad 2\Omega \frac{dA}{dt}$$

Kelvin's theorem

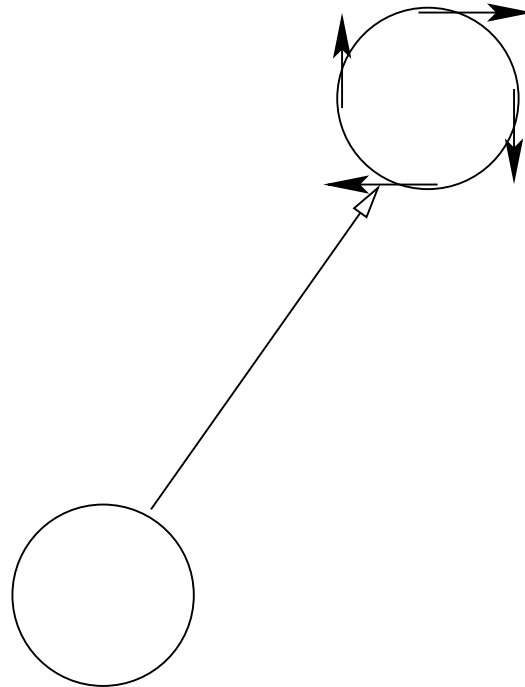If the atmosphere is barotropic (temperature constant on pressure surfaces):

$$\oint \frac{\nabla p}{\rho} \cdot \vec{dl} = \frac{1}{\rho} \oint \nabla p \cdot \vec{dl} = 0$$

If atmosphere is also frictionless ($\vec{F} = 0$), then:

$$\frac{d}{dt}\Gamma_a = - \oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl} = 0$$

The *absolute circulation is conserved*

Example

Move a circular region of air, with fixed area, north. Assume the air is initially at rest. What happens?

Example

$$\Gamma_a(t_0) = \Gamma_a(t_f)$$

$$0 + \int\int f_0 \, dA = \int\int \zeta \, dA + \int\int f_f \, dA$$

So:

$$\int\int \zeta \, dA = \int\int (f_f - f_0) \, dA > 0$$

The relative vorticity *increases*

Vorticity equation

Horizontal momentum equations (p-coords):

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,u - fv = -\frac{\partial}{\partial x}\Phi + F_x$$

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,v + fu = -\frac{\partial}{\partial y}\Phi + F_y$$

Take $\frac{\partial}{\partial x}$ of the second, subtract $\frac{\partial}{\partial y}$ of the first

Vorticity equation

Find:

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p} \right) \zeta_a$$

$$= -\zeta_a \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x} \right) + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x \right)$$

$$\zeta_a = \zeta + f$$

Vorticity equation

So absolute vorticity can change due to three terms

1) The divergence term:

$$-\zeta_a \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Divergence changes the vorticity, just like density

Divergence

Consider small area of air:

$$\delta A = \delta x\, \delta y$$

Time change in the area is:

$$\frac{\delta A}{\delta t} = \delta y \frac{\delta x}{\delta t} + \delta x \frac{\delta y}{\delta t} = \delta y\, \delta u + \delta x\, \delta v$$

Relative change is the divergence:

$$\frac{1}{\delta A} \frac{\delta A}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}$$

Divergence

So rewrite the divergence term:

$$-(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\zeta_a = -\frac{\zeta_a}{A}\frac{dA}{dt}$$

So:

$$\frac{d}{dt}\zeta_a = -\frac{\zeta_a}{A}\frac{dA}{dt} \quad \rightarrow \quad \frac{d}{dt}\zeta_a A = 0$$
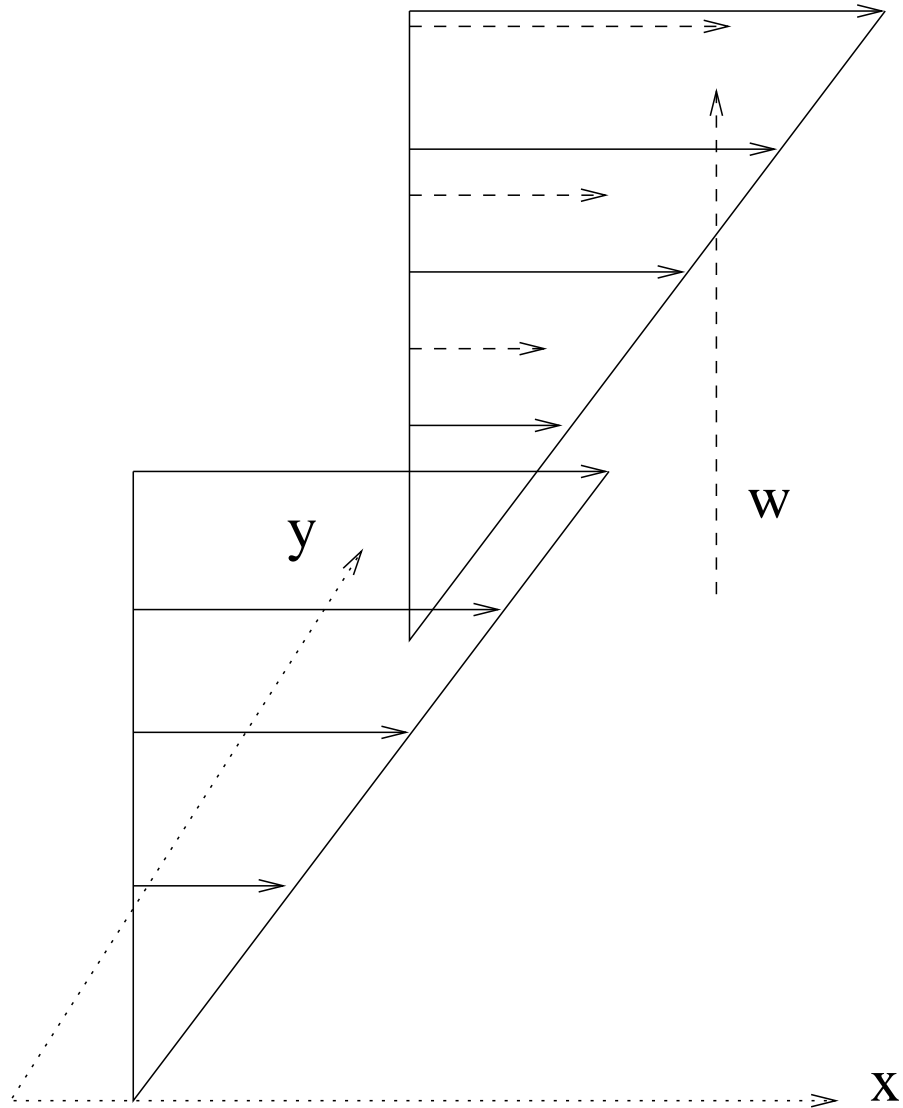
This is Kelvin's theorem again!

Vorticity equation

2) The *tilting* term:

$$\left(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}\right)$$

Difference in vertical motion can change the horizontal shear

Tilting

Vorticity equation

3) The Forcing term:

$$\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

Say frictional forcing:

$$F_x = \nu\nabla^2 u, \quad F_y = \nu\nabla^2 v$$

Friction

Then:

$$\left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x\right) = \nu \nabla^2 \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nu \nabla^2 \zeta$$

Then:

$$\frac{d}{dt}(\zeta + f) = \nu \nabla^2 \zeta$$

Friction

If $f \approx const.$:

$$\frac{d}{dt}\zeta = \nu \nabla^2 \zeta$$

Friction *diffuses* vorticity

Causes cyclones to spread out and weaken

Can occur due to friction in the *boundary layer*

Scaling

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}\right)\zeta_a = -\zeta_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}\right)$$

For synoptic scale motion:

$$U \approx 10 m/sec \quad \omega \approx 10 hPa/day \quad L \approx 10^6 m \quad \partial p \approx 100 hPa$$

$$f_0 \approx 10^{-4} sec^{-1} \quad L/U \approx 10^5 sec \quad \frac{\partial f}{\partial y} \approx 10^{-11} m^{-1} sec^{-1}$$

Scaling

$$\zeta \propto \frac{U}{L} \approx 10^{-5} sec^{-1}$$

So the Rossby number is:

$$\epsilon = \frac{\zeta}{f_0} \approx 0.1$$

So:

$$(\zeta + f) \approx f$$

Scaling

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta \;\; \propto \;\; \frac{U^2}{L^2} \;\; \approx \;\; 10^{-10}$$

$$\omega\frac{\partial}{\partial p}\zeta \;\; \propto \;\; \frac{U\omega}{LP} \;\; \approx \;\; 10^{-11}$$

$$v\frac{\partial}{\partial y}f \;\; \propto \;\; U\frac{\partial f}{\partial y} \;\; \approx \;\; 10^{-10}$$

$$\left(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}\right) \;\; \propto \;\; \frac{U\omega}{LP} \;\; \approx \;\; 10^{-11}$$

$$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \;\; \propto \;\; \frac{fU}{L} \;\; \approx \;\; 10^{-9}$$

Scaling

Divergence term unbalanced!

To understand why, we need to look at the vertical velocities...

Vertical velocity

Continuity, with constant density:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

Scaling implies:

$$\frac{U}{L} \approx \frac{W}{D}$$

or

$$\frac{W}{U} \approx \frac{D}{L}$$

Vertical velocity

Typically:

$$\frac{D}{L} \approx \frac{5km}{1000km} = 0.005$$

But:

$$\frac{W}{U} \approx \frac{1cm/sec}{10m/sec} = 0.001$$

The vertical velocity is 5 times smaller

Vertical velocity

Vorticity scales as:

$$\zeta = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \propto \frac{U}{L}$$

Divergence scales as:

$$D = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \propto \frac{U}{L} \approx \zeta$$

We have:

$$\frac{U}{L} \approx \frac{10m/sec}{10^6 m} = 10^{-5}sec^{-1}$$

Vertical velocity

From measurements:

$$|\zeta| \approx 10^{-5} sec^{-1}$$

But:

$$|D| \approx 10^{-6} sec^{-1} \; !$$

Divergence is 10 times smaller. Why?

Ageostrophic velocities

Write:

$$u = u_g + u_a, \quad v = v_g + v_a$$

From the derivation of the gradient wind:

$$\frac{u_g}{u} \approx 1 + \epsilon$$

This implies:

$$\frac{|u_a|}{|u_g|} \propto \epsilon \approx 0.1$$

Ageostrophic velocities

So we can write:

$$u = u_g + \epsilon\tilde{u}_a, \quad v = v_g + \epsilon\tilde{v}_a$$

where $\tilde{u}_a = u_a/\epsilon$. So the vorticity is:

$$\zeta = \frac{\partial}{\partial x}v_g - \frac{\partial}{\partial y}u_g + \epsilon\left(\frac{\partial}{\partial x}\tilde{v}_a - \frac{\partial}{\partial y}\tilde{u}_a\right)$$

While the divergence is:

$$D = \frac{\partial}{\partial x}u_g - \frac{\partial}{\partial y}v_g + \epsilon\left(\frac{\partial}{\partial x}\tilde{u}_a + \frac{\partial}{\partial y}\tilde{v}_a\right)\frac{1}{f}\frac{\partial}{\partial x}\left(-\frac{\partial\Phi}{\partial y}\right) + \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial x}\right)$$

$$+\epsilon\left(\frac{\partial}{\partial x}\tilde{u}_a + \frac{\partial}{\partial y}\tilde{v}_a\right) = \epsilon\left(\frac{\partial}{\partial x}\tilde{u}_a + \frac{\partial}{\partial y}\tilde{v}_a\right)$$

Vertical velocities

Also:

$$\frac{\partial}{\partial z} w = -D \approx -\epsilon \left(\frac{\partial}{\partial x} \tilde{u}_a + \frac{\partial}{\partial y} \tilde{v}_a \right)$$

So the divergence and the vertical velocity are order Rossby number

Rotation *suppresses vertical motion*

Scaling

Thus the divergence estimate is ten times smaller than we had it before. So:

$$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \propto \epsilon\frac{fU}{L} \quad \approx \quad 10^{-10}$$

So now we have proper balance

Scaling

Retaining the 10^{-10} terms in the vorticity equation:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\zeta + f) = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Is fairly accurate at synoptic scales

Can even be used for prediction

Forecasting

Used for forecasts in the 1930's and 1940's

Approach:

Assume geostrophic velocities:

$$u \approx u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}$$

$$v \approx v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

Forecasting

$$\zeta \approx \zeta_g = \frac{1}{f}\,\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f}\,(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}) = \frac{1}{f}\nabla^2 \Phi$$

The divergence vanishes:

$$(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y})\,(\zeta_g + f) = -f\,(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}) = 0$$

Implies ζ_a is *conserved following the horizontal winds*

Remember: on a pressure surface

Forecasting

Now only *one unknown*: Φ

$$\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right)(\zeta_g + f) = 0$$

becomes:

$$\left(\frac{\partial}{\partial t} - \frac{1}{f}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} + \frac{1}{f}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{1}{f}\nabla^2 \Phi + f\right) = 0$$

Forecasting

Can write equation:

$$\frac{\partial}{\partial t}\zeta_g + u_g \cdot \nabla \zeta_g + v_g \frac{\partial}{\partial y}f = 0$$

or:

$$\frac{\partial}{\partial t}\zeta_g = -u_g \cdot \nabla \zeta_g - v_g \frac{\partial}{\partial y}f$$

Can predict how ζ changes in time

Then convert $\zeta \rightarrow \Phi$

Example

Assume the atmosphere is *barotropic*: no vertical shear

Then $\Phi(x, y, p, t) = \Phi(x, y, t)$

Solve:

$$\frac{\partial}{\partial t}\zeta_g = -u_g \cdot \nabla \zeta_g - v_g \frac{\partial}{\partial y} f$$

Need to specify $\frac{\partial}{\partial y} f$

Beta-plane approximation

Assume the channel is centered at θ_0. Write f using a Taylor expansion:

$$f(\theta) = f(\theta_0) + \frac{df}{d\theta}(\theta_0)\ \theta + \frac{1}{2}\frac{d^2 f}{d\theta^2}(\theta_0)\ \theta^2 + ...$$

If range of latitudes is small, neglect the higher order terms:

$$f \approx f(\theta_0) + \frac{df}{d\theta}(\theta_0)\ \theta$$

Beta-plane approximation

Change to Cartesian coordinates:

$$R\theta \to y$$

Then:

$$f = f_0 + \beta y$$

where:

$$f_0 = 2\Omega sin(\theta_0), \quad \beta = \frac{2\Omega}{R}cos(\theta_0)$$

Beta-plane approximation

So:

$$v\frac{df}{dy} = v\frac{\partial}{\partial y}(f_0 + \beta y) = \beta v$$

In addition, approximate:

$$u_g = -\frac{1}{f}\frac{\partial}{\partial y}\Phi \approx -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi$$

$$v_g = \frac{1}{f}\frac{\partial}{\partial x}\Phi \approx \frac{1}{f_0}\frac{\partial}{\partial x}\Phi$$

Example

Assume the geopotential is:

$$\Phi = -f_0 U y + f_0 A \, sin(kx - \omega t) \, sin(ly)$$

with:

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi = U - lA \, sin(kx - \omega t) \, cos(ly)$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = kA \, cos(kx - \omega t) \, sin(ly)$$

Example

$\Phi = \sin(2x)\,\sin(\pi y)$

Example

A sin(3x−2t)

Example

A sin(3x+2t)
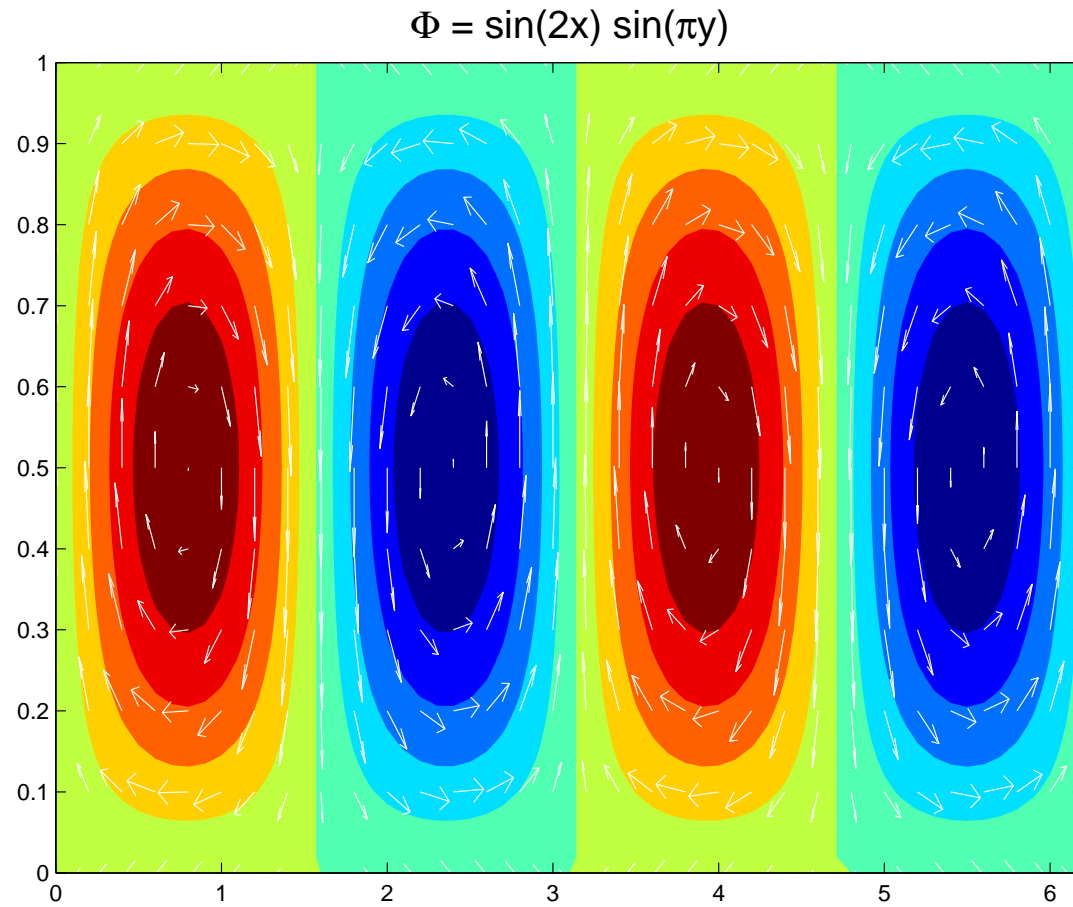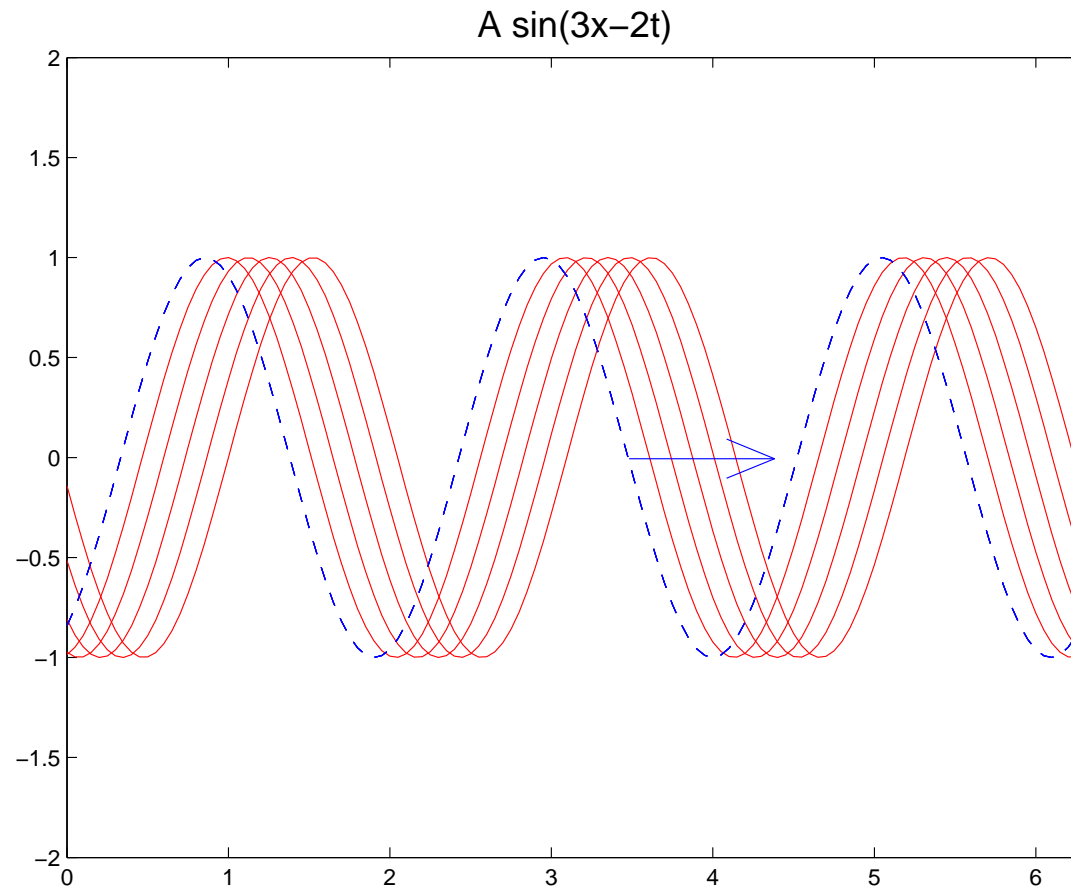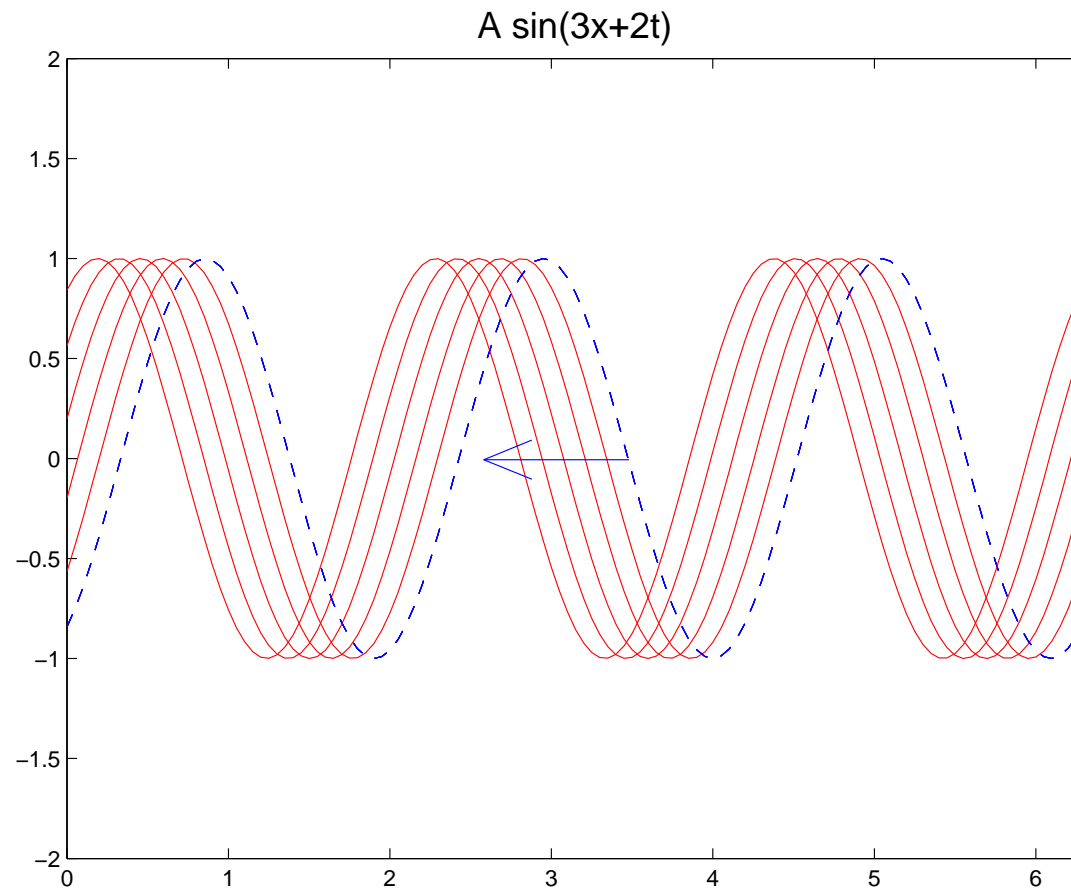
Example

The relative vorticity is:

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi = -(k^2 + l^2)A\,sin(kx - \omega t)\,sin(ly)$$

Also need the derivatives:

$$\frac{\partial}{\partial x}\zeta_g = -k(k^2 + l^2)A\,cos(kx - \omega t)\,sin(ly)$$

$$\frac{\partial}{\partial y}\zeta_g = -l(k^2 + l^2)A\,sin(kx - \omega t)\,cos(ly)$$

Example

Collect terms:

$$-u\frac{\partial}{\partial x}\zeta - v\frac{\partial}{\partial y}\zeta = [U - lA\,sin(kx - \omega t)\,cos(ly)]\times$$

$$[k(k^2 + l^2)A\,cos(kx - \omega t)\,sin(ly)] + [kA\,cos(kx - \omega t)\,sin(ly)]\times$$

$$[l(k^2 + l^2)A\,sin(kx - \omega t)\,cos(ly)]$$

$$= Uk(k^2 + l^2)A\,cos(kx - \omega t)\,sin(ly)$$

Example

Also:

$$-\beta v = -\beta k A \cos(kx - \omega t)\, sin(ly)$$

So:

$$\frac{\partial}{\partial t}\zeta = (U(k^2 + l^2) - \beta)kA\cos(kx - \omega t)\, sin(ly)$$

Also, since:

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi = -(k^2 + l^2)A\, sin(kx - \omega t)\, sin(ly)$$

Example

Then:

$$\frac{\partial}{\partial t}\zeta = \omega(k^2 + l^2)A\,cos(kx - \omega t)\,sin(ly)$$

Equate both sides:

$$\omega(k^2 + l^2)A\,cos(kx - \omega t)\,sin(y)$$

$$= (U(k^2 + l^2) - \beta)kA\,cos(kx - \omega t)\,sin(ly)$$

or:

$$\omega(k^2 + l^2) = (U(k^2 + l^2) - \beta)k$$

Example

or:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

The solution gives a specific value for the frequency!

This is called a *dispersion relation*

Relation between frequency, ω, and wavenumbers, (k, l)

Phase speed

If a travelling wave:

$$\psi \propto sin(kx - \omega t)$$

the crests move with a *phase speed*:

$$c_x = \frac{\omega}{k}$$

If $\omega > 0$, waves move toward positive x (eastward)

Phase speed

$$c = 2/3$$

A sin(3x–2t)
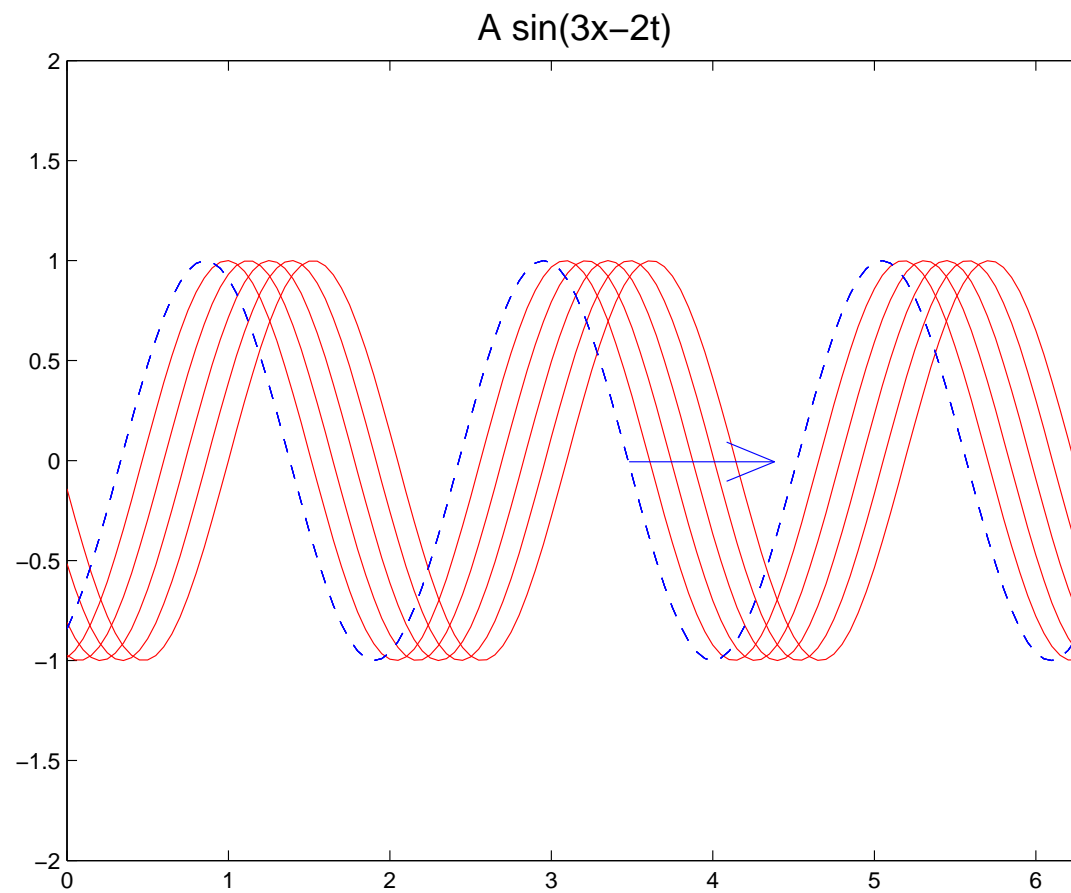
Phase speed

We have:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

so:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}$$

Phase speed

If $U = 0$, then:

$$c_x = -\frac{\beta}{k^2 + l^2}$$

All waves propagate *westward*!

Westward propagation

y=0

+

−
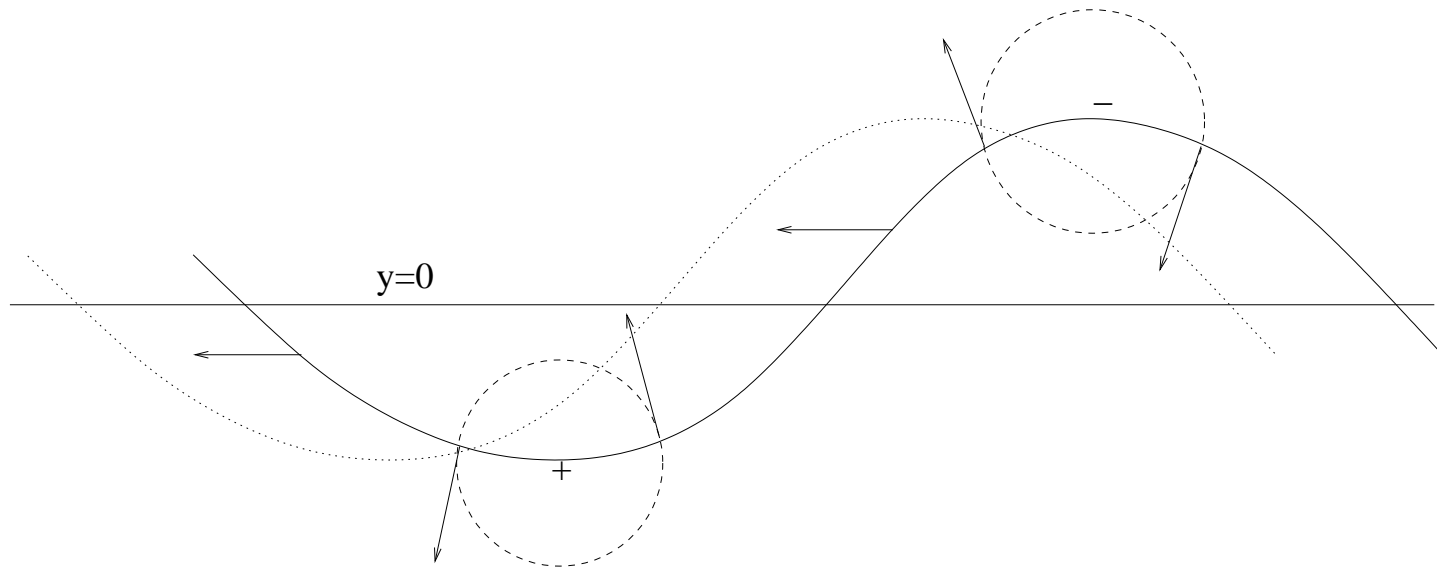
Phase speed

The wavelength in the x-direction is:

$$\lambda_x = \frac{2\pi}{k}$$

and in the y-direction:

$$\lambda_y = \frac{2\pi}{l}$$

Phase speed

Say $l = 0$. Then:

$$c_x = -\frac{\beta}{k^2} = -\frac{\beta \lambda_x^2}{4\pi^2}$$

The larger the wave, the faster it propagates

The waves are *dispersive*

If a collection of different waves initially, they separate by wavelength

Phase speed

If $U \neq 0$, then:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}$$

Longest waves go west while shorter waves are swept eastward by the zonal flow, U. If:

$$k^2 + l^2 = \frac{\beta}{U}$$

the wave is *stationary* in the background flow

Rossby waves

These solutions are called *Rossby waves*

Discovered by Carl Gustav Rossby (1936)

Are observed in the atmosphere

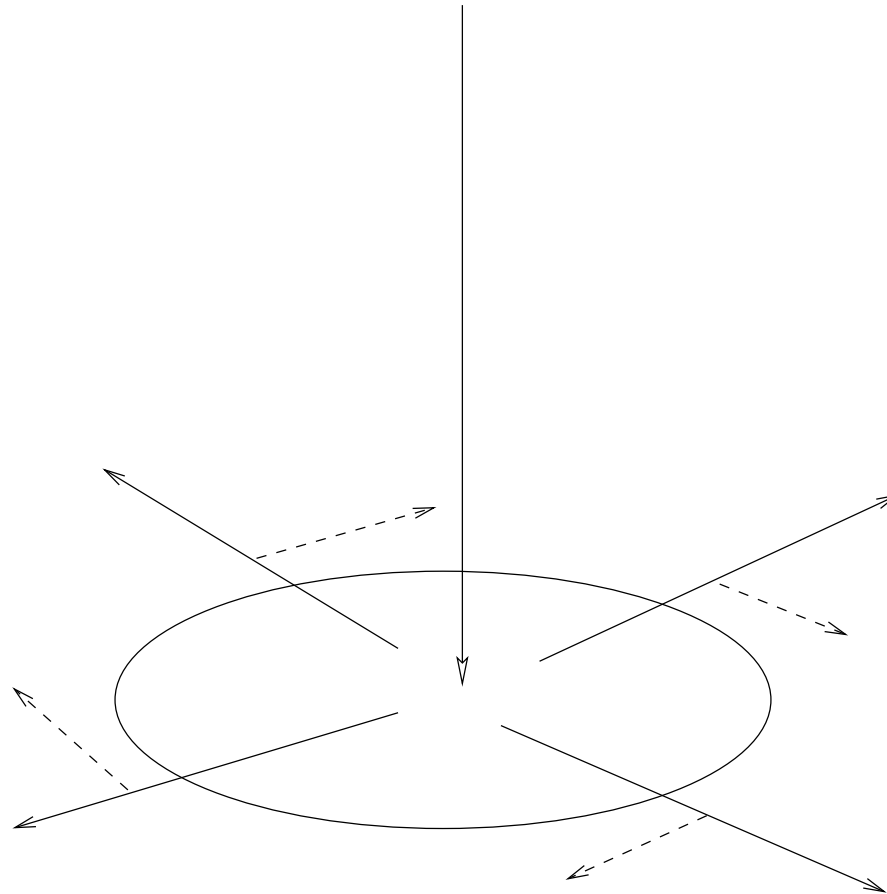Stationary Rossby waves are important for long term weather patterns

Will study more later

Divergence

Previously ignored divergence effects. But very important for the growth of unstable disturbances (storms)

$$
(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = \frac{d}{dt}(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})
$$

As air flows out of a divergent region, is deflected by the Coriolis force

Divergence

Divergence

Consider flow with constant divergence:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D > 0$$

$$\frac{d}{dt}\zeta_a = -\zeta_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -D\zeta_a$$

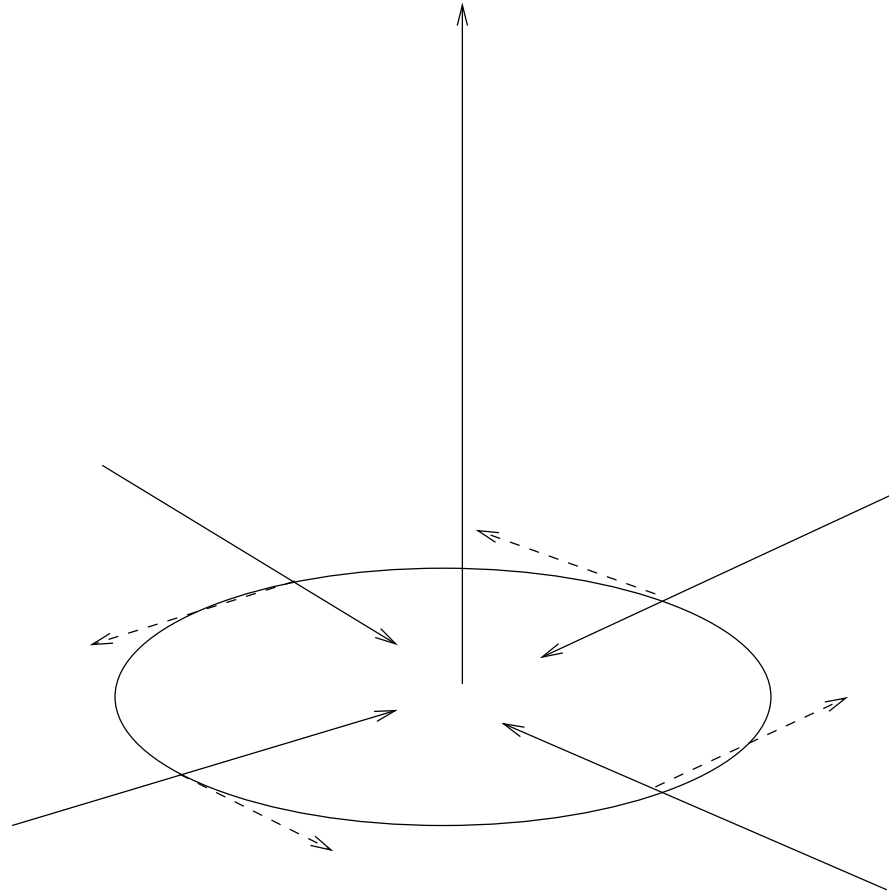$$\zeta_a(t) = \zeta_a(0)\ e^{-Dt}$$

Divergence

So:

$$\zeta_a = \zeta + f \to 0$$

$$\zeta \to -f$$

Divergent flow favors *anticyclonic* vorticity

Vorticity approaches $-f$, regardless of initial value

Convergence

Divergence

Now say $D = -C$

$$\frac{d}{dt}\zeta_a = -\zeta_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = C\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\ e^{Ct}$$

$$\zeta_a \rightarrow \pm\infty$$

But which sign?

Divergence

If the Rossby number is small, then:

$$\zeta_a(0) = \zeta(0) + f \approx f > 0$$

So:

$$\zeta \to +\infty$$

Convergent flow favors *cyclonic* vorticity

Vorticity increases *without bound*

• Why intense storms are almost always cyclonic

Summary

The vorticity equation is approximately:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

or:

$$\frac{d}{dt}\zeta + v\frac{df}{dy} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

- Vorticity changes due to meridional motion
- Vorticity changes due to divergence

Forecasting

$$\frac{\partial}{\partial t}\zeta = -u \cdot \nabla\zeta - v\frac{\partial}{\partial y}f - (\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

Write in difference form:

$$\frac{\zeta(x,y,t_1) - \zeta(x,y,t_0)}{\triangle t} = -u(x,y,t_0) \cdot \nabla\zeta(x,y,t_0)$$

$$-v(x,y,t_0)\frac{\partial}{\partial y}f(y) - [\zeta(x,y,t_0)+f(y)][\frac{\partial u(x,y,t_0)}{\partial x} + \frac{\partial v(x,y,t_0)}{\partial y}]$$

Forecasting

or:

$$\zeta_g(x, y, t_1) = \zeta_g(x, y, t_0) - (\triangle t)\, u_g(x, y, t_0) \cdot \nabla \zeta_g(x, y, t_0)$$

$$-(\triangle t)\, v_g(x, y, t_0) \frac{\partial}{\partial y} f(y) - [\triangle t)\, (\zeta(x, y, t_0) + f(y)] \times$$

$$(\frac{\partial u(x, y, t_0)}{\partial x} + \frac{\partial v(x, y, t_0)}{\partial y})$$

With fields at t_0, can predict vorticity at t_1

Forecasting

Method:

- Obtain $\Phi(x, y, t_0)$ from measurements on p-surface
- Calculate $u_g(t_0)$, $v_g(t_0)$, $\zeta_g(t_0)$
- Calculate $\zeta_g(t_1)$
- *Invert* ζ_g to get $\Phi(t_1)$
- Start over
- Obtain $\Phi(t_2)$, $\Phi(t_3)$,...

Inversion

$$\zeta_g = \frac{1}{f}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)$$

$$\nabla^2 \Phi = f\zeta_g$$

Poisson's equation

Need boundary conditions to solve

Inversion

Simple example: channel, with no flow at boundaries. Let:

$$\zeta = sin(3x)sin(\pi y)$$

$$x = [0, 2\pi], \quad y = [0, 1]$$

So:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = sin(3x)sin(\pi y)$$

Inversion

Try a particular solution:

$$\Phi = A sin(3x) sin(\pi y)$$

This solution works in a channel, because:

$$\Phi(x = 2\pi) = \Phi(x = 0)$$

Also, at $y = 0, 1$:

$$v = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} = 0$$

Inversion

Substitute into equation:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = -(9 + \pi^2)Asin(3x)sin(\pi y) = sin(3x)sin(\pi y)$$

So:

$$A = -\frac{1}{9 + \pi^2}$$

So we have Φ, and we can proceed

Inversion

Generally though, the vorticity is complicated

Solve Poisson's equation numerically

Often takes the most time during a calculation

Barotropic potential vorticity

Imagine atmosphere as a layer with constant density. Lies between two boundaries at $z = z_1, z_2$ (e.g. the surface and the tropopause)

The continuity equation is:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0$$

If density constant, then:

$$(\nabla \cdot \vec{u}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Barotropic potential vorticity

So:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Thus the vorticity equation can be written:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\zeta + f) = (\zeta + f)\frac{\partial w}{\partial z}$$

With constant density:

$$\frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = 0$$

Barotropic potential vorticity

So we can integrate vertically:

$$\int_{z1}^{z2} (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f)dz =$$

$$h(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,(w(z_2) - w(z_1))$$

where $h = z_2 - z_1$. Note that $w = Dz/Dt$. Thus:

$$w(z_2) - w(z_1) = \frac{d}{dt}(z_2 - z_1) = \frac{dh}{dt}$$

Barotropic potential vorticity

So:

$$h \frac{d}{dt}(\zeta + f) = (\zeta + f) \frac{dh}{dt}$$

or:

$$\frac{1}{\zeta + f} \frac{d}{dt}(\zeta + f) - \frac{1}{h} \frac{dh}{dt} = 0$$

$$\frac{d}{dt} ln(\zeta + f) - \frac{d}{dt} lnh = 0$$

$$\frac{d}{dt} ln \frac{\zeta + f}{h} = 0$$

Barotropic potential vorticity

Thus:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0$$

Therefore:

$$\frac{\zeta + f}{h} = const.$$

the barotropic potential vorticity (PV)

The PV takes into account changes in layer *thickness*

If h increases, either ζ or f must also increase

Layer potential vorticity
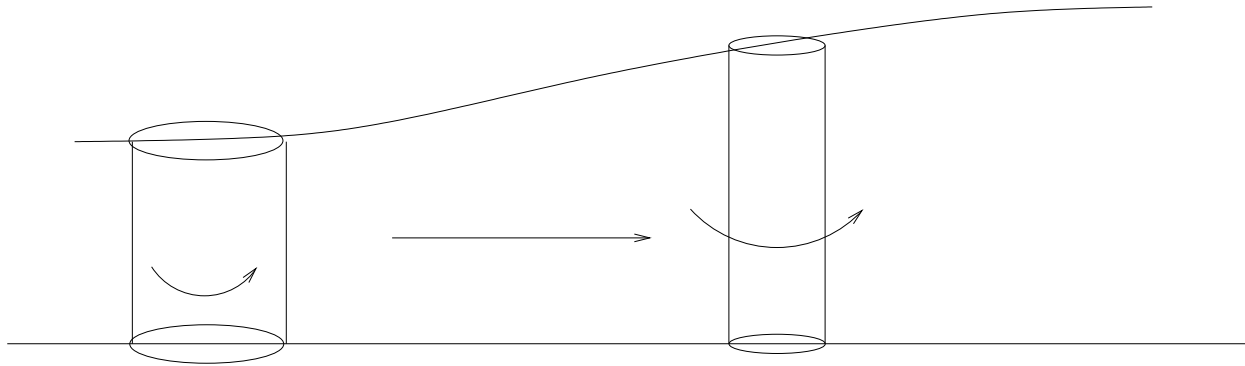
Alternate derivation

Consider a column between z_1 and z_2

As it moves, conserves its mass:

$$\frac{d}{dt}(hA) = 0$$

So:

$$A\frac{dh}{dt} + h\frac{dA}{dt} = 0$$

or:

$$\frac{1}{h}\frac{dh}{dt} + \frac{1}{A}\frac{dA}{dt} = 0$$

Alternate derivation

The relative change in area is the divergence:

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{\delta x\,\delta y}\frac{d}{dt}(\delta x \delta y)$$

$$= \frac{1}{\delta x\,\delta y}(\delta u \delta y + \delta x \delta v)$$

$$= \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} \quad \rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

So:

$$\frac{1}{h}\frac{dh}{dt} = -\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Alternate derivation

Substitute for the divergence in the vorticity equation:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x}) + \frac{\partial v}{\partial y}$$

$$= \frac{\zeta + f}{h}\frac{dh}{dt}$$

which leads to:

$$\frac{d}{dt}\frac{\zeta + f}{h} = 0$$

Adiabat

Same idea applies to motion on an *adiabatic*

The thermodynamic energy equation is:

$$c_p dT - \alpha dp = dq$$

From the ideal gas law:

$$p\alpha = RT$$

So:

$$c_p dT - \frac{RT}{p} dp = dq$$

Adiabat

$$c_p \frac{dT}{T} - R\frac{dp}{p} = c_p \, dlnT - R \, dlnp = \frac{dq}{T}$$

If no heating, $dq = 0$, and:

$$c_p \, dlnT = R \, dlnp$$

Defines an *adiabat*: a surface on which a parcel can freely move without external heating

Potential temperature

Heating changes the temperature and the pressure

Define a "temperature" to remove the pressure effect

Imagine a parcel with pressure p and temperature T

Compress or expand adiabatically to a standard pressure:

$$p_0 = 100 \ kPa \ = 1000mb$$

This is potential temperature:

$$c_p \, lnT - R \, lnp = c_p \, ln\theta - R \, lnp_0$$

Potential temperature

Rearranging and taking the exponential of both sides:

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

Potential temperature depends on T and p

If zero heating, a parcel <u>conserves its potential temperature</u>

Call a surface with constant potential temperature an *isentropic* surface

Layer potential vorticity

Flow between two isentropic surfaces trapped if zero heating

So mass in a column between two surfaces is conserved:

$$A\delta z = const.$$

From the hydrostatic relation:

$$-\frac{A\delta p}{\rho g} = const.$$

where δp is the spacing between surfaces

Layer potential vorticity

$$\theta + \delta\theta$$

$$\delta\rho$$

$$\theta$$
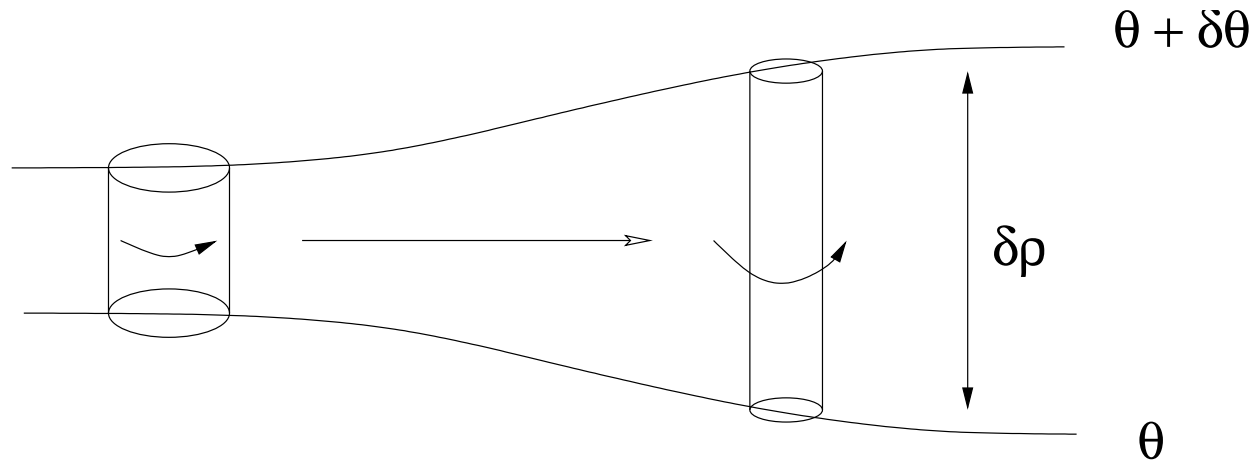
Layer potential vorticity

Relate δp to $\delta\theta$ using the static stability:

$$\delta p = \left(\frac{\partial\theta}{\partial p}\right)^{-1}\delta\theta$$

The stronger the stratification, the smaller the pressure difference between surfaces. So:

$$A\left(\frac{\partial\theta}{\partial p}\right)^{-1}\frac{\delta\theta}{g} = const.$$

Layer potential vorticity

We have $\rho \approx const.$ between the surfaces. Also $\delta\theta$ and g are constant. So:

$$\frac{d}{dt}[A(\frac{\partial\theta}{\partial p})^{-1}] = 0$$

or:

$$(\frac{\partial\theta}{\partial p})^{-1}\frac{d}{dt}A + A\frac{d}{dt}(\frac{\partial\theta}{\partial p})^{-1}$$

$$= (\frac{\partial\theta}{\partial p})^{-1}\frac{d}{dt}A - A(\frac{\partial\theta}{\partial p})^{-2}\frac{d}{dt}(\frac{\partial\theta}{\partial p})$$

Layer potential vorticity

Rearranging:

$$\frac{1}{A}\frac{d}{dt}A - (\frac{\partial \theta}{\partial p})^{-1}\frac{d}{dt}(\frac{\partial \theta}{\partial p}) = 0$$

From before:

$$\frac{1}{A}\frac{d}{dt}A = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v$$

so:

$$(\frac{\partial \theta}{\partial p})^{-1}\frac{d}{dt}(\frac{\partial \theta}{\partial p}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Layer potential vorticity

Substituting again from the vorticity equation:

$$\frac{1}{\zeta + f} \frac{d}{dt}(\zeta + f) + (\frac{\partial \theta}{\partial p})^{-1} \frac{d}{dt}(\frac{\partial \theta}{\partial p}) = 0$$

or:

$$\frac{d}{dt}[(\zeta + f)\frac{\partial \theta}{\partial p}] = 0$$

This is the *isentropic potential vorticity*, first derived by Ertel (1942)

Layer potential vorticity

Remember: ζ evaluted on potential temperature surface

Very useful quantity: can label air by its PV

Can distinguish air in the troposphere which comes from stratosphere

Layer potential vorticity

Can be used for prediction:

$$\frac{\partial}{\partial t}\left[(\zeta + f)\frac{\partial \theta}{\partial p}\right] = -\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left[(\zeta + f)\frac{\partial \theta}{\partial p}\right]$$

Can predict how the PV changes on the isentrope

Then invert the PV to determine the winds

Not as accurate as solving the full primitive equations, but very useful for understanding/visualization

Planetary boundary layer

z=0
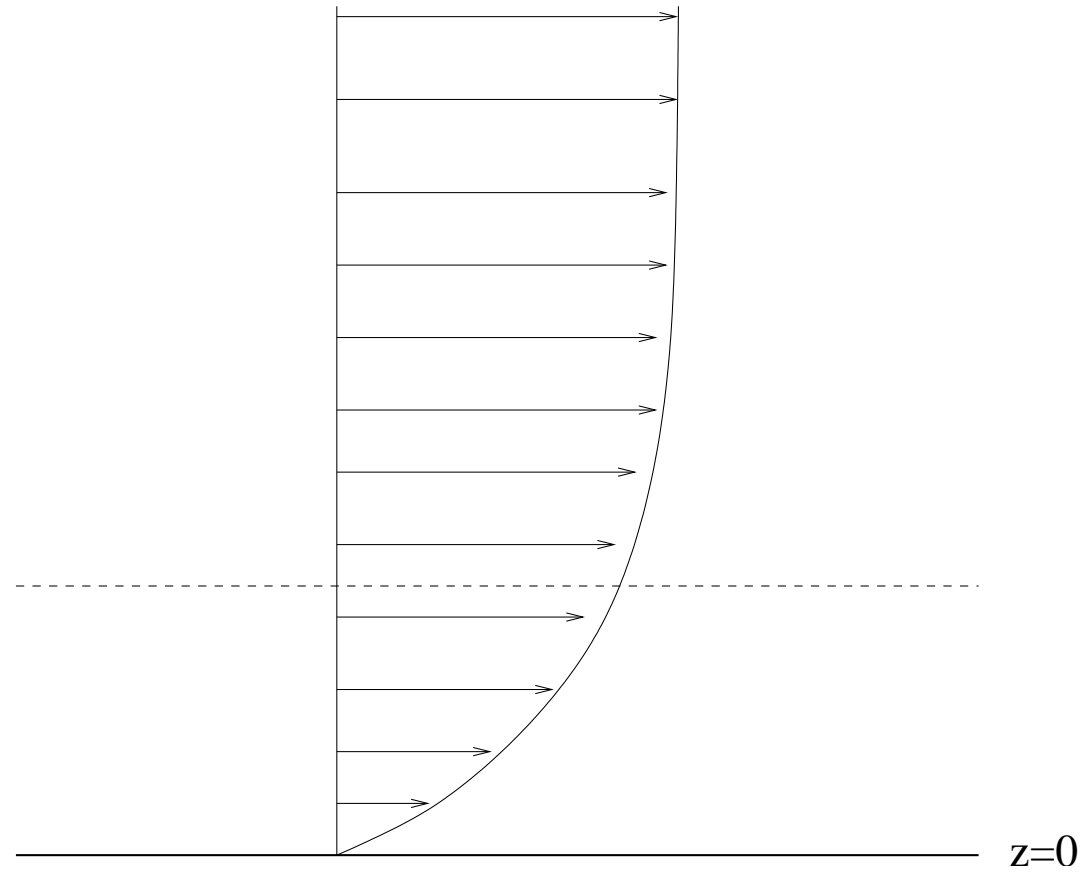
Turbulence

There is a *continuum* of eddy scales

Largest resolved by our models, but the smallest are not.
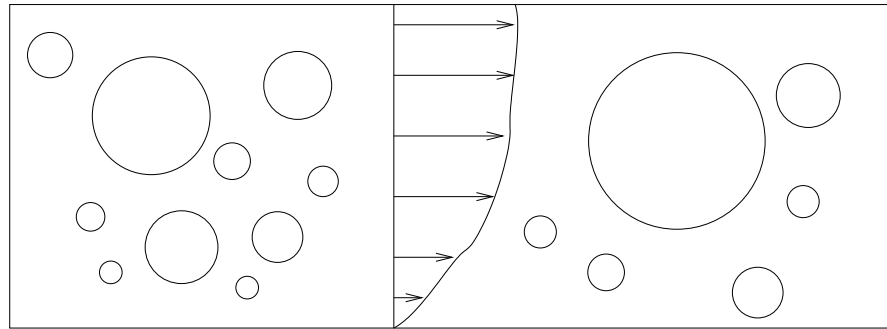
Boussinesq equations

Assume we can split the velocity into a mean (over some period) and a perturbation:

$$\overline{u} = \overline{u} + u'$$

Use the momentum equations with no friction:

$$\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

Boussinesq equations

Assume density in the boundary layer approximately constant, so that:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

Substitute the partitioned velocities into the momentum equations and then average:

$$\frac{\partial}{\partial t}(\overline{u}+u') + (\overline{u}+u')\frac{\partial}{\partial x}(\overline{u}+u') + (\overline{v}+v')\frac{\partial}{\partial y}(\overline{u}+u') - f(\overline{v}+v')$$

$$+(\overline{w}+w')\frac{\partial}{\partial z}(\overline{u}+u') = \frac{1}{\rho}\frac{\partial}{\partial x}(\overline{p}+p')$$

Boussinesq equations

Now average. Note that:

$$\overline{\overline{u} + u'} = \overline{u}$$

so:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{u'\frac{\partial}{\partial x}u'} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{v'\frac{\partial}{\partial y}u'} +$$

$$+\overline{w}\frac{\partial}{\partial z}\overline{u} + \overline{w'\frac{\partial}{\partial z}u'} + -f\overline{v} = \frac{1}{\rho}\frac{\partial}{\partial x}\overline{p}$$

Boussinesq equations

Because of the continuity equation, we can write:

$$\overline{u'\frac{\partial}{\partial x}u'} + \overline{v'\frac{\partial}{\partial y}u'} + \overline{w'\frac{\partial}{\partial z}u'} = \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}$$

So:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} =$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - (\frac{\partial}{\partial x}\rho\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'})$$

Boussinesq equations

Similarly:

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} =$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - (\frac{\partial}{\partial x}\rho\overline{v'u'} + \frac{\partial}{\partial y}\overline{v'v'} + \frac{\partial}{\partial z}\overline{v'w'})$$

Terms on the RHS are the "eddy stresses"

PBL equations

Assume the eddy stresses don't vary horizontally. Then:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} = -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} = -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

Outside the boundary layer, assume geostrophy. In the layer, we have geostrophic terms *plus* vertical mixing. So turbulence *breaks geostrophy*:

$$-f\overline{v} = -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$= -f\overline{v}_g - \frac{\partial}{\partial z}\overline{u'w'}$$

$$f\overline{u} = -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - \frac{\partial}{\partial z}\overline{v'w'} = f\overline{u}_g - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

But too many unknowns! : $\overline{u}, \overline{v}, u', v', w'$

Must *parameterize* the eddy stresses.

Two cases:

- Stable boundary layer: stratified
- Unstable boundary layer: convective mixing

Convective boundary layer

Layer is vertically *mixed* by convection

Temperature and velocity are constant with height

From measurements, surface fluxes:

$$\overline{u'w'}\big|_s = -C_d \mathcal{V}\, u, \qquad \overline{v'w'}\big|_s = -C_d \mathcal{V}\, v$$

C_d is the "drag coefficient" and

$$\mathcal{V} \equiv (u^2 + v^2)^{1/2}$$

Convective boundary layer

Integrate vertically:

$$\int_0^h -f(\overline{v} - \overline{v}_g)\, dz = -\int_0^h \frac{\partial}{\partial z}\overline{u'w'}\, dz$$

Mixing vanishes outside of the layer, so:

$$\overline{u'w'}\big|_h = 0$$

Thus:

$$fh(\overline{v} - \overline{v}_g) = -\overline{u'w'}\big|_s$$

Convective boundary layer

With the parameterization:

$$f h(\overline{v} - \overline{v}_g) = C_d \mathcal{V}\,\overline{u}$$

Likewise:

$$-f h(\overline{u} - \overline{u}_g) = C_d \mathcal{V}\,\overline{v}$$

Convective boundary layer

Say $v_g = 0$; then:

$$v = \frac{C_d}{fh}\,\mathcal{V}\,u,$$

$$u = u_g - \frac{C_d}{fh}\,\mathcal{V}\,v$$

Solving equations not so simple because $\mathcal{V} = \sqrt{u^2 + v^2}$

But can use iterative methods

Convective boundary layer

If $u > 0$, then $v > 0$
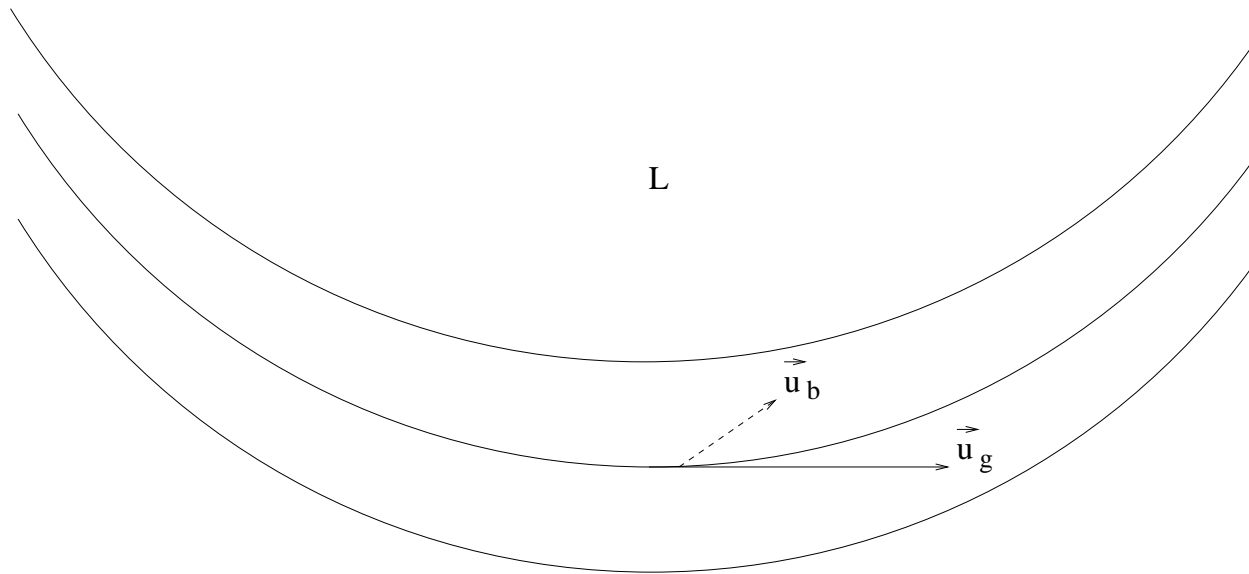
• Flow down the pressure gradient

Stable boundary layer

Now assume no large scale vertical mixing

Wind speed and direction can vary with height

Specify turbulent velocities using *mixing length theory*:

$$u' = -l' \frac{\partial}{\partial z} \overline{u}$$

where $l' > 0$ if up.

Mixing length

u(z)

l'
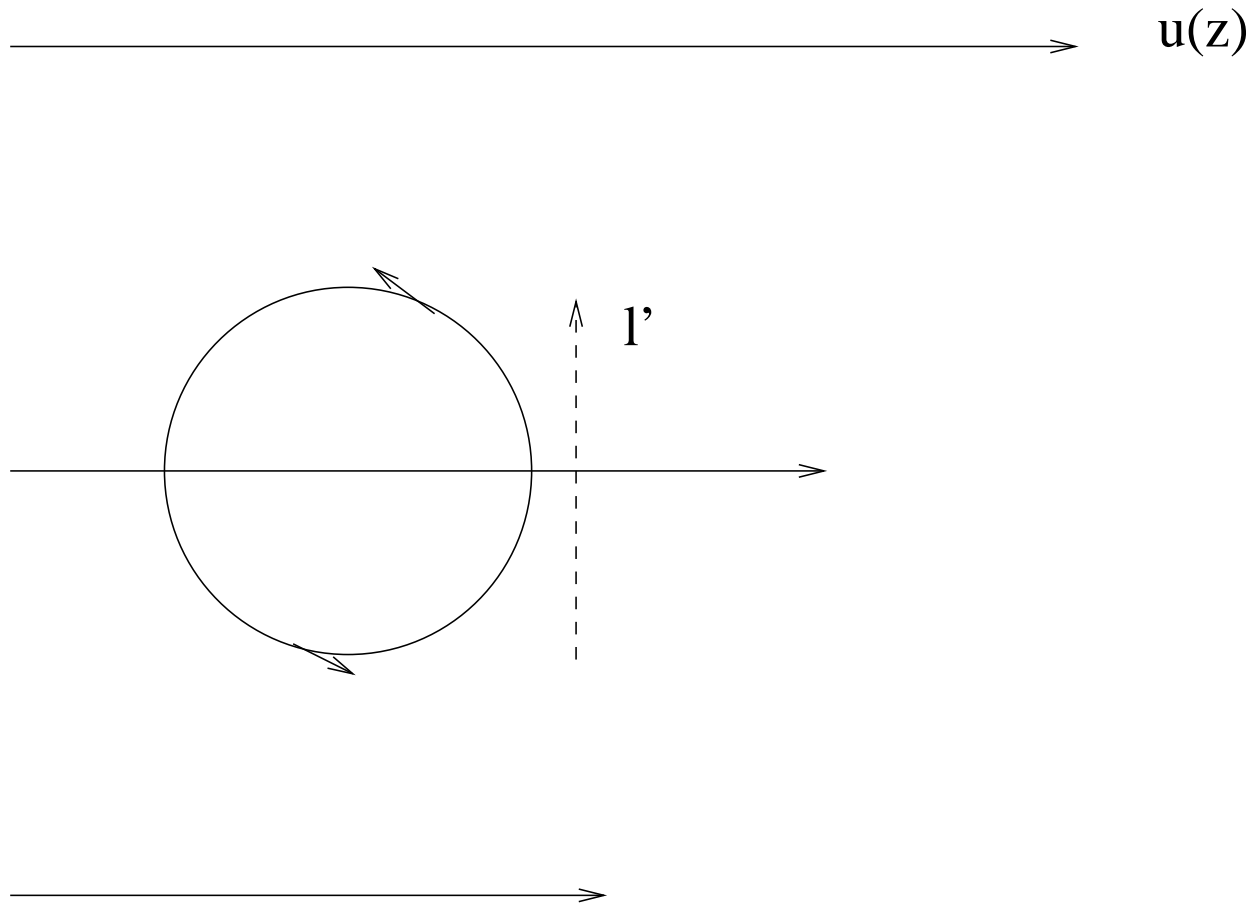
Stable boundary layer

So:

$$-\overline{u'w'} = \overline{w'l'}\,\frac{\partial}{\partial z}\overline{u}$$

Assume the vertical and horizontal eddy scales are comparable

$$w' = l'\frac{\partial}{\partial z}\mathcal{V}$$

where again $\mathcal{V} = \sqrt{u^2 + v^2}$

Notice $w' > 0$ if $l' > 0$.

Stable boundary layer

So:

$$-\rho \, \overline{u'w'} = (\rho \, \overline{l'^2} \, \frac{\partial}{\partial z} \mathcal{V}) \, \frac{\partial}{\partial z} \overline{u} \equiv A_z \, \frac{\partial}{\partial z} \overline{u}$$

Same argument:

$$-\rho \, \overline{v'w'} = A_z \, \frac{\partial}{\partial z} \overline{v}$$

where A_z is the "eddy exchange coefficient"

Depends on the size of turbulent eddies and mean shear

Ekman layer

An important example of a stably stratified PBL

Studied by Swedish oceanographer V. W. Ekman (1905)

Assume A_z is constant

Simplest version of a stratified layer

Ekman layer

So we have:

$$A_z \frac{\partial^2}{\partial z^2} u + f(v - v_g) = 0$$

$$A_z \frac{\partial^2}{\partial z^2} v - f(u - u_g) = 0$$

For boundary conditions, use the "no-slip condition":

$$u = 0, v = 0 \quad at \ z = 0$$

Ekman layer

Assume far from the surface, the velocities approach their geostrophic values:

$$u \to u_g, v \to v_g \quad z \to \infty$$

Assume the geostrophic flow is zonal and independent of height:

$$u_g = U, \qquad v_g = 0$$

Ekman layer

Boundary layer velocities vary only in the vertical:

$$u = u(z) , \quad v = v(z) , \quad w = w(z)$$

From continuity:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = \frac{\partial}{\partial z}w = 0 .$$

With a flat bottom, this implies:

$$w = 0$$

Ekman layer

The system is linear, so can decompose the horizontal velocities:

$$u = U + \hat{u}$$

$$v = 0 + \hat{v}$$

Then:

$$-f\hat{v} = A_z \frac{\partial^2}{\partial z^2}\hat{u}$$

$$f\hat{u} = A_z \frac{\partial^2}{\partial z^2}\hat{v} \ .$$

Ekman layer

Boundary conditions:

$$\hat{u} = -U, \hat{v} = 0 \quad at \ z = 0$$

Introduce a new variable:

$$\chi \equiv \hat{u} + i\hat{v}$$

Then:

$$\frac{\partial^2}{\partial z^2}\chi = i\frac{f}{A_z}\chi \ .$$

Ekman layer

The solution is:

$$\chi = A\,exp(\sqrt{\frac{if}{A_z}}z) + B\,exp(-\sqrt{\frac{if}{A_z}}z)$$

Constants A and B are possibly complex. Rewrite:

$$\sqrt{i} = \sqrt{exp(\frac{i\pi}{2})} = exp(\frac{i\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

then:

$$\chi = A\,exp(\frac{z}{\delta_E})\,exp(i\frac{z}{\delta_E}) + B\,exp(-\frac{z}{\delta_E})\,exp(-i\frac{z}{\delta_E})\ ,$$

Ekman layer

where:

$$\delta_E = \sqrt{\frac{2A_z}{f}}$$

This is the "Ekman depth"

Corrections should decay going up, so:

$$A = 0$$

Ekman layer

Take the real part of the horizontal velocities:

$$u = Re\{\chi\} = Re\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

and

$$v = Im\{\chi\} = -Re\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

Ekman layer

For zero flow at $z = 0$, require $Re\{B\} = -U$ and $Im\{B\} = 0$. So:

$$u = U + \hat{u} = U - U\,exp(-\frac{z}{\delta_E})\,cos(\frac{z}{\delta_E})$$

$$v = \hat{v} = U\,exp(-\frac{z}{\delta_E})\,sin(\frac{z}{\delta_E})\ ,$$

Ekman layer
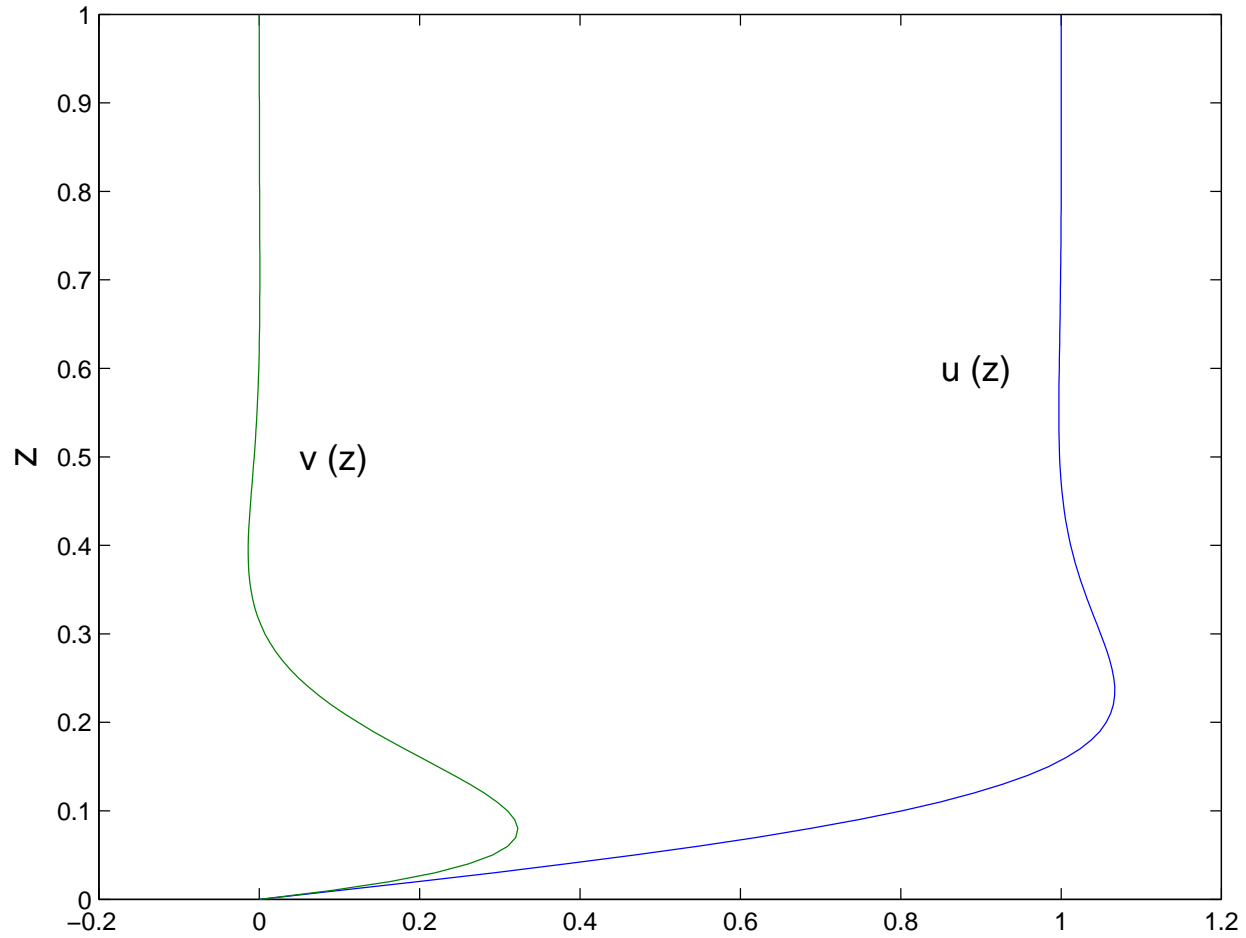
Ekman spiral

The Ekman spiral ($\delta_E = 0.1$)
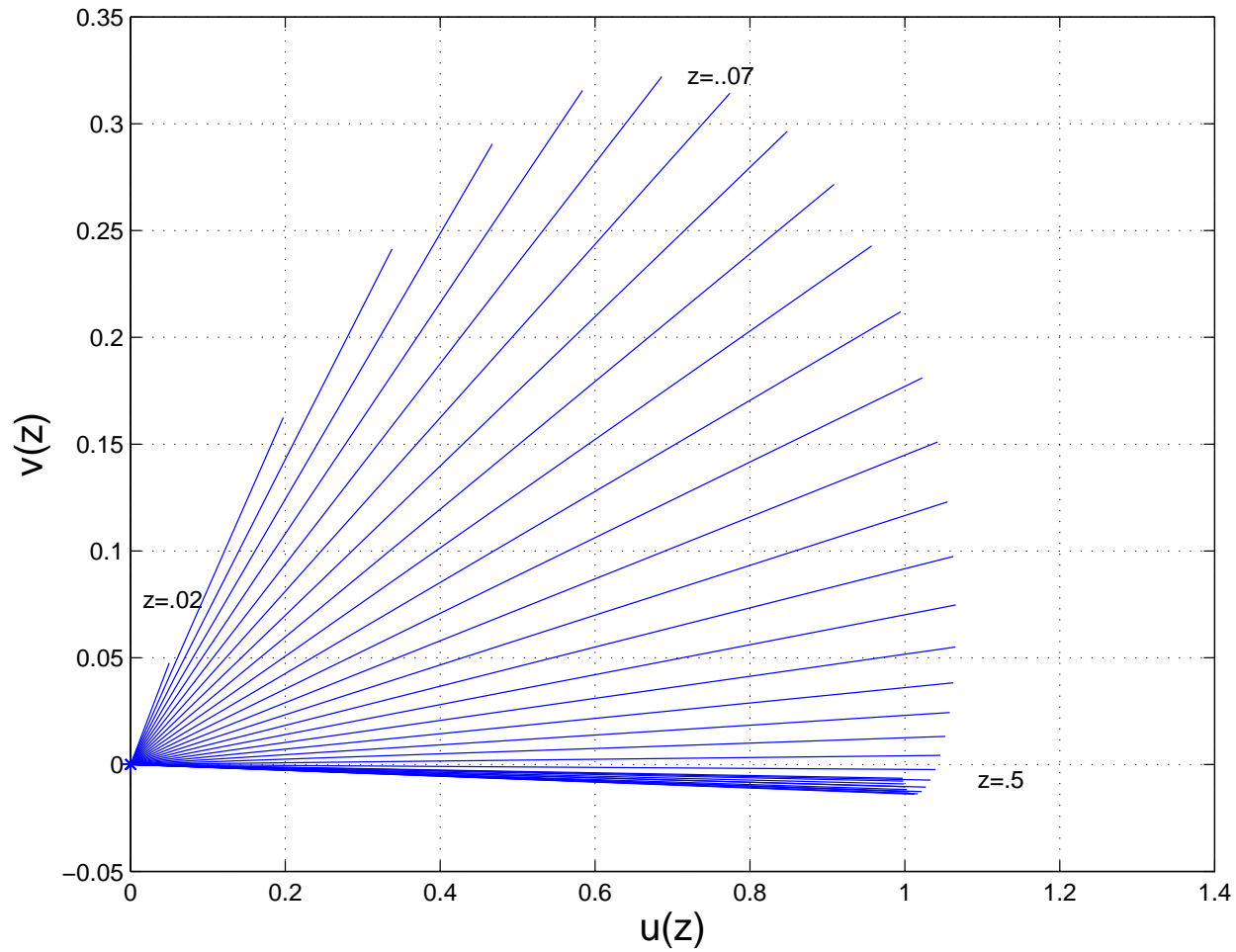
Ekman spiral

The velocity veers to the *left* as it decays

Observations suggest $u \to u_g$ at $z = 1$ km.

With $f = 10^{-4}/sec$, we have:

$$A_z \approx 5\, m^2/sec$$

If $\frac{\partial}{\partial z}\mathcal{V}| = 5 \times 10^{-3}$, the mixing length $l \approx 30$ m.

As in the convective boundary layer, turbulence allows flow from high pressure to low pressure.

Surface layer

Ekman layer cannot hold near surface: can't have 30 m eddies 10 m from surface. Introduce a *surface layer* where:

$$l = kz$$

Then:

$$A_z = k^2 z^2 \frac{\partial}{\partial z} \mathcal{V}$$

So:

$$A_z \frac{\partial}{\partial z} u = k^2 z^2 |\frac{\partial}{\partial z} V| \frac{\partial}{\partial z} u \approx k^2 z^2 (\frac{\partial}{\partial z} u)^2$$

Surface layer

Measurements suggest the turbulent momentum flux is approximately constant in the surface layer:

$$\frac{\partial}{\partial z}\overline{u'w'} \approx u_*^2$$

where u_* is the "friction velocity". So:

$$\frac{\partial}{\partial z}u = \frac{u_*}{kz} \quad \rightarrow \quad u = \frac{u_*}{k}\,ln(\frac{z}{z_0})$$

Here:

- $k \approx 0.4$ is von Karman's constant
- z_0 is the "roughness length"

Surface layer

Match the velocity at the top of the surface layer to that at the base of the Ekman layer.

Comparisons with observations are so-so (see Fig. 5.5 of Holton)

Ekman spiral is often unstable, generating eddies that mix away the signal

Spin-down

Turbulence in both stable and convective boundary layers causes the winds to slow down

Both have flow down pressure gradient

This weakens the gradient and the geostrophic wind

Convergence/divergence in the Ekman layer causes a vertical velocity at the top of the layer

Spin-down

Illustrate using the barotropic vorticity equation:

$$\frac{D}{Dt}\left(\zeta + f\right) \approx f\,\frac{\partial w}{\partial z}$$

Integrate from the top of boundary layer ($z = d$) to the tropopause:

$$(H - d)\frac{D}{Dt}\left(\zeta + f\right) = f(w(H) - w(d)) = -fw(d)$$

Spin-down

Because the boundary layer is much thinner than the troposphere, this is approximately:

$$\frac{D}{Dt}\left(\zeta + f\right) = -\frac{f}{H}w(d)$$

So vertical velocity into/out of the boundary layer changes the vorticity in the troposphere

Ekman pumping

Example: the Ekman layer. The continuity equation is:

$$\frac{\partial}{\partial z}w = -\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v$$

Integrating over the layer, we get:

$$w(d) - 0 = -\int_0^d (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\,dz \equiv -\frac{\partial}{\partial x}M_x - \frac{\partial}{\partial y}M_y$$

where M_x and M_y are the horizontal *transports*

Spin-down

Can show:

$$M_y \approx \frac{U d}{2}$$

and:

$$M_x \approx -\frac{V d}{2}$$

So:

$$w(d) = \frac{d}{2}\left(\frac{\partial}{\partial x}V - \frac{\partial}{\partial y}U\right) = \frac{d}{2}\,\zeta$$

Spin-down

Thus:

$$\frac{D}{Dt}\left(\zeta + f\right) = -\frac{fd}{2H}\,\zeta$$

If assume $f = const.$, then:

$$\frac{D}{Dt}\,\zeta = -\frac{fd}{2H}\,\zeta$$

So that:

$$\zeta(t) = \zeta(0)\,exp(-t/\tau_E)$$

Spin-down

where:

$$\tau_E \equiv \frac{2H}{fd}$$

is the Ekman *spin-down time*. Typical values:

$$H = 10km, \ \ f = 10^{-4}sec^{-1}, \ \ d = 0.5km$$

yield:

$$\tau_E \approx 5 \ days$$

Spin-down

Compare to molecular dissipation. Then:

$$\frac{\partial}{\partial t} u = K_m \frac{\partial^2}{\partial z^2} u$$

From scaling:

$$T_d \approx \frac{H^2 U}{U K_m} = \frac{H^2}{K_m} \approx 100 \; days$$

So the Ekman layer is much more effective at damping motion

Spin-down

The vertical velocity is part of the *secondary circulation*

The primary flow is horizontal, (u_g, v_g)

The vertical velocities, though smaller, are extremely important nevertheless

Stratification reduces the effective H. So the geostrophic velocity over Ekman layer spins down more rapidly, leaving winds aloft alone.

Model Spin-up

Consider an atmospheric model

Atmosphere initially at rest

"Turn on" solar heating and see what happens...

Spin-up

cooling
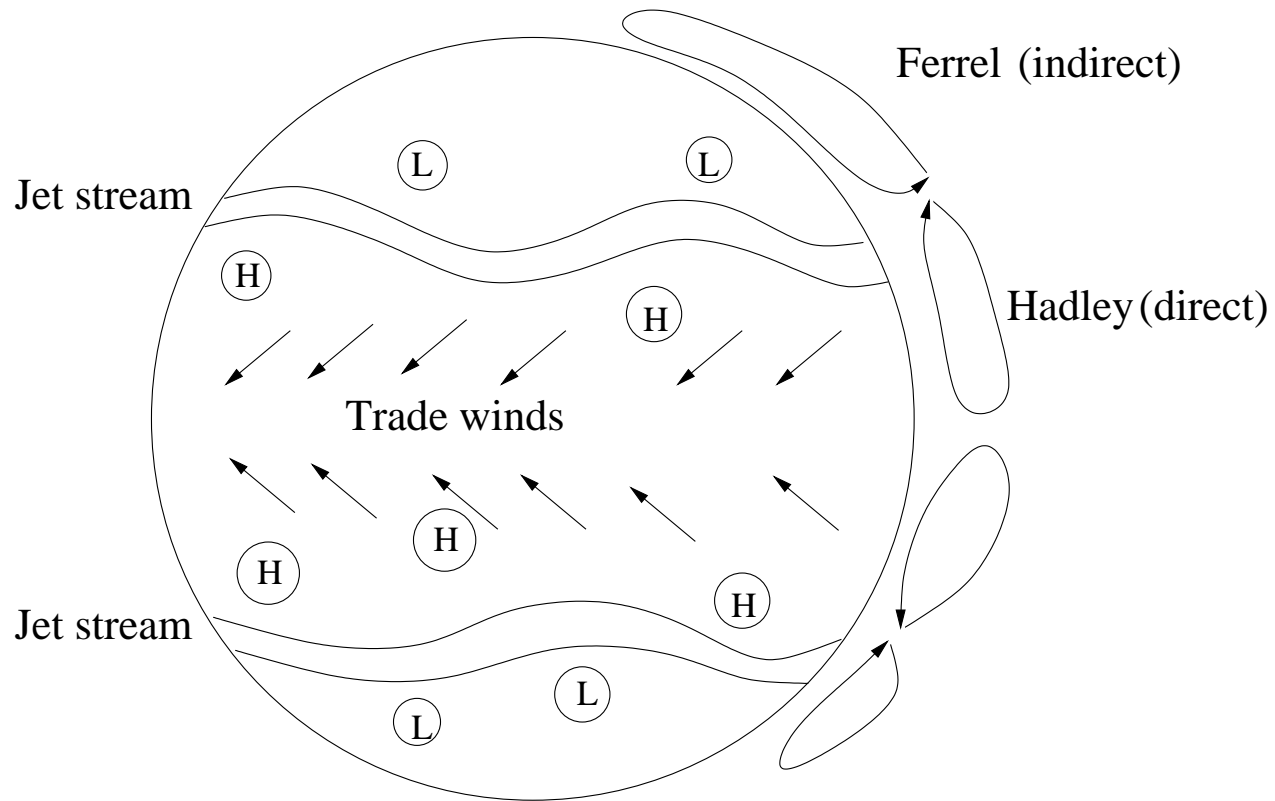
heating

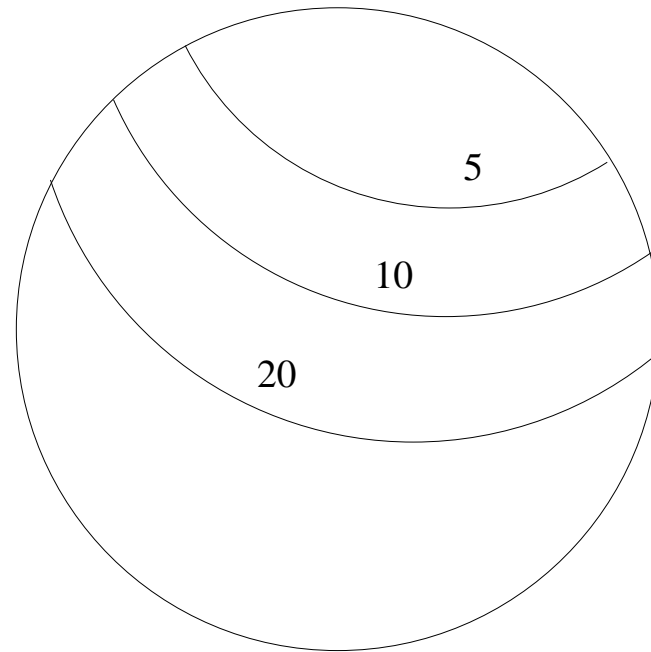cooling
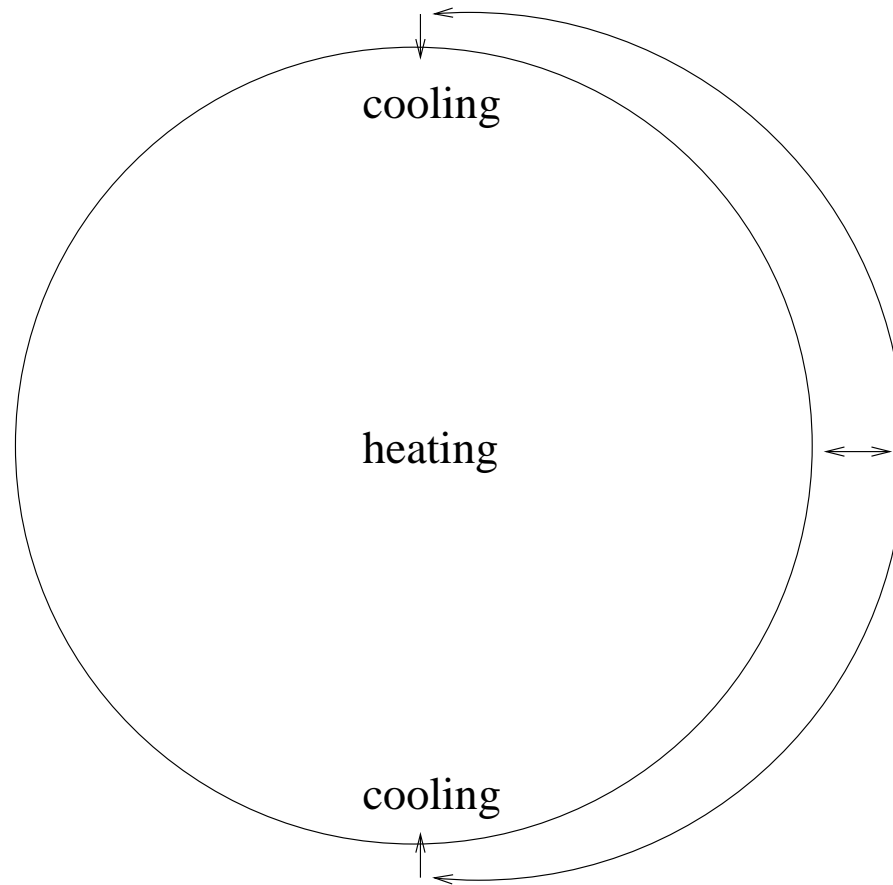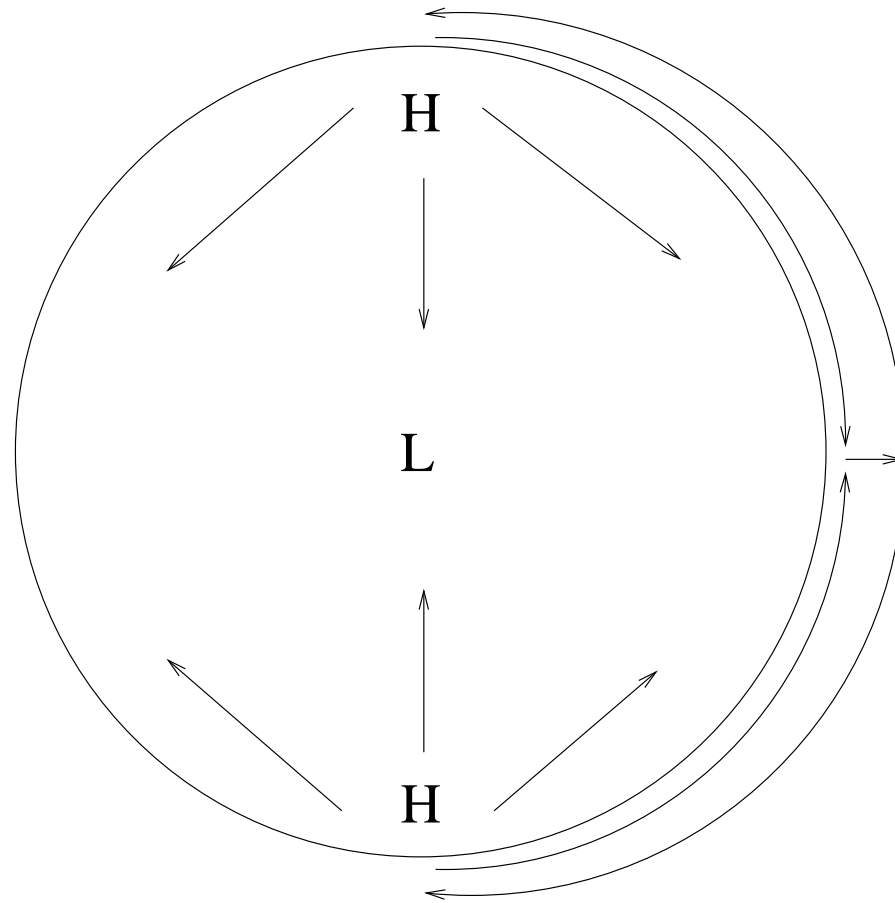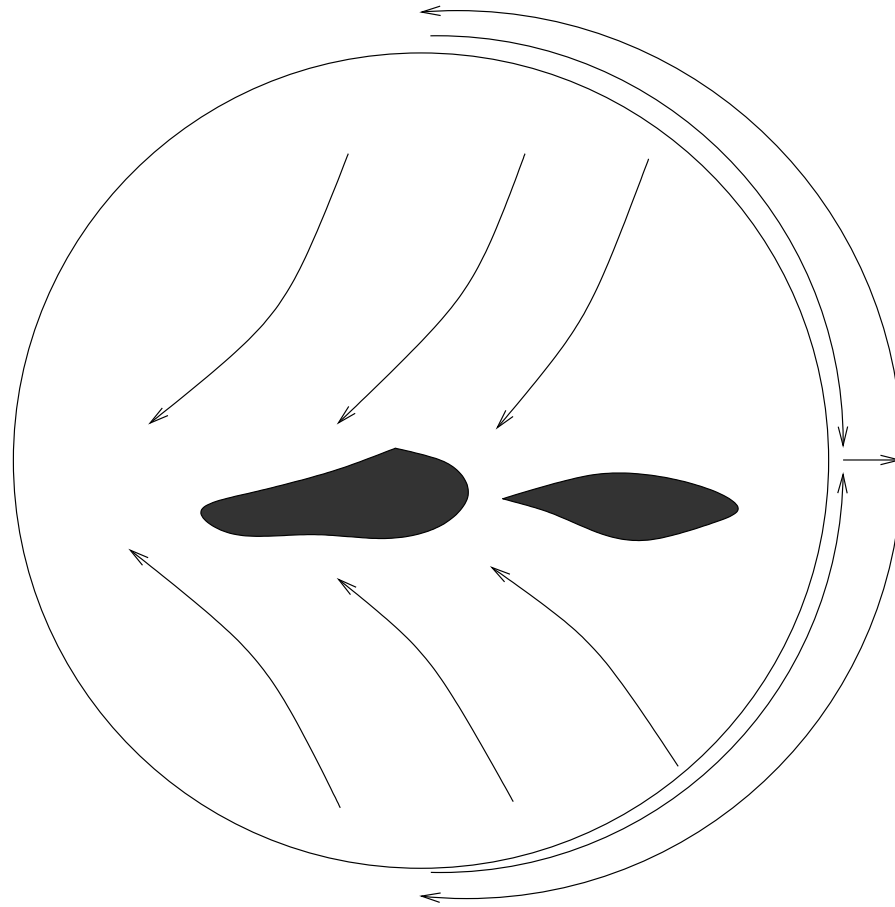
Spin-up

Spin-up

Spin-up
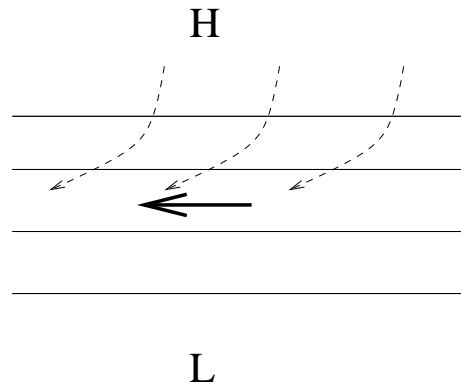
surface
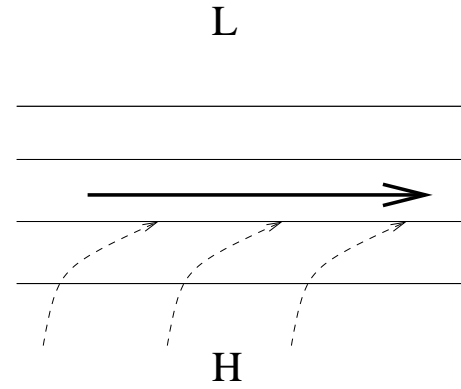
upper troposphere

Spin-up
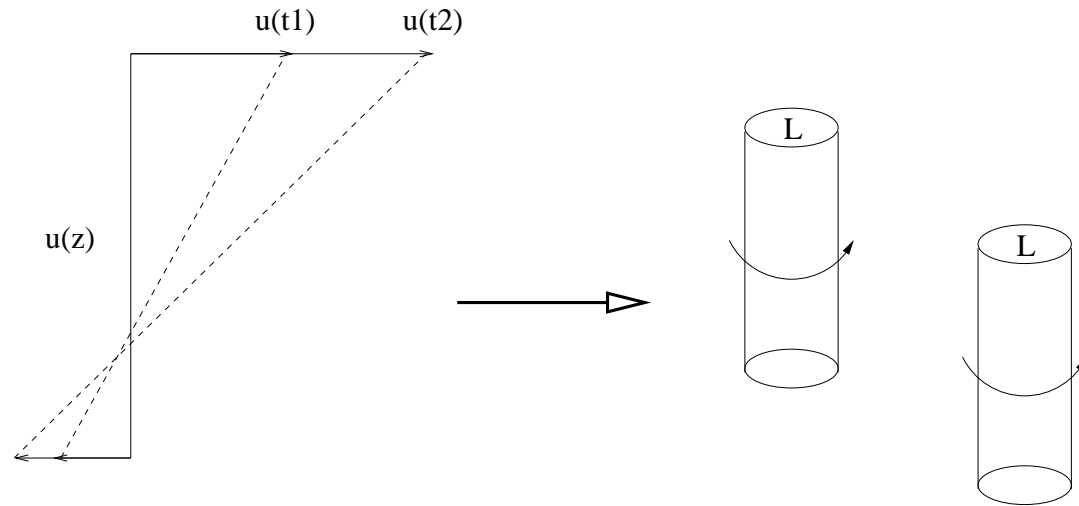
Spin-up
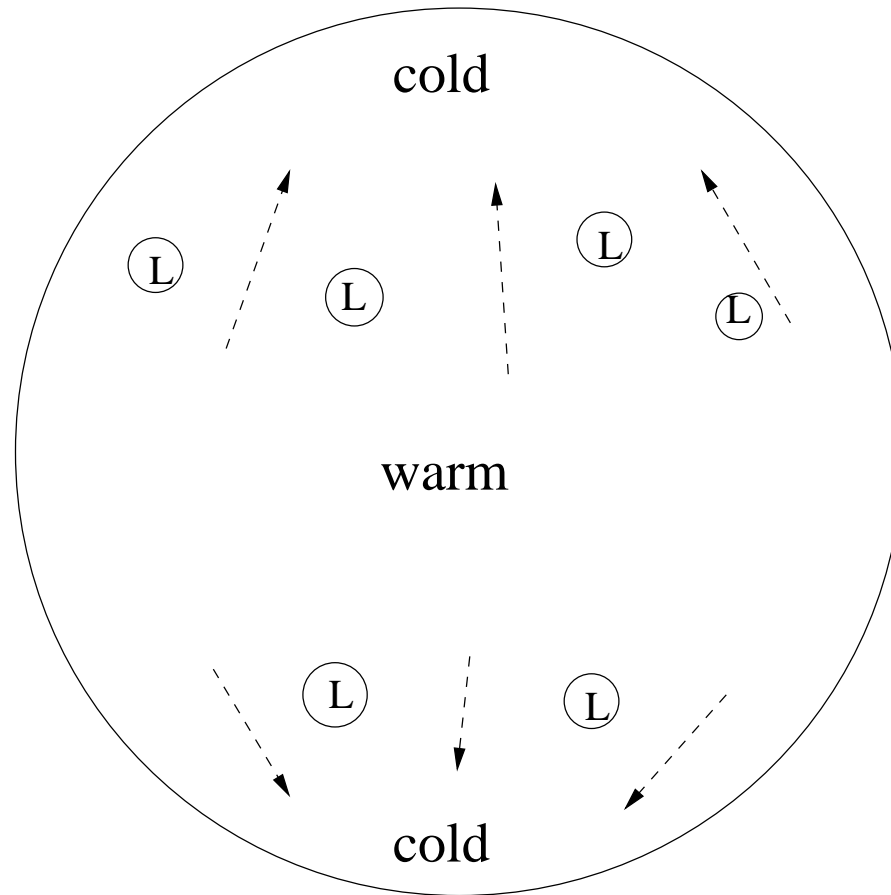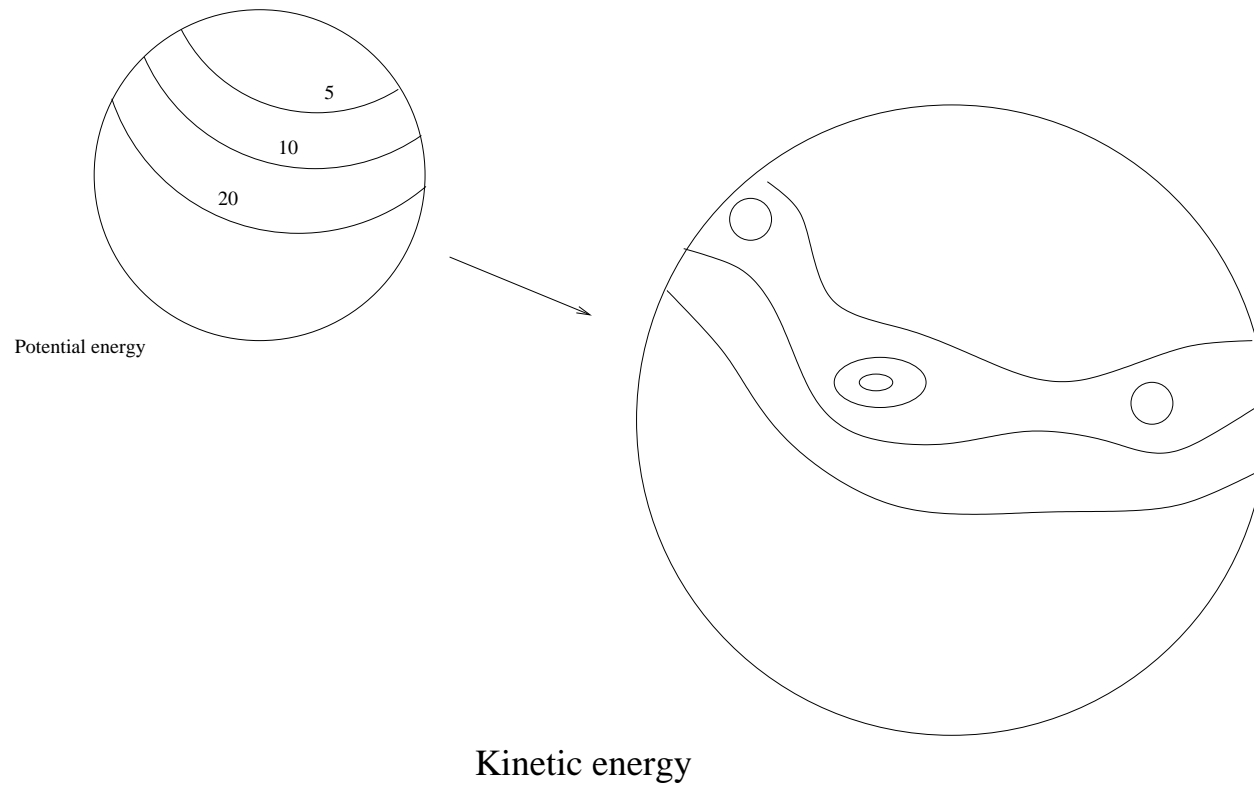
General circulation

Energy cycle

Heating

5

10

20
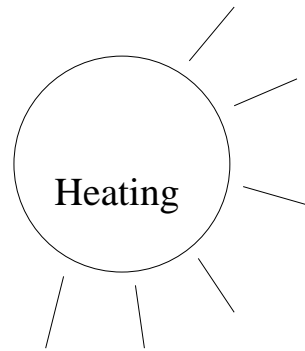
Potential energy

Energy cycle

Potential energy

Kinetic energy

Energy cycle

Turbulence

Friction