GEF 2220: Dynamics

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Primitive equations

Momentum:

$$\frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \frac{dq}{dt}$$

Momentum equations

Two types of forces:

1) Real   2) Apparent

Two ways to write the derivative:

1) Lagrangian   2) Eulerian

Derivatives

Example: derive the continuity equation using the Eulerian approach.

Then derive it using the Lagrangian approach.

Then derive it in pressure coordinates. What's best for this, Lagrangian or Eulerian?

Rotation

Acceleration:

$$(\frac{d\vec{u}_R}{dt})_R = (\frac{d\vec{u}_F}{dt})_F - 2\vec{\Omega} \times \vec{u}_R - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Two additional terms:

- Coriolis acceleration $\rightarrow \quad -2\vec{\Omega} \times \vec{u}_R$
- Centrifugal acceleration $\rightarrow \quad -\vec{\Omega} \times \vec{\Omega} \times \vec{r}$

Estimating forces

1) What is the centrifugal force for a parcel at the Equator?

Show that it is small compared to g.

2) What is the Coriolis force on a parcel moving eastward at $10\ m/sec$ at 45 N?

Find the direction and magnitude. Which component matters?

What happens in the Southern Hemisphere?

First law of thermodynamics

heat added = change in internal energy + work done:

$$dq = de + dw$$

At constant volume:

$$dq = c_v dT + p\, d\alpha$$

or, at constant pressure:

$$dq = c_p dT - \alpha dp$$

Scaling

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \qquad fU \qquad \frac{\triangle_H P}{\rho L}$$

$$10^{-4} \qquad 10^{-4} \qquad 10^{-4} \qquad 10^{-5} \qquad 10^{-6} \qquad 10^{-3} \qquad 10^{-3}$$

Geostrophy

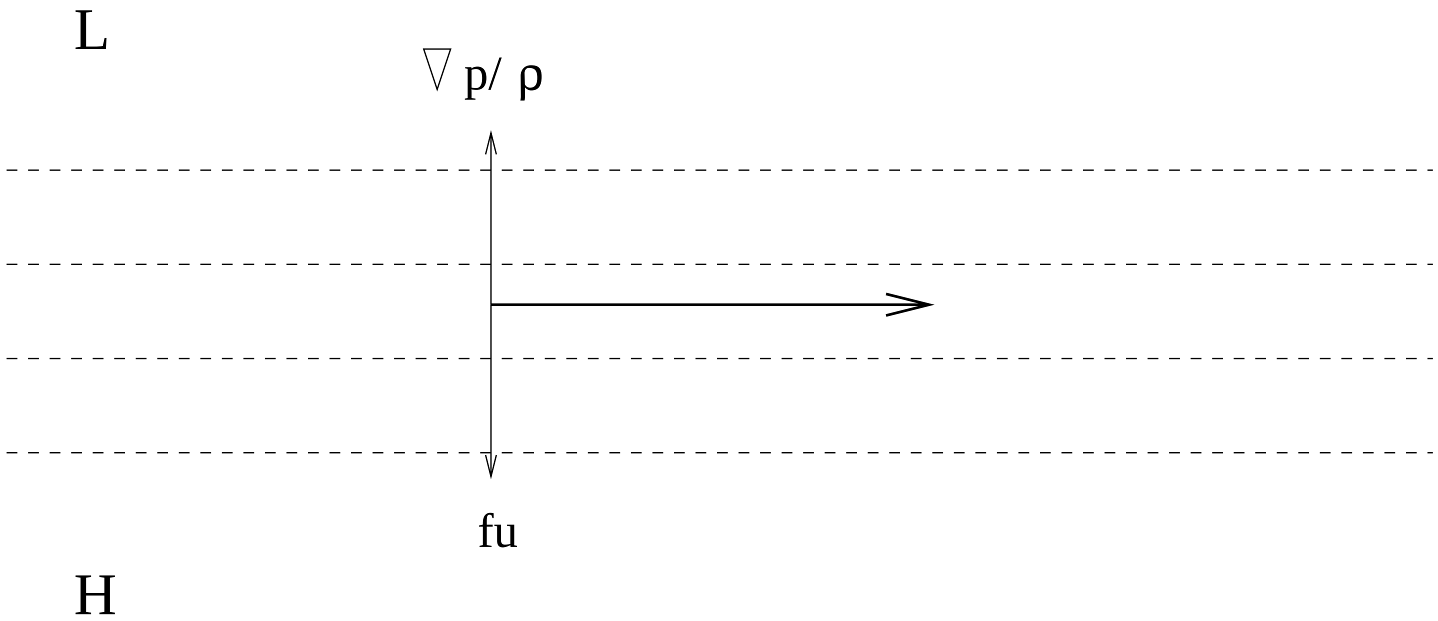
From scaling the horizontal momentum equations at weather scales:

$$f_z v = \frac{1}{\rho}\frac{\partial}{\partial x}p, \qquad f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

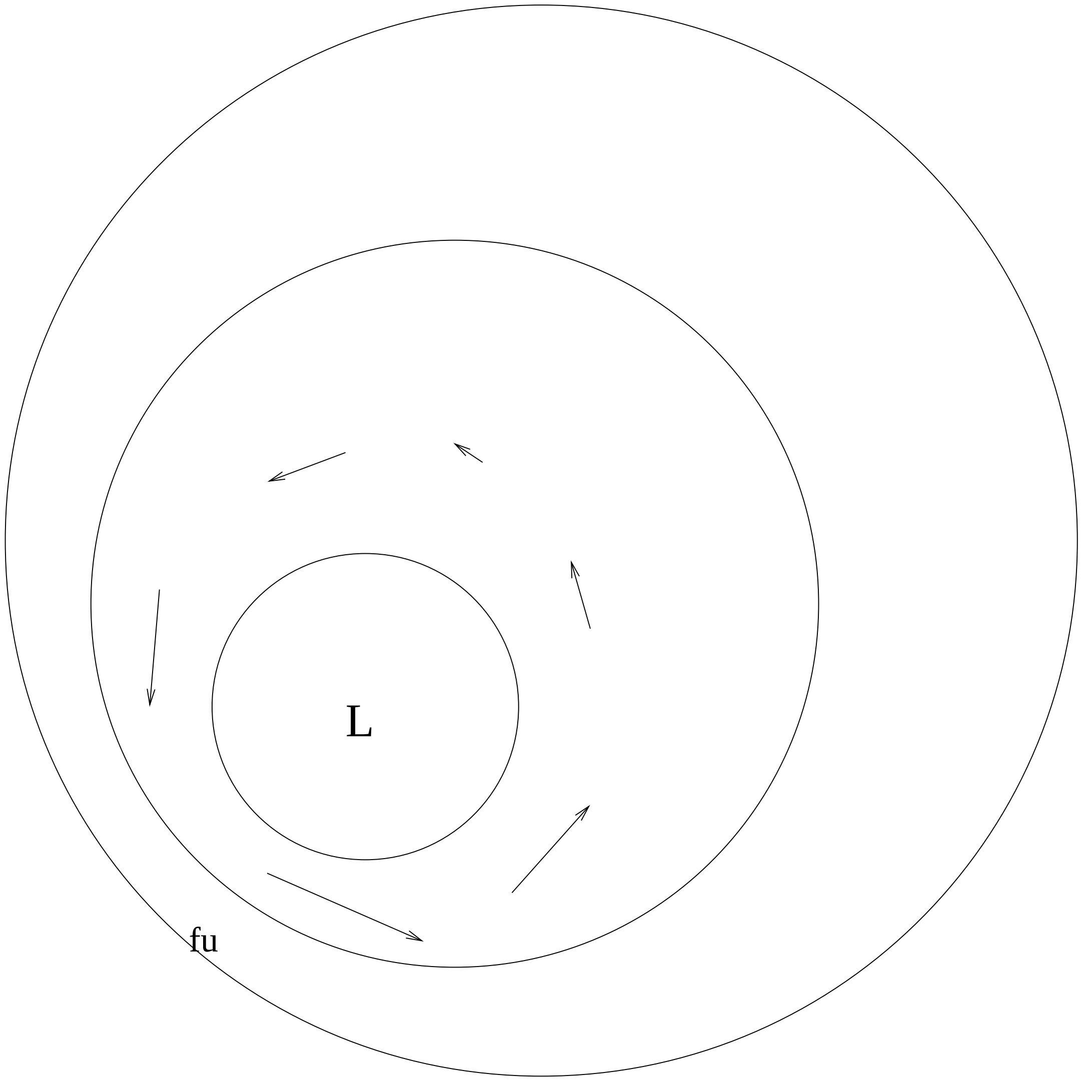
or, in pressure coordinates:

$$fv = \frac{\partial}{\partial x}\Phi, \qquad fu = -\frac{\partial}{\partial y}\Phi$$

Geostrophy



Geostrophy



Geostrophy

Given the pressure or geopotential, can derive the winds

Example: What is the pressure gradient required at the earth's surface at 45 N to maintain a geostrophic wind of 30 m/sec?

- Low geopotential height to left of the wind in Northern Hemisphere
- Lies to the *right* in Southern Hemisphere

Balance fails at equator, because $f_z = 2\Omega sin(0) = 0$

Geostrophy

Is a *diagnostic relation*

- Given the pressure, can calculate the horizontal velocities

But geostrophy cannot be used for *prediction*

Prediction

Must retain the 10^{-4} terms:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

The equations are *quasi-horizontal*: neglect vertical motion

Other momentum balances

Constant, circular motion:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho} \frac{\partial}{\partial r} p$$

Scaling:

$$\frac{U}{fR} \qquad 1 \qquad \frac{\triangle_H P}{\rho f U R}$$

First parameter is the *Rossby number*, ϵ

Other momentum balances

If $\epsilon \ll 1$, geostrophic balance:

$$f u_\theta = \frac{1}{\rho} \frac{\partial}{\partial r} p$$

If $\epsilon \ll 1$, cyclostrophic balance:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho} \frac{\partial}{\partial r} p$$

or:

$$u_\theta = \pm \left(\frac{r}{\rho} \frac{\partial}{\partial r} p\right)^{1/2}$$

Inertial oscillations

If $\frac{\partial}{\partial r} p = 0$, inertial motion:

$$\frac{u_\theta^2}{r} + f u_\theta = 0$$

or:

$$u_\theta = -fr$$

Gradient wind

If $\epsilon \approx 1$, gradient wind balance:

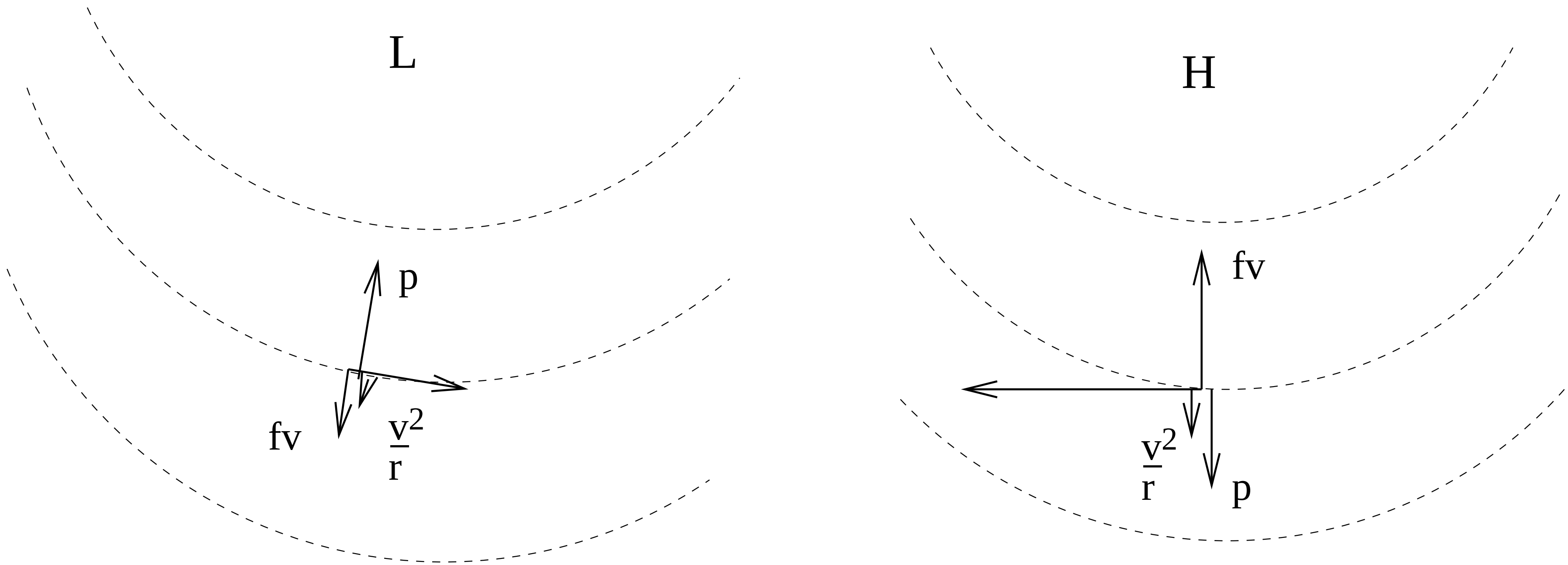
$$u_\theta = -\frac{1}{2} f r \pm \frac{1}{2}(f^2 r^2 + 4\, r\, f\, u_g)^{1/2}$$

If $u_g < 0$ (anticyclone), require:

$$|u_g| < \frac{f r}{4}$$

If $u_g > 0$ (cyclone), there is *no limit*

Gradient wind balance



Hydrostatic balance

Scale the vertical momentum equation:

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad fU \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

$$10^{-7} \qquad 10^{-7} \qquad 10^{-7} \qquad 10^{-10} \qquad 10^{-3} \qquad 10 \qquad 10$$

Hydrostatic balance

Remove the static contributions:

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \quad 10^{-1} \quad 10^{-1}$$

Vertical accelerations unimportant at synoptic scales.

Pressure coordinates

Use the hydrostatic balance to simplify equations

$$\frac{\partial p}{\partial x}|_z = \rho g \frac{dz}{dx}|_p \equiv \rho \, \frac{\partial \Phi}{\partial x}|_p$$

where:

$$\Phi \equiv \int_0^z g \, dz$$

Vertical velocities

Different vertical velocities:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\,\frac{\partial}{\partial x} + \frac{dy}{dt}\,\frac{\partial}{\partial y} + \frac{dp}{dt}\,\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p}$$

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

The flow is *incompressible* in pressure coordinates

$$\omega = -\int_{p*}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\, dp$$

Vertical motion

Showed that:

$$\omega \approx -\rho g w$$

Note that upward motion corresponds to $\omega < 0$

Thermal wind

Geostrophy + hydrostatic balance $\rightarrow$ velocity *shear*

$$v_g(p_1) - v_g(p_0) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \equiv \frac{g}{f}\frac{\partial}{\partial x} Z_{10}$$

and:

$$u_g(p_1) - u_g(p_0) = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \equiv -\frac{g}{f}\frac{\partial}{\partial y} Z_{10}$$

- Shear proportional to layer thickness, Z_{10}

Thermal wind

Alternate expression:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f}\, ln(\frac{p_1}{p_0})\, \frac{\partial \overline{T}}{\partial x}$$

$$u_g(p_1) - u_g(p_0) = \frac{R}{f}\, ln(\frac{p_1}{p_0})\, \frac{\partial \overline{T}}{\partial y}$$

- Shear proportional to the temperature gradient

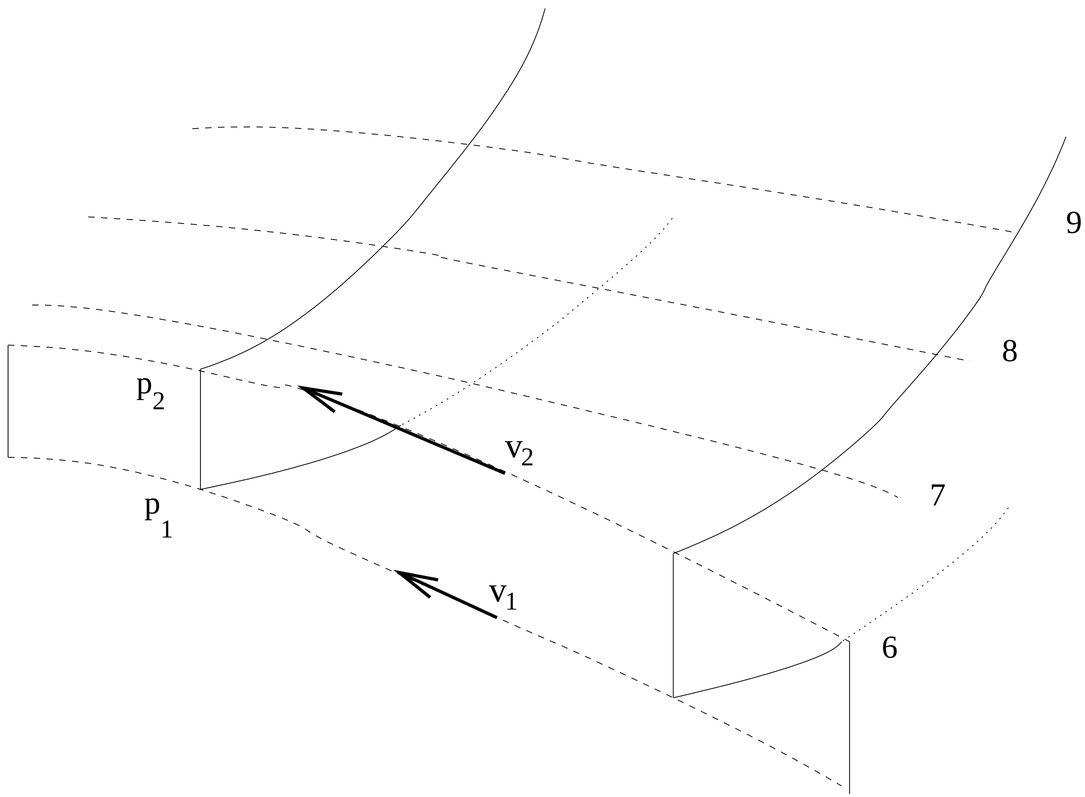
Thermal wind

Thus:

$$Z_{10} = \frac{R}{g}\,\overline{T}\,ln(\frac{p_1}{p_0})$$

- Layer thickness proportional to its mean temperature

Layer thickness

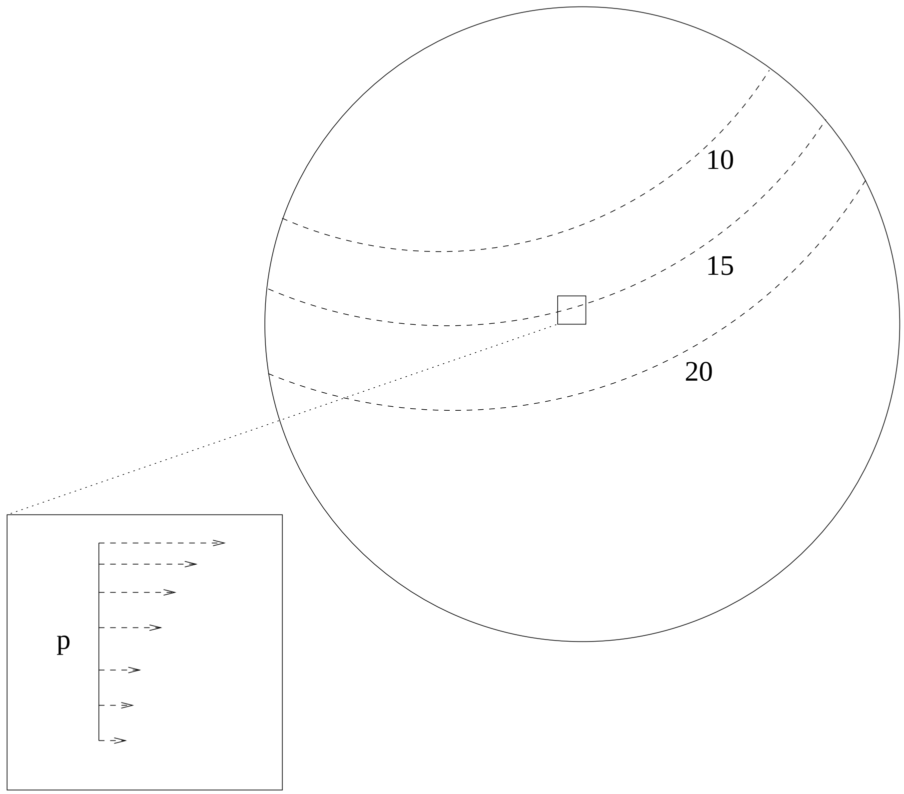


Definitions

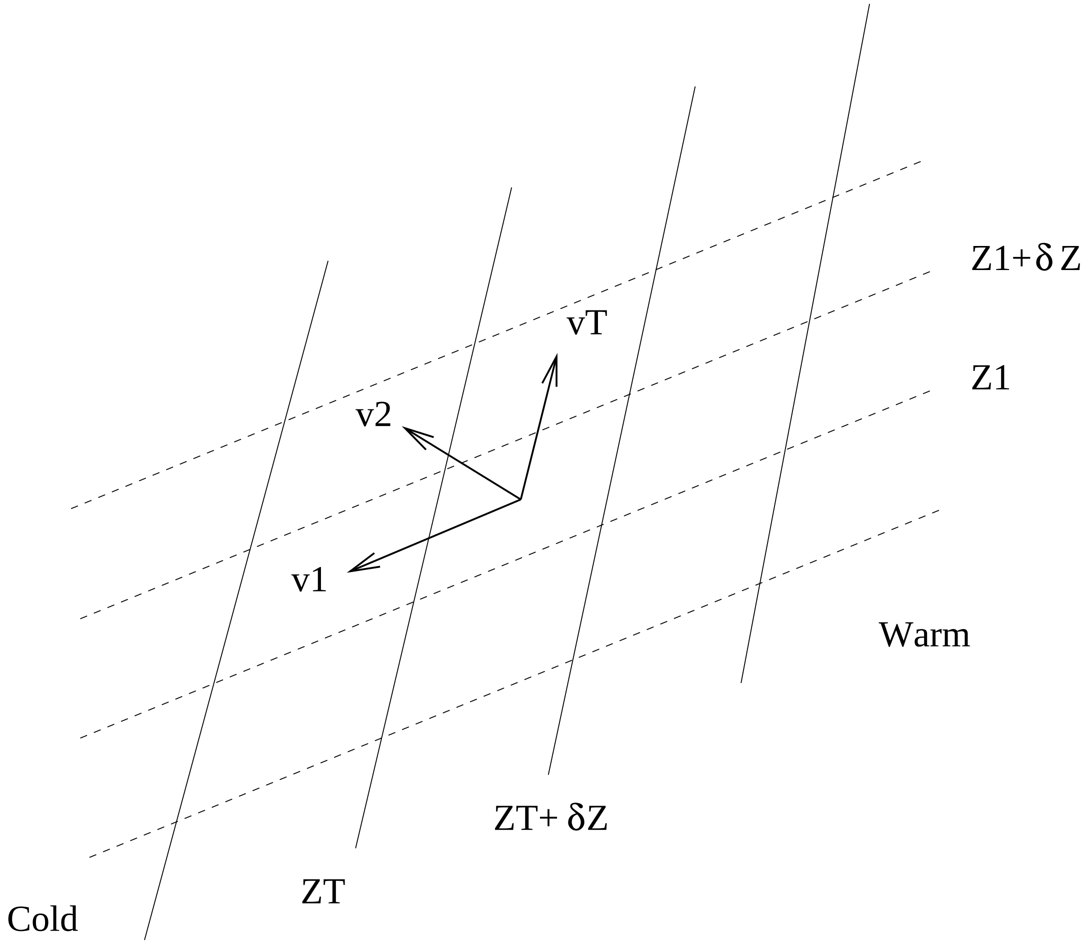
- Barotropic atmosphere: no vertical shear
- Equivalent barotropic: constant direction with height
- Baroclinic atmosphere: speed and direction change

Jet stream

Example: At 30N, the zonally-averaged temperature gradient is $0.75\ Kdeg^{-1}$, and the average wind is zero at the earth's surface. What is the mean zonal wind at the level of the jet stream ($250\ hPa$)?



Temperature advection



Temperature advection

Warm advection $\quad \longrightarrow \quad$ *veering*

- Anticyclonic (clockwise) rotation with height

Cold advection $\quad \longrightarrow \quad$ *backing*

- Cyclonic (counter-clockwise) rotation with height

Geostrophy vs. thermal wind

Geostrophic wind parallel to geopotential contours

- Wind with high pressure to the right (North Hemisphere)

Thermal wind parallel to thickness (mean temperature) contours

- Wind with high thickness to the right

Divergence and vorticity

$$D = \nabla \cdot \vec{u}$$

- The divergence in a region is constant and positive. What happens to the density of an air parcel?

$$\zeta = \nabla \times \vec{u}$$

- What is the vorticity of a typical tornado? Assume *solid body rotation*, with a velocity of 100 m/sec, 20 m from the center.

Vorticity equation

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,\zeta_a$$

$$= -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}) + (\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

where the *absolute vorticity* is:

$$\zeta_a = \zeta + f$$

Scaling

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta \quad \propto \frac{U^2}{L^2} \quad \approx \quad 10^{-10}$$

$$\omega\frac{\partial}{\partial p}\zeta \quad \propto \frac{U\omega}{LP} \quad \approx \quad 10^{-11}$$

$$v\frac{\partial}{\partial y}f \quad \propto U\frac{\partial f}{\partial y} \quad \approx \quad 10^{-10}$$

$$(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}) \quad \propto \frac{U\omega}{LP} \quad \approx \quad 10^{-11}$$

$$(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \approx f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \quad \propto \frac{fU}{L} \quad \approx \quad 10^{-9}$$

Scaling

Why is the divergence term unbalanced? Argued:

$$u = u_g + \epsilon u_a, \quad v = v_g + \epsilon v_a$$

with $\epsilon \approx 0.1$. So the divergence is:

$$D = \frac{\partial}{\partial x} u_g - \frac{\partial}{\partial y} v_g + \epsilon(\frac{\partial}{\partial x} u_a + \frac{\partial}{\partial y} v_a) \frac{1}{f}\frac{\partial}{\partial x}(-\frac{\partial \Phi}{\partial y}) + \frac{1}{f}\frac{\partial}{\partial y}(\frac{\partial \Phi}{\partial x})$$

$$+\epsilon(\frac{\partial}{\partial x} u_a + \frac{\partial}{\partial y} v_a) = \epsilon(\frac{\partial}{\partial x} u_a + \frac{\partial}{\partial y} v_a)$$

Scaling

Thus the divergence estimate is ten times smaller

Retain only the 10^{-10} terms:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = -f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

Rossby waves

No divergence:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = 0$$

Linearize, with a constant mean flow U:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\,\zeta + v\frac{\partial}{\partial y}f = 0$$

Beta-plane approximation: $f = f_0 + \beta y$.

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\,\zeta + \beta v = 0$$

Rossby waves

Write in terms of geopotential:

$$u = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi, \quad v = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi$$

$$\zeta = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = \frac{1}{f_0}\nabla^2\Phi$$

So:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\,\nabla^2\Phi + \beta\frac{\partial}{\partial x}\Phi = 0$$

Rossby waves

Substitute a wave solution:

$$\Phi = A\, exp(ikx + ily - i\omega t)$$

Get:

$$(-i\omega + ikU)(-k^2 - l^2)A + ikA = 0$$

or the *dispersion relation*:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

Rossby waves

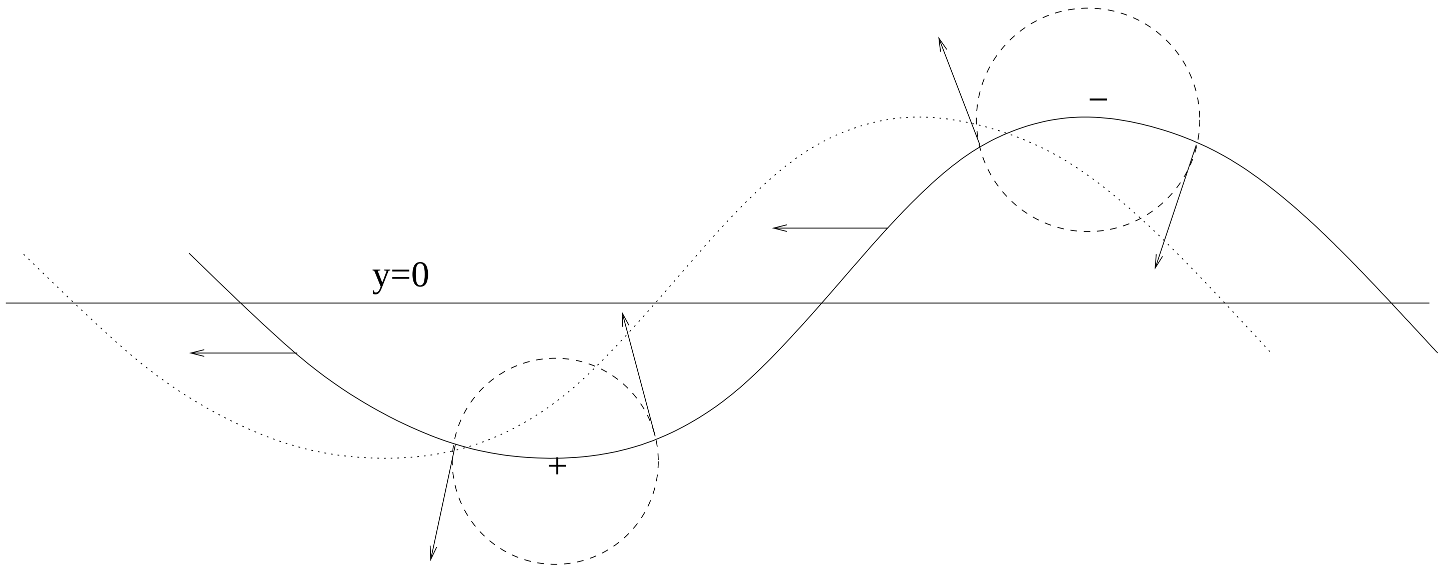
Has a phase speed of:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}$$

Waves move west relative to mean flow

Larger waves move faster than smaller waves (waves are *dispersive*)

Westward propagation



Divergence

Example: Consider small area of air:

$$\delta A = \delta x \, \delta y$$

Show that:

$$\frac{d}{dt} \zeta_a A = 0$$

This is *Kelvin's theorem*

Divergence

Important for storm development.

Example: Consider flow with constant divergence:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D$$

Example: show that divgence favors anticyclonic vorticity, convergence favors cyclonic. Show why anticyclonic vorticity is bounded and cyclonic not. This is why cyclones are more intense.

Forecasting

Method:

- Obtain $\Phi(x, y, t_0)$ from measurements on p-surface
- Calculate $u_g(t_0)$, $v_g(t_0)$, $\zeta_g(t_0)$
- Calculate $\zeta_g(t_1)$
- *Invert* ζ_g to get $\Phi(t_1)$
- Start over
- Obtain $\Phi(t_2)$, $\Phi(t_3)$,...

Barotropic potential vorticity

Imagine atmosphere as a layer with constant density. Then:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

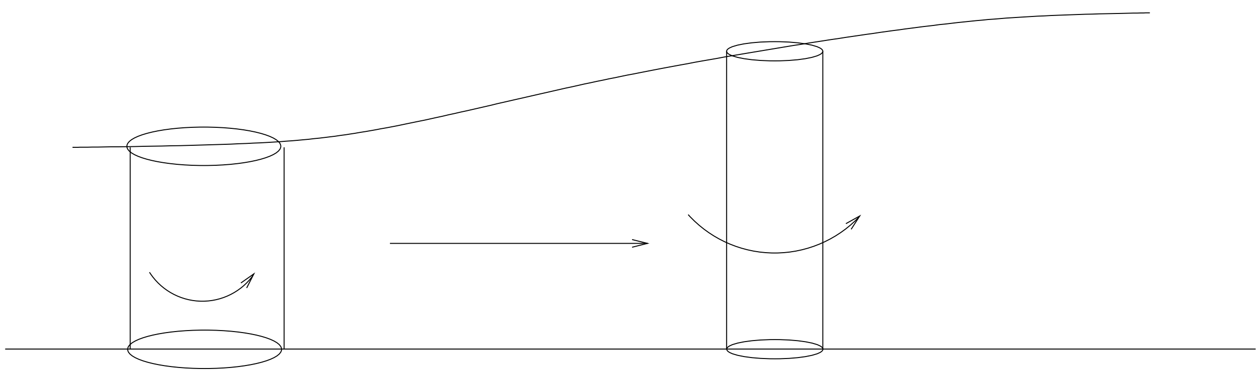
So:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,\frac{\partial w}{\partial z}$$

Derive:

$$\frac{d}{dt}(\frac{\zeta + f}{h}) = 0$$

Layer potential vorticity



Adiabat

Same idea applies to motion on an *adiabatic*

Flow between two isentropic surfaces trapped if zero heating

Derive Ertel's potential vorticity:

$$\frac{d}{dt}[(\zeta + f)\frac{\partial \theta}{\partial p}] = 0$$

Vorticity summary

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

or:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0$$

- Vorticity changes due to meridional motion
- Vorticity changes due to divergence
- Change in layer thickness

Planetary boundary layer

Small scale (unresolved) eddies mix momentum vertically:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} = -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} = -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

Too many unknowns, so parameterize the eddy stresses

Two cases:

- Stable boundary layer: stratified, sheared flow
- Unstable boundary layer: convective mixing, no shear

Convective boundary layer

With empirical surface stress:

$$f h(\overline{v} - \overline{v}_g) = C_d \mathcal{V}\, \overline{u}$$

and:

$$-f h(\overline{u} - \overline{u}_g) = C_d \mathcal{V}\, \overline{v}$$

Stable boundary layer

$$f(\overline{v} - \overline{v}_g) = \frac{\partial}{\partial z}(A_z \frac{\partial}{\partial z}\overline{u})$$

and:

$$-f(\overline{u} - \overline{u}_g) = \frac{\partial}{\partial z}(A_z \frac{\partial}{\partial z}\overline{v})$$

- Example 1: the Ekman layer ($A_z = \text{const.}$)
- Example 2: the Surface layer ($A_z = k^2 z^2 \frac{\partial}{\partial z}\mathcal{V}$)

Boundary layer effects

Most important is how PBL affects troposphere

Permits *downgradient* winds

Divergence/convergence causes vertical motion into troposphere

Weakens pressure system, causing them to *spin down*

Spin-down

Example: Ekman layer. Can derive:

$$\frac{D}{Dt}(\zeta + f) = -\frac{fd}{2H}\zeta$$

If $f = const.$, then:

$$\zeta(t) = \zeta(0)\, exp(-t/\tau_E)$$

where:

$$\tau_E \equiv \frac{2H}{fd}$$

General circulation