GEF 2220: Dynamics

J. H. LaCasce

Department for Geosciences

University of Oslo

Primitive equations

Momentum:

$$\frac{\partial}{\partial t}u + \vec{u}\cdot\nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u}\cdot\nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u}\cdot\nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w$$

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \frac{dq}{dt}$$

Primitive equations

Six equations, six unknowns:

- (u, v, w) — velocities
- p — pressure
- ρ — density
- T — temperature

Primitive equations

Momentum equations $\quad \leftarrow \quad F = ma$

Continuity $\quad \leftrightarrow \quad \rho$

Thermodynamic energy equation $\quad \leftrightarrow \quad T$

Ideal gas law relates ρ, p and T

Prediction

Solve the equations numerically with weather models

Issues:

- Numerical resolution
- Vertical coordinate
- Small scale mixing
- Convection
- Clouds

Goal: forecasting

Dynamics

Solve a simplified set of equations

- Identify dominant balances
- Simplify the equations
- Obtain solutions (analytical, numerical)
- Look for similiarities with observations

Goal: *understanding* the atmosphere

Momentum equations

Take the x-momentum equation:

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial t^2}x = \frac{1}{\rho}\sum_i F_i$$

which is like:

$$a_x = \frac{1}{m}\sum_i F_i$$

Momentum equations

Two types of forces:

1) Real   2) Apparent

Two ways to write the derivative:

1) Lagrangian   2) Eulerian

Derivatives

Consider an air parcel, with temperature T

$T = T(x, y, z, t)$

The change in temperature, from the chain rule:

$$dT = \frac{\partial}{\partial t} T \, dt + \frac{\partial T}{\partial x} \, dx + \frac{\partial T}{\partial y} \, dy + \frac{\partial T}{\partial z} \, dz$$

Derivatives

So:

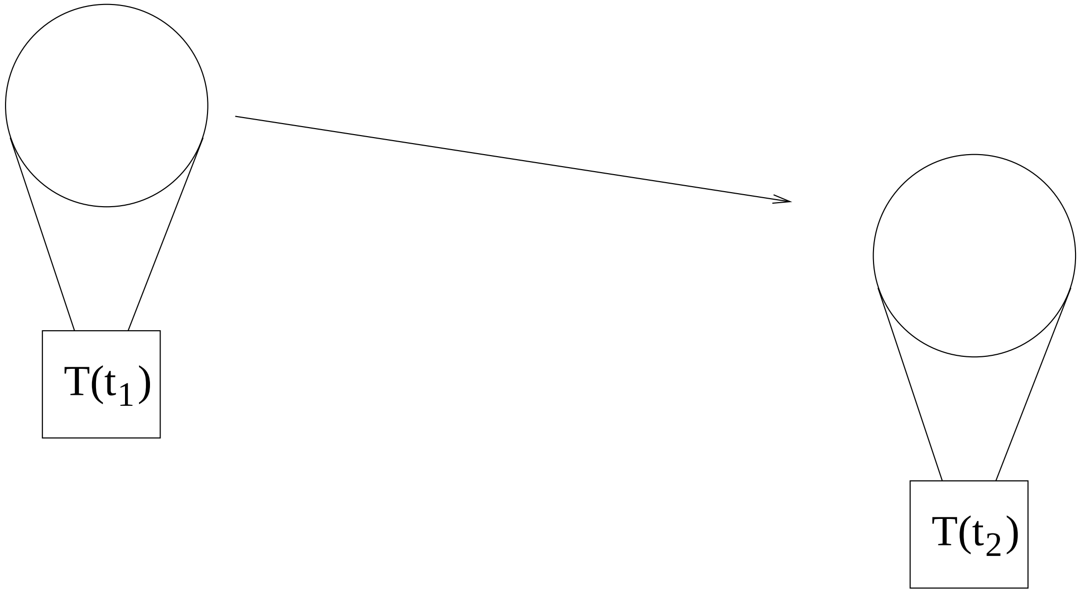
$$\frac{dT}{dt} = \frac{\partial}{\partial t}T + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z}$$

$$= \frac{\partial}{\partial t}T + \vec{u}\cdot\nabla T$$

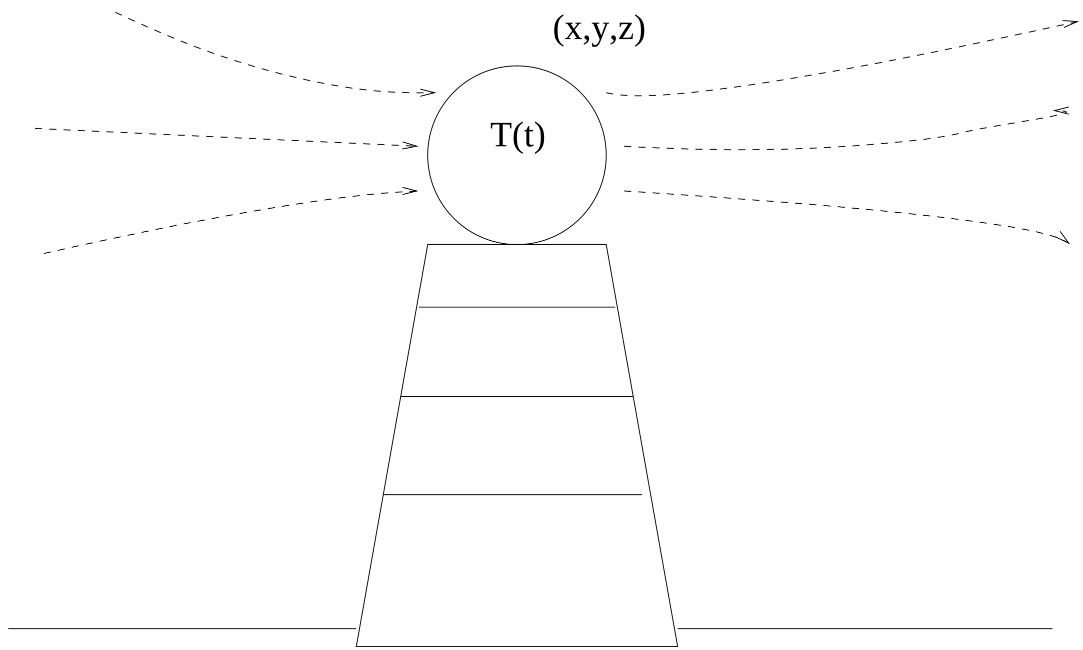
$\frac{d}{dt}$ is the “Lagrangian” derivative

$\frac{\partial}{\partial t} + \vec{u}\cdot\nabla$ is the “Eulerian” representation

Lagrangian



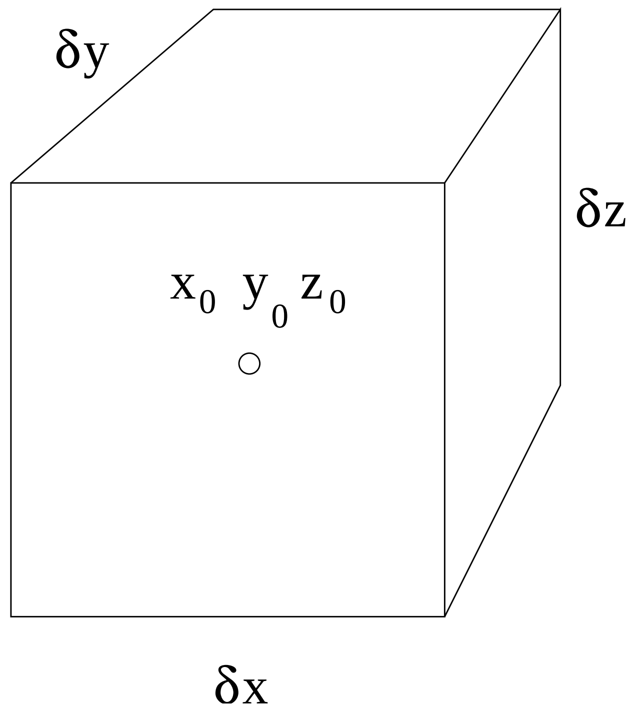
Eulerian



Real forces

- Pressure gradients
- Gravity
- Friction

Pressure gradient



$$\delta V = \delta x\ \delta y\ \delta z$$

Pressure gradient

Use *Taylor series*:

$$G(x_0 + \delta x) = G(x_0) + \frac{\partial G}{\partial x}\,\delta x + \frac{1}{2}\,\frac{\partial^2 G}{\partial x^2}\,\delta x^2 + ...$$

Pressure on the right side of the box:

$$p = p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2} + ...$$

Pressure on left side of the box:

$$p = p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2} + ...$$

Pressure gradient

The force on the right hand side (directed inwards):

$$p = -[p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

On left side:

$$p = [p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

So the net force is:

$$F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

Pressure gradient

The volume weighs:

$$m = \rho \, \delta x \, \delta y \, \delta z$$

So:

$$a_x = \frac{F_x}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

Same derivation for the y and z directions.

Momentum equations

Momentum:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}p + ...$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial y}p + ...$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p + ...$$

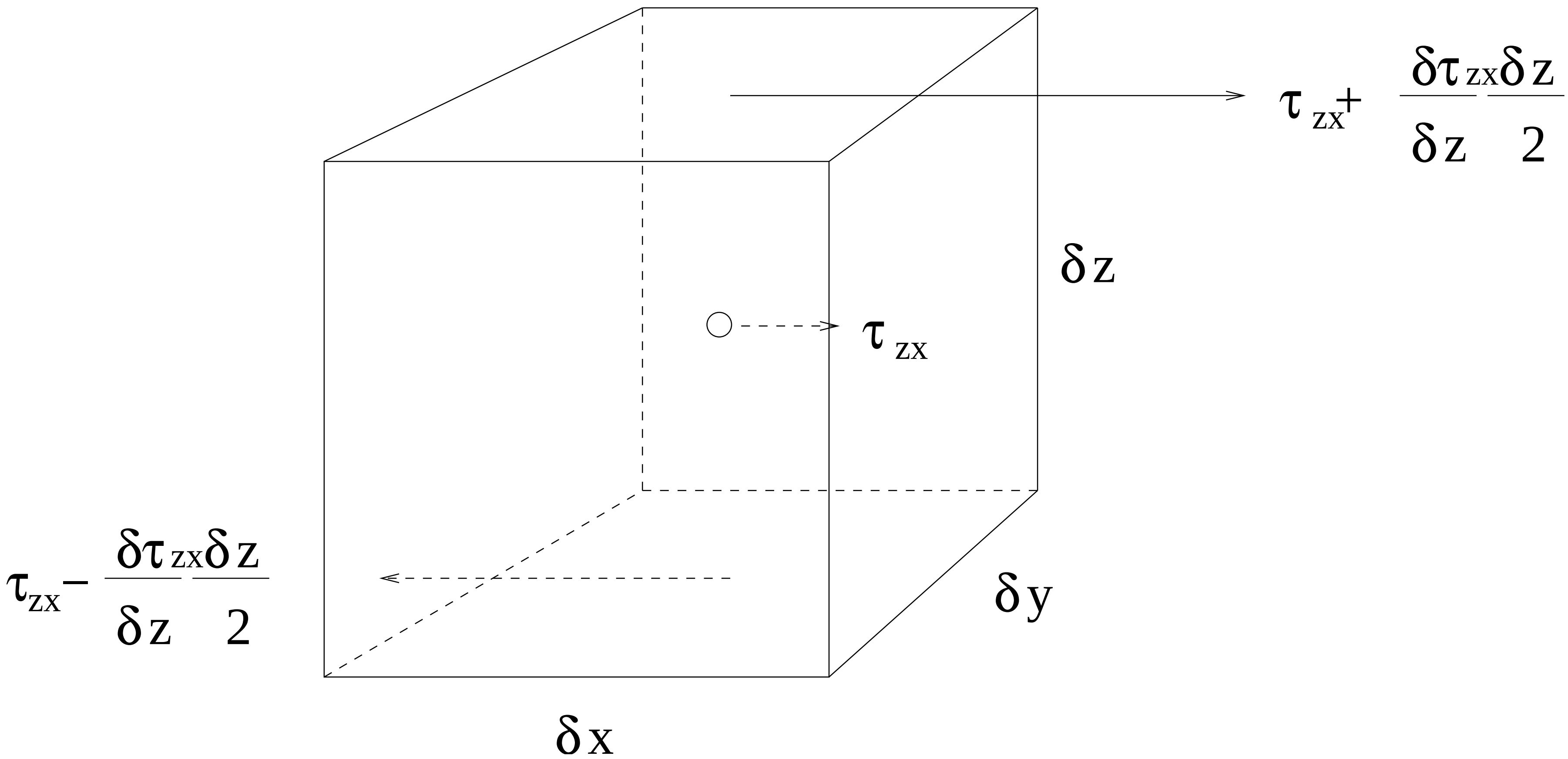
Gravity

Acts downward (toward the center of the earth):

$$a_z = \frac{F_z}{m} = -g$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

Friction



$\tau_{zx} + \frac{\delta \tau_{zx}}{\delta z} \frac{\delta z}{2}$
δz
τ_{zx}
$\tau_{zx} - \frac{\delta \tau_{zx}}{\delta z} \frac{\delta z}{2}$
δy
δx

Friction

Net viscous force (stress $\times$ area) of the boundaries acting *on the fluid* :

$$(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2})\,\delta x\,\delta y - (\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2})\,\delta x\,\delta y$$

$$= \frac{\partial \tau_{zx}}{\partial z}\,\delta x\,\delta y\,\delta z$$

Divide by the mass of the box:

$$F_{zx} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$$

Friction

Similar derivations for τ_{zy}, τ_{xx}, ...

Applied to x-direction:

$$\frac{du}{dt} = \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

Problem: we don’t know the stresses (τ_{xx}, etc.)!

Friction

So we *parameterize* the stress, assuming molecular mixing:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \equiv \frac{1}{\rho}\frac{\partial}{\partial z}(\mu \frac{\partial u}{\partial z})$$

If μ is constant:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \nu \frac{\partial^2}{\partial z^2} u$$

where the molecular *viscosity* is:

$$\nu = \frac{\mu}{\rho} = 1.46 \times 10^{-5} \ m^2/sec$$

Friction

Applied to the x-direction:

$$\frac{du}{dt} = \nu \frac{\partial^2}{\partial z^2} u$$

- Friction acts to *diffuse momentum*
- Reduces the velocity *shear*.

Momenutum equations

With the friction terms, have:

$$\frac{du}{dt} = \nu\,(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u) = \nu\nabla^2 u$$

$$\frac{dv}{dt} = \nu\,(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v) = \nu\nabla^2 v$$

$$\frac{dw}{dt} = \nu\,(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}w + \frac{\partial^2}{\partial z^2}w) = \nu\nabla^2 w$$

Momentum equations

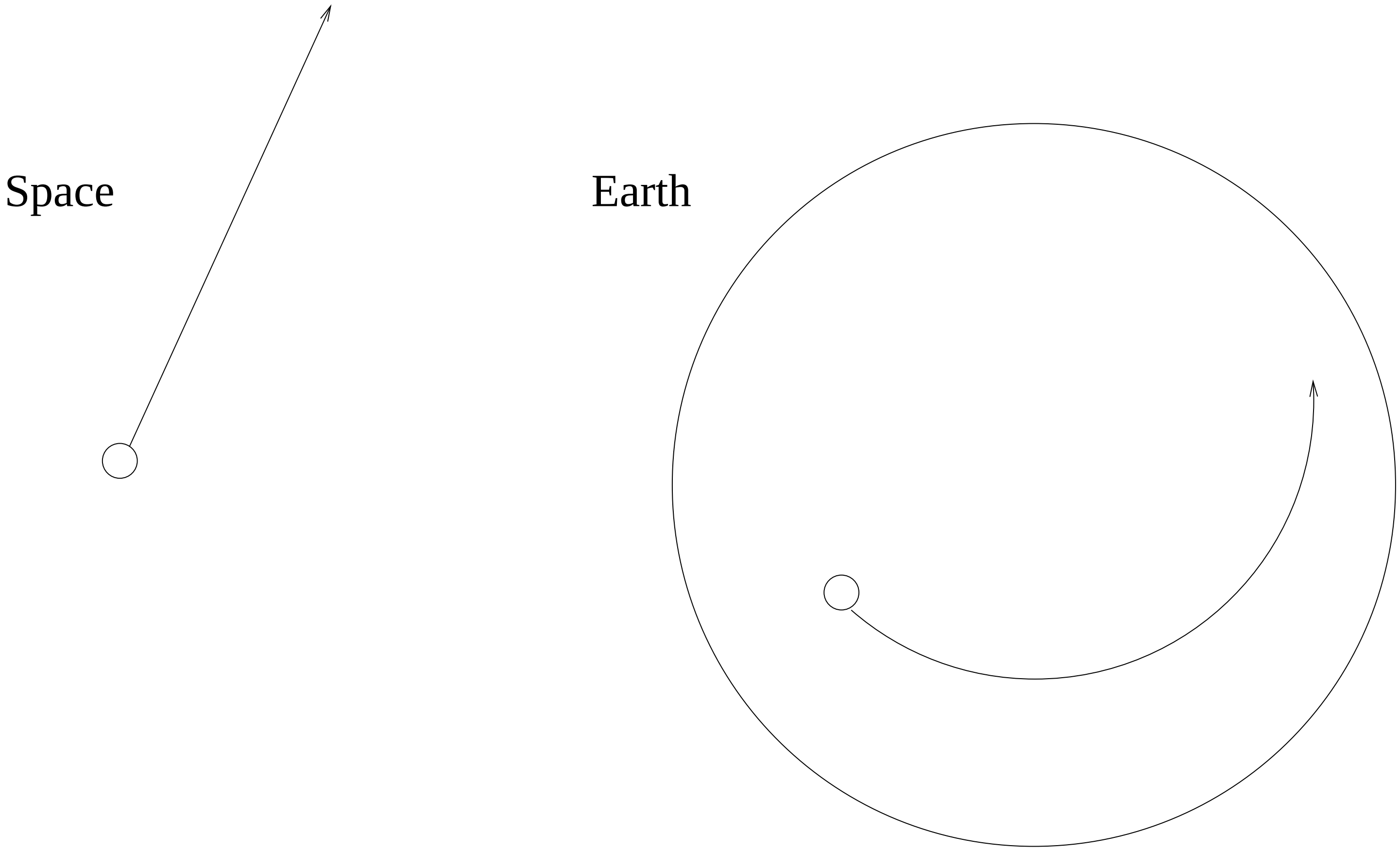
So far:

$$\frac{du}{dt} = \frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

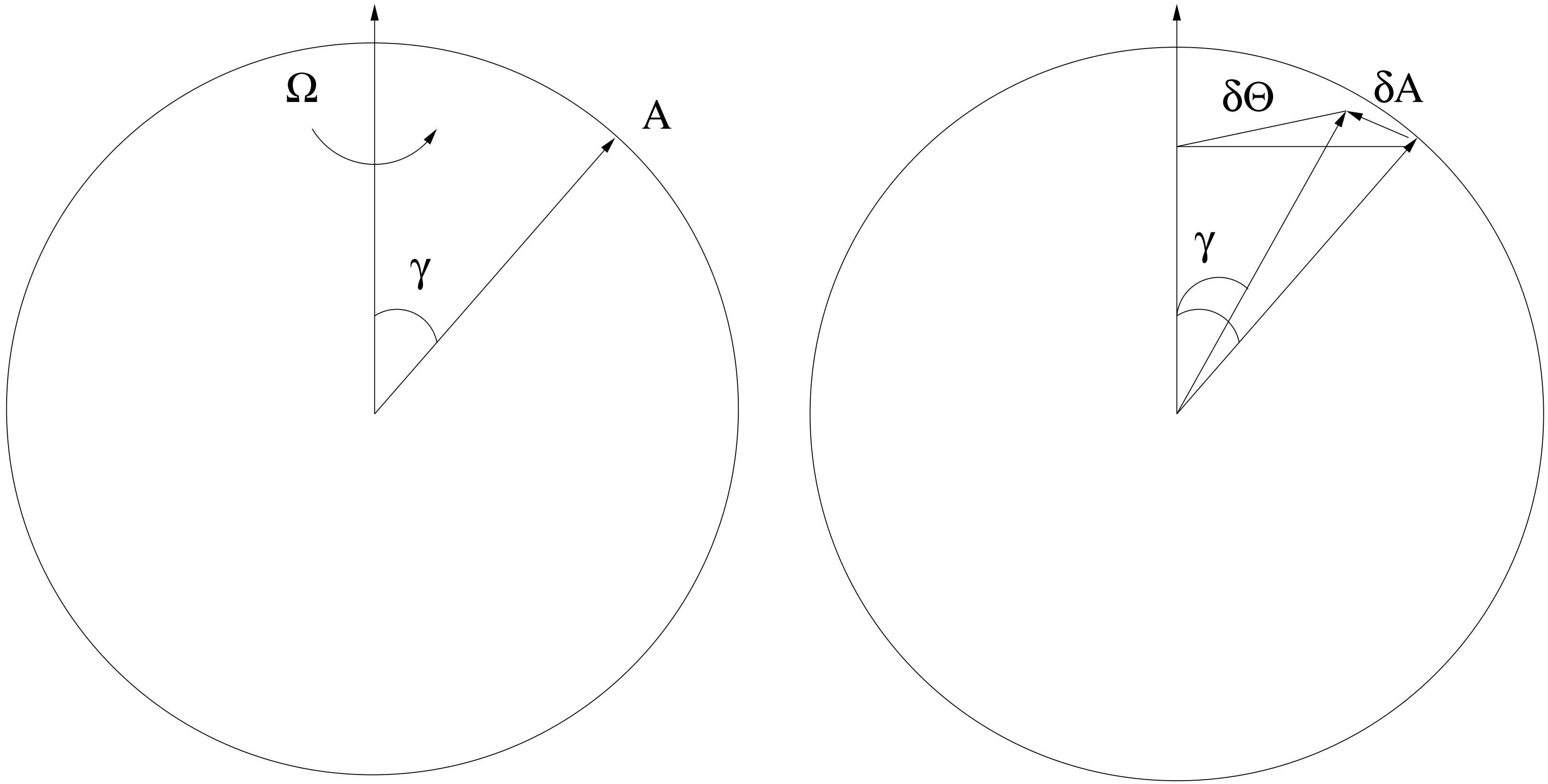
$$\frac{dv}{dt} = \frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{dw}{dt} = \frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Apparent forces



Rotation



Rotation

$$\delta\Theta = \Omega\delta t$$

Assume $\Omega = const.$ (reasonable for the earth)

Change in A is δA, the arc-length:

$$\delta\vec{A} = |\vec{A}|sin(\gamma)\delta\Theta = \Omega|\vec{A}|sin(\gamma)\delta t = (\vec{\Omega} \times \vec{A})\,\delta t$$

Rotation

So:

$$\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A}$$

This is the motion of a *fixed* vector. For a moving vector:

$$\left(\frac{d\vec{A}}{dt}\right)_F = \left(\frac{d\vec{A}}{dt}\right)_R + \vec{\Omega} \times \vec{A}$$

So the velocity in the fixed frame is equal to that in the rotating frame plus the rotational movement

Rotation

If $\vec{A} = \vec{r}$, the position vector, then:

$$(\frac{d\vec{r}}{dt})_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

If $\vec{A} = \vec{r}$, we get the acceleration:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_F}{dt})_R + \vec{\Omega} \times \vec{u}_F = [\frac{d}{dt}(u_R + \vec{\Omega} \times \vec{r})]_R + \vec{\Omega} \times \vec{u}_F$$

$$= (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Rotation

Rearranging:

$$(\frac{d\vec{u}_R}{dt})_R = (\frac{d\vec{u}_F}{dt})_F - 2\vec{\Omega} \times \vec{u}_R - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Two additional terms:

- Coriolis acceleration $\rightarrow \quad -2\vec{\Omega} \times \vec{u}_R$
- Centrifugal acceleration $\rightarrow \quad -\vec{\Omega} \times \vec{\Omega} \times \vec{r}$

Centrifugal acceleration

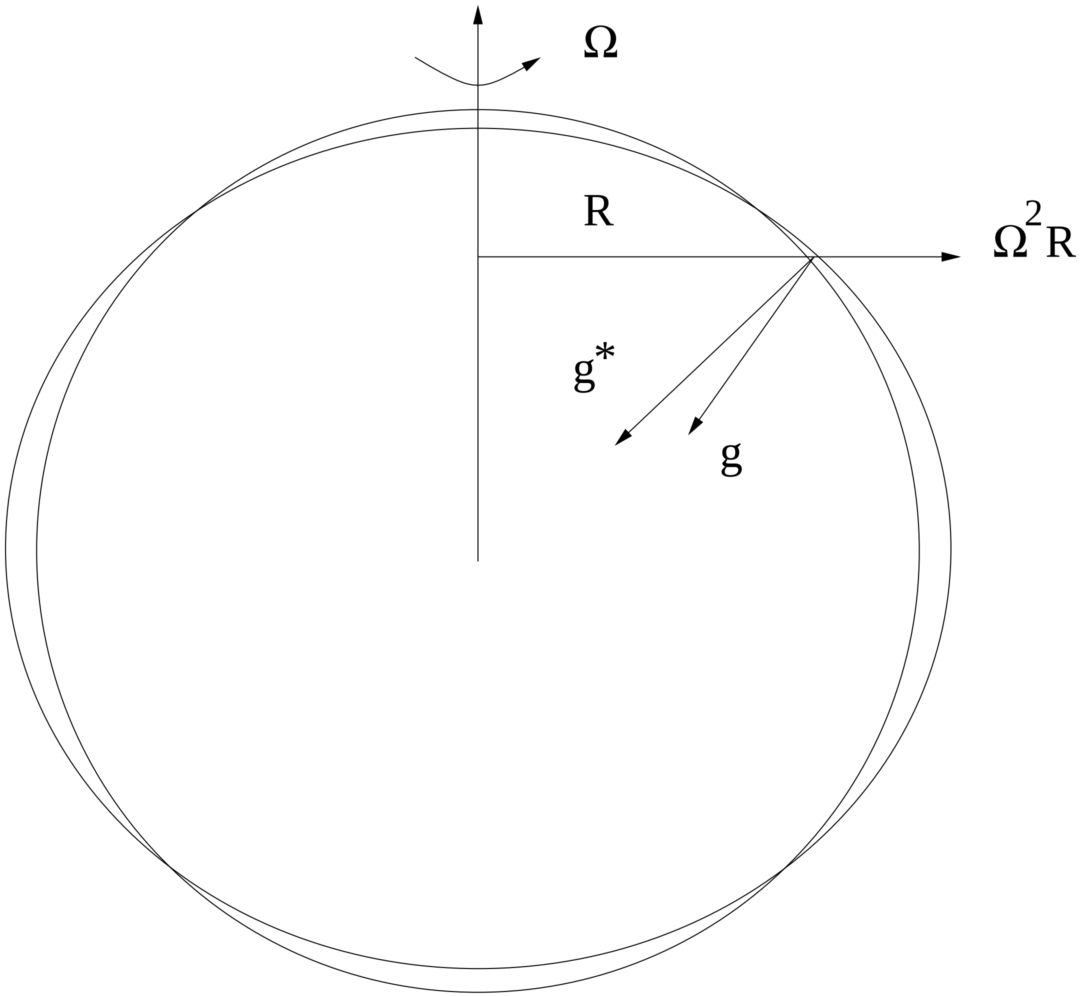
Rotation requires a force towards the center of rotation—the *centripetal acceleration*

From the rotating frame, the sign is opposite—the *centrifugal acceleration*

Acceleration points out from the earth's radius of rotation

So has components in the radial and N-S directions

Centrifugal



Centrifugal

The earth is not spherical, but has deformed into an oblate spheroid

As a result, there is a component of gravity which exactly balances the centrifugal force in the N-S direction

Defines surfaces of constant *geopotential*

The locally vertical centrifugal acceleration can be absorbed into gravity:

$$g' \approx g - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Centrifugal

Example: What is the centrifugal force for a parcel of air at the Equator?

$$-\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\Omega \times (\Omega r) = \Omega^2 r$$

with:

$$r_e = 6.378 \times 10^6 \; m$$

and:

$$\Omega = \frac{2\pi}{3600(24)} \; sec$$

Centrifugal

So:

$$\Omega^2 r_e = 0.034\ m/sec^2$$

This is much smaller than $g = 9.8\ m^2/sec$

- Only a minor change to absorb into g'

Geopotential

Gravity represented as the gradient of a potential:

$$\nabla \phi = -\vec{g}$$

Because $\vec{g} = -g\hat{k}$, then $\phi = \phi(z)$

If we set $\phi = 0$ at sea level, then:

$$\phi(z) = \int_0^z g\, dz$$

Cartesian coordinates

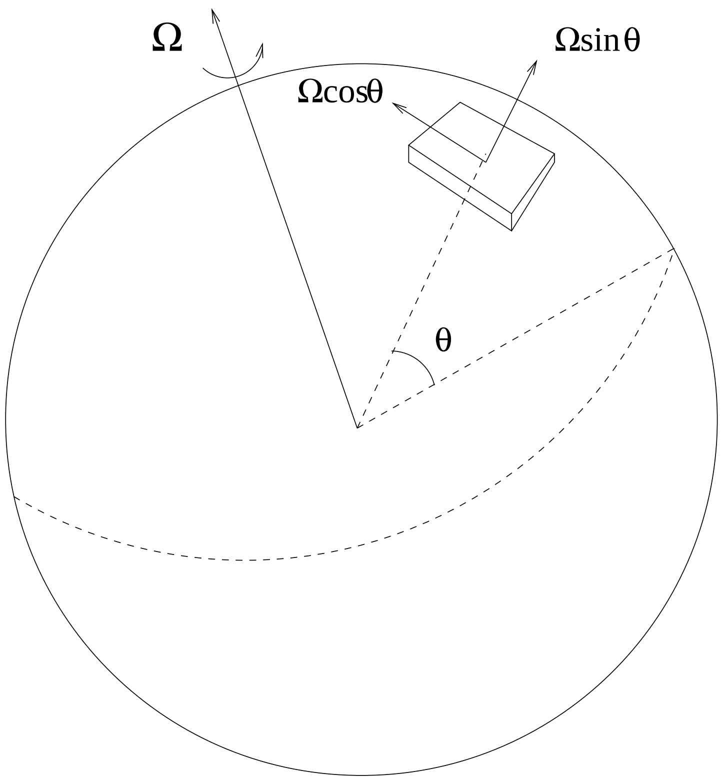
Earth radius at equator is only 21 km larger than at the poles

So can use spherical coordinates

However, we will primarily use *Cartesian* coordinates

- Simplifies the math
- Neglected terms are *unimportant* at weather scales

Cartesian coordinates



Coriolis force

Rotation vector projects onto local vertical and meridional directions:

$$2\vec{\Omega} = 2\Omega cos\theta \ \hat{j} + 2\Omega sin\theta \ \hat{k} \equiv f_y \ \hat{j} + f_z \ \hat{k}$$

So the Coriolis force is:

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, v, w)$$

$$= -(f_y w - f_z v, f_z u, -f_y u)$$

Momentum equations

Move Coriolis terms to the LHS:

$$\frac{\partial}{\partial t}u + \vec{u}\cdot\nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u}\cdot\nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u}\cdot\nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w$$

Coriolis force

Example: What is the Coriolis force on a parcel moving eastward at $10\ m/sec$ at 45 N?

We have:

$$f_y = f_z = 2\Omega cos(45) = (7.292 \times 10^{-5})(0.7071)$$

$$= 5.142 \times 10^{-5}\ sec^{-1}$$

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, 0, 0) = -f_z u\ \hat{j} + f_y u \hat{k}$$

$$= (0, -5.142 \times 10^{-4}, 5.142 \times 10^{-4})\ m/sec^2$$

Coriolis force

Vertical acceleration is negligible compared to gravity ($g = 9.8\ m/sec^2$), so has little effect in z

Horizontal acceleration is to the *south*

- Coriolis acceleration is most important in the horizontal direction
- Acts *to the right* in the Northern Hemisphere

Coriolis force

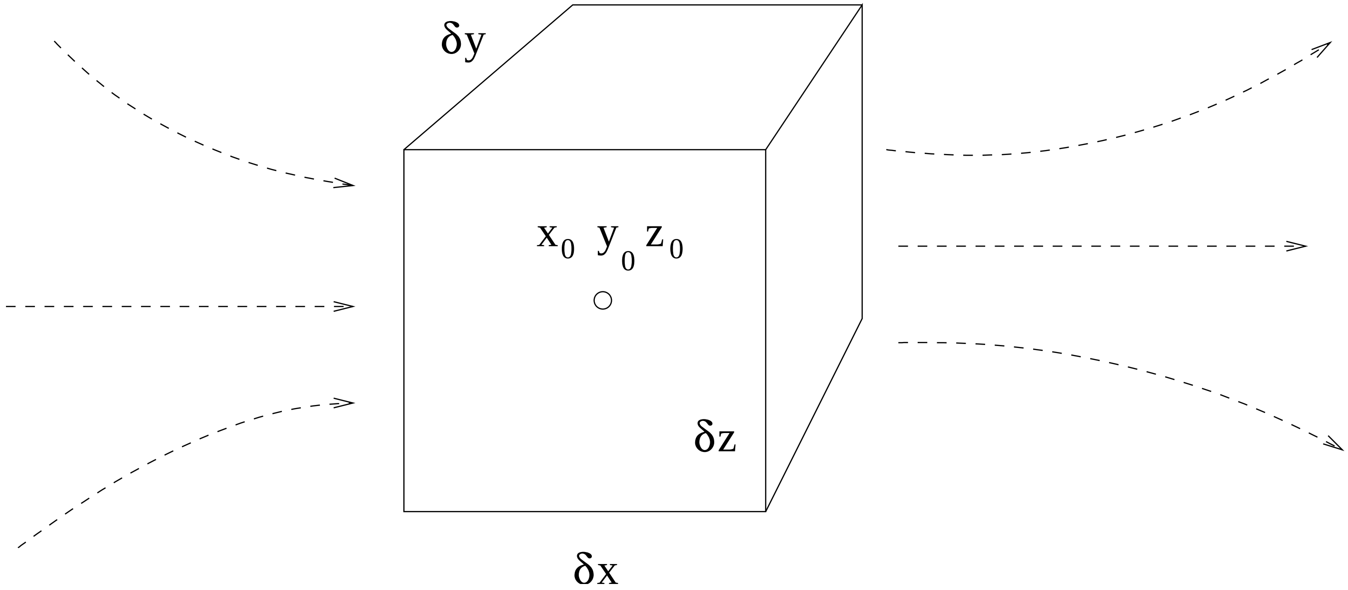
In the Southern hemisphere, $\theta < 0$. Same problem, at 45 S:

$$f_y = 2\Omega cos(-45) = -5.142 \times 10^{-5}\ sec^{-1} = -f_z$$

$$-2\vec{\Omega} \times \vec{u} = (0, +5.142 \times 10^{-4}, -5.142 \times 10^{-4})\ m/sec^2$$

Acceleration is to the north, to the *left* of the parcel velocity.

Continuity



Continuity

Consider a fixed volume

Density flux through the left side:

$$[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Through the right side:

$$[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Continuity

So the net rate of change in mass is:

$$\frac{\partial}{\partial t}(\rho\,\partial x\,\partial y\,\partial z) = [\rho u - \frac{\partial}{\partial x}(\rho u]\frac{\partial x}{2})\,\partial y\,\partial z$$

$$-[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\partial y\,\partial z = -\frac{\partial}{\partial x}(\rho u)\partial x\,\partial y\,\partial z$$

The volume δV is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u)$$

Continuity

Taking the other sides of the box:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{u})$$

Can rewrite:

$$\nabla \cdot (\rho \vec{u}) = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho \ .$$

So:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho (\nabla \cdot \vec{u}) = 0$$

Continuity

Can also derive using a Lagrangian box

As the box moves, it conserves it mass. So:

$$\frac{1}{\partial M}\frac{d}{dt}(\partial M) = \frac{1}{\rho\delta V}\frac{d}{dt}(\rho\delta V) = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d\delta V}{dt} = 0$$

Expand the volume term:

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{\partial}{\delta x}\frac{\delta x}{dt} + \frac{\partial}{\delta y}\frac{\delta y}{dt} + \frac{\partial}{\delta z}\frac{\delta z}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Continuity

So:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{u} = 0$$

Same as before

The change in density is proportional to the velocity *divergence*.

If the volume changes, the density changes to keep the mass constant.

Ideal Gas Law

Five of the equations are *prognostic*: they describe the time evolution of fields.

But one “diagnostic” relation.

Relates the density, pressure and temperature

Ideal Gas Law

For dry air:

$$p = \rho R T$$

where

$$R = 287\ Jkg^{-1}K^{-1}$$

Moist air

Law moist air, can write (Chp. 3):

$$p = \rho R T_v$$

where the *virtual temperature* is:

$$T_v \equiv \frac{T}{1 - e/p(1-\epsilon)}$$

$$\epsilon \equiv \frac{R_d}{R_v} = 0.622$$

Hereafter, we neglect moisture effects

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

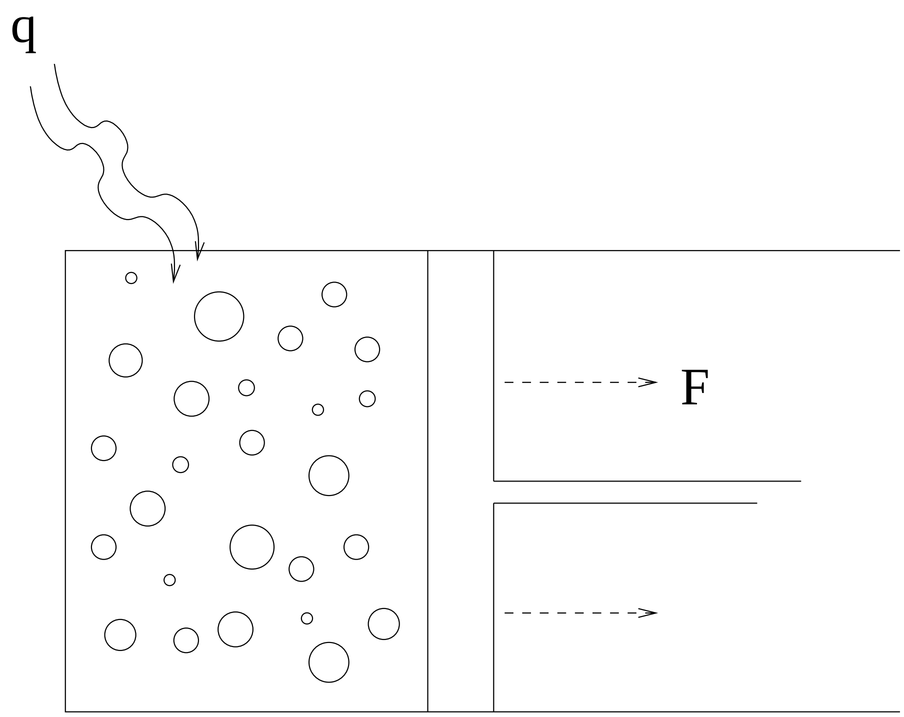
Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \frac{dq}{dt}$$

Thermodynamic equation



First law of thermodynamics

Change in internal energy = heat added - work done:

$$de = dq - dw$$

Work is done by expanding against external forces:

$$dw = F dx = pA dx = p dV$$

If $dV > 0$, the volume is doing the work

First law of thermodynamics

If volume has a unit mass, then:

$$\rho V = 1$$

so:

$$dV = d(\frac{1}{\rho}) = d\alpha$$

where α is the *specific volume*. So:

$$dq = p\, d\alpha + de$$

First law of thermodynamics

Add heat to the volume, the temperature rises. The *specific heat* determines how much. At constant volume:

$$c_v \equiv \frac{dq}{dT}\Big|_v$$

This is also the change in internal energy:

$$c_v = \frac{de}{dT}\Big|_v$$

First Law of thermodynamics

Joule's Law: e only depends on temperature for an ideal gas. So even if V changes:

$$c_v = \frac{de}{dT}$$

Result is the First Law:

$$dq = c_v dT + p\, d\alpha$$

First law of thermodynamics

At constant pressure:

$$c_p \equiv \frac{dq}{dT}\Big|_p$$

Volume expands keeping p constant. Requires more heat to raise the temperature. Rewrite the first law:

$$dq = c_v dT + d(p\alpha) - \alpha dp$$

First law of thermodynamics

The ideal gas law is:

$$p = \rho RT = \alpha^{-1} RT$$

So:

$$d(p\alpha) = RdT$$

Thus:

$$dq = (c_v + R)dT - \alpha dp$$

First law of thermodynamics

At constant pressure, $dp = 0$, so:

$$\frac{dq}{dT}|_p = c_p = c_v + R$$

So the specific heat at constant pressure is *greater* than at constant volume. For dry air:

$$c_v = 717 Jkg^{-1}K^{-1}, \qquad c_p = 1004 Jkg^{-1}K^{-1}$$

so:

$$R = 287 \; Jkg^{-1}K^{-1}$$

First law of thermodynamics

So First Law can also be written:

$$dq = c_p dT - \alpha dp$$

Obtain the thermodynamic equation by dividing by dt:

$$\frac{dq}{dt} = c_v \frac{dT}{dt} + p\frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}$$

Basic balances

Not all terms in the momentum equations are equally important for weather systems.

Will simplify the equations by identifying primary balances (throw out as many terms as possible).

Begin with horizontal momentum equations.

Scaling

General technique: *scale* equations using estimates of the various parameters. Take the x-momentum equation, without friction:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \qquad fU \qquad \frac{\triangle_H P}{\rho L}$$

Scaling

Now use typical values. Length scales:

$$L \approx 10^6 m, \quad D \approx 10^4 m$$

Horizontal scale is 1000 km, the *synoptic scale* (of weather systems).

Velocities:

$$U \approx V \approx 10\, m/sec, \quad W \approx 1\, cm/sec$$

Winds are *quasi-horizontal*

Scaling

Pressure term:

$$\triangle_H P/\rho \approx 10^3 m^2/sec^2$$

A typical *horizontal* difference.

Time scale:

$$T = L/U \approx 10^5 sec$$

Called an "advective time scale" (≈ 1 day).

Scaling

Coriolis terms:

$$(f_y, f_z) = 2\Omega(cos\theta, sin\theta)$$

with

$$\Omega = 2\pi(86400)^{-1}sec^{-1}$$

Assume at mid-latitudes:

$$f_y \approx f_z \approx 2\Omega sin(45) \approx 10^{-4}sec^{-1}$$

Scaling

Plug in:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \qquad fU \qquad \frac{\triangle_H P}{\rho L}$$

$$10^{-4} \qquad 10^{-4} \qquad 10^{-4} \qquad 10^{-5} \qquad 10^{-6} \qquad 10^{-3} \qquad 10^{-3}$$

Geostrophy

Keeping only the 10^{-3} terms:

$$f_z v = \frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

These are the *geostrophic* relations.

Balance between the pressure gradient and Coriolis force.

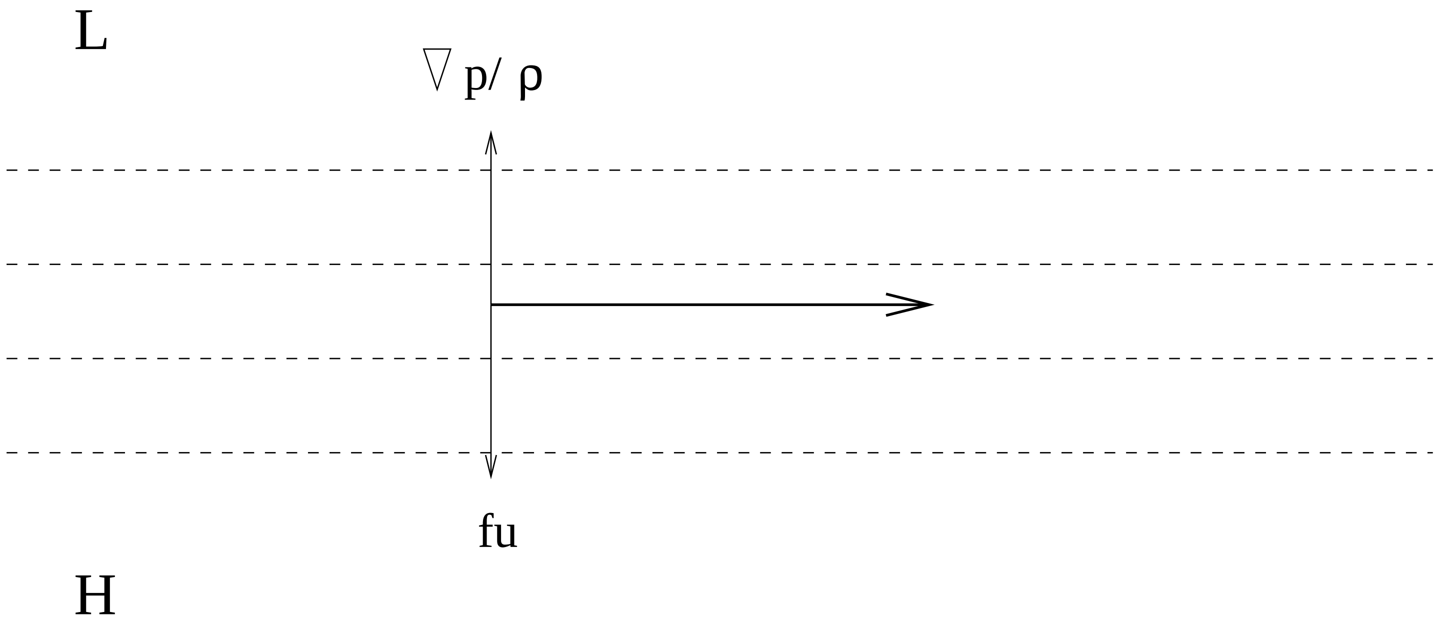
Geostrophy

Fundamental momentum balance at synoptic scales

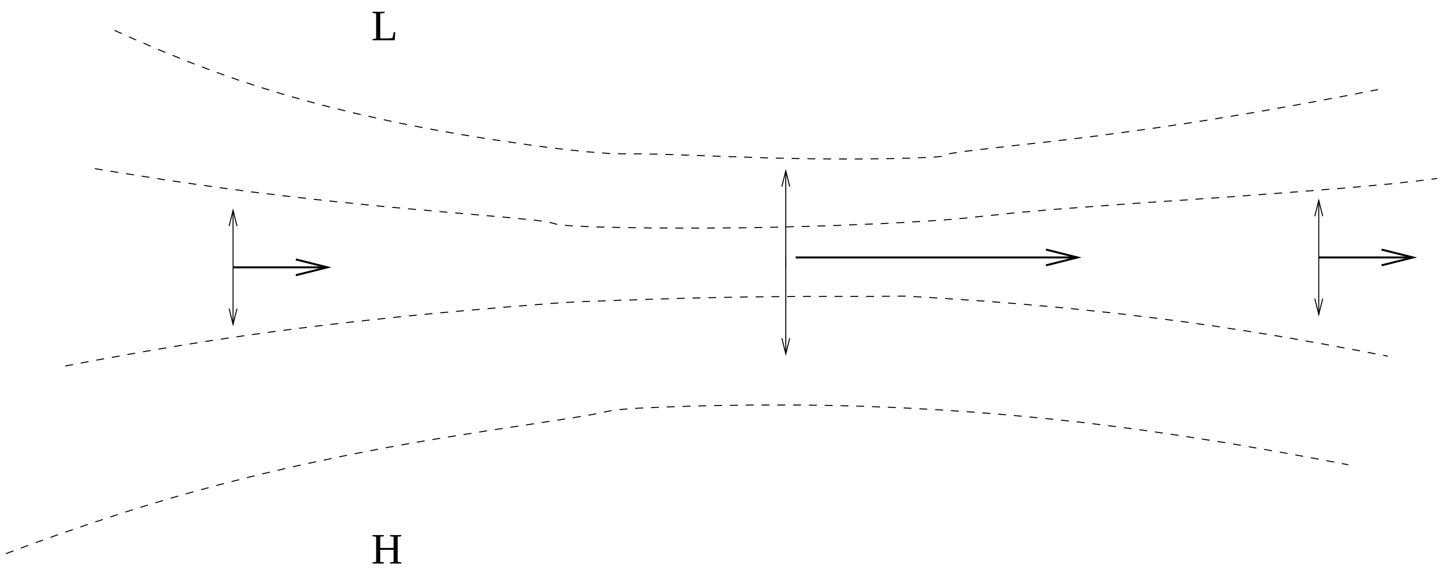
- Low pressure to left of the wind in Northern Hemisphere
- Low pressure to *right* in Southern Hemisphere

But balance fails at equator, because $f_z = 2\Omega sin(0) = 0$

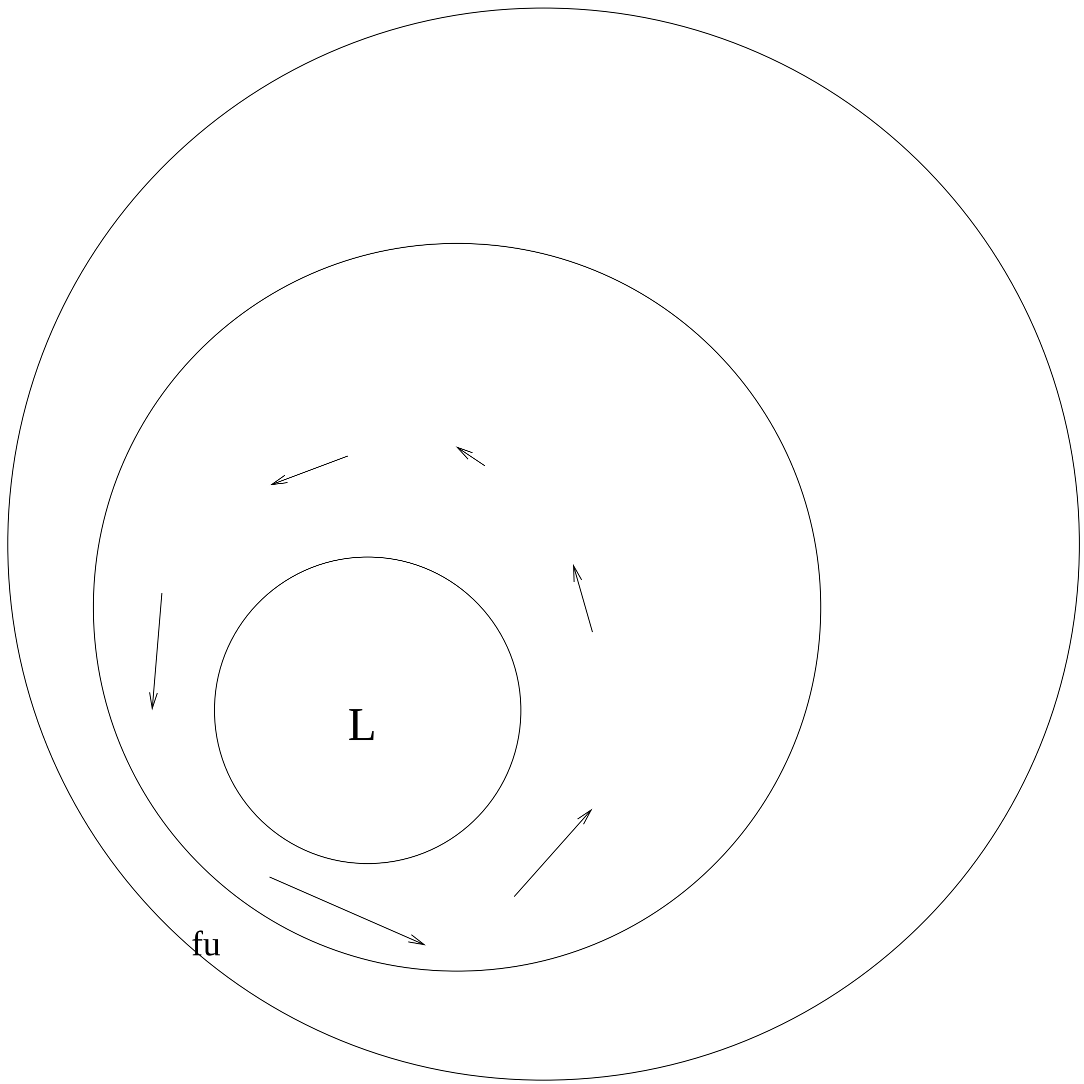
Geostrophy



Geostrophy



Geostrophy



Geostrophy

Example: What pressure gradient is required at the surface at 45 N to maintain a geostrophic wind of 30 m/sec?

$$f_z = 2\Omega sin(45) = 1.414*(7.27\times 10^{-5})\ sec^{-1} = 1.03\times 10^{-4}\ sec^{-1}$$

$$\frac{\partial p}{\partial l} = \rho_0 f_z u = (1.2\ kg/m^3)(1.03 \times 10^{-4}\ sec^{-1})(30\ m/sec)$$

$$= 3.7 \times 10^{-3}\ N/m^3 = .37\ kPa/100km$$

Geostrophy

Is a *diagnostic relation*

- Given the pressure, can calculate the horizontal velocities

But geostrophy cannot be used for *prediction*

Approximate horizontal momentum

So we must also retain the 10^{-4} terms:

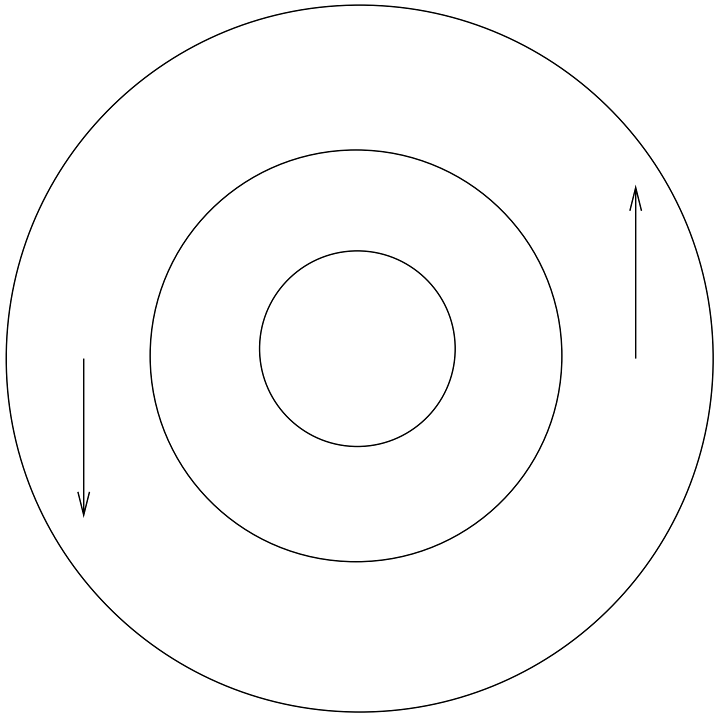
$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

The equations are *quasi-horizontal*: neglect vertical motion

Other momentum balances

Geostrophy most important balance at synoptic scales. But other balances possible. Consider purely circular flow:



Other momentum balances

Consider an air parcel in cylindrical coordinates.

The radial acceleration is:

$$\frac{d}{dt}u_r - \frac{u_\theta^2}{r} - f u_\theta = -\frac{1}{\rho}\frac{\partial}{\partial r}p$$

u_θ^2 is the *cyclostrophic* term – this is related to centripetal acceleration.

Other momentum balances

If constant circulation, $\frac{d}{dt}u_r = 0$. Then:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

Scale this:

$$\frac{U^2}{R} \qquad fU \qquad \frac{\triangle_H P}{\rho R}$$

Other momentum balances

The ratio of the first and second terms is the *Rossby number*:

$$\frac{U}{fR} \equiv \epsilon$$

If $\epsilon \ll 1$, we recover the geostrophic balance:

$$f u_\theta = \frac{1}{\rho} \frac{\partial}{\partial r} p$$

Other momentum balances

If $\epsilon \gg 1$, the first term dominates. Happens at smaller scales and with stronger winds.

A typical tornado at mid-latitudes has:

$$U \approx 30m/s, \quad f = 10^{-4} sec^{-1}, \quad R \approx 300m$$

so that:

$$\epsilon = 1000$$

Cyclostrophic wind balance

Have:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

or:

$$u_\theta = \pm(\frac{r}{\rho}\frac{\partial}{\partial r}p)^{1/2}$$

- Rotation does not enter.
- Circulation can go *either way*.

Inertial oscillations

Third possibility: there is no radial pressure gradient:

$$\frac{u_\theta^2}{r} + f u_\theta = 0$$

then:

$$u_\theta = -fr$$

Rotation is clockwise (*anticyclonic*) in the Northern Hemisphere.

Inertial oscillations

The time for a fluid parcel to complete a loop is:

$$\frac{2\pi r}{u_\theta} = \frac{2\pi}{f} = \frac{0.5\ day}{|sin\theta|} \ ,$$

Called the “inertial period”.

Inertial oscillations are seen in the surface layer of the ocean, but are rarer in the atmosphere.

Gradient wind balance

Fourth possibility: all terms are important ($\epsilon = O|1|$).

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

Solve using the quadratic formula:

$$u_\theta = -\frac{1}{2} f r \pm \frac{1}{2}(f^2 r^2 + \frac{4r}{\rho}\frac{\partial}{\partial r}p)^{1/2}$$

$$= -\frac{1}{2} f r \pm \frac{1}{2}(f^2 r^2 + 4\, r\, f\, u_g)^{1/2}$$

Gradient wind balance

If $u_g < 0$ (anticyclone), we require:

$$|u_g| < \frac{fr}{4}$$

If $u_g > 0$ (cyclone), there is *no limit*

Wind gradients are *stronger* in cyclones than in anticyclones

Gradient wind balance

Alternately:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r} p = f u_g$$

Then:

$$\frac{u_g}{u_\theta} = 1 + \frac{u_\theta}{fr} \approx 1 + \epsilon$$

If $\epsilon = 0.1$, the gradient wind estimate differs by 10 %.

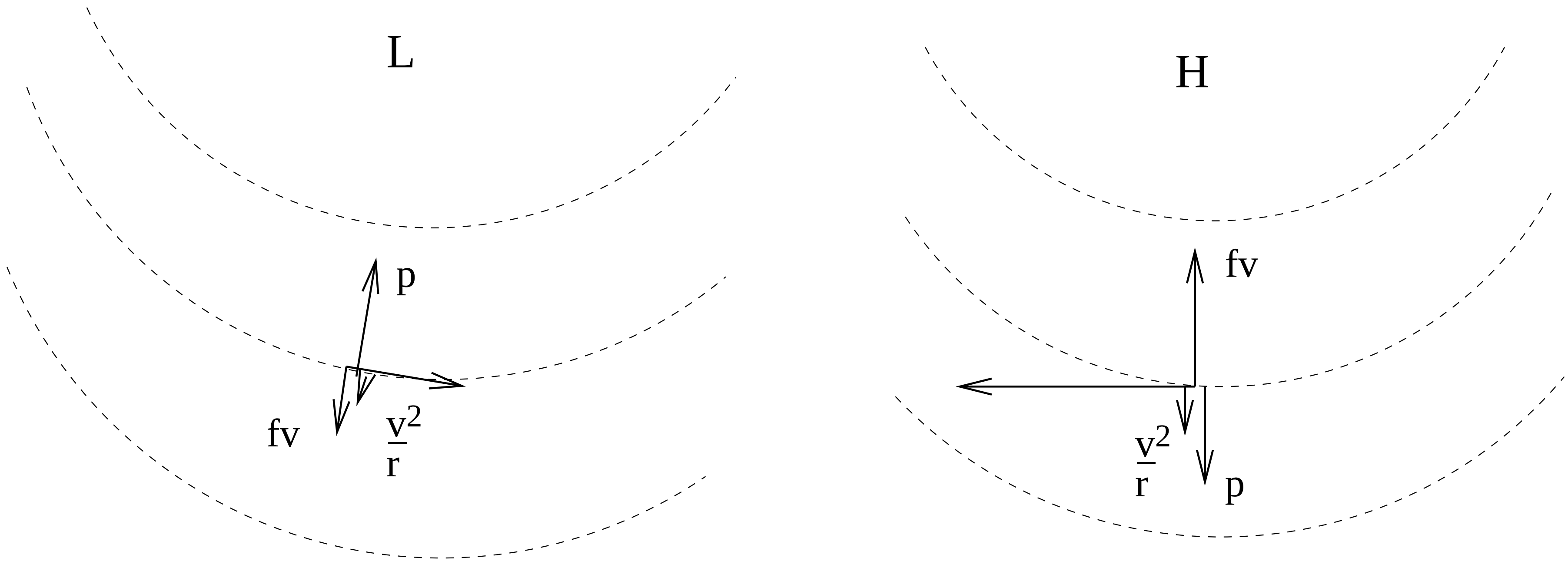
Gradient wind balance

At low latitudes, where ϵ can be 1-10, the gradient wind estimate is more accurate.

Geostrophy is *symmetric to sign changes*: no difference between cyclones and anticyclones

The gradient wind balance is *not* symmetric to sign change.

Gradient wind balance



Gradient wind balance

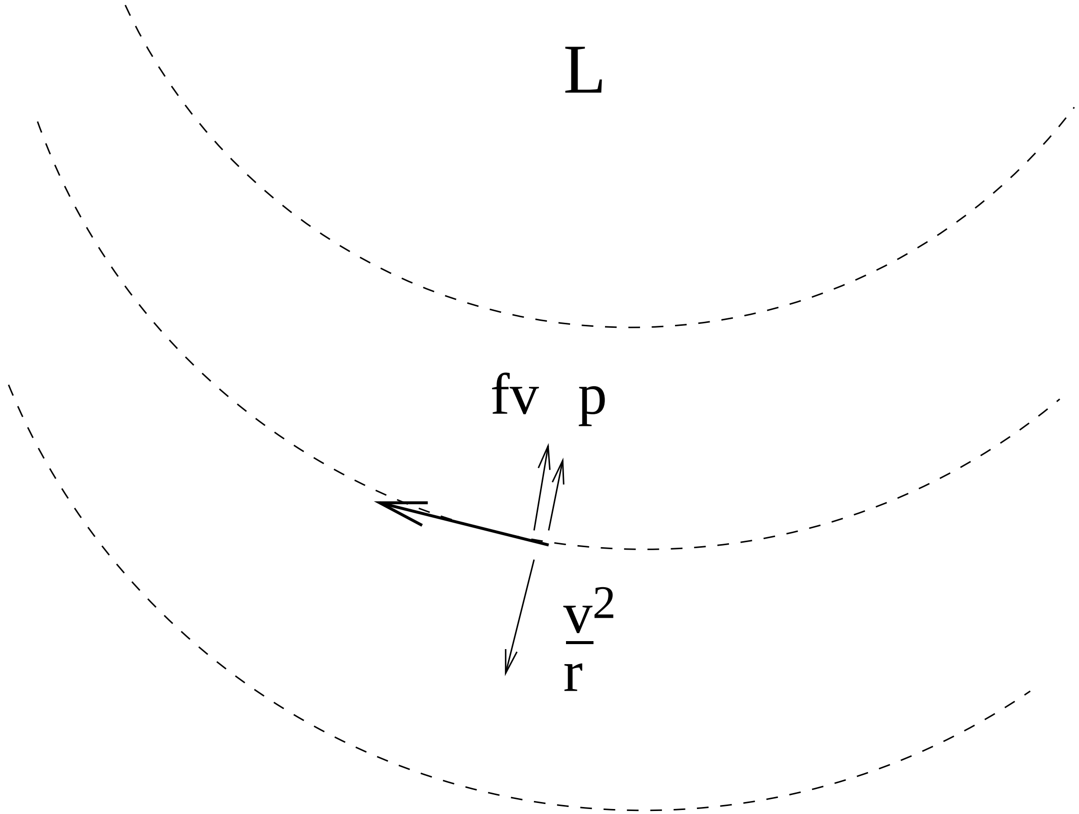
Geostrophic motion has Coriolis and pressure gradient forces opposed.

If cyclostrophic term large enough, gradient wind vortices can have the have pressure gradient and Coriolis forces in the *same direction*.

Called an *anomalous low*: low pressure with clockwise flow

Usually only found near the equator

Gradient wind balance



Hydrostatic balance

Now scale the vertical momentum equation

We must scale:

$$\frac{1}{\rho}\frac{\partial}{\partial z}p$$

The vertical variation of pressure much greater than the horizontal variation:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Hydrostatic balance

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \quad \frac{UW}{L} \quad \frac{UW}{L} \quad \frac{W^2}{D} \quad fU \quad \frac{\triangle_V P}{\rho D} \quad g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \quad 10 \quad 10$$

Static atmosphere

Dominant balance is between the vertical pressure gradient and gravity

However, same balance *if there no motion at all* !

Setting $(u, v, w) = 0$ in the equations of motion yields:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p = \frac{1}{\rho}\frac{\partial}{\partial y}p = \frac{\partial}{\partial t}\rho = \frac{dT}{dt} = 0$$

Static atmosphere

Two equations left:

$$\frac{\partial}{\partial z} p = -\rho g$$

the *hydrostatic balance* and

$$p = \rho R T$$

Equations describe a non-moving atmosphere

Static atmosphere

Integrate the hydrostatic relation:

$$p(z) = \int_z^\infty \rho g \, dz \ .$$

The pressure at any point is equal to the weight of air above it. Sea level pressure is:

$$p(0) = 101.325 \ kPa \ (1013.25mb)$$

The average weight per square meter of the entire atmospheric column(!)

Static atmosphere

Say the temperature is constant (*isothermal*):

$$\frac{\partial}{\partial z} p = -\frac{pg}{RT}$$

This implies:

$$ln(p) = -\frac{gz}{RT}$$

Static atmosphere

So that:

$$p = p_0 \, e^{-z/H}$$

Pressure decays exponentially. The e-folding scale is the “scale height”:

$$H \equiv \frac{RT}{g}$$

Scaling

Static hydrostatic balance not interesting for weather. Separate the pressure and density into static and non-static (moving) components:

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

Assume:

$$|p'| \ll |p_0|, \quad |\rho'| \ll |\rho_0|$$

Scaling

Then:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(p_0 + p') - g$$

$$\approx -\frac{1}{\rho_0}\,(1 - \frac{\rho'}{\rho_0})\,\frac{\partial}{\partial z}(p_0 + p') - g$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + (\frac{\rho'}{\rho_0})\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$\rightarrow$ Neglect $(\rho' p')$

Scaling

Use these terms in the vertical momentum equation

But how to scale?

Vertical variation of the perturbation pressure comparable to the horizontal perturbation:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2$$

Scaling

Also:

$$|\rho'| \approx 0.001|\rho_0|$$

So:

$$\frac{\rho'}{\rho_0}g \approx 10^{-1} m/sec^2$$

Scaling

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \quad 10^{-1} \quad 10^{-1}$$

Hydrostatic perturbations

Dominant balance still hydrostatic, but with perturbations:

$$\frac{\partial}{\partial z} p' = -\rho' g$$

thus vertical acceleration unimportant at synoptic scales

But we lost the vertical velocity! Deal with this later.

Coriolis parameter

So all terms with f_y are unimportant

From now on, neglect f_y and write f_z simply as f:

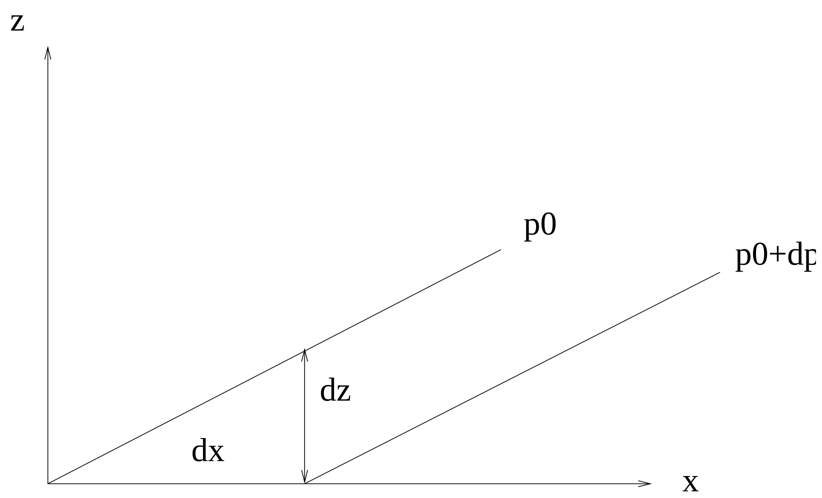
$$f \equiv 2\Omega sin(\theta)$$

f_y only important near the equator

Pressure coordinates

Can use the hydrostatic balance to simplify equations

Constant *pressure surfaces* (in two dimensions):



Pressure coordinates

On a pressure surface:

$$dp = \frac{\partial p}{\partial x}\, dx + \frac{\partial p}{\partial z}\, dz = 0$$

Substitute hydrostatic relation:

$$dp = \frac{\partial p}{\partial x}\, dx - \rho g\, dz = 0$$

So:

$$\frac{\partial p}{\partial x}\Big|_z = \rho g \frac{dz}{dx}\Big|_p \equiv \rho\, \frac{\partial \Phi}{\partial x}\Big|_p$$

Geopotential

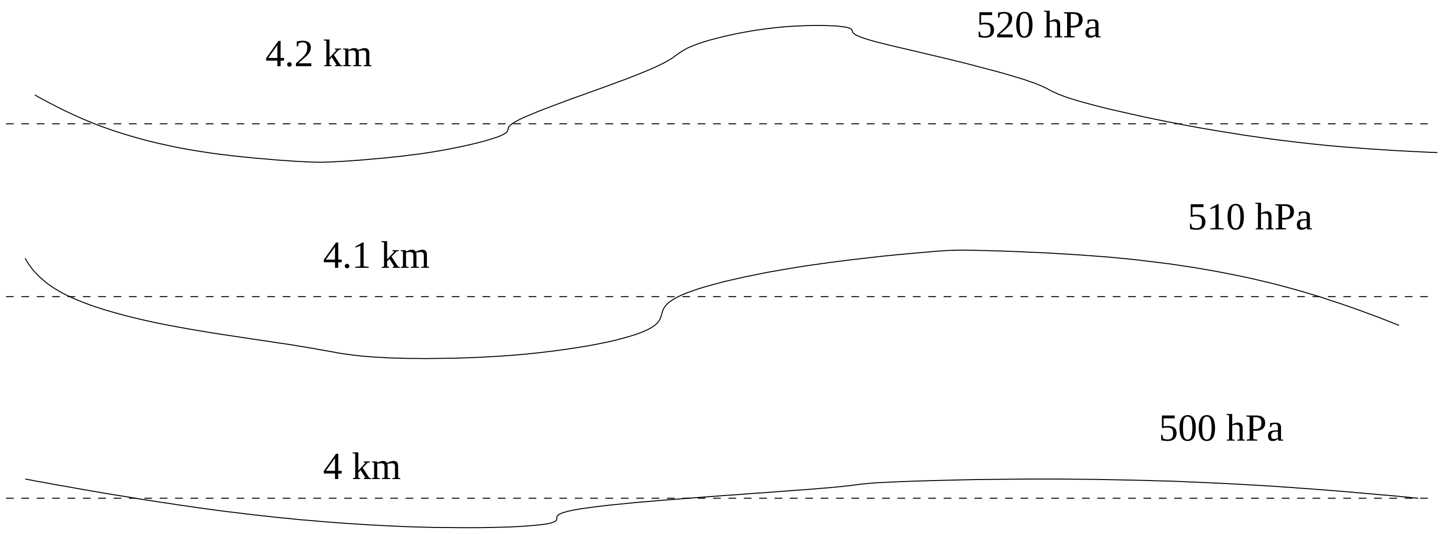
where:

$$\Phi \equiv \int_0^z g\,dz$$

Instead of pressure at a certain height, think:

Height of a certain pressure field

Geopotential



Geostrophy

Removes density from the momentum equation!

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x}$$

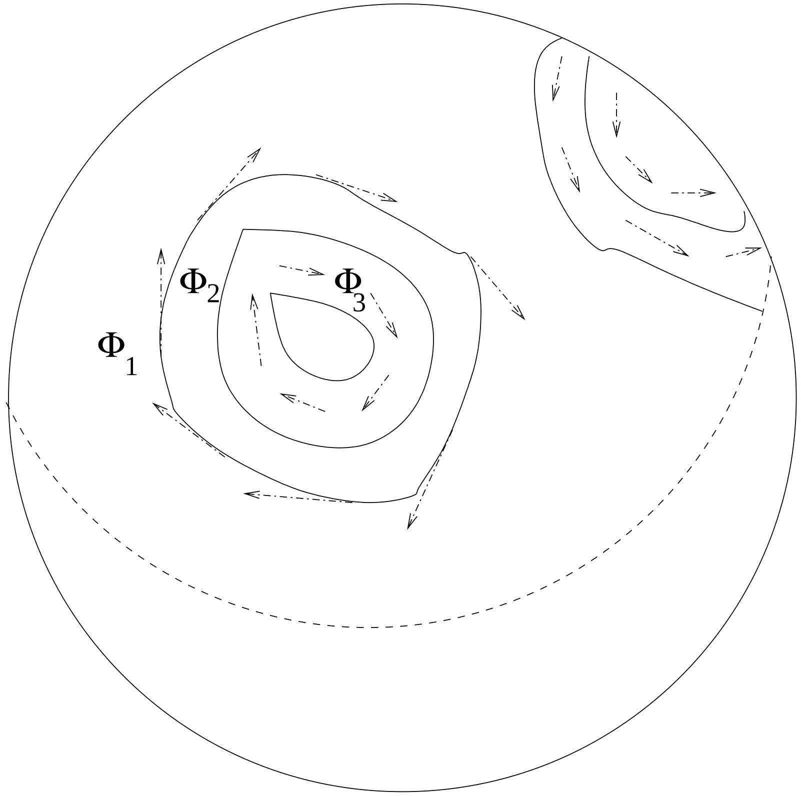
Now the geostrophic balance is:

$$fv = \frac{\partial}{\partial x}\Phi$$

$$fu = -\frac{\partial}{\partial y}\Phi$$

Geostrophy

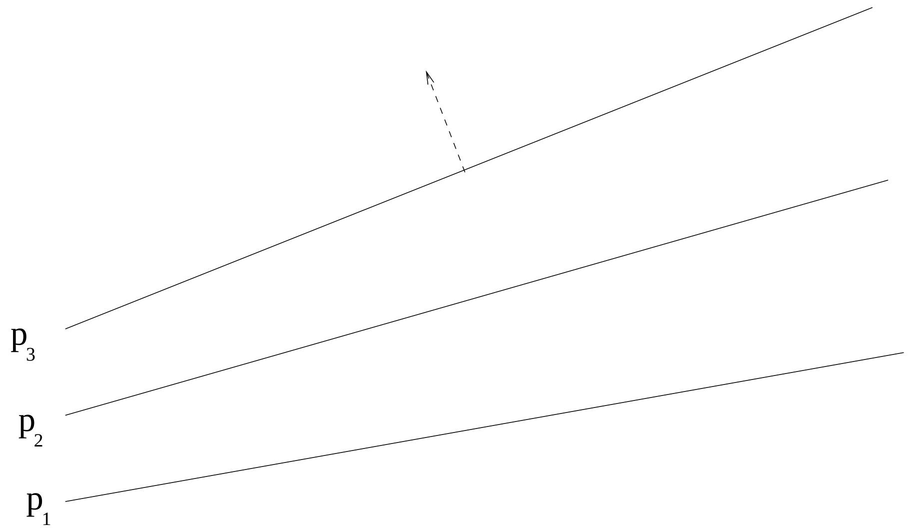
500 hPa



Vertical velocities

Different vertical velocities:

$$w = \frac{dz}{dt} \quad \longrightarrow \quad \omega = \frac{dp}{dt}$$



Geopotential

Lagrangian derivative is now:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p}$$

Continuity

Lagrangian box:

$$\delta V = \delta x \, \delta y \, \delta z = -\delta x \, \delta y \, \frac{\delta p}{\rho g}$$

with a mass:

$$\rho \delta V = -\delta x \, \delta y \, \delta p / g$$

Continuity

Conservation of mass:

$$\frac{1}{\delta M}\frac{d}{dt}\delta M = \frac{g}{\delta x\delta y\delta p}\frac{d}{dt}(\frac{\delta x\delta y\delta p}{g}) = 0$$

Rearrange:

$$\frac{1}{\delta x}\delta(\frac{dx}{dt}) + \frac{1}{\delta y}\delta(\frac{dy}{dt}) + \frac{1}{\delta p}\delta(\frac{dp}{dt}) = 0$$

Continuity

Let $\delta \to 0$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

The flow is *incompressible* in pressure coordinates

Much simpler to work with

Hydrostatic balance

$$\frac{dp}{dz} = -\rho g$$

$$dp = -\rho g dz = -\rho d\Phi$$

So:

$$\frac{d\Phi}{dp} = -\frac{1}{\rho} = -\frac{RT}{p}$$

using the Ideal Gas Law.

Summary

Geostrophy:

$$fv = \frac{\partial}{\partial x}\Phi, \qquad fu = -\frac{\partial}{\partial y}\Phi$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Hydrostatic:

$$\frac{d\Phi}{dp} = -\frac{RT}{p}$$

Diagnosing vertical motion

Lost the vertical acceleration. But can find the velocity, ω, by integrating the continuity equation:

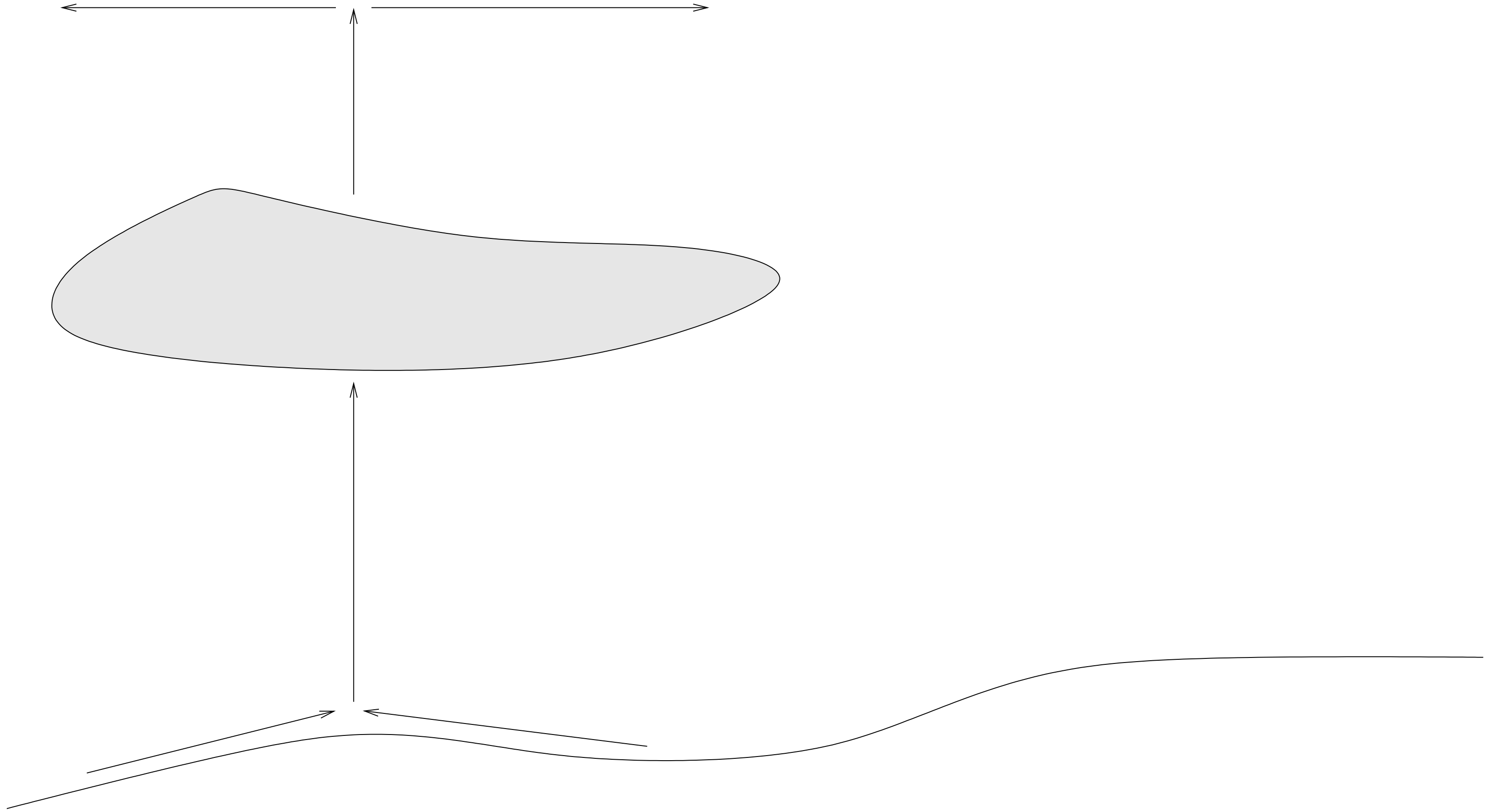
$$\omega = -\int_{p*}^{p} (\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v)\, dp$$

If the top of the atmosphere, $p* = 0$, so:

$$\omega = -\int_{0}^{p} (\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v)\, dp$$

So vertical motion occurs *when there is horizontal divergence*.

Divergence



Vertical motion

How does ω relate to the actual vertical velocity?

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p + w\frac{\partial}{\partial z}p$$

Using the hydrostatic relation:

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p - \rho g w$$

For geostrophic motion:

$$u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p = -\frac{1}{\rho f}\frac{\partial}{\partial y}p(\frac{\partial}{\partial x}p) + \frac{1}{\rho f}\frac{\partial}{\partial x}p(\frac{\partial}{\partial y}p) = 0$$

Vertical motion

So

$$\omega \approx \frac{\partial}{\partial t} p - \rho g w$$

Also:

$$\frac{\partial}{\partial t} p \approx 10 hPa/day$$

$$\rho g w \approx (1.2 kg/m^3)\,(9.8 m/sec^2)(0.01 m/sec) \approx 100 hPa/day$$

Vertical motion

So:

$$\omega \approx -\rho g w$$

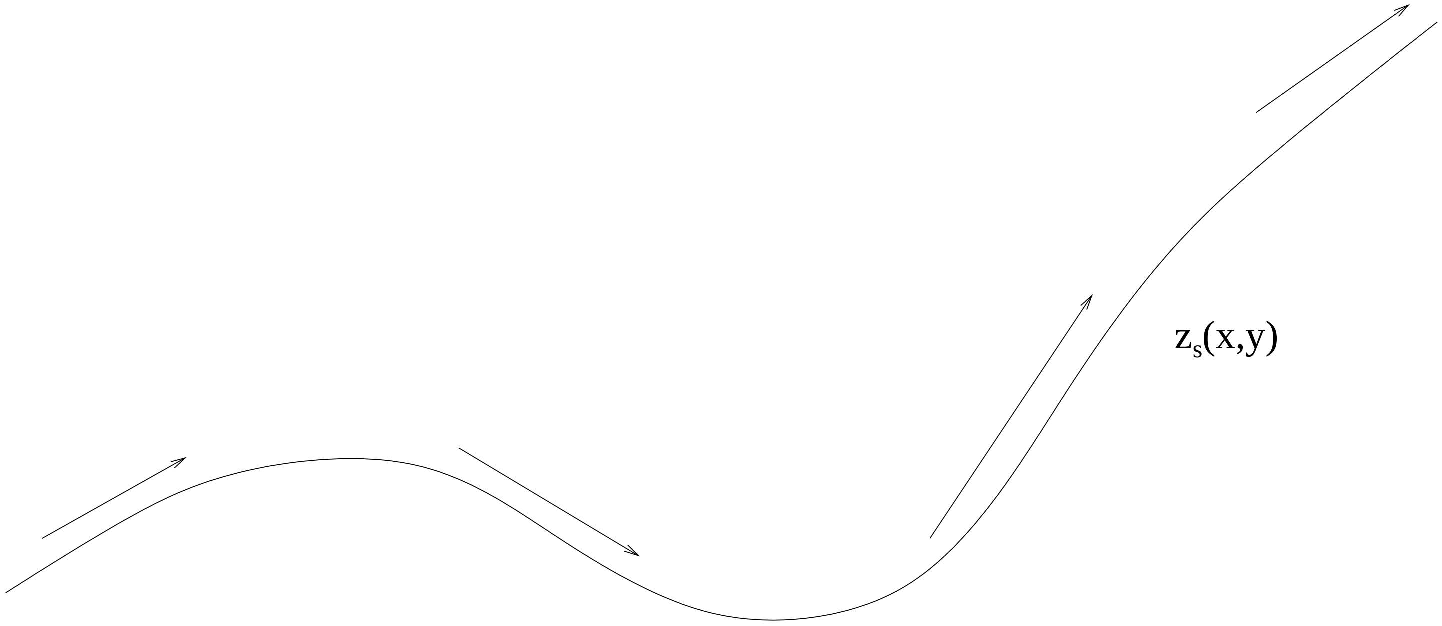
This is accurate within 10 % in the mid-troposphere

In the lowest 1-2 km, topography alters the balances

At the surface:

$$w_s = u \frac{\partial}{\partial x} z_s + v \frac{\partial}{\partial y} z_s$$

Vertical motion



Thermal wind

Geostrophy tells us what the velocities are if we know the geopotential on a pressure surface

What about the velocities on other pressure surfaces?

Need to know the velocity *shear*

Shear is determined by the thermal wind relation

Thermal wind

Can use geostrophy to calculate the shear between two pressure surfaces:

$$v_g(p_1) - v_g(p_0) = \frac{1}{f}\,\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \equiv \frac{g}{f}\,\frac{\partial}{\partial x} Z_{10}$$

and:

$$u_g(p_1) - u_g(p_0) = -\frac{1}{f}\,\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \equiv -\frac{g}{f}\,\frac{\partial}{\partial y} Z_{10}$$

Thermal wind

where:

$$Z_{10} = \frac{1}{g}\left(\Phi_1 - \Phi_0\right)$$

is the layer *thickness* between p_0 and p_1.

- Shear proportional to gradients of layer thickness

Thermal wind II

From the hydrostatic balance:

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

Now take the derivative wrt pressure of the geostrophic relation:

$$\frac{\partial}{\partial p}\left(f v_g = \frac{\partial \Phi}{\partial x}\right)$$

But:

$$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

Thermal wind II

So:

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

Or:

$$\frac{\partial v_g}{\partial\, ln(p)} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

Also:

$$\frac{\partial u_g}{\partial\, ln(p)} = \frac{R}{f}\,\frac{\partial T}{\partial y}$$

Thermal wind II

- Shear proportional to temperature gradient on p-surface

If we know the velocity at p_0, can calculate it at p_1

Integrate between two pressure levels:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} \frac{\partial T}{\partial x}\, d\, ln(p)$$

$$= -\frac{R}{f} \frac{\partial}{\partial x} \int_{p_0}^{p_1} T\, d\, ln(p)$$

Mean temperature

Define the *mean temperature* in the layer bounded by p_0 and p_1:

$$\overline{T} \equiv \frac{\int_{p_0}^{p_1} T\, d(lnp)}{\int_{p_0}^{p_1}\, d(lnp)} = \frac{\int_{p_0}^{p_1} T\, d(lnp)}{ln(\frac{p_1}{p_0})}$$

Then:

$$v_g(p_1) - v_g(p_0) = \frac{R}{f}\, ln(\frac{p_0}{p_1})\, \frac{\partial \overline{T}}{\partial x}$$

Thermal wind

From before:

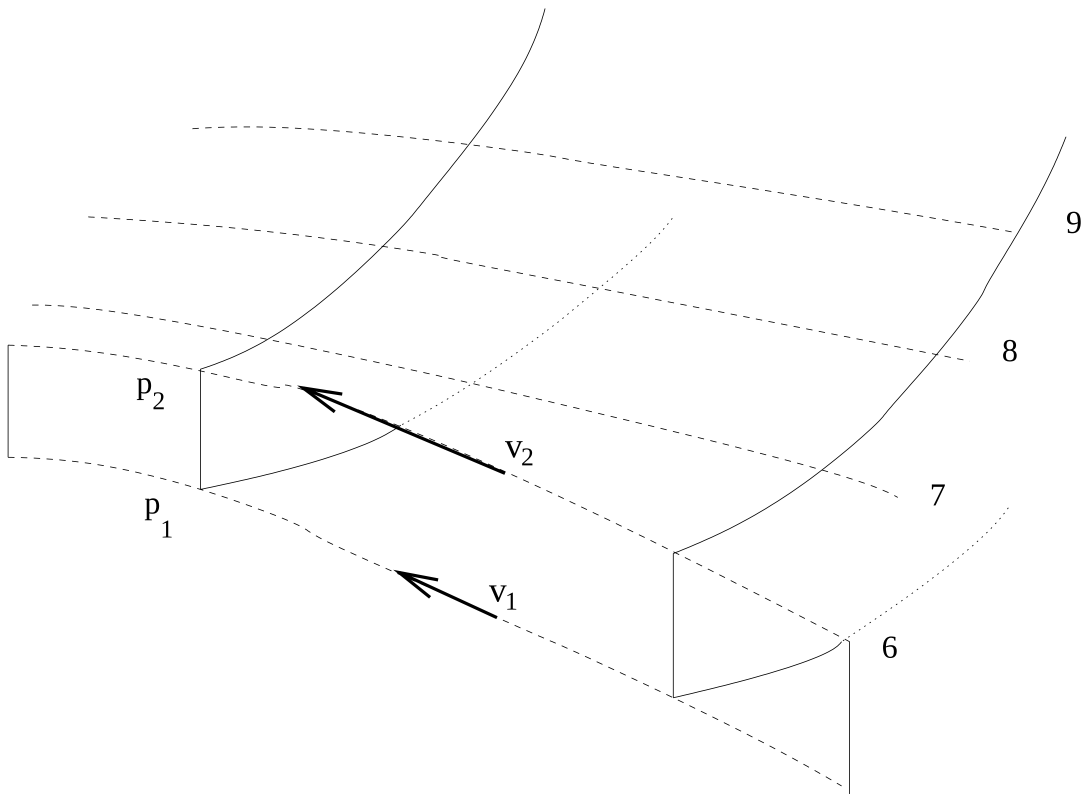
$$v_g(p_1) - v_g(p_0) = \frac{g}{f}\frac{\partial}{\partial x} Z_{10}$$

so:

$$Z_{10} = \frac{R}{g}\overline{T}\, ln(\frac{p_0}{p_1})$$

- Layer thickness *proportional to its mean temperature*

Layer thickness



Barotropic atmosphere

What if temperature constant on all pressure surfaces?

Then $\nabla T = 0 \quad \rightarrow \quad$ *no vertical shear*

Velocities don't change with height

Also: all layers have *equal thickness*

Stacked like pancakes

Equivalent barotropic

If temperature and geopotential contours are parallel:

$$\frac{\partial}{\partial p}\vec{u}_g \parallel \vec{u}_g$$

Wind changes magnitude but *not direction* with height

Geostrophic wind *increases* with height if

- Warm high pressure
- Cold low pressure

Equivalent barotropic

Consider the zonal-average temperature :

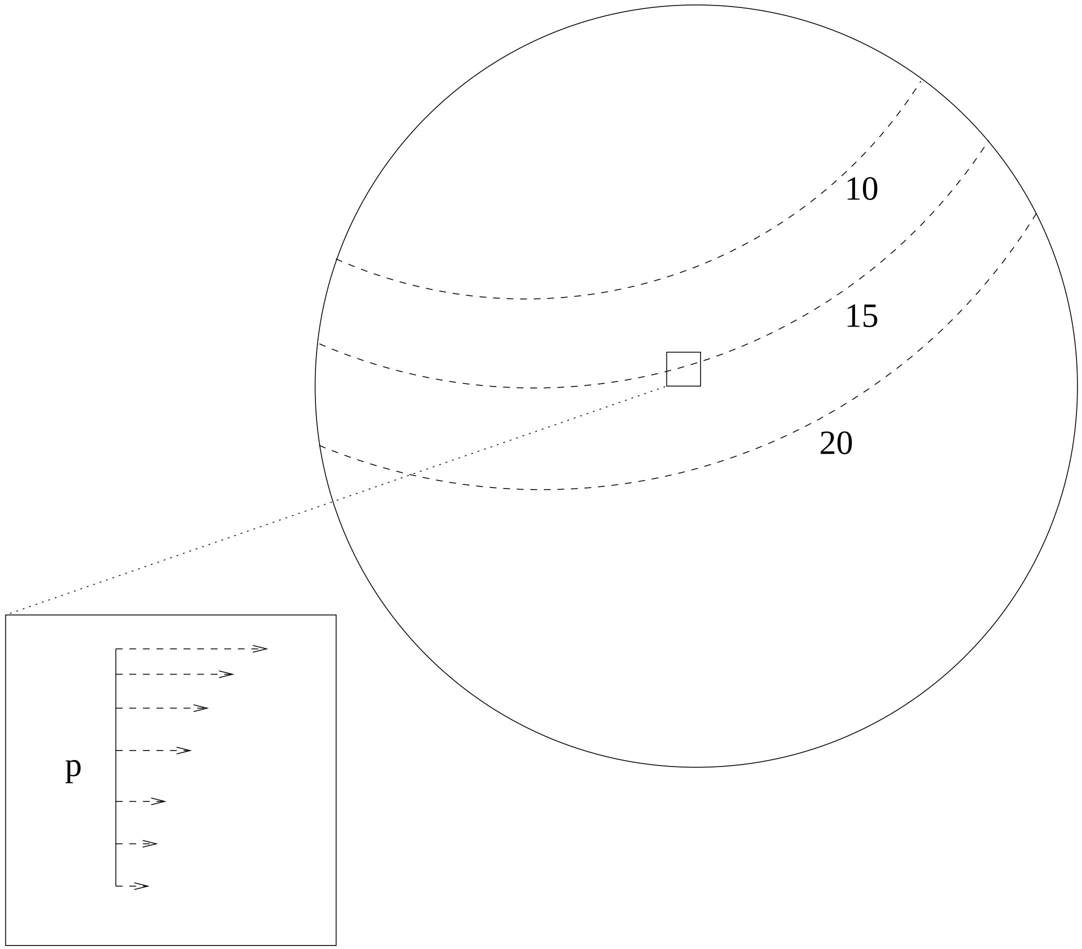
$$\frac{1}{2\pi}\int_0^{2\pi} T\, d\phi$$

Decreases from the equator to the pole

So $\frac{\partial}{\partial y}T < 0$

Thermal wind $\longrightarrow$ winds increase with height

Jet stream



Thermal wind

Example: At 30N, the zonally-averaged temperature gradient is $0.75\ Kdeg^{-1}$, and the average wind is zero at the earth's surface. What is the mean zonal wind at the level of the jet stream ($250\ hPa$)?

$$u_g(p_1) - u_g(p_0) = u_g(p_1) = -\frac{R}{f}\, ln(\frac{p_0}{p_1})\, \frac{\partial \overline{T}}{\partial y}$$

$$u_g(250) = -\frac{287}{2\Omega sin(30)}\, ln(\frac{1000}{250})\, (-\frac{0.75}{1.11 \times 10^5\, m}) = 36.8\ m/sec$$

Baroclinic atmosphere

Usually:

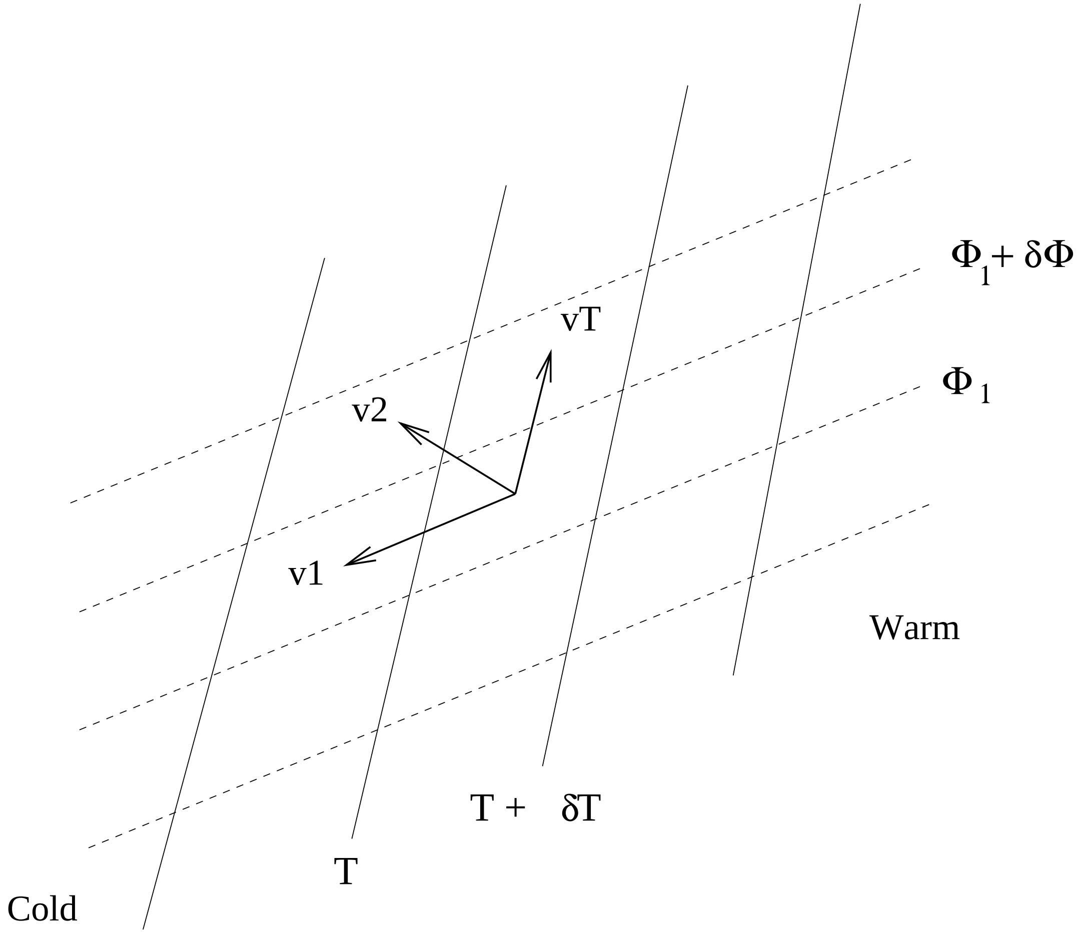
$$T \not\parallel \Phi$$

Geostrophic wind has a component normal to the temperature contours (isotherms)

Produces *geostrophic temperature advection*

Winds blow from warm to cold or vice versa

Temperature advection



Temperature advection

Warm advection $\longrightarrow$ *veering*

- Anticyclonic (clockwise) rotation with height

Cold advection $\longrightarrow$ *backing*

- Cyclonic (counter-clockwise) rotation with height

Summary

Geostrophic wind parallel to geopotential contours

- Wind with high pressure to the right (North Hemisphere)

Thermal wind parallel to thickness (mean temperature) contours

- Wind with high thickness to the right

Divergence

Continuity equation:

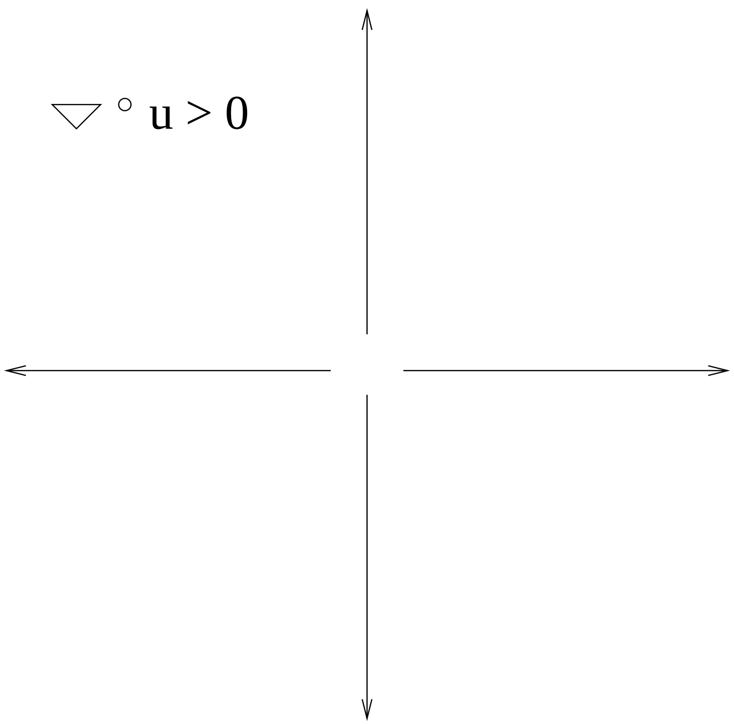
$$\frac{d\rho}{dt} + \rho \nabla \cdot \underline{u} = 0$$

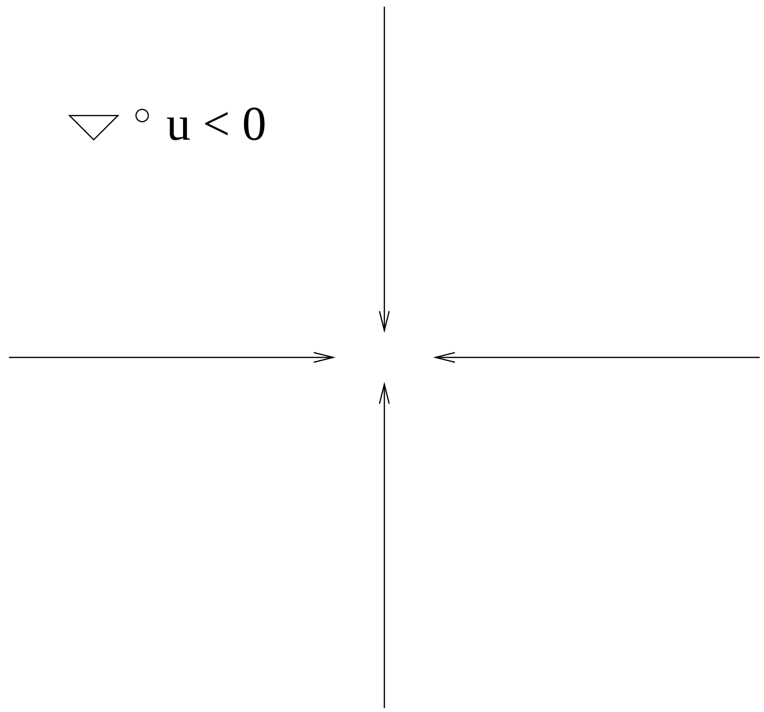
or:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{u} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$$

- Density changes due to divergence

Divergence





Example

The divergence in a region is constant and positive:

$$\nabla \cdot \vec{u} = D > 0$$

What happens to the density of an air parcel?

Example

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{u} = -D$$

$$\frac{d\rho}{dt} = -\rho D$$

$$\rho(t) = \rho(0)\ e^{-Dt}$$

Density decreases exponentially in time

Vorticity

Central quantity in dynamics

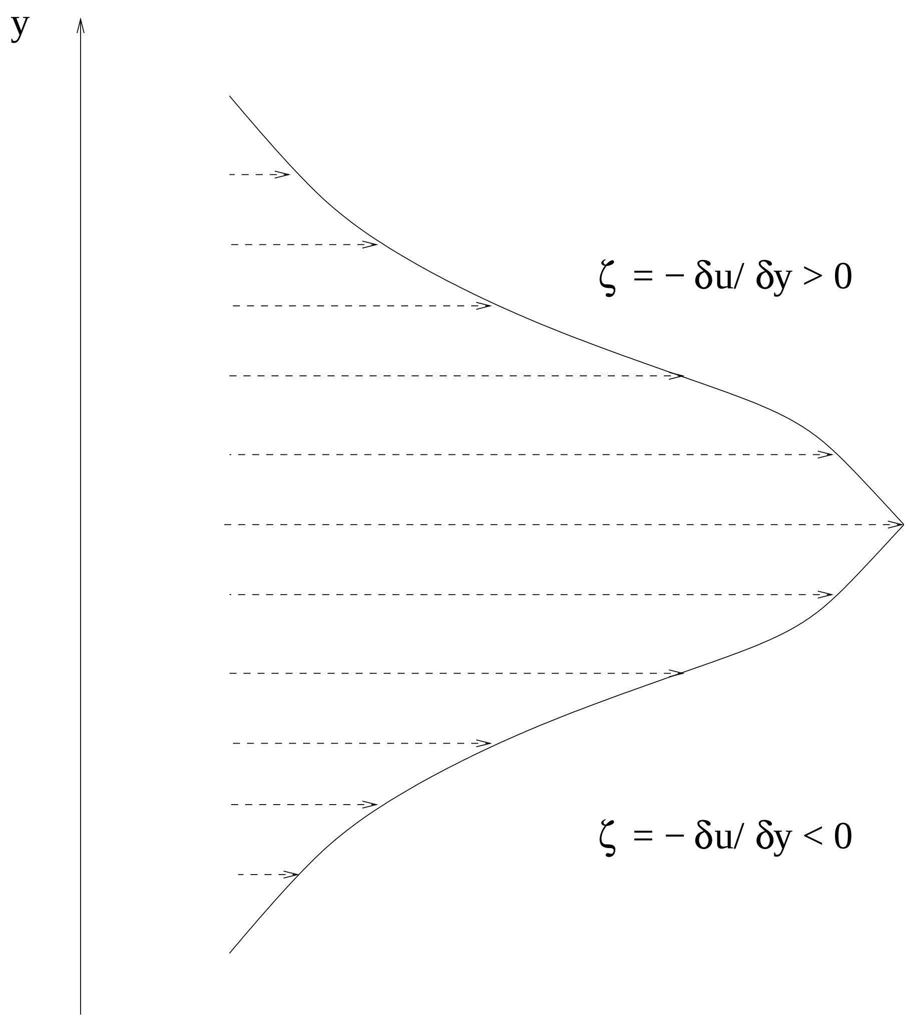
$$\vec{\zeta} \equiv \nabla \times \vec{u}$$

$$\vec{\zeta} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

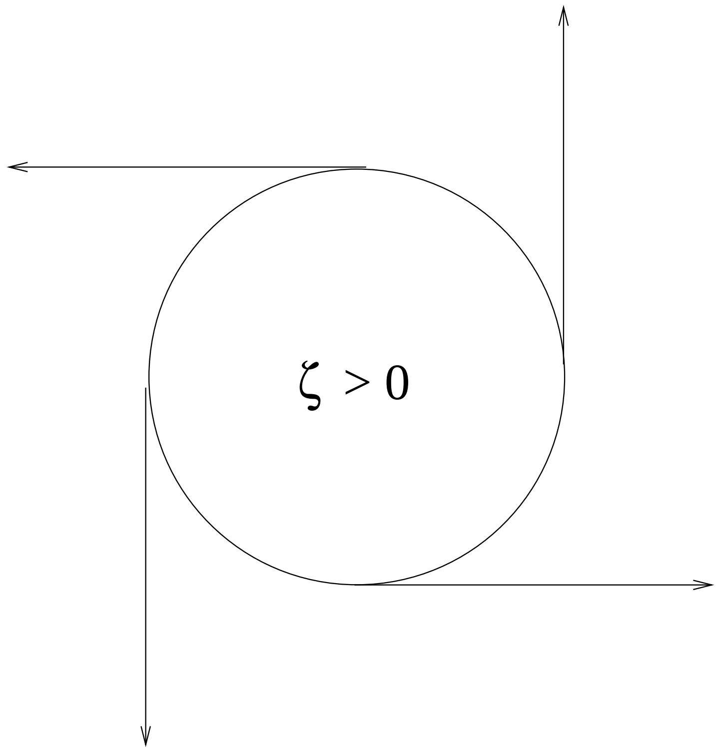
Most important at synoptic scales is vertical component:

$$\vec{\zeta} = \zeta \hat{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity



Vorticity



Example

What is the vorticity of a typical tornado? Assume *solid body rotation*, with a velocity of 100 m/sec, 20 m from the center.

In cylindrical coordinates, the vorticity is:

$$\zeta = \frac{1}{r}\frac{\partial\, r v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

For solid body rotation, $v_r = 0$ and

$$v_\theta = \omega r$$

Vorticity

with $\omega =$const. So:

$$\zeta = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = \frac{1}{r}\frac{\partial \omega r^2}{\partial r} = 2\omega$$

We have $v_\theta = 100$ m/sec at $r = 20$ m:

$$\omega = \frac{v_\theta}{r} = \frac{100}{20} = 5\, rad/sec$$

So:

$$\zeta = 10\, rad/sec$$

Absolute vorticity

Now add rotation. The velocity in the fixed frame is:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

So:

$$\vec{\zeta}_a = \nabla \times (\vec{u} + \vec{\Omega} \times \vec{r}) = \vec{\zeta} + 2\vec{\Omega}$$

We have an extra component because the earth is in solid body rotation!

Absolute vorticity

Two components:

- $\nabla \times \vec{u}$ — the *relative vorticity*
- 2Ω — the *planetary vorticity*

Vertical component is the most important:

$$\zeta_a \cdot \hat{k} = (\frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u) + 2\Omega sin(\theta) \equiv \zeta + f$$

(ζ now refers to vertical relative vorticity)

Absolute vorticity

Scaling:

$$\zeta \propto \frac{U}{L}$$

So:

$$\frac{|\zeta|}{f} \approx \frac{U}{fL} = \epsilon$$

The Rossby number

Absolute vorticity

- $\epsilon \ll 1$

Geostrophic velocities

Planetary vorticity dominates

- $\epsilon \gg 1$

Cyclostrophic velocities

Relative vorticity dominates

Circulation

Circulation is the integral of vorticity over an area:

$$\Gamma \equiv \int\int \zeta dA$$

Due to Stoke's theorem, we can rewrite this as an integral of the velocity around the circumference:

$$\Gamma = \int\int \nabla \times \vec{u}\, dA = \oint \vec{u} \cdot \hat{n}\, dl$$

So we can derive an equation for the circulation by integrating the momentum equations around a closed curve.

Circulation

First write momentum equations in vector form. Turns out to be simpler using the fixed frame velocity:

$$\frac{d}{dt}\vec{u}_F = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F}$$

Integrate around a closed area:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{g} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl}$$

Circulation

Gravity vanishes because can write as a potential:

$$\vec{g} = -g\hat{k} = \frac{\partial}{\partial z}(-gz) \equiv \nabla \Phi_g$$

and the closed integral of a potential vanishes:

$$\oint \nabla \Phi_g \cdot \vec{dl} = \oint d\Phi_g = 0$$

Circulation

So:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

Put rotation back in. The fixed velocity is:

$$\vec{u}_F = \vec{u}_R + \Omega \times r$$

So:

$$\Gamma_F = \oint (\vec{u}_R + \Omega \times r) \cdot \vec{dl}$$

Circulation

Rewrite using Stoke's theorem:

$$\oint (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot \vec{dl} = \int \int \nabla \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot \hat{n}\, dA$$

From before:

$$\nabla \times (\vec{\Omega} \times \vec{r}) = 2\Omega$$

If the motion is quasi-horizontal, then $\hat{n} = \hat{k}$:

$$\Gamma_F = \int \int [\zeta + 2\Omega sin(\theta)] dA = \int \int (\zeta + f) dA$$

Kelvin's theorem

Thus:

$$\frac{d}{dt}\Gamma_a = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

where

$$\Gamma_a = \int\int (\zeta + f) dA$$

is the *absolute circulation*, the sum of relative and planetary circulation

Kelvin's theorem

If the atmosphere is barotropic (temperature constant on pressure surfaces):

$$\oint \frac{dp}{\rho} = \frac{1}{\rho} \oint dp = 0$$

If atmosphere is also frictionless ($\vec{F} = 0$), then:

$$\frac{d}{dt}\Gamma_a = 0$$

The *absolute circulation is conserved* on the parcel

Kelvin's theorem

Notice that if the area is small, so that the vorticity is approximately constant over the area, then:

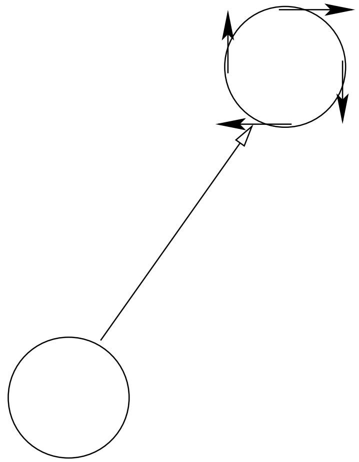
$$\frac{d}{dt}\Gamma_a \approx \frac{d}{dt}(\zeta + f)A = 0$$

which implies:

$$(\zeta + f)A = const.$$

on a parcel. Thus if a parcel's area or latitude changes, it's vorticity must change to compensate.

Kelvin's theorem



Move a parcel north, where f is larger. Either:

- Vorticity decreases
- Area decreases

Vorticity equation

Now we will derive an equation for the vorticity.

Horizontal momentum equations (p-coords):

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,u - fv = -\frac{\partial}{\partial x}\Phi + F_x$$

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,v + fu = -\frac{\partial}{\partial y}\Phi + F_y$$

Take $\frac{\partial}{\partial x}$ of the second, subtract $\frac{\partial}{\partial y}$ of the first

Vorticity equation

Find:

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,\zeta_a$$

$$= -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}) + (\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

where:

$$\zeta_a = \zeta + f$$

Vorticity equation

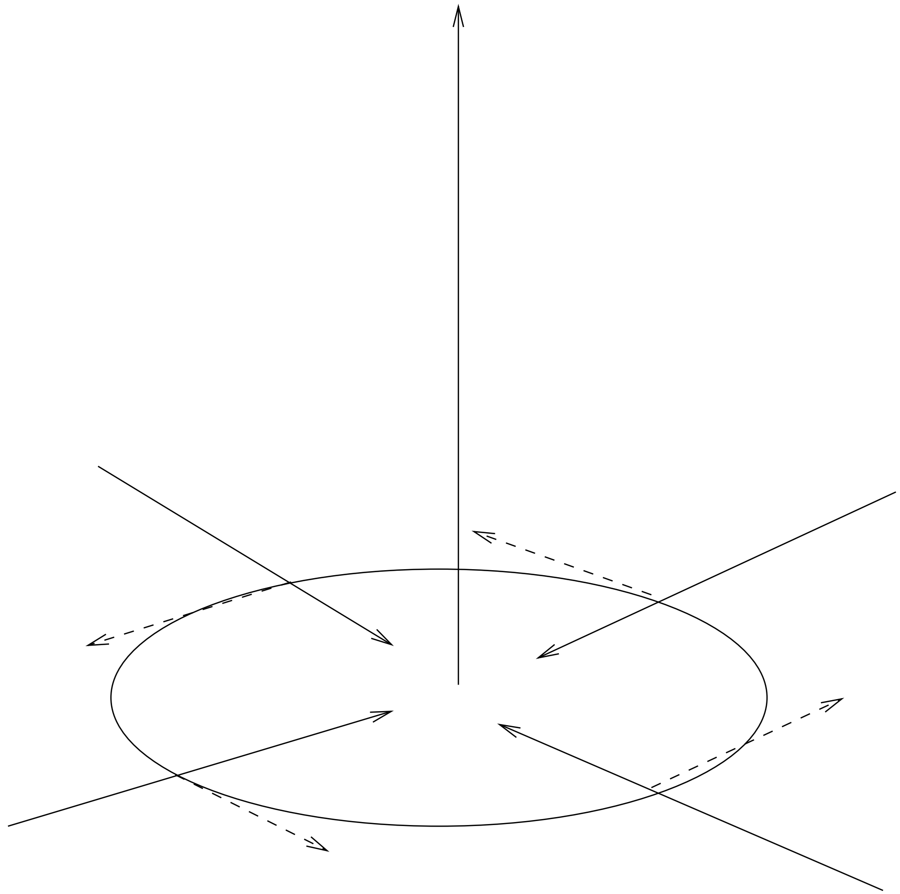
The absolute vorticity can change due to three terms

1) Divergence:

$$-\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

Divergence changes the vorticity, just like density

Convergence



Divergence

Can absorb the divergence into the left side. Consider small area of air:

$$\delta A = \delta x \, \delta y$$

Time change in the area is:

$$\frac{\delta A}{\delta t} = \delta y \frac{\delta x}{\delta t} + \delta x \frac{\delta y}{\delta t} = \delta y \, \delta u + \delta x \, \delta v$$

Relative change is the divergence:

$$\frac{1}{\delta A} \frac{\delta A}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}$$

Divergence

So rewrite the divergence term:

$$-(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\zeta_a = -\frac{\zeta_a}{A}\frac{dA}{dt}$$

So:

$$\frac{d}{dt}\zeta_a = -\frac{\zeta_a}{A}\frac{dA}{dt} \quad \rightarrow \quad \frac{d}{dt}\zeta_a A = 0$$

This is just Kelvin's theorem again!

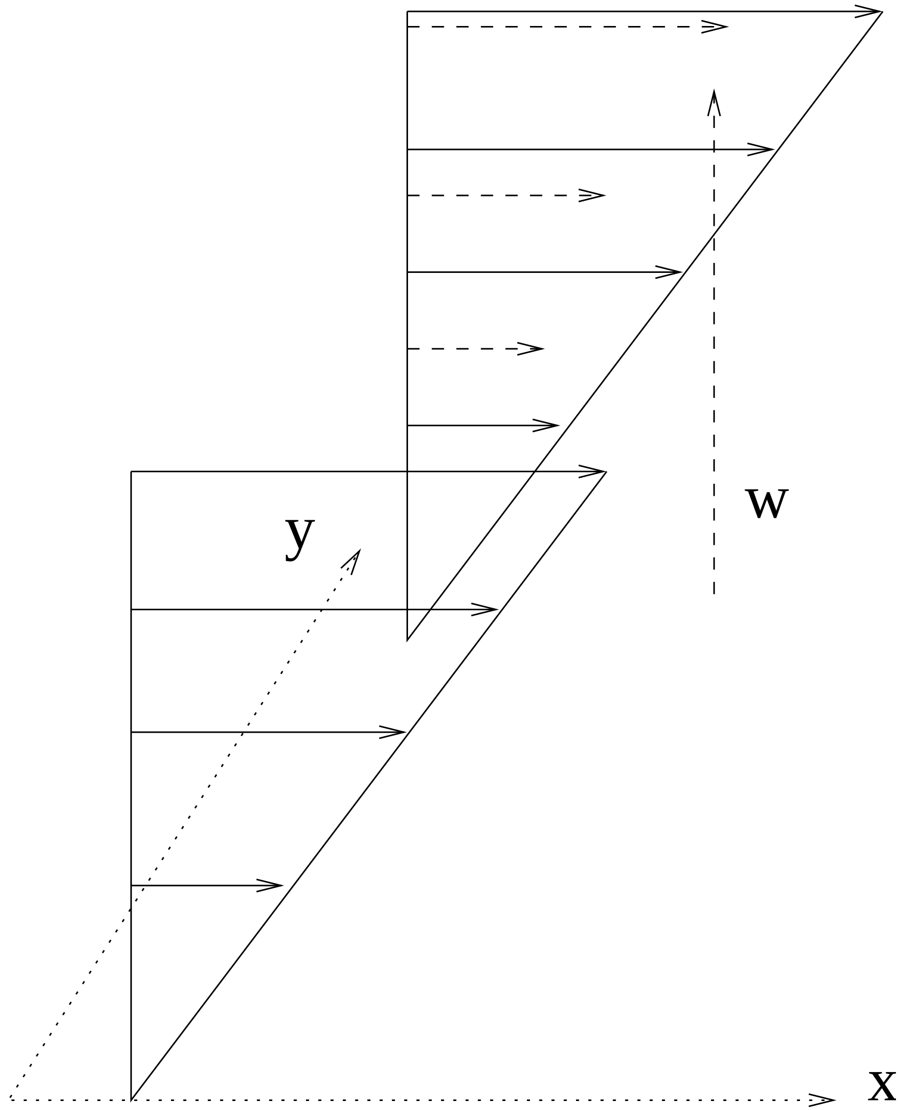
Vorticity equation

2) The *tilting* term:

$$(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x})$$

Differences in ω can affect the horizontal shear

Tilting



Vorticity equation

3) The Forcing term:

$$(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

Say frictional forcing:

$$F_x = \nu\nabla^2 u, \quad F_y = \nu\nabla^2 v$$

Friction

Then:

$$\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right) = \nu\nabla^2\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nu\nabla^2\,\zeta$$

Then:

$$\frac{d}{dt}(\zeta + f) = \nu\nabla^2\,\zeta$$

Friction

If $f \approx const.$:

$$\frac{d}{dt}\zeta = \nu \nabla^2 \zeta$$

Friction *diffuses* vorticity

Causes cyclones to spread out and weaken

Can occur due to friction in the *boundary layer*

Scaling

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,\zeta_a = -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x})$$

For synoptic scale motion, away from boundary layer:

$$U \approx 10m/sec \quad \omega \approx 10hPa/day \quad L \approx 10^6 m \quad \partial p \approx 100hPa$$

$$f_0 \approx 10^{-4} sec^{-1} \quad L/U \approx 10^5 sec \quad \frac{\partial f}{\partial y} \approx 10^{-11} m^{-1} sec^{-1}$$

Scaling

$$\zeta \propto \frac{U}{L} \approx 10^{-5} sec^{-1}$$

So the Rossby number is:

$$\epsilon = \frac{\zeta}{f_0} \approx 0.1$$

So:

$$(\zeta + f) \approx f$$

Scaling

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta \quad \propto \frac{U^2}{L^2} \quad \approx \quad 10^{-10}$$

$$\omega\frac{\partial}{\partial p}\zeta \quad \propto \frac{U\omega}{LP} \quad \approx \quad 10^{-11}$$

$$v\frac{\partial}{\partial y}f \quad \propto U\frac{\partial f}{\partial y} \quad \approx \quad 10^{-10}$$

$$(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}) \quad \propto \frac{U\omega}{LP} \quad \approx \quad 10^{-11}$$

$$(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \approx f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \quad \propto \frac{fU}{L} \quad \approx \quad 10^{-9}$$

Scaling

Divergence term is unbalanced! But it's actually smaller than it appears. We can write:

$$u = u_g + u_a, \quad v = v_g + v_a$$

From the derivation of the gradient wind:

$$\frac{u_g}{u} \approx 1 + \epsilon$$

This implies:

$$\frac{|u_a|}{|u_g|} \propto \epsilon \approx 0.1$$

Ageostrophic velocities

So we can write:

$$u = u_g + \epsilon \tilde{u}_a, \quad v = v_g + \epsilon \tilde{v}_a$$

where $\tilde{u}_a = u_a/\epsilon$. So the vorticity is:

$$\zeta = \frac{\partial}{\partial x} v_g - \frac{\partial}{\partial y} u_g + \epsilon \left(\frac{\partial}{\partial x} \tilde{v}_a - \frac{\partial}{\partial y} \tilde{u}_a\right)$$

While the divergence is:

$$D = \frac{\partial}{\partial x} u_g - \frac{\partial}{\partial y} v_g + \epsilon \left(\frac{\partial}{\partial x} \tilde{u}_a + \frac{\partial}{\partial y} \tilde{v}_a\right) \frac{1}{f} \frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial y}\right) + \frac{1}{f} \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x}\right)$$

$$+ \epsilon \left(\frac{\partial}{\partial x} \tilde{u}_a + \frac{\partial}{\partial y} \tilde{v}_a\right) = \epsilon \left(\frac{\partial}{\partial x} \tilde{u}_a + \frac{\partial}{\partial y} \tilde{v}_a\right)$$

Vertical velocities

Also:

$$\frac{\partial}{\partial z}w = -D \approx -\epsilon(\frac{\partial}{\partial x}\tilde{u}_a + \frac{\partial}{\partial y}\tilde{v}_a)$$

So the divergence and the vertical velocity are order Rossby number

Rotation *suppresses vertical motion*

Scaled equation

Thus the divergence estimate is ten times smaller than we had it before. So:

$$(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \approx f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \quad \propto \epsilon \frac{fU}{L} \quad \approx \quad 10^{-10}$$

Retaining the 10^{-10} terms yields the approximate vorticity equation:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = -f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

Forecasting

Used for forecasts in the 1930's and 1940's

Approach:

Assume geostrophic velocities:

$$u \approx u_g = -\frac{1}{f}\,\frac{\partial \Phi}{\partial y}$$

$$v \approx v_g = \frac{1}{f}\,\frac{\partial \Phi}{\partial x}$$

Forecasting

$$\zeta \approx \zeta_g = \frac{1}{f}\,\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f}\,(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}) = \frac{1}{f}\nabla^2\Phi$$

The divergence vanishes:

$$(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y})\,(\zeta_g + f) = -f\,(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}) = 0$$

Implies ζ_a is *conserved following the horizontal winds*

Remember: on a pressure surface

Forecasting

Now only *one unknown*: Φ

$$(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y})\,(\zeta_g + f) = 0$$

becomes:

$$(\frac{\partial}{\partial t} - \frac{1}{f}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} + \frac{1}{f}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y})\,(\frac{1}{f}\nabla^2 \Phi + f) = 0$$

Forecasting

Can write equation:

$$\frac{\partial}{\partial t}\zeta_g + u_g \cdot \nabla\zeta_g + v_g\frac{\partial}{\partial y}f = 0$$

or:

$$\frac{\partial}{\partial t}\zeta_g = -u_g \cdot \nabla\zeta_g - v_g\frac{\partial}{\partial y}f$$

Can predict how ζ changes in time

Then convert $\zeta \rightarrow \Phi$ by *inversion*

Forecasting

Method:

- Obtain $\Phi(x, y, t_0)$ from measurements on p-surface
- Calculate $u_g(t_0)$, $v_g(t_0)$, $\zeta_g(t_0)$
- Calculate $\zeta_g(t_1)$
- *Invert* ζ_g to get $\Phi(t_1)$
- Start over
- Obtain $\Phi(t_2)$, $\Phi(t_3)$,...

Inversion

$$\zeta_g = \frac{1}{f} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)$$

$$\nabla^2 \Phi = f \zeta_g$$

Poisson's equation

Need boundary conditions to solve

Usually do this numerically

Inversion

Simple analytical example: a channel, with zero flow at northern and southern boundaries. Let:

$$\zeta = sin(3x)sin(\pi y)$$

$$x = [0, 2\pi], \quad y = [0, 1]$$

So:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = sin(3x)sin(\pi y)$$

Inversion

Try a particular solution:

$$\Phi = Asin(3x)sin(\pi y)$$

This solution works in a channel, because:

$$\Phi(x = 2\pi) = \Phi(x = 0)$$

Also, at $y = 0, 1$:

$$v = \frac{1}{f_0}\frac{\partial\Phi}{\partial x} = 0$$

Inversion

Substitute into equation:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = -(9+\pi^2)A sin(3x) sin(\pi y) = sin(3x) sin(\pi y)$$

So:

$$\Phi = -\frac{1}{9+\pi^2}\, sin(3x) sin(\pi y)$$

Then we can proceed (calculate u_g, v_g, etc.)

Analytical example

Assume a barotropic atmosphere (no vertical shear) with:

$$\Phi = -f_0 U y + f_0 A \, sin(kx - \omega t) \, sin(ly)$$

so that:

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi = U - lA \, sin(kx - \omega t) \, cos(ly)$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = kA \, cos(kx - \omega t) \, sin(ly)$$

Describe how the field evolves in time.

Example

We must solve:

$$\frac{\partial}{\partial t}\zeta_g = -u_g \cdot \nabla\zeta_g - v_g\frac{\partial}{\partial y}f$$

To simplify things, we make the β-*plane approximation*:

$$f \approx f_0 + \beta y$$

where:

$$f_0 = 2\Omega sin(\theta_0), \quad \beta = \frac{2\Omega}{R}cos(\theta_0)$$

Example

So:

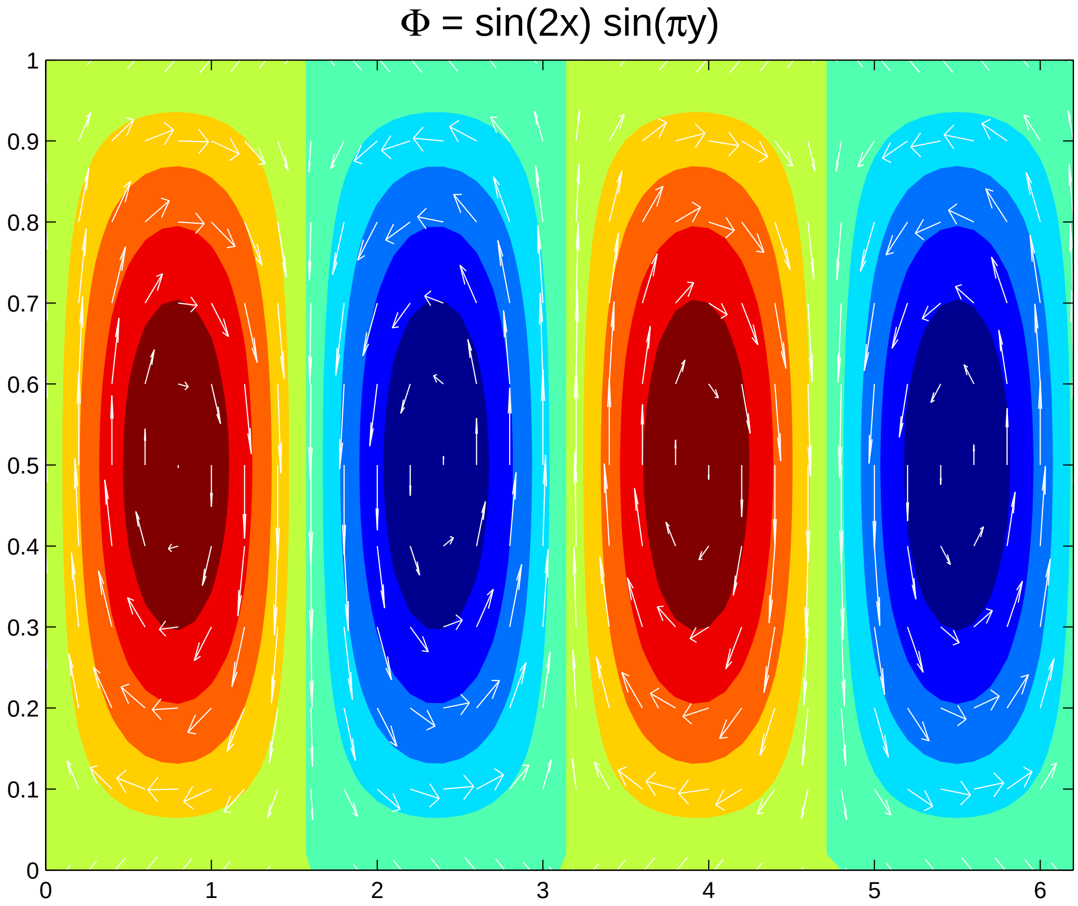
$$v\frac{df}{dy} = v\frac{\partial}{\partial y}(f_0 + \beta y) = \beta v$$

In addition, we approximate:

$$u_g = -\frac{1}{f}\frac{\partial}{\partial y}\Phi \approx -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi$$

$$v_g = \frac{1}{f}\frac{\partial}{\partial x}\Phi \approx \frac{1}{f_0}\frac{\partial}{\partial x}\Phi$$

Initial geopotential



Example

The relative vorticity is:

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi = -(k^2 + l^2)A\,sin(kx - \omega t)\,sin(ly)$$

Also need the derivatives:

$$\frac{\partial}{\partial x}\zeta_g = -k(k^2 + l^2)A\,cos(kx - \omega t)\,sin(ly)$$

$$\frac{\partial}{\partial y}\zeta_g = -l(k^2 + l^2)A\,sin(kx - \omega t)\,cos(ly)$$

Example

Collect terms:

$$-u\frac{\partial}{\partial x}\zeta - v\frac{\partial}{\partial y}\zeta = [U - lA\,sin(kx-\omega t)\,cos(ly)]\times$$

$$[k(k^2+l^2)A\,cos(kx-\omega t)\,sin(ly)] + [kA\,cos(kx-\omega t)\,sin(ly)]\times$$

$$[l(k^2+l^2)A\,sin(kx-\omega t)\,cos(ly)]$$

$$= Uk(k^2+l^2)A\,cos(kx-\omega t)\,sin(ly)$$

Example

Also:

$$-\beta v = -\beta k A\, cos(kx - \omega t)\, sin(ly)$$

So:

$$\frac{\partial}{\partial t}\zeta = (U(k^2 + l^2) - \beta)kA\, cos(kx - \omega t)\, sin(ly)$$

Also, since:

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi = -(k^2 + l^2)A\, sin(kx - \omega t)\, sin(ly)$$

Example

Then:

$$\frac{\partial}{\partial t}\zeta = \omega(k^2 + l^2)A\,cos(kx - \omega t)\,sin(ly)$$

Equate both sides:

$$\omega(k^2 + l^2)A\,cos(kx - \omega t)\,sin(y)$$

$$= (U(k^2 + l^2) - \beta)kA\,cos(kx - \omega t)\,sin(ly)$$

We can cancel the $A\,cos(kx - \omega t)\,sin(y)$, leaving:

$$\omega(k^2 + l^2) = (U(k^2 + l^2) - \beta)k$$

Example

or:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

So the solution is:

$$\Phi = A\, cos(kx - \omega t)\, sin(y)$$

with ω given above. Thus, for a given size wave, the frequency is determined.

This is called a *dispersion relation*

Phase speed

If a travelling wave:

$$\psi \propto sin(kx - \omega t)$$

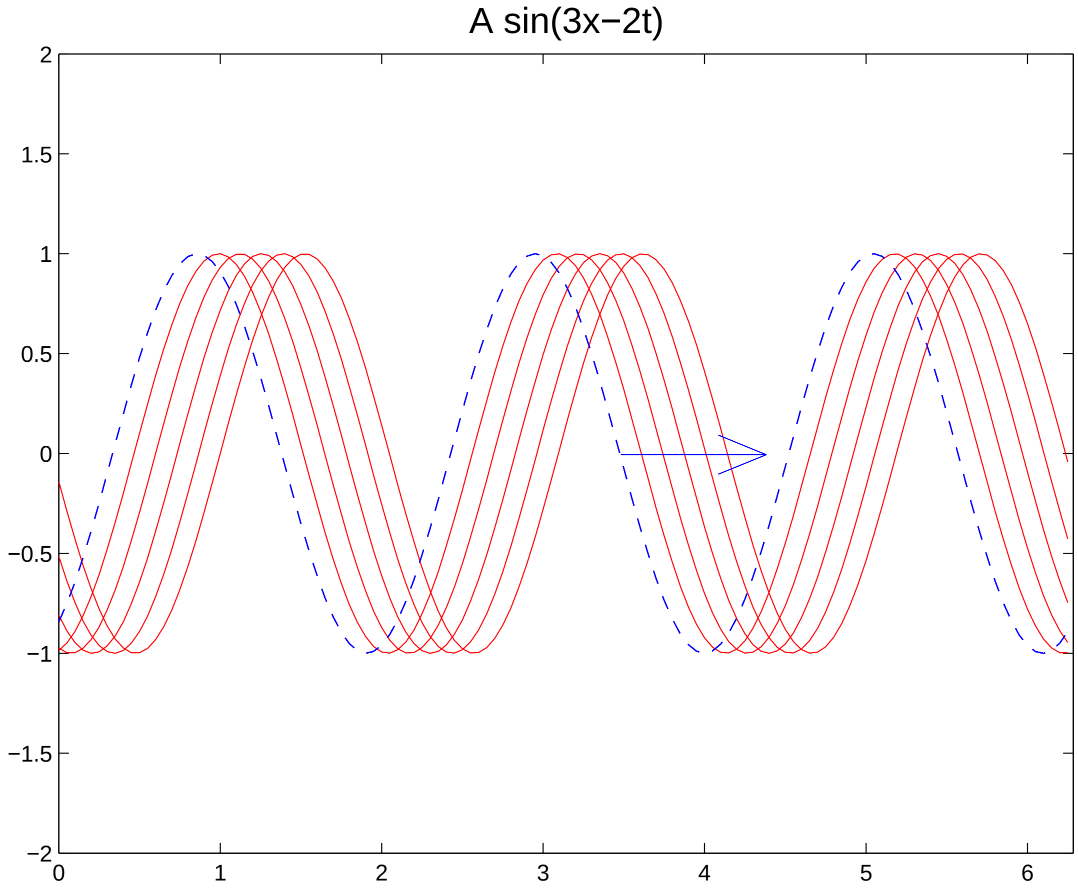
the crests move with a *phase speed*:

$$c_x = \frac{\omega}{k}$$

If $\omega > 0$, waves move toward positive x (eastward)

Phase speed

$$c = 2/3$$



Phase speed

We have:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

so:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}$$

If $U = 0$:

$$c_x = -\frac{\beta}{k^2 + l^2}$$

$\rightarrow$ All waves propagate *westward*!

Phase speed

The wavelengths in both directions are:

$$\lambda_x = \frac{2\pi}{k}, \quad \lambda_y = \frac{2\pi}{l}$$

So:

$$c_x = -\frac{\beta}{k^2} = -\frac{\beta}{4\pi^2}(\lambda_x^2 + \lambda_y^2)$$

Larger waves propagate faster

$\rightarrow$ The waves are *dispersive*

Phase speed

If $U \neq 0$, then:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}$$

Longest waves go west while shorter waves are swept eastward by the zonal flow, U. If:

$$k^2 + l^2 = \frac{\beta}{U}$$

the wave is *stationary* in the background flow

Phase speed

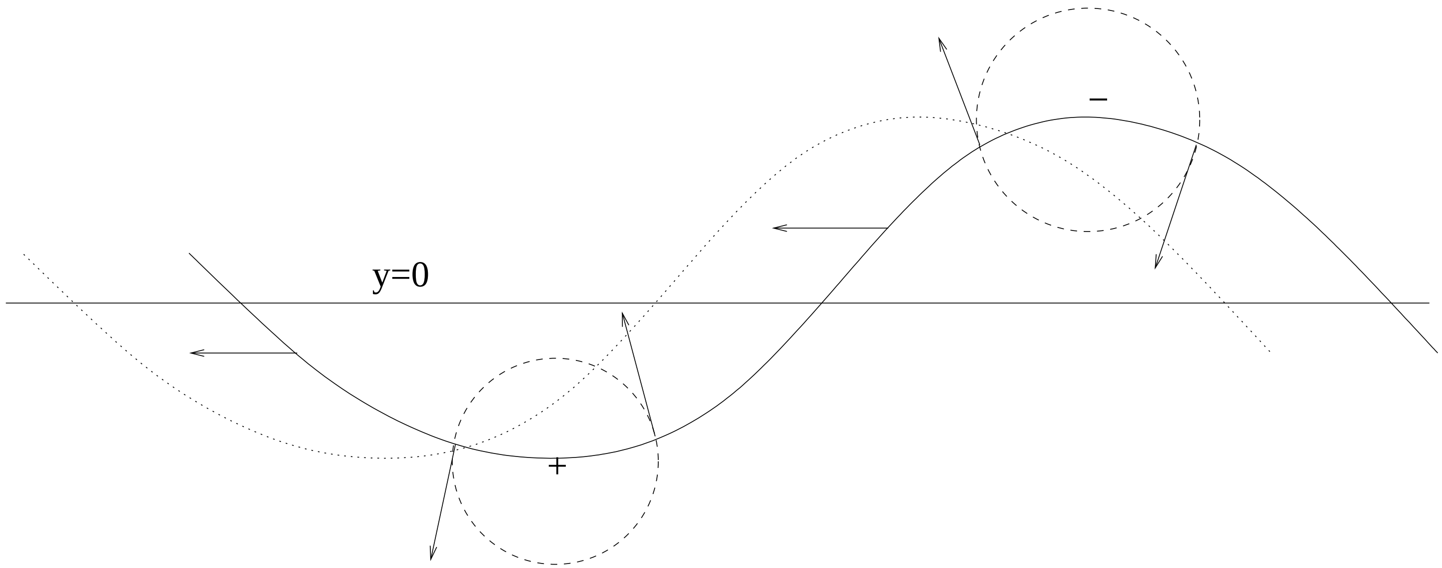
The westward propagation is actually a consequence of Kelvin's theorem

Parcels advected north/south acquire relative vorticity

The parcels then advect neighboring parcels around them

Leads to a westward shift of the wave

Westward propagation



Rossby waves

Solutions are called *Rossby waves*

Discovered by Carl Gustav Rossby (1936)

Observed in the atmosphere

Stationary Rossby waves are important for long term weather patterns

Study more later (GEF4500)

Divergence

Previously ignored divergence effects. But very important for the growth of unstable disturbances (storms)

The approximate vorticity equation is:

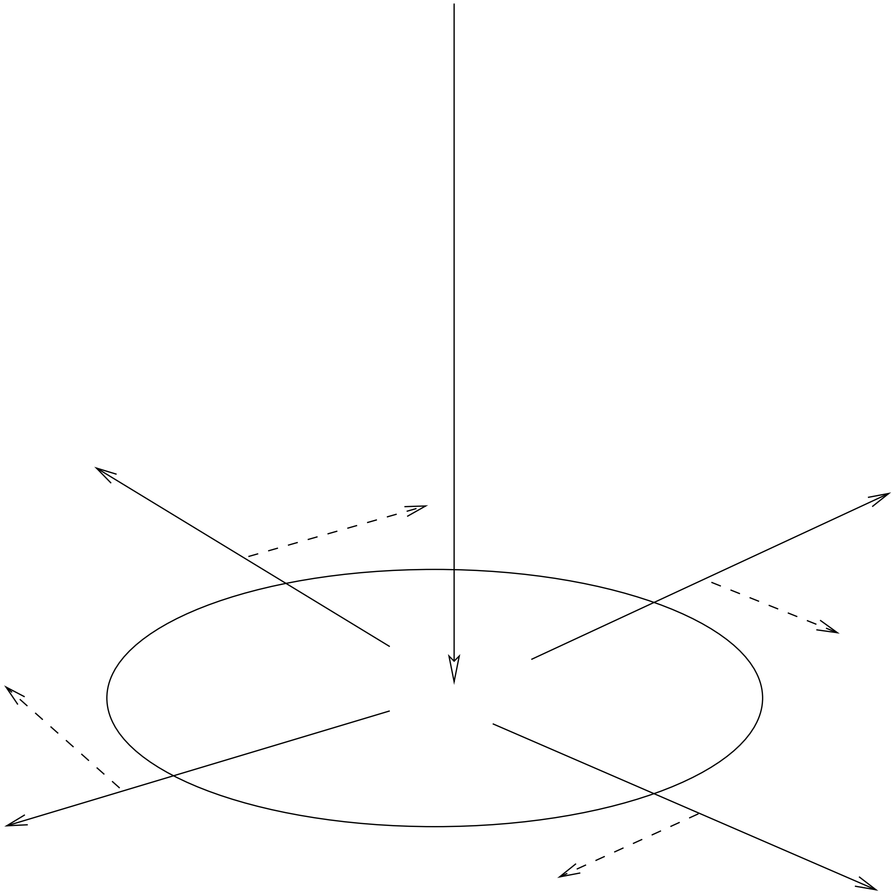
$$\frac{d_H}{dt}(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

where:

$$\frac{d_H}{dt} = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})$$

is the Lagrangian derivative following the horizontal flow

Divergence



Divergence

Consider flow with constant divergence:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D > 0$$

$$\frac{d}{dt}\zeta_a = -\zeta_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -D\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\ e^{-Dt}$$

Divergence

So:

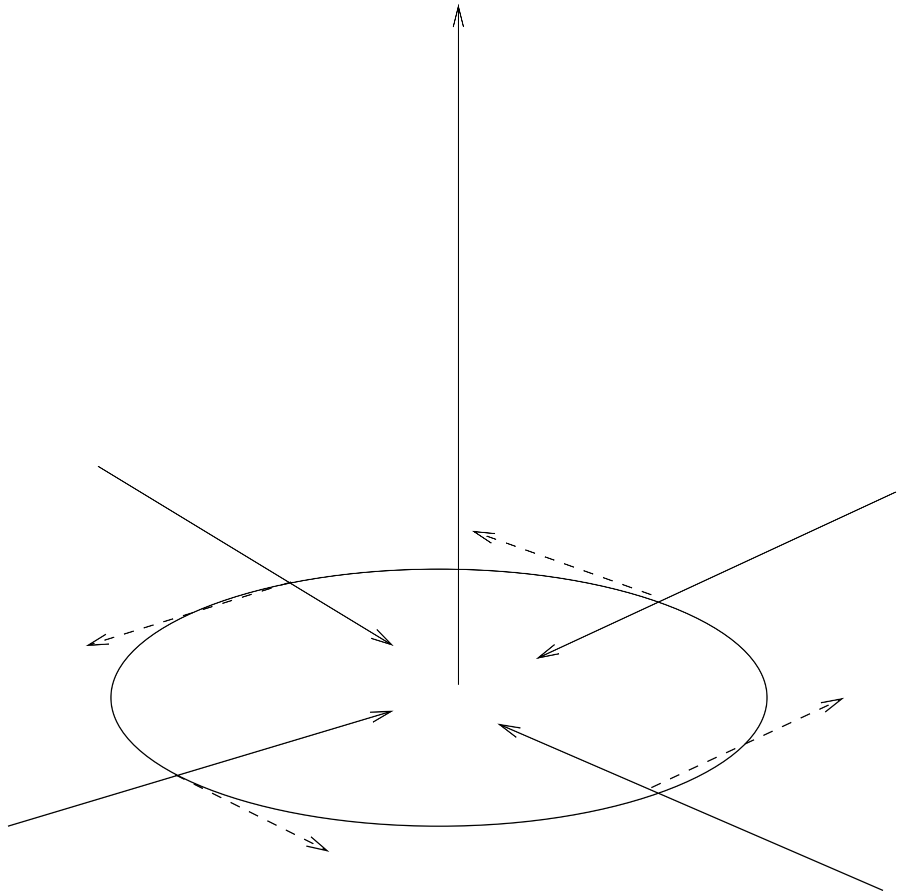
$$\zeta_a = \zeta + f \longrightarrow 0$$

$$\zeta \longrightarrow -f$$

Divergent flow favors *anticyclonic* vorticity

Vorticity approaches $-f$, regardless of initial value

Convergence



Divergence

Now say $D = -C$

$$\frac{d}{dt}\zeta_a = -\zeta_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = C\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\ e^{Ct}$$

$$\zeta_a \longrightarrow \pm\infty$$

But which sign?

Divergence

If the Rossby number is small, then:

$$\zeta_a(0) = \zeta(0) + f \approx f > 0$$

So:

$$\zeta \rightarrow +\infty$$

Convergent flow favors *cyclonic* vorticity

Vorticity increases *without bound*

- Why intense storms are cyclonic

Summary

The vorticity equation is approximately:

$$\frac{d}{dt}_H(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

or:

$$\frac{d}{dt}_H\zeta + v\frac{df}{dy} = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

- Vorticity changes due to meridional motion
- Vorticity changes due to divergence

Barotropic potential vorticity

Consider an atmospheric layer with constant density, between two surfaces, at $z = z_1, z_2$ (e.g. the surface and the tropopause)

The continuity equation is:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0$$

If density constant, then:

$$(\nabla \cdot \vec{u}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Barotropic potential vorticity

So:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Thus the vorticity equation can be written:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,\frac{\partial w}{\partial z}$$

Taylor-Proudman Theorem

The constant density assumption affects the shear

$$\frac{d}{dt}u - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

Taking a z-derivative:

$$\frac{d}{dt}(\frac{\partial}{\partial z}u) - f(\frac{\partial}{\partial z}v) = -\frac{1}{\rho}\frac{\partial}{\partial x}(\frac{\partial}{\partial z}p) = \frac{\rho}{\rho}\frac{\partial}{\partial x}g = 0$$

$\rightarrow$ If there is no shear initially, have no shear at any time.
With constant density:

$$\frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = 0$$

Barotropic potential vorticity

So the integral of the vorticity equation is simply:

$$\int_{z1}^{z2} (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) dz =$$

$$h(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,[w(z_2) - w(z_1)]$$

where $h = z_2 - z_1$. Note that $w = Dz/Dt$. Thus:

$$w(z_2) - w(z_1) = \frac{d}{dt}(z_2 - z_1) = \frac{dh}{dt}$$

Barotropic potential vorticity

So:

$$h\,\frac{d}{dt}(\zeta + f) = (\zeta + f)\,\frac{dh}{dt}$$

or:

$$\frac{1}{\zeta + f}\,\frac{d}{dt}(\zeta + f) - \frac{1}{h}\,\frac{dh}{dt} = 0$$

$$\frac{d}{dt}ln(\zeta + f) - \frac{d}{dt}lnh = 0$$

$$\frac{d}{dt}ln\frac{\zeta + f}{h} = 0$$

Barotropic potential vorticity

Thus:

$$\frac{d}{dt}(\frac{\zeta + f}{h}) = 0$$

So the barotropic potential vorticity (PV):

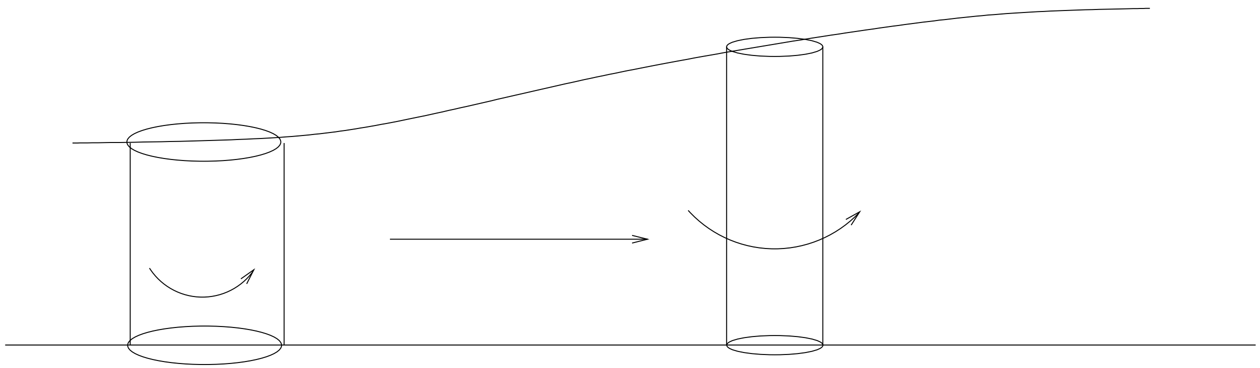
$$\frac{\zeta + f}{h} = const.$$

is conserved on a fluid parcel.

Similar to Kelvin's theorem, except includes layer *thickness*

If h increases, either ζ or f must also increase

Layer potential vorticity



Alternate derivation

Consider a fluid column between z_1 and z_2. As it moves, conserves mass:

$$\frac{d}{dt}(hA) = 0$$

So:

$$hA = const.$$

Because the density is constant, we can apply Kelvin's theorem:

$$\frac{d}{dt}(\zeta + f)A \propto \frac{d}{dt}\frac{\zeta + f}{h} = 0$$

Potential temperature

But the atmosphere is not constant density. What use is the potential vorticity?

As move upward in atmosphere, both temperature and pressure change—neither is absolute.

But can define the *potential temperature* which is absolute—accounts for pressure change.

The potential vorticity can then be applied in layers *between potential temperature surfaces*

Potential temperature

The thermodynamic energy equation is:

$$c_p dT - \alpha dp = dq$$

With zero heating:

$$c_p dT = \alpha dp = \frac{RT}{p} dp$$

using the ideal gas law. Rewriting:

$$c_p \, dlnT = R \, dlnp$$

Potential temperature

If move a parcel upward from the surface, both its temperature and pressure change. But using the surface pressure, we can define:

$$c_p\, lnT - R\, lnp = c_p\, ln\theta - R\, lnp_0$$

where p_0 is the surface pressure:

$$p_0 = 100\ kPa\ = 1000mb$$

Rearranging:

$$\theta = T\, (\frac{p_0}{p})^{R/c_p}$$

Potential temperature

If zero heating, a parcel conserves its potential temperature, θ

Call a surface with constant potential temperature an isentropic surface or an “adiabat”

θ is the temperature a parcel has if we move it adiabatically back to the surface

Note potential temperature depends on *both* T and p

Layer potential vorticity

Flow between two isentropic surfaces trapped if zero heating

So mass in a column between two surfaces is conserved:

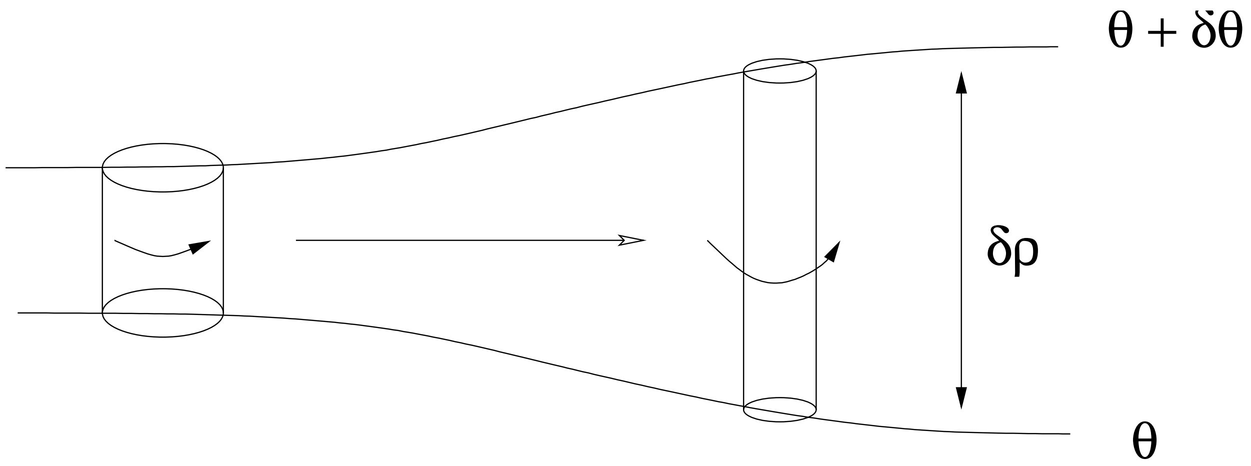
$$A\delta z = const.$$

From the hydrostatic relation:

$$-\frac{A\delta p}{\rho g} = const.$$

where δp is the spacing between surfaces

Layer potential vorticity



Layer potential vorticity

Rewrite δp thus:

$$\delta p = (\frac{\partial \theta}{\partial p})^{-1} \, \delta \theta$$

Here, $\frac{\partial \theta}{\partial p}$ is the *stratification*. The stronger the stratification, the smaller the pressure difference between temperature surfaces. Thus:

$$\frac{A \delta p}{\rho g} = A(\frac{\partial \theta}{\partial p})^{-1} \, \frac{\delta \theta}{g} = const.$$

Layer potential vorticity

From the Ideal Gas Law and the definition of potential temperature, we can write:

$$\rho = p^{c_v/c_p}(R\theta)^{-1}p_s^{R/c_p}$$

So the density is only a function of pressure. This means that:

$$\oint \frac{dp}{\rho} \propto \oint dp^{1-c_v/c_p} = 0$$

So Kelvin's theorem applies in the layer

Layer potential vorticity

Thus:

$$\frac{d}{dt}[(\zeta + f)A] = 0$$

implies:

$$\frac{d}{dt}[(\zeta + f)\frac{\partial \theta}{\partial p}] = 0$$

This is Ertel's (1942) "isentropic potential vorticity equation"

Layer potential vorticity

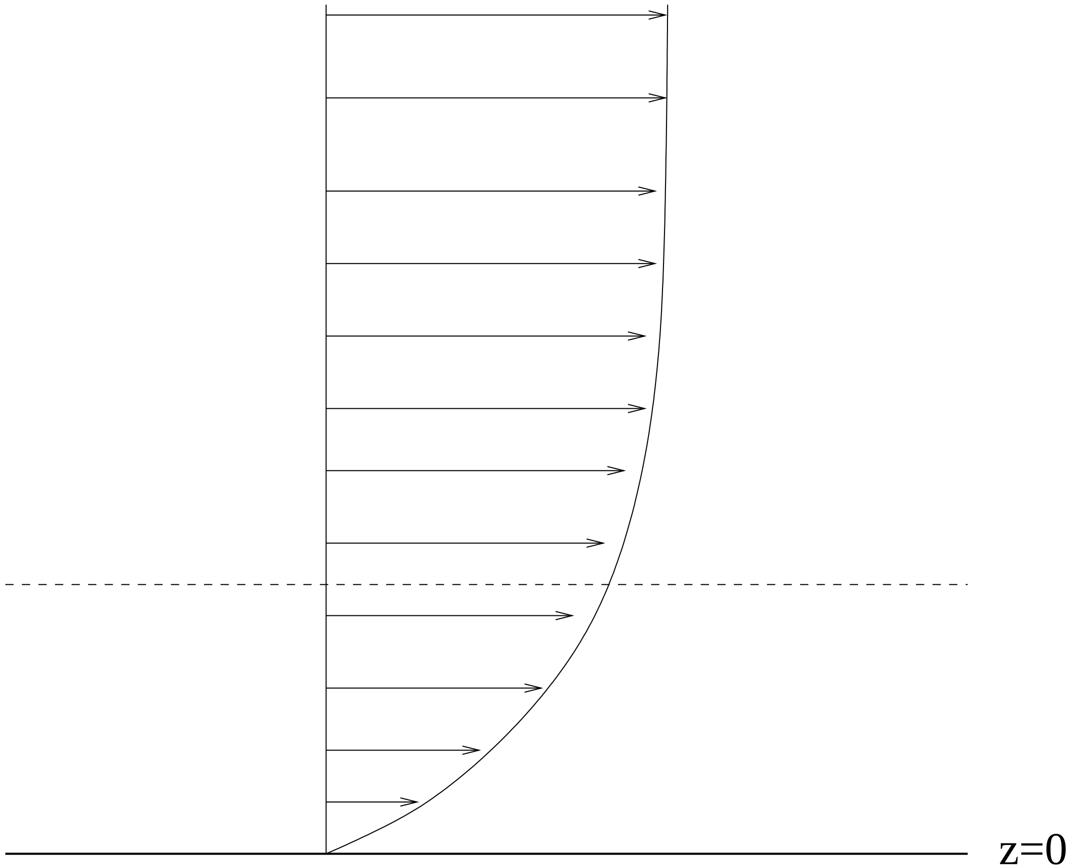
Remember: ζ evaluted on potential temperature surface

Very useful quantity: can label air by its PV

Can distinguish air in the troposphere which comes from stratosphere

Ertel's equation can also be used for prediction

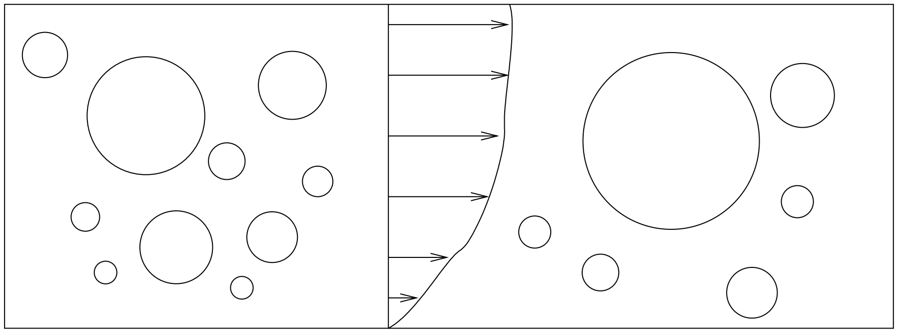
Planetary boundary layer



Turbulence

There is a *continuum* of eddy scales

Largest resolved by our models, but the smallest are not.



Boussinesq equations

Assume we can split the velocity into a mean (over some period) and a perturbation:

$$\overline{u} = \overline{u} + u'$$

Use the momentum equations with no friction:

$$\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

Boussinesq equations

Assume density in the boundary layer approximately constant, so that:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

Substitute the partitioned velocities into the momentum equations and then average:

$$\frac{\partial}{\partial t}(\overline{u} + u') + (\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u') + (\overline{v} + v')\frac{\partial}{\partial y}(\overline{u} + u') - f(\overline{v} + v')$$

$$+(\overline{w} + w')\frac{\partial}{\partial z}(\overline{u} + u') = \frac{1}{\rho}\frac{\partial}{\partial x}(\overline{p} + p')$$

Boussinesq equations

Now average. Note that:

$$\overline{\overline{u} + u'} = \overline{u}$$

so:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{u'\frac{\partial}{\partial x}u'} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{v'\frac{\partial}{\partial y}u'} +$$

$$+\overline{w}\frac{\partial}{\partial z}\overline{u} + \overline{w'\frac{\partial}{\partial z}u'} + -f\overline{v} = \frac{1}{\rho}\frac{\partial}{\partial x}\overline{p}$$

Boussinesq equations

Because of the continuity equation, we can write:

$$\overline{u'\frac{\partial}{\partial x}u'} + \overline{v'\frac{\partial}{\partial y}u'} + \overline{w'\frac{\partial}{\partial z}u'} = \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}$$

So:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} =$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - (\frac{\partial}{\partial x}\rho\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'})$$

Boussinesq equations

Similarly:

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} =$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - (\frac{\partial}{\partial x}\rho\overline{v'u'} + \frac{\partial}{\partial y}\overline{v'v'} + \frac{\partial}{\partial z}\overline{v'w'})$$

Terms on the RHS are the “eddy stresses”

PBL equations

Assume the eddy stresses don't vary horizontally. Then:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} = -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} = -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

Outside the boundary layer, assume geostrophy. In the layer, we have geostrophic terms *plus* vertical mixing. So turbulence *breaks geostrophy*:

$$-f\overline{v} = -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$= -f\overline{v}_g - \frac{\partial}{\partial z}\overline{u'w'}$$

$$f\overline{u} = f\overline{u}_g - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

But too many unknowns! : $\overline{u}, \overline{v}, u', v', w'$

Must *parameterize* the eddy stresses.

Two cases:

- Stable boundary layer: stratified
- Convective boundary layer: vertically mixed

Convective boundary layer

Due to vertical mixing, temperature and velocity are constant with height. So we can integrate the momentum equation vertically:

$$\int_0^h -f(\overline{v} - \overline{v}_g)\, dz = -fh(\overline{v} - \overline{v}_g) =$$

$$-\int_0^h \frac{\partial}{\partial z} \overline{u'w'}\, dz = \overline{u'w'}|_h - \overline{u'w'}|_0$$

We can assume mixing vanishes outside of the layer:

$$\overline{u'w'}|_h = 0$$

Convective boundary layer

Thus:

$$f h(\overline{v} - \overline{v}_g) = -\overline{u'w'}|_s$$

From surface measurements, can parameterize the fluxes:

$$\overline{u'w'}|_0 = -C_d \mathcal{V}\, u, \qquad \overline{v'w'}|_0 = -C_d \mathcal{V}\, v$$

where C_d is the "drag coefficient" and

$$\mathcal{V} \equiv (u^2 + v^2)^{1/2}$$

Convective boundary layer

Thus:

$$f h(\overline{v} - \overline{v}_g) = C_d \mathcal{V}\, \overline{u}$$

and:

$$-f h(\overline{u} - \overline{u}_g) = C_d \mathcal{V}\, \overline{v}$$

Convective boundary layer

Say $v_g = 0$; then:

$$v = \frac{C_d}{fh} \mathcal{V} u,$$

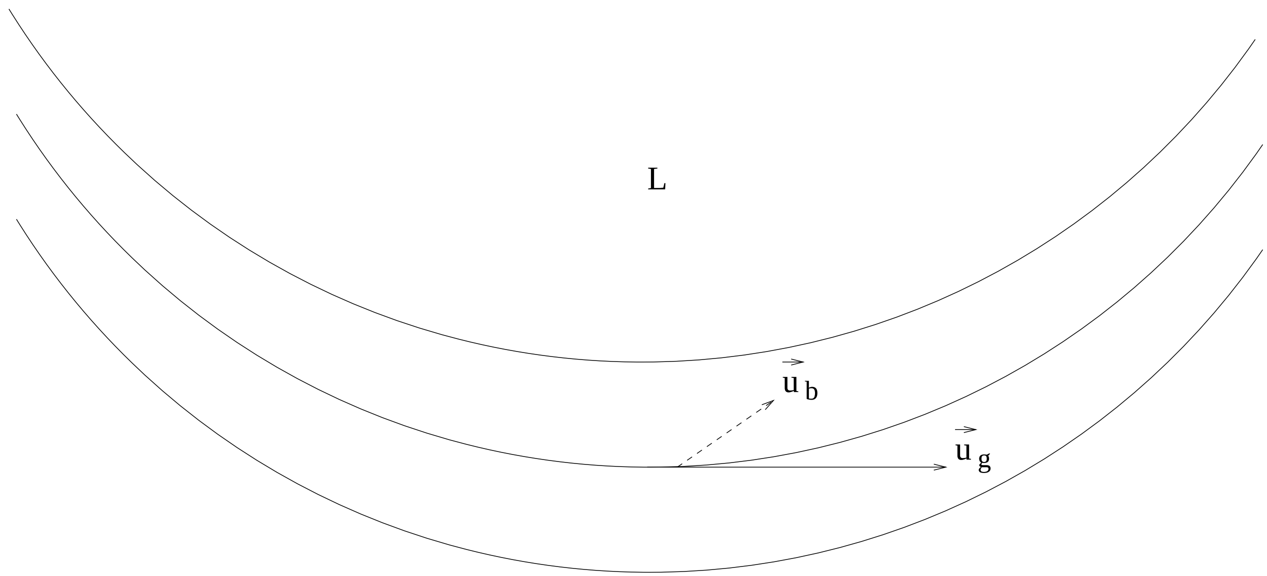
$$u = u_g - \frac{C_d}{fh} \mathcal{V} v$$

Solving equations not so simple because $\mathcal{V} = \sqrt{u^2 + v^2}$

But can use iterative methods

Convective boundary layer

If $u > 0$, then $v > 0$



- Flow down the pressure gradient

Stable boundary layer

Now assume no large scale vertical mixing

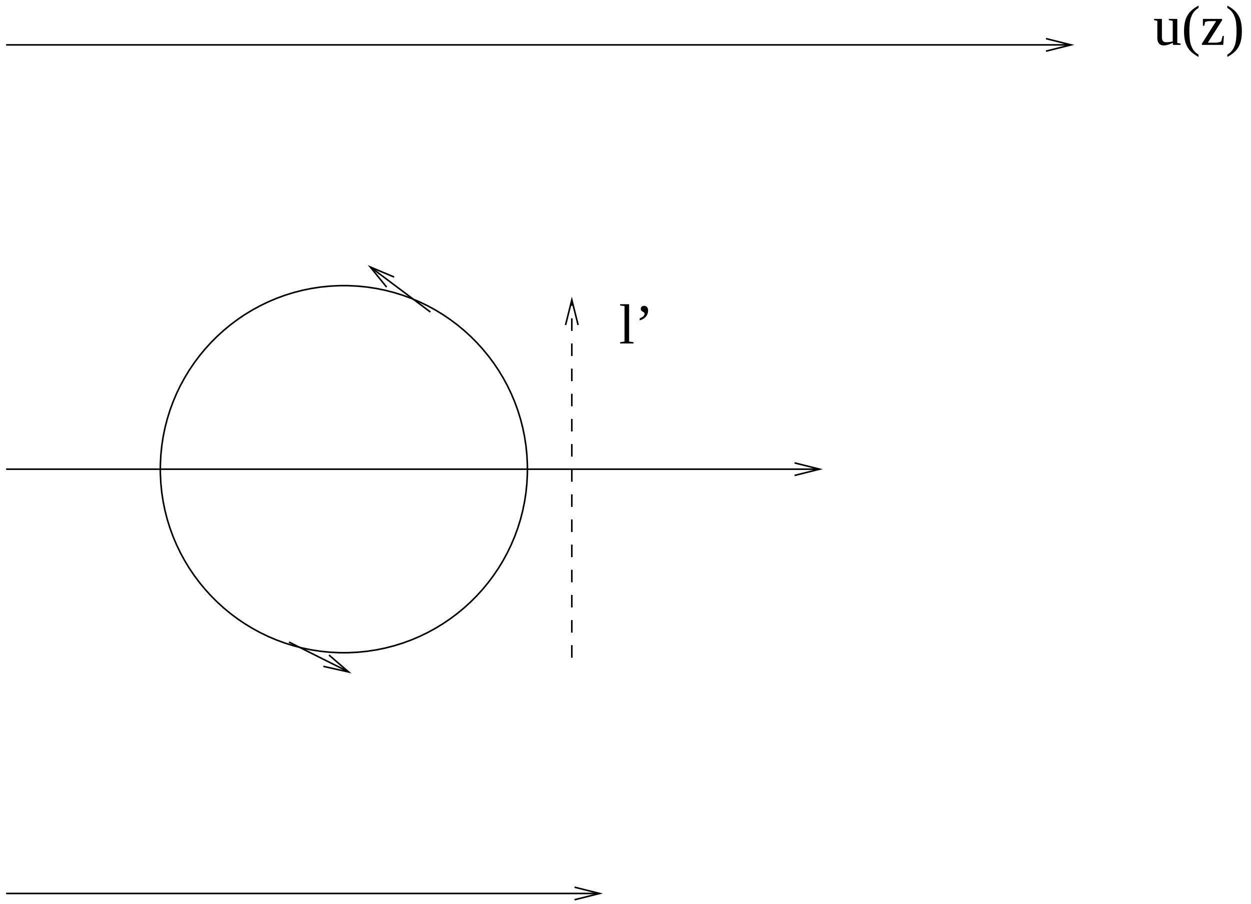
Wind speed and direction can vary with height

Specify turbulent velocities using *mixing length theory*:

$$u' = -l' \frac{\partial}{\partial z}\overline{u}$$

where $l' > 0$ if up.

Mixing length



Stable boundary layer

So:

$$-\overline{u'w'} = \overline{w'l'}\,\frac{\partial}{\partial z}\overline{u}$$

Assume the vertical and horizontal eddy scales are comparable

$$w' = l'\frac{\partial}{\partial z}\mathcal{V}$$

where again $\mathcal{V} = \sqrt{u^2 + v^2}$

Notice $w' > 0$ if $l' > 0$.

Stable boundary layer

So:

$$-\overline{u'w'} = (\overline{l'^2} \frac{\partial}{\partial z}\mathcal{V}) \frac{\partial}{\partial z}\overline{u} \equiv A_z \frac{\partial}{\partial z}\overline{u}$$

Same argument:

$$-\overline{v'w'} = A_z \frac{\partial}{\partial z}\overline{v}$$

where A_z is the "eddy exchange coefficient"

Depends on the size of turbulent eddies and mean shear

Stable boundary layer

So we have:

$$f(v - v_g) = \frac{\partial}{\partial z}[A_z(z)\frac{\partial}{\partial z}u]$$

$$-f(u - u_g) = \frac{\partial}{\partial z}[A_z(z)\frac{\partial}{\partial z}v]$$

Simplest case is if $A_z(z)$ is constant

Studied by Swedish oceanographer V. W. Ekman (1905)

Ekman layer

Boundary conditions: use the “no-slip condition”:

$$u = 0, v = 0 \quad at \ z = 0$$

Far from the surface, the velocities approach their geostrophic values:

$$u \longrightarrow u_g, v \longrightarrow v_g \qquad z \longrightarrow \infty$$

Assume the geostrophic flow is zonal and independent of height:

$$u_g = U, \qquad v_g = 0$$

Ekman layer

Boundary layer velocities vary only in the vertical:

$$u = u(z) \; , \quad v = v(z) \; , \quad w = w(z)$$

From continuity:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = \frac{\partial}{\partial z}w = 0 \; .$$

With a flat bottom, this implies:

$$w = 0$$

Ekman layer

The system is linear, so can decompose the horizontal velocities:

$$u = U + \hat{u}, \quad v = 0 + \hat{v}$$

Then:

$$-f\hat{v} = A_z \frac{\partial^2}{\partial z^2}\hat{u}$$

$$f\hat{u} = A_z \frac{\partial^2}{\partial z^2}\hat{v} \ .$$

Ekman layer

Boundary conditions:

$$\hat{u} = -U, \hat{v} = 0 \quad at \ z = 0$$

Introduce a new variable:

$$\chi \equiv \hat{u} + i\hat{v}$$

Then:

$$\frac{\partial^2}{\partial z^2}\chi = i\frac{f}{A_z}\chi$$

Ekman layer

The solution is:

$$\chi = A\,exp(\frac{z}{\delta_E})\,exp(i\frac{z}{\delta_E}) + B\,exp(-\frac{z}{\delta_E})\,exp(-i\frac{z}{\delta_E})\ ,$$

where:

$$\delta_E = \sqrt{\frac{2A_z}{f}}$$

This is the “Ekman depth”

Corrections should decay going up, so:

$$A = 0$$

Ekman layer

Take the real part of the horizontal velocities:

$$u = Re\{\chi\} = Re\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

and

$$v = Im\{\chi\} = -Re\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

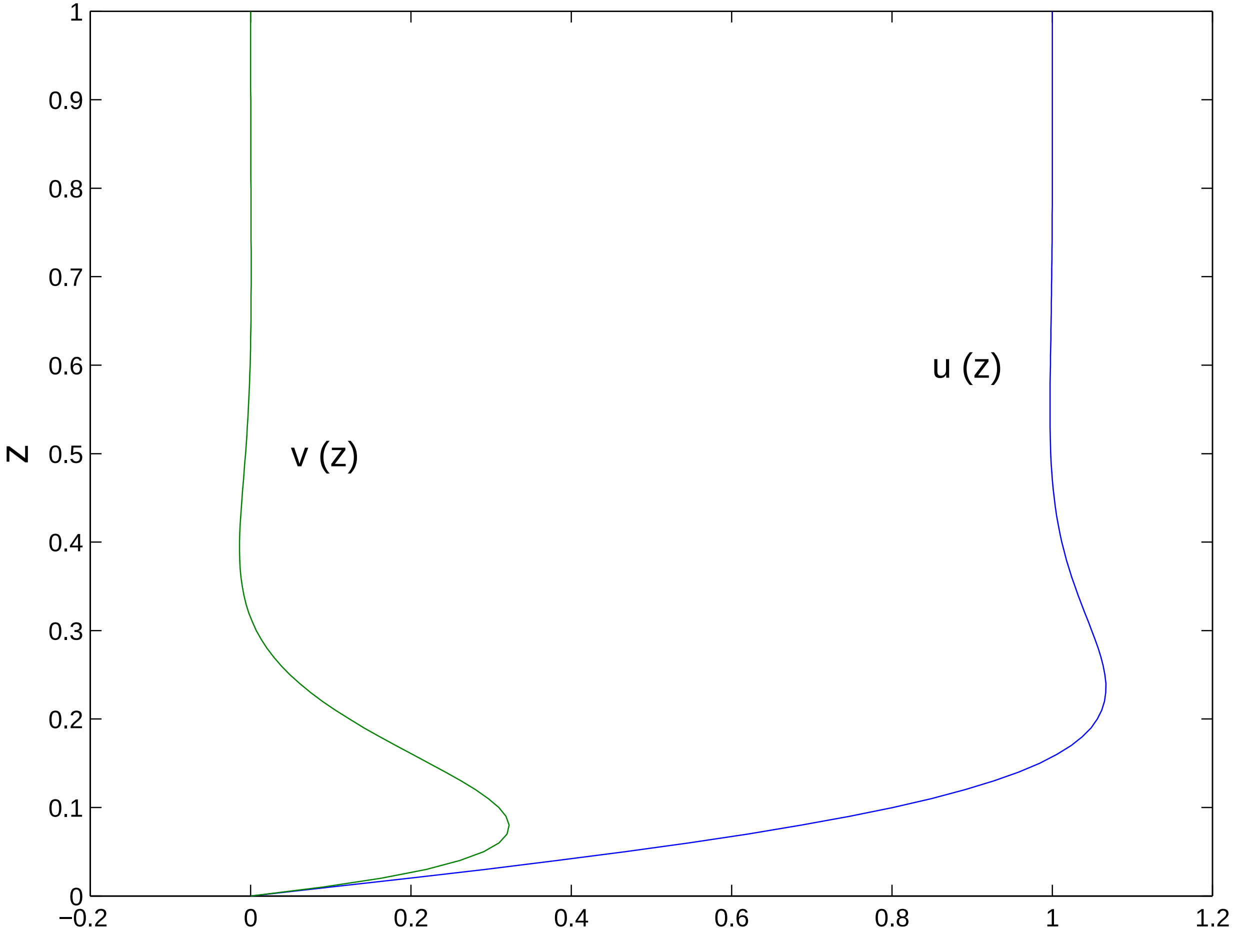
Ekman layer

For zero flow at $z = 0$, require $Re\{B\} = -U$ and $Im\{B\} = 0$. So:

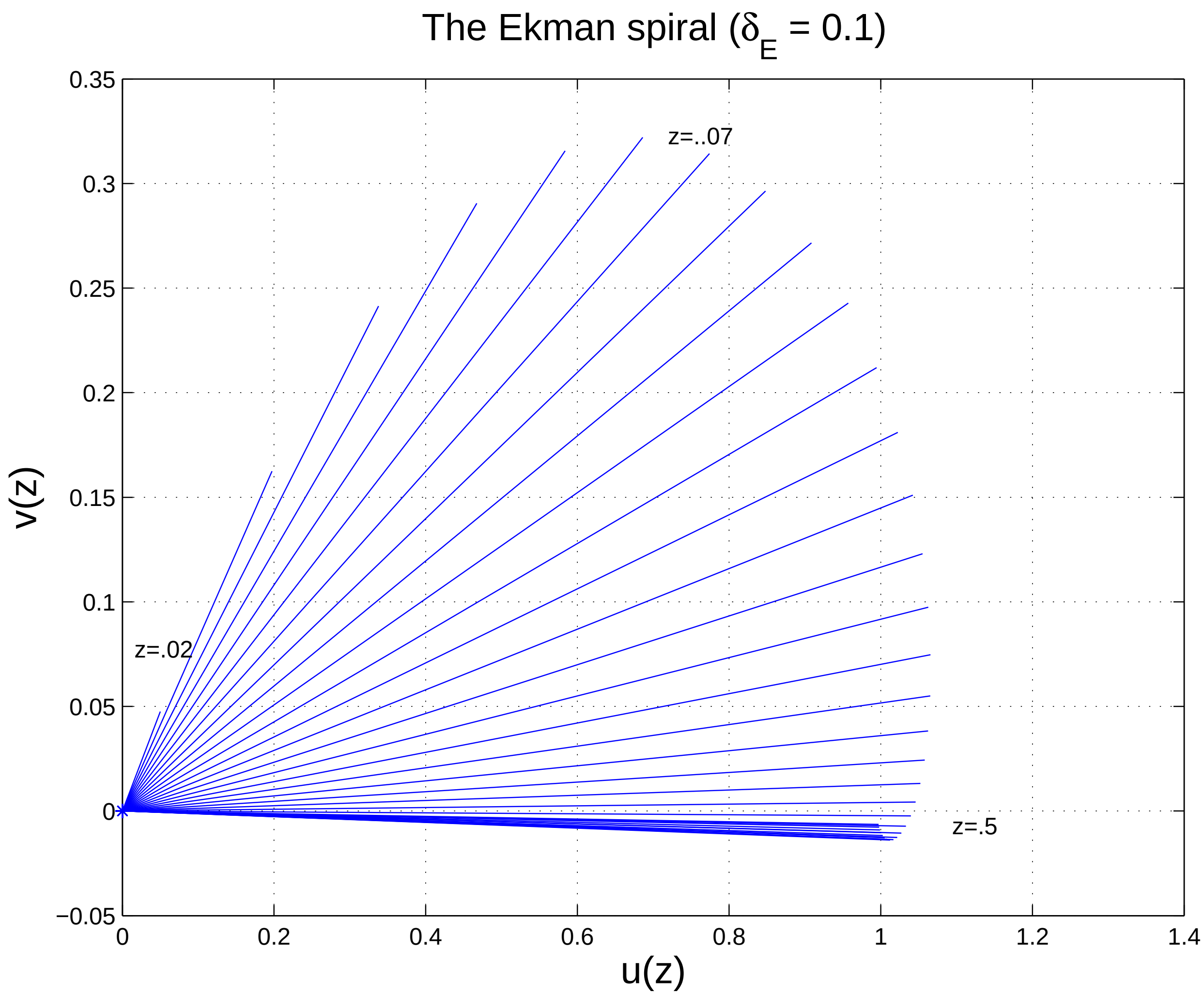
$$u = U + \hat{u} = U - U\,exp(-\frac{z}{\delta_E})\,cos(\frac{z}{\delta_E})$$

$$v = \hat{v} = U\,exp(-\frac{z}{\delta_E})\,sin(\frac{z}{\delta_E})\ ,$$

Ekman layer



Ekman spiral



Ekman spiral

The velocity veers to the *left* in the layer

Observations suggest $u \to u_g$ at $z = 1$ km.

With $f = 10^{-4}/sec$, we have:

$$A_z \approx 5\, m^2/sec$$

If $\frac{\partial}{\partial z}\mathcal{V}| = 5 \times 10^{-3}$, the mixing length $l \approx 30$ m.

As in the convective boundary layer, turbulence allows flow from high pressure to low pressure.

Surface layer

Ekman layer cannot hold near surface: can't have 30 m eddies 10 m from surface. Introduce a *surface layer* where:

$$l' = kz$$

Then:

$$A_z = k^2\, z^2\, \frac{\partial}{\partial z}\mathcal{V}$$

So:

$$A_z \frac{\partial}{\partial z} u = k^2 z^2 |\frac{\partial}{\partial z} V| \frac{\partial}{\partial z} u \approx k^2 z^2 (\frac{\partial}{\partial z} u)^2$$

Surface layer

Measurements suggest the turbulent momentum flux is approximately constant in the surface layer:

$$\frac{\partial}{\partial z}\overline{u'w'} \approx u_*^2$$

where u_* is the "friction velocity". So:

$$\frac{\partial}{\partial z}u = \frac{u_*}{kz} \quad \rightarrow \quad u = \frac{u_*}{k}ln(\frac{z}{z_0})$$

Here:

- $k \approx 0.4$ is von Karman's constant
- z_0 is the "roughness length"

Surface layer

Match the velocity at the top of the surface layer to that at the base of the Ekman layer.

Comparisons with observations are only fair (see Fig. 5.5 of Holton)

Ekman spiral is often unstable, generating eddies that mix away the signal

Spin-down

Turbulence in both stable and convective boundary layers causes the winds to slow down

Both have flow down pressure gradient

This weakens the gradient and the geostrophic wind

Convergence/divergence in the Ekman layer causes a vertical velocity at the top of the layer

Spin-down

Illustrate using the barotropic vorticity equation:

$$\frac{D}{Dt}(\zeta + f) \approx f\,\frac{\partial w}{\partial z}$$

Integrate from the top of boundary layer ($z = d$) to the tropopause:

$$(H - d)\frac{D}{Dt}(\zeta + f) = f(w(H) - w(d)) = -fw(d)$$

Spin-down

Because the boundary layer is much thinner than the troposphere, this is approximately:

$$\frac{D}{Dt}(\zeta + f) = -\frac{f}{H}w(d)$$

So vertical velocity into/out of the boundary layer changes the vorticity in the troposphere

Ekman pumping

Example: the Ekman layer. The continuity equation is:

$$\frac{\partial}{\partial z}w = -\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v$$

Integrating over the layer, we get:

$$w(d) - 0 = -\int_0^d (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\,dz \equiv -\frac{\partial}{\partial x}M_x - \frac{\partial}{\partial y}M_y$$

where M_x and M_y are the horizontal *transports*

Spin-down

Can show:

$$M_y \approx \frac{Ud}{2}$$

and:

$$M_x \approx -\frac{Vd}{2}$$

So:

$$w(d) = \frac{d}{2}\,(\frac{\partial}{\partial x}V - \frac{\partial}{\partial y}U) = \frac{d}{2}\,\zeta$$

Spin-down

Thus:

$$\frac{D}{Dt}(\zeta + f) = -\frac{fd}{2H}\zeta$$

If assume $f = const.$, then:

$$\frac{D}{Dt}\zeta = -\frac{fd}{2H}\zeta$$

So that:

$$\zeta(t) = \zeta(0)\, exp(-t/\tau_E)$$

Spin-down

where:

$$\tau_E \equiv \frac{2H}{fd}$$

is the Ekman *spin-down time*. Typical values:

$$H = 10km, \;\; f = 10^{-4}sec^{-1}, \;\; d = 0.5km$$

yield:

$$\tau_E \approx \; 5 \; days$$

Spin-down

Compare to molecular dissipation. Then:

$$\frac{\partial}{\partial t}u = K_m \frac{\partial^2}{\partial z^2}u$$

From scaling:

$$T_d \approx \frac{H^2 U}{U K_m} = \frac{H^2}{K_m} \approx 100 \; days$$

The Ekman layer is much more effective at damping motion

Spin-down

The vertical velocity is part of the *secondary circulation*

The primary flow is horizontal, (u_g, v_g)

The vertical velocities, though smaller, are extremely important nevertheless

Stratification reduces the effective H. So the geostrophic velocity over Ekman layer spins down more rapidly, leaving winds aloft alone.