GEF 2220: Dynamics

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Course

<u>Part 1</u>: Dynamics: *LaCasce*

Chapter 7, Wallace and Hobbs + my notes

<u>Part 2</u>: Weather systems: *Røsting*

Chapter 8, Wallace and Hobbs + extra articles + DIANA

Dynamics

1) Derive the equations which describe atmospheric motion

2) Derive approximate balances

3) Understand pressure systems, temperature gradients

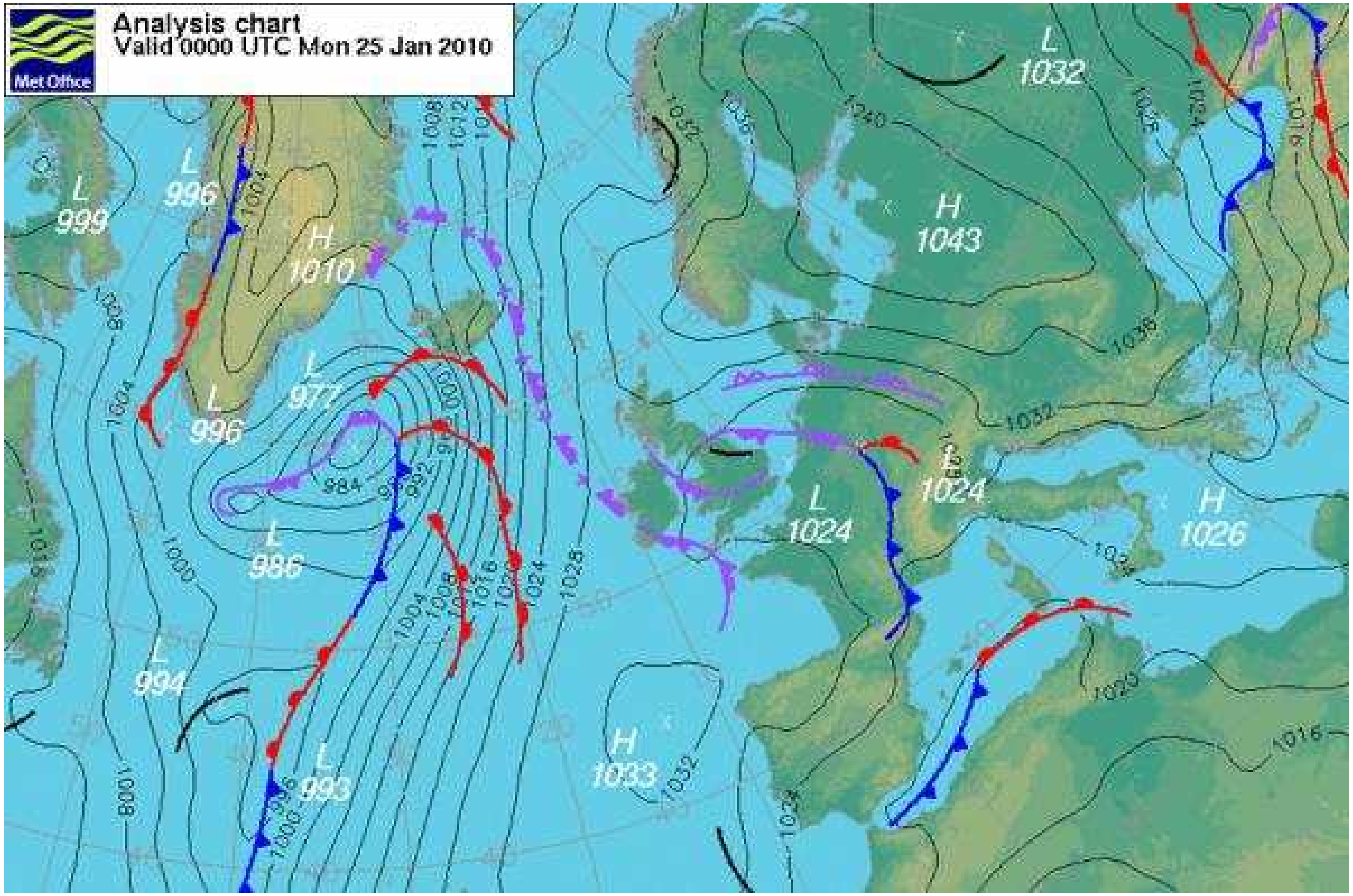
4) Introduce the general circulation

Variables

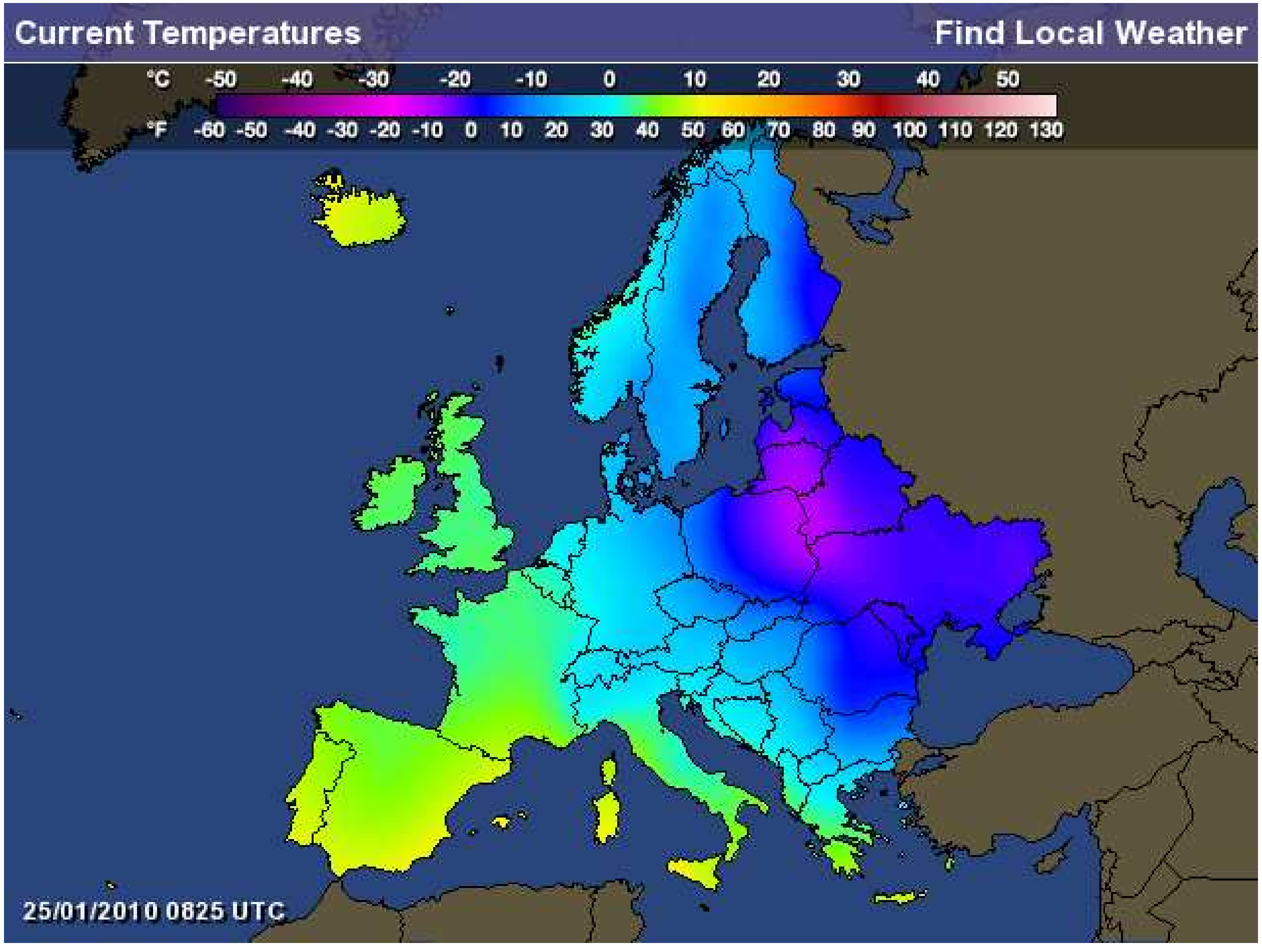
Six unknowns:

- (u, v, w) — Wind velocities
- p — Pressure
- T — Temperature
- ρ — Density

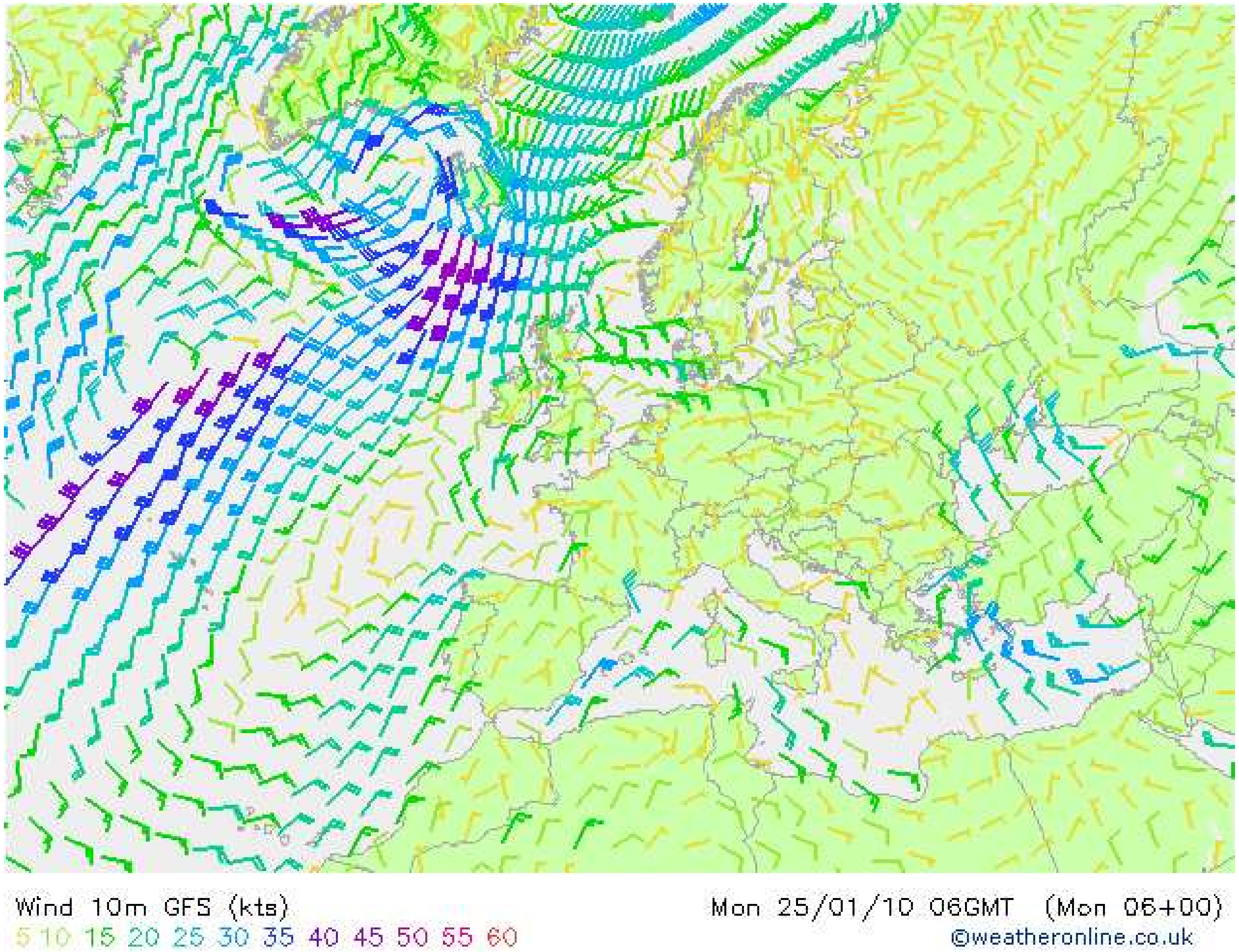
Pressure


Met Office
Analysis chart
Valid 0000 UTC Mon 25 Jan 2010
L 999
L 996
H 1010
L 977
L 996
L 986
L 994
L 993
L 1032
H 1043
L 1024
L 1024
H 1026
H 1033

Temperature



Winds



Wind 10m GFS (kts)
5 10 15 20 25 30 35 40 45 50 55 60
Mon 25/01/10 06GMT (Mon 06+00)
©weatheronline.co.uk

Primitive equations

Momentum equations $\longleftarrow$ $F = ma$

Thermodynamic energy equation $\longleftrightarrow$ T

Continuity $\longleftrightarrow$ ρ

Ideal gas law

Primitive equations

Momentum:

$$\frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w$$

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$$

Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \frac{dq}{dt}$$

Prediction

Solve the equations numerically with weather models

Issues:

- Numerical resolution
- Vertical coordinate
- Small scale mixing
- Convection
- Clouds

Goal: forecasting

Dynamics

Solve a simplified set of equations

- Identify dominant balances
- Simplify the equations
- Obtain solutions (analytical, numerical)
- Look for similiarities with observations

Goal: *understanding* the atmosphere

Derivatives

Consider an air parcel, with temperature $T = T(x, y, z, t)$

The change in temperature, from the chain rule:

$$dT = \frac{\partial T}{\partial t}\,dt + \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz$$

So:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T$$

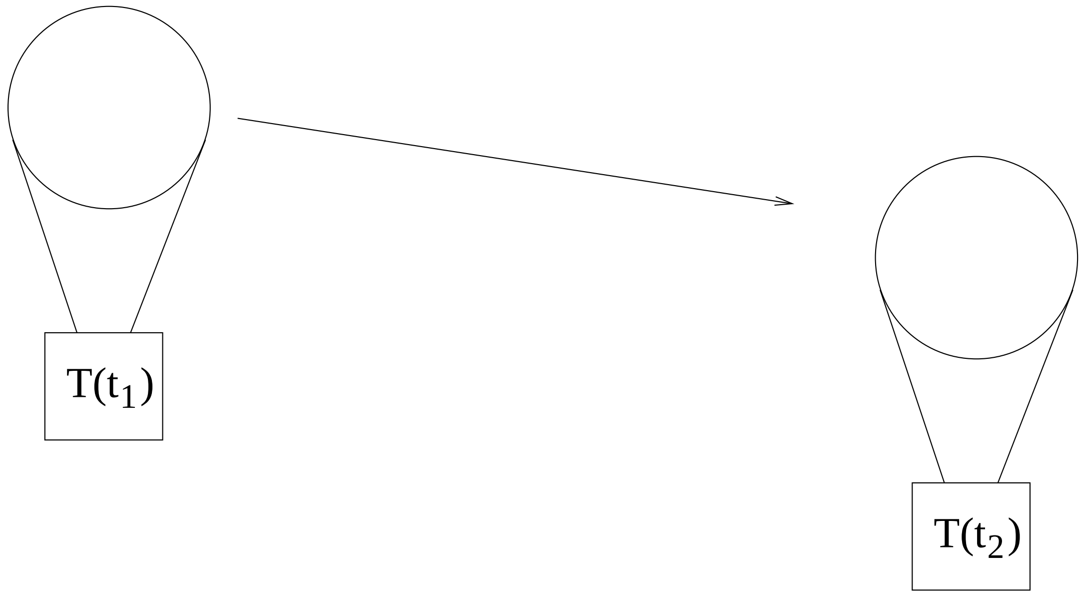
Derivatives

(u, v, w) are the wind velocities in the (x, y, z) directions

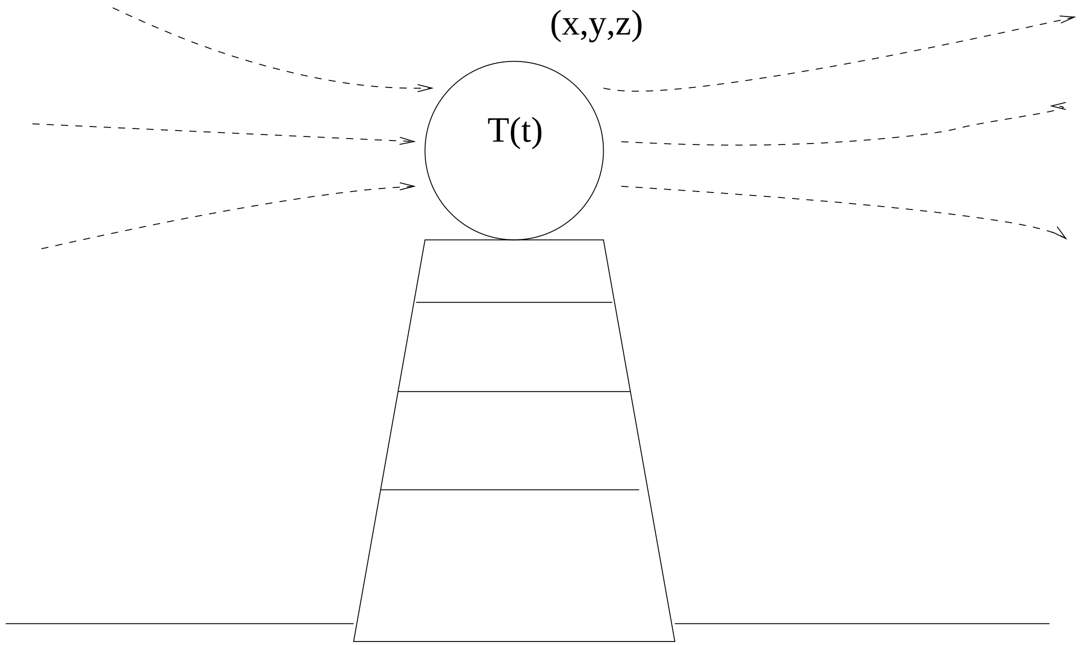
$\frac{d}{dt}$ is the “Lagrangian” derivative

$\frac{\partial}{\partial t} + \vec{u} \cdot \nabla$ is the “Eulerian” derivative

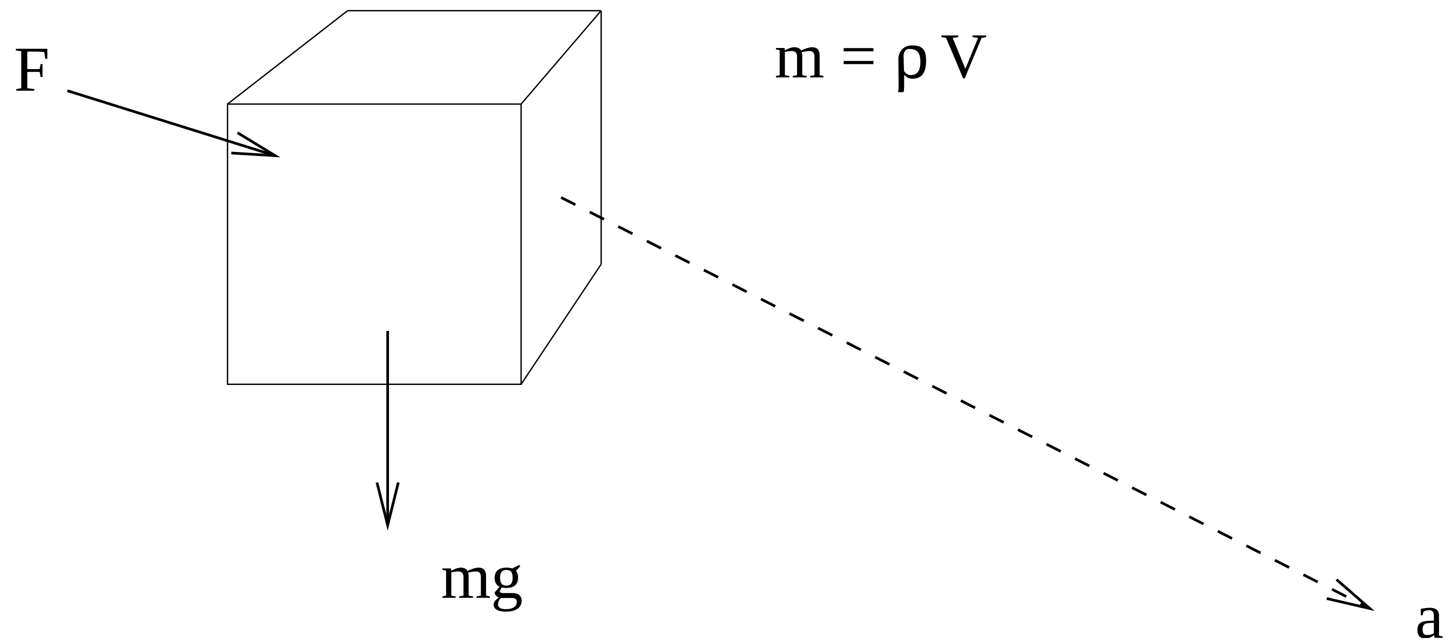
Lagrangian



Eulerian



Momentum equations



Momentum equations

The acceleration in the x-direction is:

$$a_x = \frac{1}{m} \sum_i F_i$$

Two types of force:

- Real
- Apparent

Forces

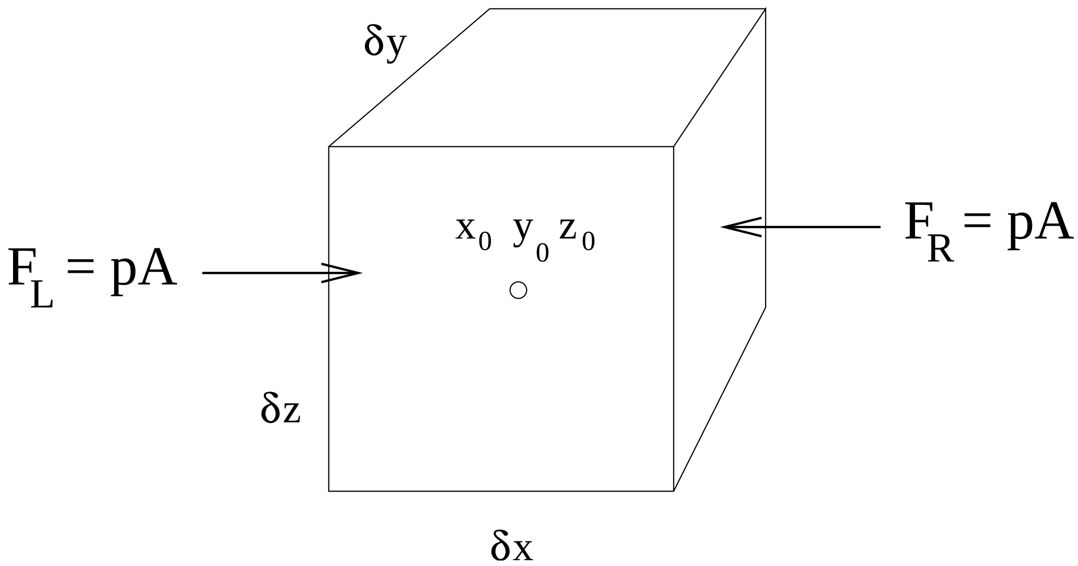
Real forces

- Pressure gradient
- Gravity
- Friction

Apparent forces

- Coriolis
- Centrifugal

Pressure gradient



$$\delta V = \delta x \; \delta y \; \delta z$$

Pressure gradient

Using a *Taylor series*, we can write the pressure on the right side of the box:

$$p_R = p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2} + ...$$

Similarly, the pressure on left side of the box is:

$$p_L = p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2} + ...$$

Pressure gradient

The force on the right hand side (directed inwards):

$$F_R = -p_R A = [p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

On left side:

$$F_L = p_L A = [p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

So the net force is:

$$F_x = F_L + F_R = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

Pressure gradient

The volume weighs:

$$m = \rho\, \delta x\, \delta y\, \delta z$$

So:

$$a_x \equiv \frac{du}{dt} = \frac{F_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Same derivation for the y and z directions.

Note this is a *Lagrangian* derivative

Momentum equations

Momentum with pressure gradients:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p$$

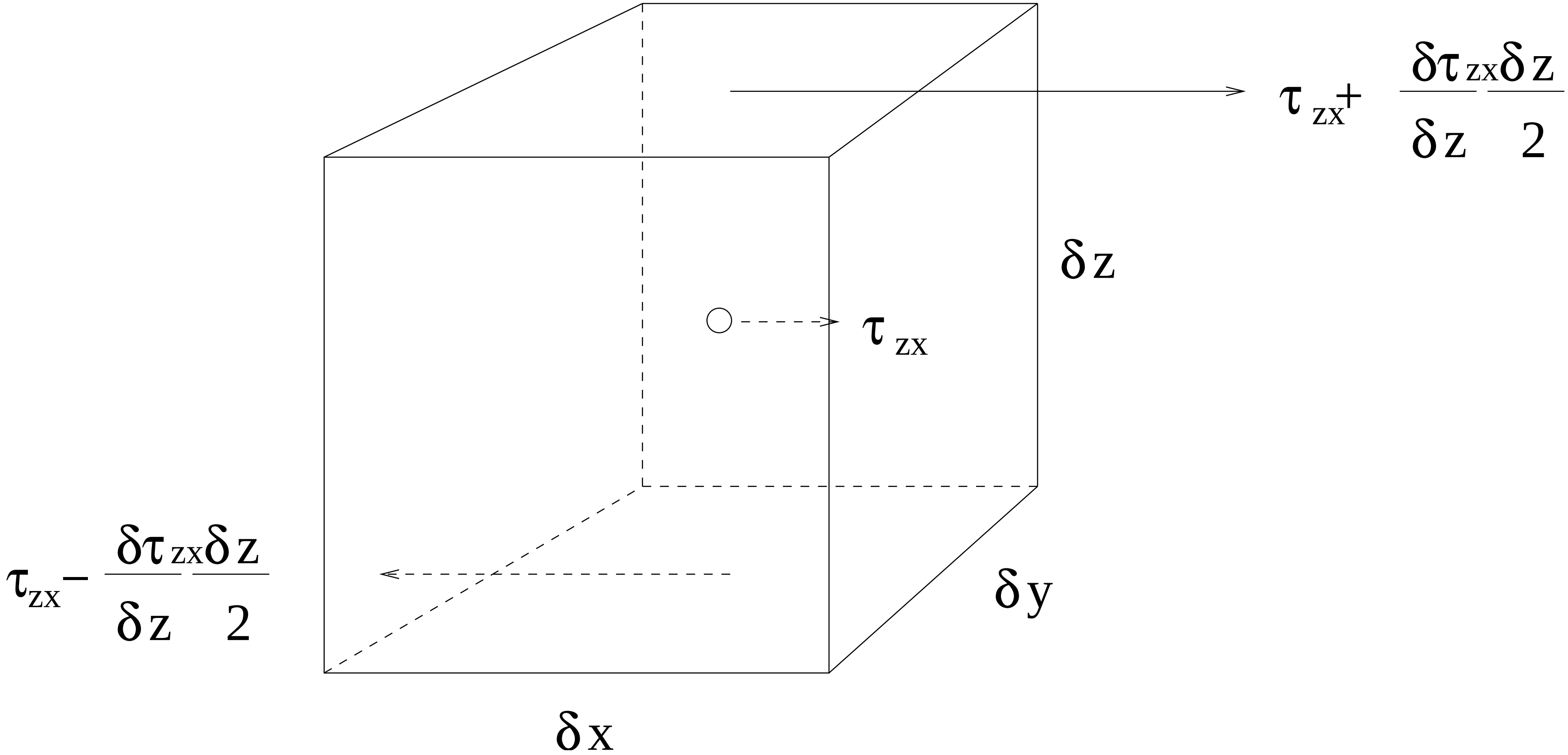
Gravity

Acts downward (toward the center of the earth):

$$a_z = \frac{F_z}{m} = -g$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

Friction



$\tau_{zx} + \frac{\delta\tau_{zx}}{\delta z}\frac{\delta z}{2}$
δz
τ_{zx}
$\tau_{zx} - \frac{\delta\tau_{zx}}{\delta z}\frac{\delta z}{2}$
δy
δx

Friction

The stress causes an accerleration:

$$\frac{du}{dt} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$$

We don't know the stress. So we *parameterize* it:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \nu \frac{\partial^2}{\partial z^2} u$$

(for example with molecular mixing). In 3 dimensions:

$$\frac{du}{dt} = \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u = \nu \nabla^2 u$$

Momenutum equations

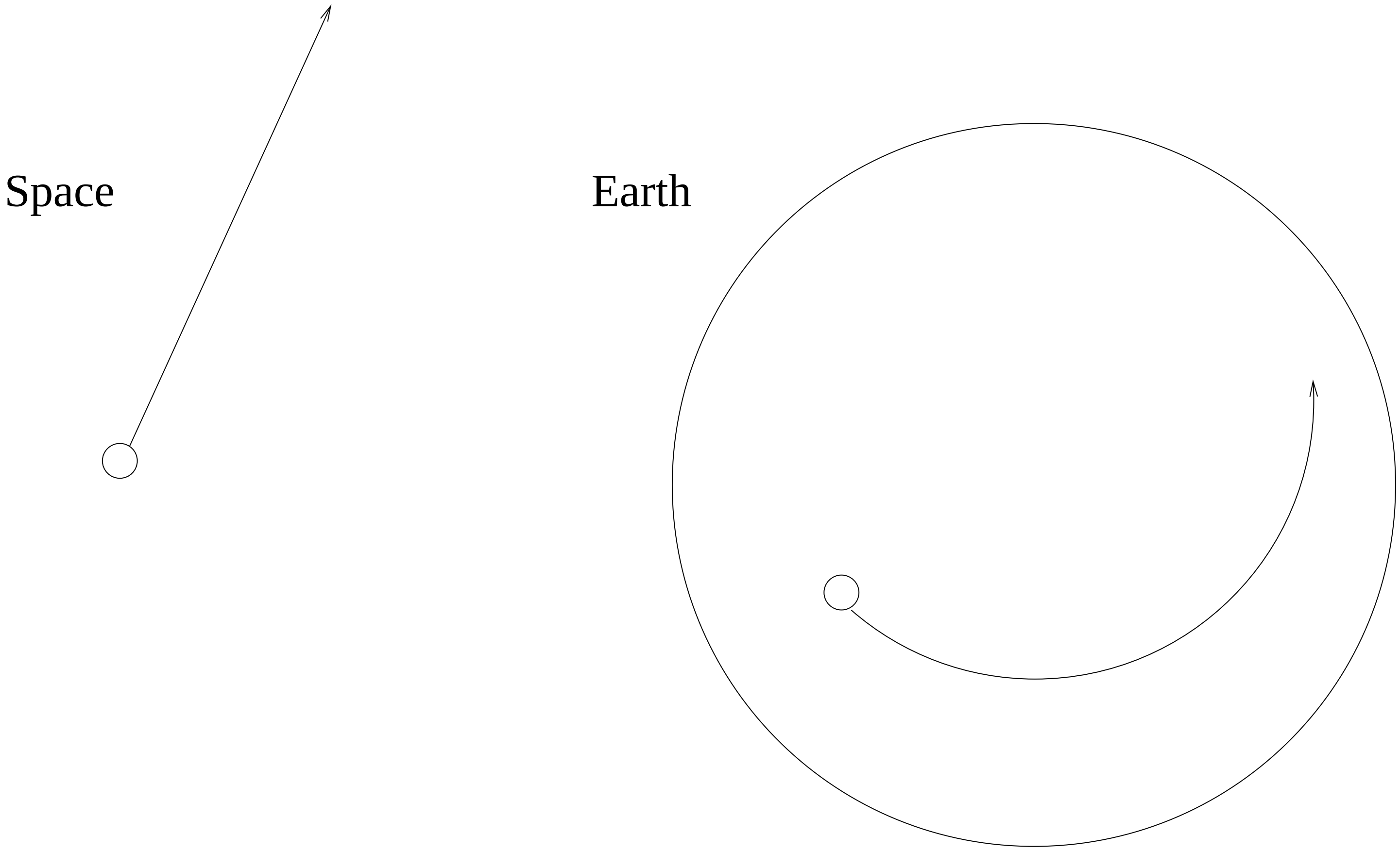
With all the real forces, we have:

$$\frac{du}{dt} = \frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

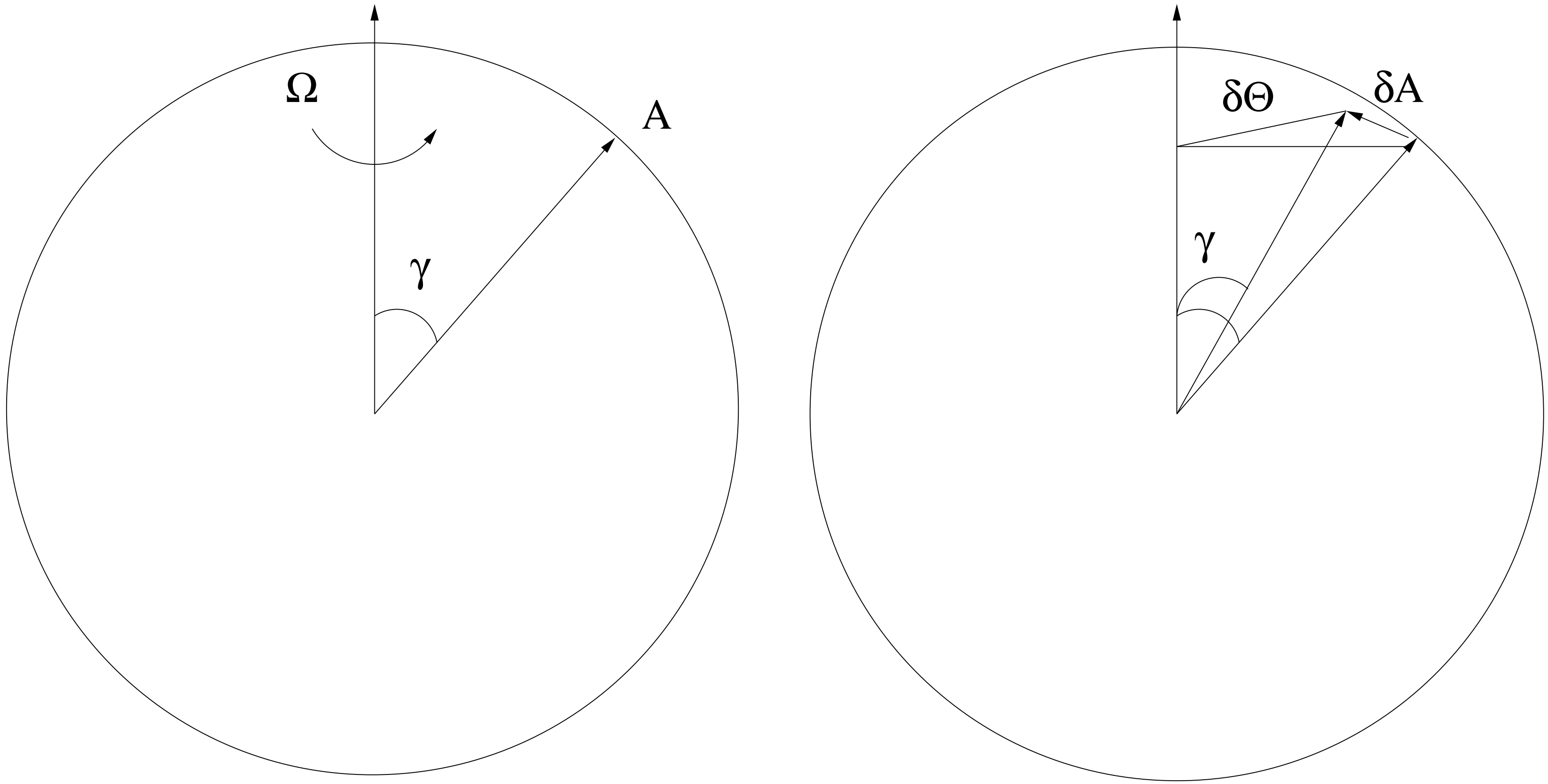
$$\frac{dv}{dt} = \frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{dw}{dt} = \frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Apparent forces



Rotation



Rotation

$$\delta\Theta = \Omega \delta t$$

Assume $\Omega = const.$ (reasonable for the earth)

Change in A is δA, the arc-length:

$$\delta \vec{A} = |\vec{A}| sin(\gamma) \delta\Theta = \Omega |\vec{A}| sin(\gamma) \delta t = (\vec{\Omega} \times \vec{A})\, \delta t$$

Rotation

So:

$$\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A}$$

This is the motion of a *fixed* vector. For a moving vector:

$$(\frac{d\vec{A}}{dt})_F = (\frac{d\vec{A}}{dt})_R + \vec{\Omega} \times \vec{A}$$

So the velocity in the fixed frame is equal to that in the rotating frame plus the rotational movement

Rotation

If $\vec{A} = \vec{r}$, the position vector, then:

$$(\frac{d\vec{r}}{dt})_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

If $\vec{A} = \vec{r}$, we get the acceleration:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_F}{dt})_R + \vec{\Omega} \times \vec{u}_F = [\frac{d}{dt}(u_R + \vec{\Omega} \times \vec{r})]_R + \vec{\Omega} \times \vec{u}_F$$

$$= (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Rotation

Rearranging:

$$(\frac{d\vec{u}_R}{dt})_R = (\frac{d\vec{u}_F}{dt})_F - 2\vec{\Omega} \times \vec{u}_R - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Two additional terms:

- Coriolis acceleration $\rightarrow \quad -2\vec{\Omega} \times \vec{u}_R$
- Centrifugal acceleration $\rightarrow \quad -\vec{\Omega} \times \vec{\Omega} \times \vec{r}$

Centrifugal acceleration

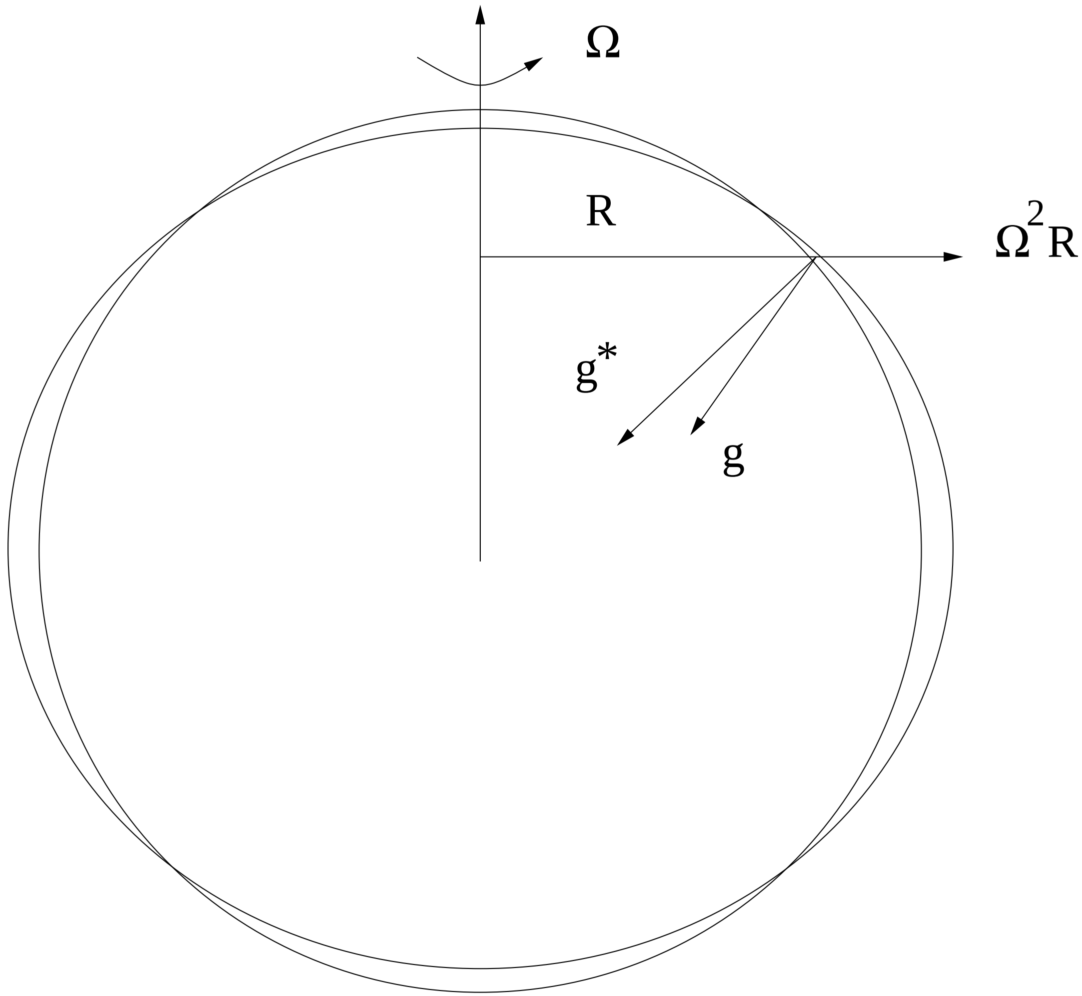
Rotation requires a force towards the center of rotation—the *centripetal acceleration*

From the rotating frame, the sign is opposite—the *centrifugal acceleration*

Acceleration points out from the earth’s radius of rotation

So has components in the radial and N-S directions

Centrifugal



Centrifugal

The earth is not spherical, but has deformed into an *oblate spheroid*

There is a component of gravity which exactly balances the centrifugal force in the N-S direction

Defines surfaces of constant *geopotential*

The locally vertical centrifugal acceleration can be absorbed into gravity:

$$g' = g - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Centrifugal

Example: What is the centrifugal acceleration for a parcel of air at the Equator?

$$-\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\Omega \times (\Omega r) = \Omega^2 r$$

with:

$$r_e = 6.378 \times 10^6 \ m$$

and:

$$\Omega = \frac{2\pi}{3600(24)} \ sec^{-1}$$

Centrifugal

So:

$$\Omega^2 r_e = 0.034 \; m/sec^2$$

This is much smaller than $g = 9.8 \; m^2/sec$

- Only a minor change to absorb into g'

Cartesian coordinates

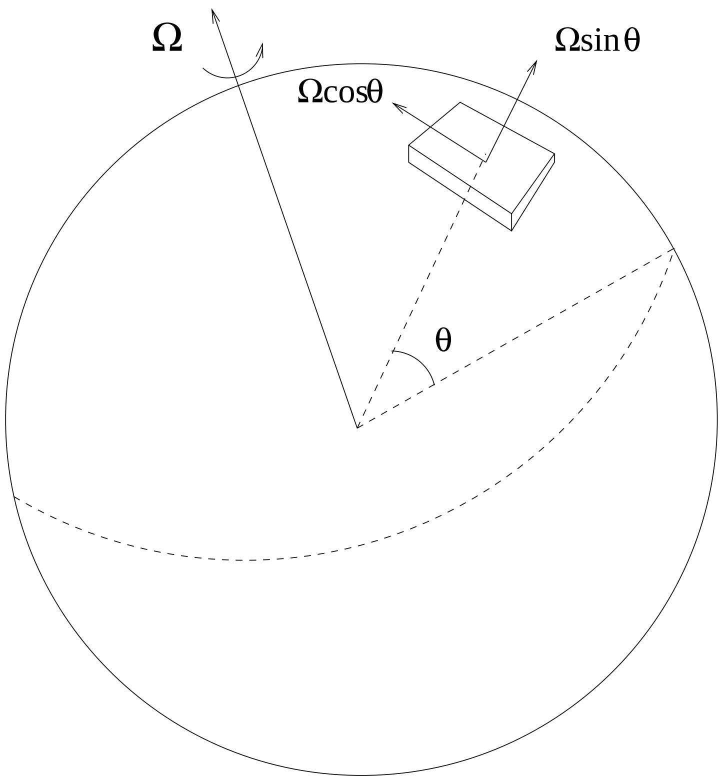
Equatorial radius is only 21 km larger than at poles

So can use spherical coordinates

However, we will use *Cartesian* coordinates

- Simplifies the math
- Neglected terms are *unimportant* at weather scales

Cartesian coordinates



Coriolis force

Rotation vector projects onto local vertical and meridional directions:

$$2\vec{\Omega} = 2\Omega cos\theta\ \hat{j} + 2\Omega sin\theta\ \hat{k} \equiv f_y\ \hat{j} + f_z\ \hat{k}$$

So the Coriolis force is:

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, v, w)$$

$$= -(f_y w - f_z v, f_z u, -f_y u)$$

Coriolis force

Example: What is the Coriolis acceleration on a parcel moving eastward at $10\ m/sec$ at 45 N?

We have:

$$f_y = 2\Omega cos(45) = 5.142 \times 10^{-5}\ sec^{-1}$$

$$f_z = 2\Omega sin(45) = 5.142 \times 10^{-5}\ sec^{-1}$$

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, 0, 0) = -f_z u\ \hat{j} + f_y u\ \hat{k}$$

$$= (0, -5.142 \times 10^{-4}, 5.142 \times 10^{-4})\ m/sec^2$$

Coriolis force

Vertical acceleration is negligible compared to gravity ($g = 9.8\ m/sec^2$), so has little effect in z

But unbalanced in the horizontal direction

Note acceleration is to the *south*

- Coriolis acceleration is most important in the *horizontal*
- Acts *to the right* in the Northern Hemisphere

Coriolis force

In the Southern hemisphere, $\theta < 0$. Same problem, at 45 S:

$$f_y = 2\Omega cos(-45) = 5.142 \times 10^{-5}\ sec^{-1}$$

$$f_z = 2\Omega sin(-45) = -5.142 \times 10^{-5}\ sec^{-1}$$

$$-2\vec{\Omega} \times \vec{u} = -f_z u\ \hat{j} + f_y u\ \hat{k}$$

$$= (0, +5.142 \times 10^{-4}, 5.142 \times 10^{-4})\ m/sec^2$$

Acceleration to the north, to the *left* of the parcel velocity.

Momentum equations

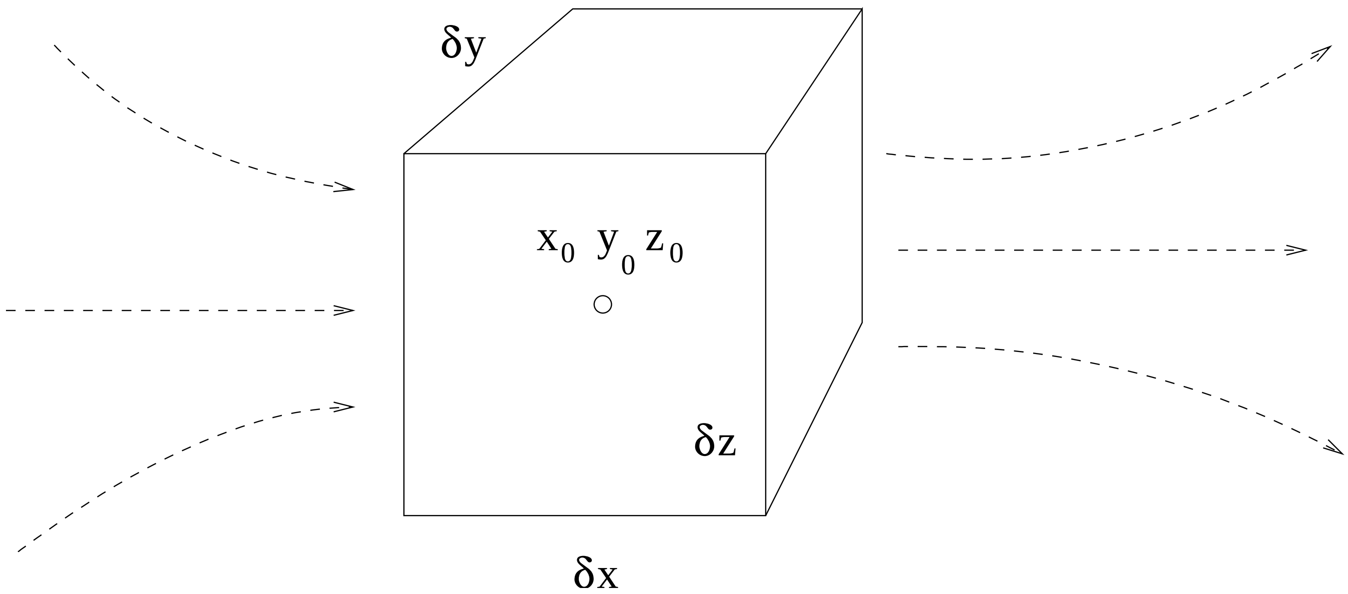
Move Coriolis terms to the LHS:

$$\frac{\partial}{\partial t}u + \vec{u}\cdot\nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u}\cdot\nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u}\cdot\nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w$$

Continuity



Continuity

Consider a fixed volume

Density flux through the left side:

$$[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Through the right side:

$$[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Continuity

So the net rate of change in mass is:

$$\frac{\partial}{\partial t}m = \frac{\partial}{\partial t}(\rho\,\partial x\,\partial y\,\partial z) = [\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\partial y\,\partial z$$

$$-[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\partial y\,\partial z = -\frac{\partial}{\partial x}(\rho u)\partial x\,\partial y\,\partial z$$

The volume δV is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u)$$

Continuity

Taking the other sides of the box:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = -\nabla \cdot (\rho \vec{u})$$

Can rewrite:

$$\nabla \cdot (\rho \vec{u}) = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho \ .$$

So:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho (\nabla \cdot \vec{u}) = 0$$

Continuity

Can also derive using a *Lagrangian* box

As the box moves, it conserves it mass. So:

$$\frac{1}{m}\frac{d}{dt}(\partial m) = \frac{1}{\rho\delta V}\frac{d}{dt}(\rho\delta V) = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d\delta V}{dt} = 0$$

Expand the volume term:

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{1}{\delta x}\frac{d}{dt}\delta x + \frac{1}{\delta y}\frac{d}{dt}\delta y + \frac{1}{\delta z}\frac{d}{dt}\delta z$$

$$= \frac{1}{\delta x}\delta\frac{dx}{dt} + \frac{1}{\delta y}\delta\frac{dy}{dt} + \frac{1}{\delta z}\delta\frac{dz}{dt} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}$$

Continuity

As $\delta \to 0$:

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} \longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

So:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{u} = 0$$

Change in density proportional to the velocity *divergence*.

If volume changes, density changes to keep mass constant.

Ideal Gas Law

Five of the equations are *prognostic*: they describe the time evolution of fields.

But we have one *diagnostic* relation.

This relates the density, pressure and temperature

Ideal Gas Law

For dry air:

$$p = \rho RT$$

where

$$R = 287\ Jkg^{-1}K^{-1}$$

Moist air

Law moist air, can write (Chp. 3):

$$p = \rho R T_v$$

where the *virtual temperature* is:

$$T_v \equiv \frac{T}{1 - e/p(1-\epsilon)}$$

$$\epsilon \equiv \frac{R_d}{R_v} = 0.622$$

We will ignore moisture. But remember that we *can* take it into account in this way.

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

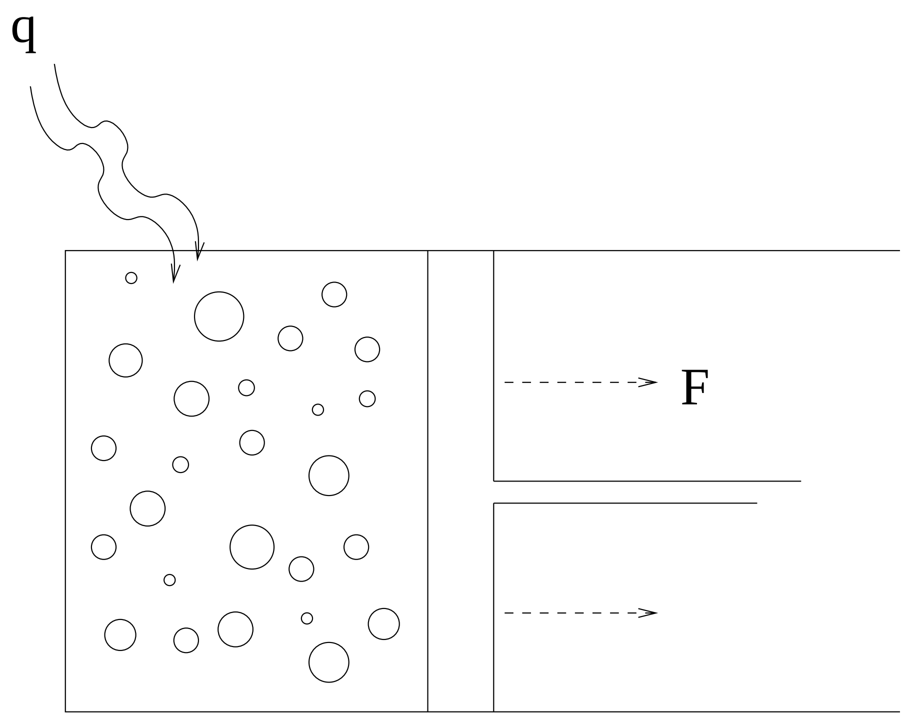
Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \frac{dq}{dt}$$

Thermodynamic equation



First law of thermodynamics

Change in internal energy = heat added - work done:

$$de = dq - dw$$

Work is done by expanding against external forces:

$$dw = F dx = pA dx = p dV$$

If $dV > 0$, the volume is doing the work

First law of thermodynamics

Assume the volume has a unit mass, so that:

$$\rho V = 1$$

Then:

$$dV = d(\frac{1}{\rho}) = d\alpha$$

where α is the *specific volume*. So:

$$de = dq - p\, d\alpha$$

First law of thermodynamics

Add heat to the volume, the temperature rises. The *specific heat* (c_v) determines how much. If the volume is held constant:

$$dq_v = c_v dT$$

With $dV = 0$, equas the change in internal energy:

$$dq_v = de_v = c_v dT$$

First Law of thermodynamics

Joule's Law: e only depends on temperature for an ideal gas. So even if V changes:

$$de = c_v dT$$

So:

$$dq = c_v dT + p\, d\alpha$$

Divide by dt to find the theromdynamic energy equation:

$$\frac{dq}{dt} = c_v \frac{dT}{dt} + p\, \frac{d\alpha}{dt}$$

First law of thermodynamics

Now imagine we keep the pressure constant:

$$dq_p = c_p dT$$

We let the volume expand while keeping p constant. This requires more heat to raise the temperature. Rewrite the work term:

$$p\, d\alpha = d(p\alpha) - \alpha dp$$

So:

$$dq = c_v dT + d(p\alpha) - \alpha dp$$

First law of thermodynamics

The ideal gas law is:

$$p = \rho RT = \alpha^{-1} RT$$

So:

$$d(p\alpha) = RdT$$

Thus:

$$dq = (c_v + R)dT - \alpha dp$$

First law of thermodynamics

At constant pressure, $dp = 0$, so:

$$dq_p = (c_v + R)dT = c_p dT$$

So the specific heat at constant pressure is *greater* than at constant volume. For dry air:

$$c_v = 717 Jkg^{-1}K^{-1}, \qquad c_p = 1004 Jkg^{-1}K^{-1}$$

so:

$$R = 287\ Jkg^{-1}K^{-1}$$

First law of thermodynamics

So we can also write:

$$dq = c_p dT - \alpha dp$$

Dividing by dt, we have:

$$\frac{dq}{dt} = c_v \frac{dT}{dt} + p\frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}$$

Basic balances

Not all terms in the momentum equations are equally important for weather systems.

Will simplify the equations by identifying primary balances (throw out as many terms as possible).

Begin with horizontal momentum equations.

Scaling

General technique: *scale* equations using estimates of the various parameters. Take the x-momentum equation, without friction:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad f_y W \qquad f_z U \qquad \frac{\triangle_H P}{\rho L}$$

Scaling

Now use typical values. Length scales:

$$L \approx 10^6 m, \quad D \approx 10^4 m$$

Horizontal scale is 1000 km, the *synoptic scale* (of weather systems).

Velocities:

$$U \approx V \approx 10\, m/sec, \quad W \approx 1\, cm/sec$$

Notice the winds are *quasi-horizontal*

Scaling

Pressure term, from measurements:

$$\triangle_H P/\rho \approx 10^3 m^2/sec^2$$

Time scale:

$$T = L/U \approx 10^5 sec$$

Called an “advective time scale” (≈ 1 day).

Scaling

Coriolis terms:

$$(f_y, f_z) = 2\Omega(cos\theta, sin\theta)$$

with

$$\Omega = 2\pi(86400)^{-1} sec^{-1}$$

Assume at mid-latitudes:

$$f_y \approx f_z \approx 10^{-4} sec^{-1}$$

Scaling

Plug in:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \qquad fU \qquad \frac{\triangle_H P}{\rho L}$$

$$10^{-4} \qquad 10^{-4} \qquad 10^{-4} \qquad 10^{-5} \qquad 10^{-6} \qquad 10^{-3} \qquad 10^{-3}$$

Geostrophy

Keeping only the 10^{-3} terms:

$$f_z v = \frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

These are the *geostrophic* relations.

Balance between the pressure gradient and Coriolis force.

Geostrophy

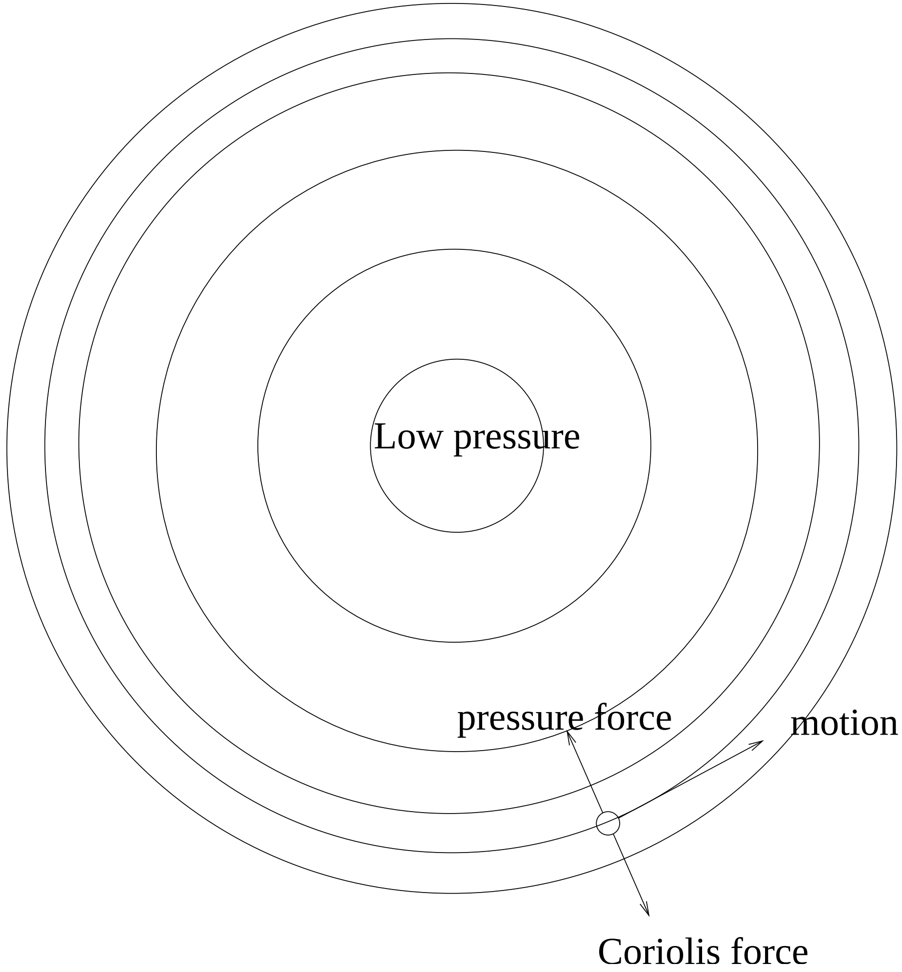
Fundamental momentum balance at synoptic scales

- Low pressure to left of the wind in Northern Hemisphere
- Low pressure to *right* in Southern Hemisphere

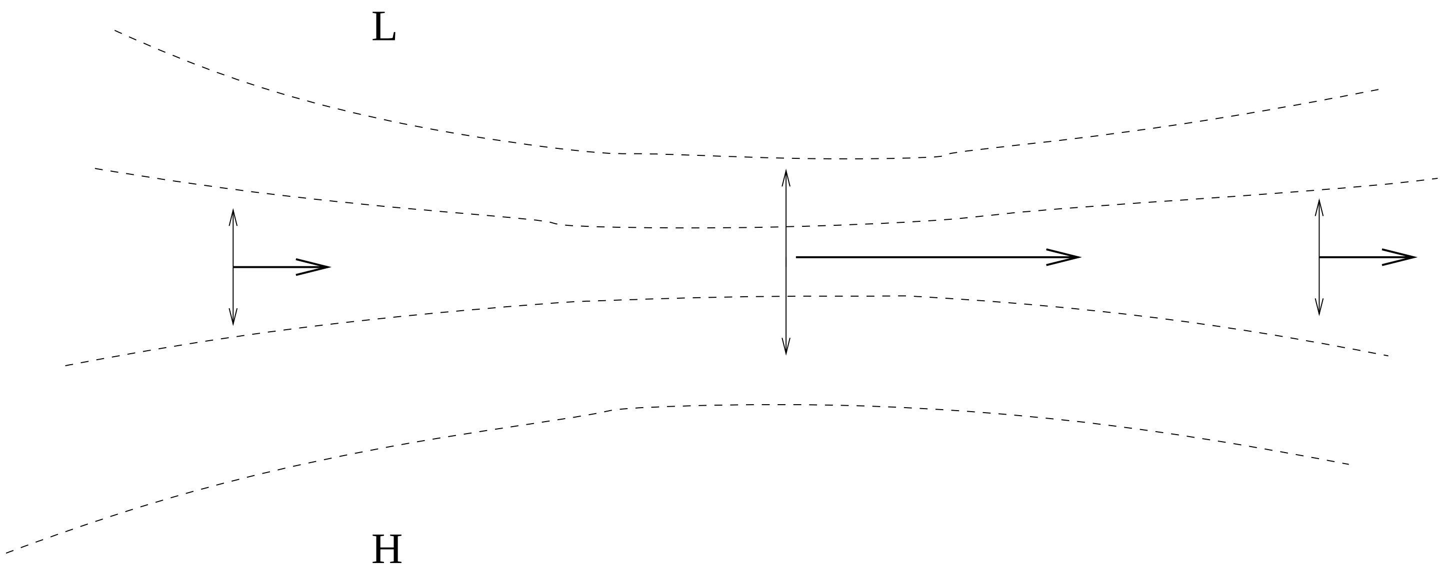
But balance *fails* at equator, because $f_z = 2\Omega sin(0) = 0$

There we must keep other terms

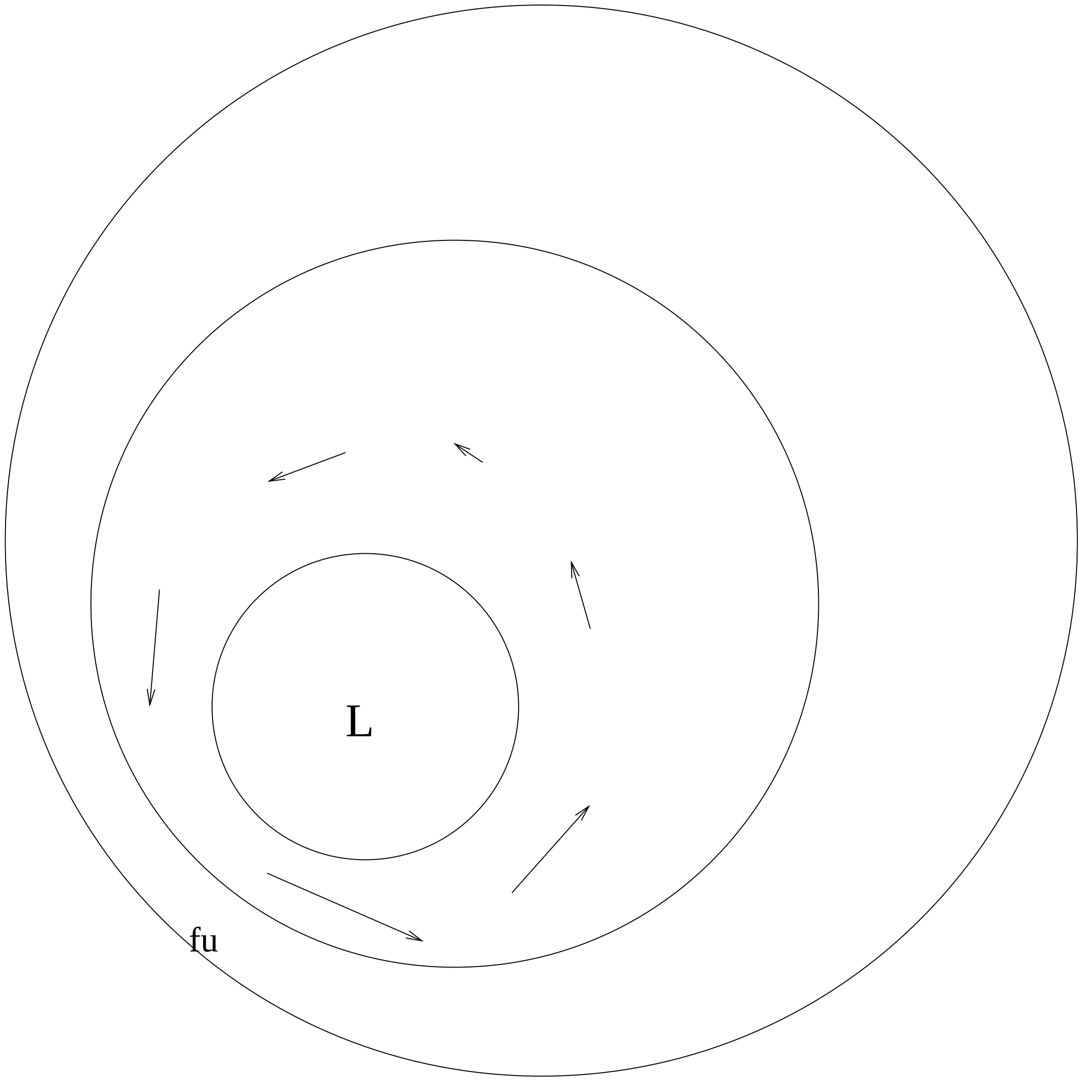
Geostrophy



Geostrophy



Geostrophy



Geostrophy

Example: If the pressure difference is 0.37 kPa over 100 km, how strong are the winds? Imagine we're at 45 N.

$$f_z = 2\Omega sin(45) = 1.414 * (7.27 \times 10^{-5})\ sec^{-1} = 1.03 \times 10^{-4}\ sec^{-1}$$

$$\frac{\partial p}{\partial l} = \frac{0.37 \times 10^3\ N/m^2}{10^5\ m} = 3.7 \times 10^{-3}\ N/m^3$$

So:

$$u = \frac{1}{\rho_0 f_z}\frac{\partial p}{\partial l} = \frac{1}{(1.2\ kg/m^3)(1.03 \times 10^{-4} sec^{-1})}(3.7 \times 10^{-3}\ N/m^3)$$

= 29.9 m/sec (Strong!)

Geostrophy

Is a *diagnostic relation*

- Given the pressure, can calculate the horizontal velocities

But geostrophy cannot be used for *prediction*

Means that we must also retain the 10^{-4} terms in the scaling

Approximate horizontal momentum

So:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

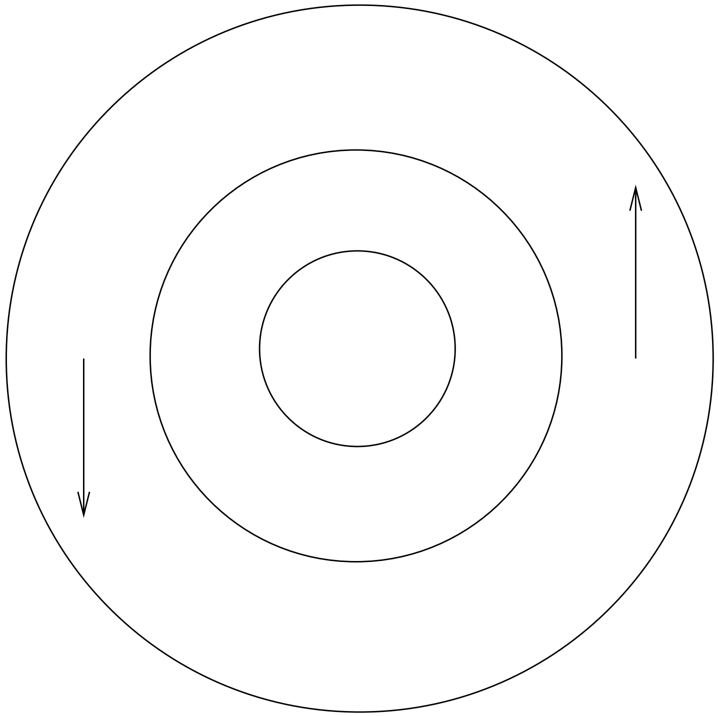
$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

These equations are *quasi-horizontal*: neglect vertical motion

Explains why the horizontal winds are so much larger than in the vertical

Other momentum balances

Geostrophy most important balance at synoptic scales. But other balances possible. Consider purely circular flow:



Other momentum balances

Must use cylindrical coordinates. From standard text books, can find that the acceleration in the radial direction is given by:

$$\frac{d}{dt}u_r - \frac{u_\theta^2}{r} - fu_\theta = -\frac{1}{\rho}\frac{\partial}{\partial r}p$$

u_θ^2/r is the *cyclostrophic* term

This is related to centripetal acceleration.

Other momentum balances

Assume no radial motion: $u_r = 0$. Then:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r} p$$

Scaling we get:

$$\frac{U^2}{R} \qquad fU \qquad \frac{\triangle_H P}{\rho R}$$

Or:

$$\frac{U}{fR} \qquad 1 \qquad \frac{\triangle_H P}{\rho f U R}$$

Other momentum balances

The ratio:

$$\frac{U}{fR} \equiv \epsilon$$

is called the *Rossby number*. If $\epsilon \ll 1$, the first term is very small. So we have:

$$fu_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

The geostrophic relation.

Other momentum balances

If $\epsilon \gg 1$, the first term dominates.

A tornado at mid-latitudes has:

$$U \approx 30m/s, \; f = 10^{-4}sec^{-1}, \; R \approx 300m \rightarrow \epsilon \approx 1000$$

Cyclostrophic wind balance

Then we have:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

or:

$$u_\theta = \pm\left(\frac{r}{\rho}\frac{\partial}{\partial r}p\right)^{1/2}$$

- Rotation does not enter.
- Winds can go *either way*.

Inertial oscillations

Third possibility: there is no radial pressure gradient:

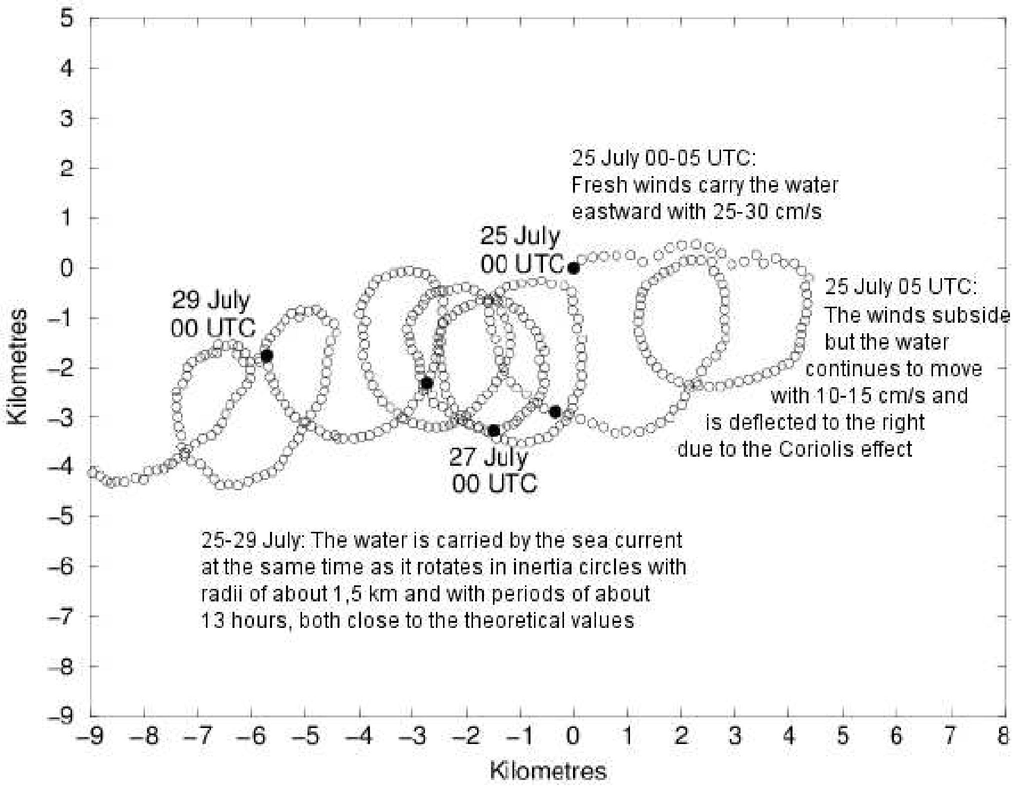
$$\frac{u_\theta^2}{r} + f u_\theta = 0$$

then:

$$u_\theta = -fr$$

Rotation is clockwise (*anticyclonic*) in the Northern Hemisphere.

Inertial oscillations



A drifting buoy in the Baltic Sea, July 1969. Courtesy Persson and Broman.

Inertial oscillations

The time for a fluid parcel to complete a loop is:

$$\frac{2\pi r}{u_\theta} = \frac{2\pi}{f} = \frac{0.5\ day}{|sin\theta|}$$

Called the “inertial period”

Strong effect in the surface ocean

Less frequently observed in the atmosphere

Gradient wind balance

Fourth possibility: all terms are important ($\epsilon \approx 1$)

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r} p$$

Solve using the quadratic formula:

$$u_\theta = -\frac{1}{2} f r \pm \frac{1}{2}(f^2 r^2 + \frac{4r}{\rho}\frac{\partial}{\partial r} p)^{1/2}$$

$$= -\frac{1}{2} f r \pm \frac{1}{2}(f^2 r^2 + 4\, r\, f\, u_g)^{1/2}$$

Gradient wind balance

If $u_g < 0$ (anticyclone), we require:

$$|u_g| < \frac{fr}{4}$$

If $u_g > 0$ (cyclone), there is *no limit*

Wind gradients can be *much stronger* in cyclones than in anticyclones

Gradient wind balance

Alternately can write:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p = f u_g$$

Divide through by $f u_\theta$:

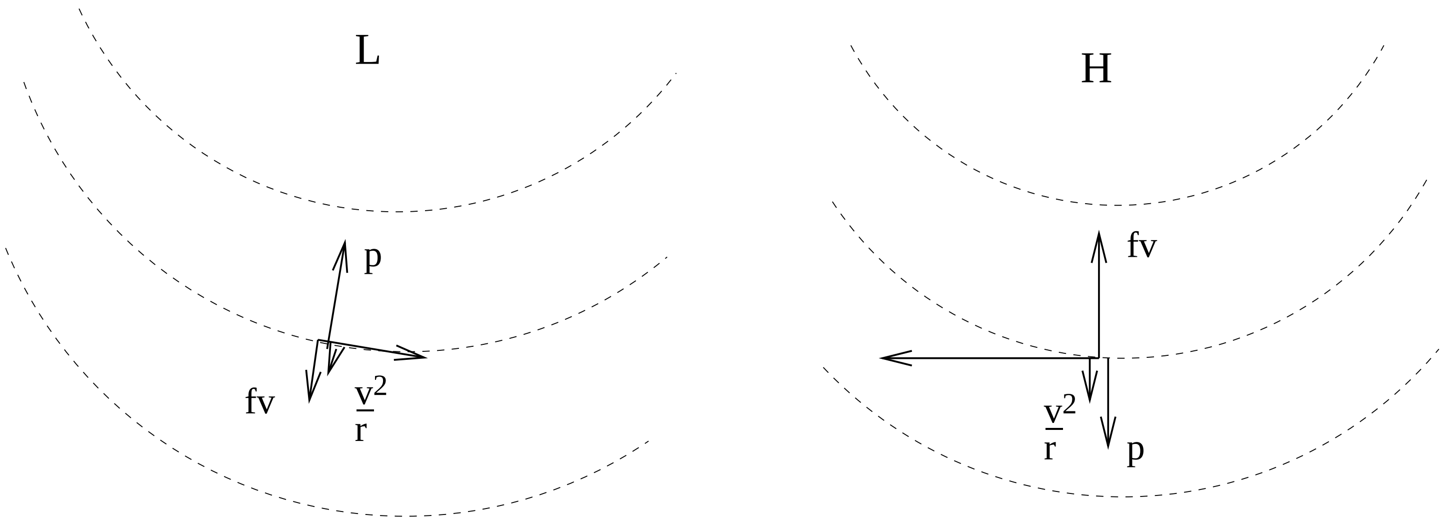
$$\frac{u_\theta}{f r} + 1 = \epsilon + 1 = \frac{u_g}{u_\theta}$$

So if $\epsilon = 0.1$, the gradient wind estimate differs by 10 %

Gradient wind balance

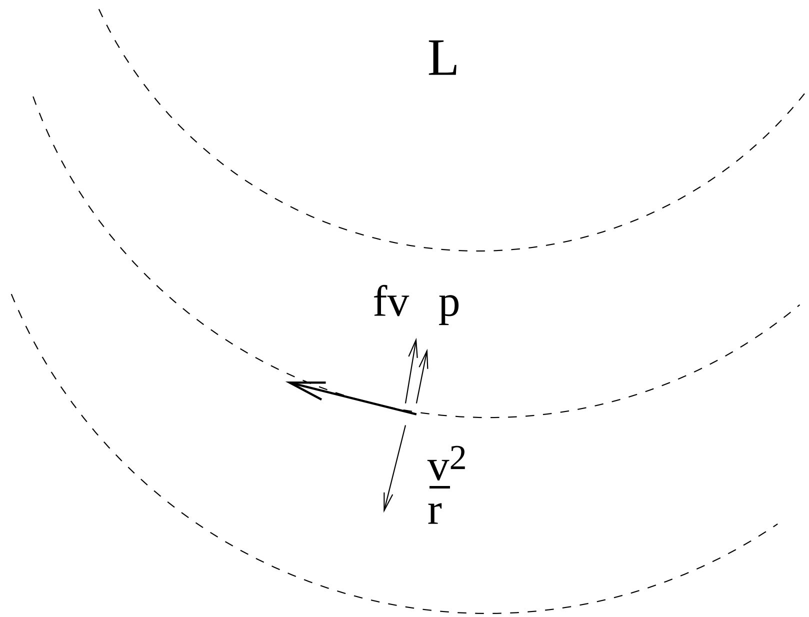
- At low latitudes, ϵ can be 1-10. Then the gradient wind estimate is more accurate.
- Geostrophy is *symmetric to sign changes*: no difference between cyclones and anticyclones
- The gradient wind balance is *not* symmetric to sign change. Cyclones can be stronger.

Gradient wind balance



Winds weaker than geostrophic for a low pressure system; they are stronger for a high pressure system.

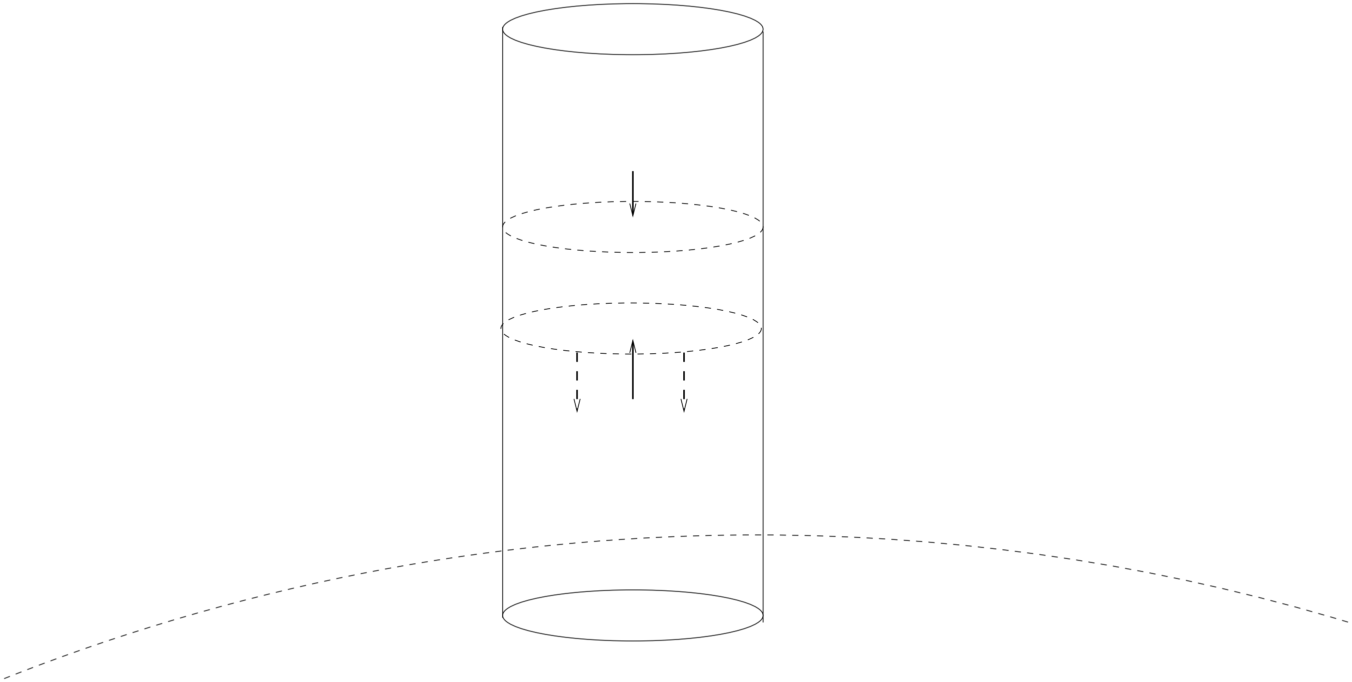
Gradient wind balance



An *anomalous low*: low pressure with clockwise flow

Usually only occurs at low latitudes, where Coriolis weak

Hydrostatic balance



Hydrostatic balance

Now scale the vertical momentum equation

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad fU \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

Hydrostatic balance

We must scale:

$$\frac{1}{\rho}\frac{\partial}{\partial z}p$$

The vertical variation of pressure much greater than the horizontal variation:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Hydrostatic balance

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad fU \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

$$10^{-7} \qquad 10^{-7} \qquad 10^{-7} \qquad 10^{-10} \qquad 10^{-3} \qquad 10 \qquad 10$$

Static atmosphere

Dominant balance is between the vertical pressure gradient and gravity

However, same balance *if there no motion at all* !

Setting $(u, v, w) = 0$ in the equations of motion yields:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p = \frac{1}{\rho}\frac{\partial}{\partial y}p = \frac{\partial}{\partial t}\rho = \frac{dT}{dt} = 0$$

Which implies:

$$\rho = \rho(z), \quad p = p(z), \quad T = T(z)$$

Static atmosphere

Two equations left:

$$\frac{\partial}{\partial z}p = -\rho g$$

the *hydrostatic balance* and

$$p = \rho RT$$

Equations describe a non-moving atmosphere

Static atmosphere

Integrate the hydrostatic relation:

$$p(z) = \int_z^{\infty} \rho g \, dz \; .$$

The pressure at any point is equal to the weight of air above it. Sea level pressure is:

$$p(0) = 101.325 \; kPa \; (1013.25 mb)$$

The average weight per square meter of the entire atmospheric column

Static atmosphere

Say the $T = const.$ (an *isothermal* atmosphere):

$$\frac{\partial}{\partial z} p = -\frac{pg}{RT}$$

This implies:

$$ln(p) = -\frac{gz}{RT}$$

Static atmosphere

So that:

$$p = p_0\, e^{-z/H}$$

Pressure decays exponentially. The e-folding scale is the "scale height":

$$H \equiv \frac{RT}{g}$$

Scaling

Static hydrostatic balance not interesting for weather. Separate the pressure and density into static and non-static (moving) components:

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

Assume:

$$|p'| \ll |p_0|, \quad |\rho'| \ll |\rho_0|$$

Scaling

Then:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0+\rho'}\frac{\partial}{\partial z}(p_0+p') - g$$

$$\approx -\frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}(p_0+p') - g$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + \left(\frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$\rightarrow$ Neglect $(\rho' p')$

Scaling

Use these terms in the vertical momentum equation

But how to scale?

Vertical variation of the perturbation pressure comparable to the horizontal perturbation:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2$$

Scaling

Also:

$$|\rho'| \approx 0.001|\rho_0|$$

So:

$$\frac{\rho'}{\rho_0} g \approx 10^{-1} m/sec^2$$

Scaling

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \quad 10^{-1} \quad 10^{-1}$$

Hydrostatic perturbations

Dominant balance still hydrostatic, but with perturbations:

$$\frac{\partial}{\partial z} p' = -\rho' g$$

thus vertical acceleration unimportant at synoptic scales

But we lost the vertical velocity! Deal with this later.

Coriolis parameter

So all terms with f_y are unimportant

From now on, neglect f_y and write f_z simply as f:

$$f \equiv 2\Omega sin(\theta)$$

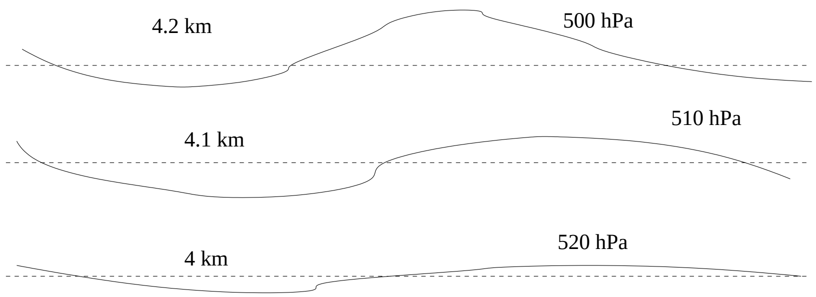
f_y only important near the equator

Pressure coordinates

The hydrostatic balance implies an equivalence between changes in pressure and z

Can use it to change vertical coordinates

Consider constant pressure surfaces (here in two dimensions):



Pressure coordinates

On a pressure surface:

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial z}\,dz = 0$$

Substitute hydrostatic relation:

$$dp = \frac{\partial p}{\partial x}\,dx - \rho g\,dz = 0$$

So:

$$\frac{\partial p}{\partial x} = \rho g \frac{dz}{dx} \equiv \rho\,\frac{\partial \Phi}{\partial x}$$

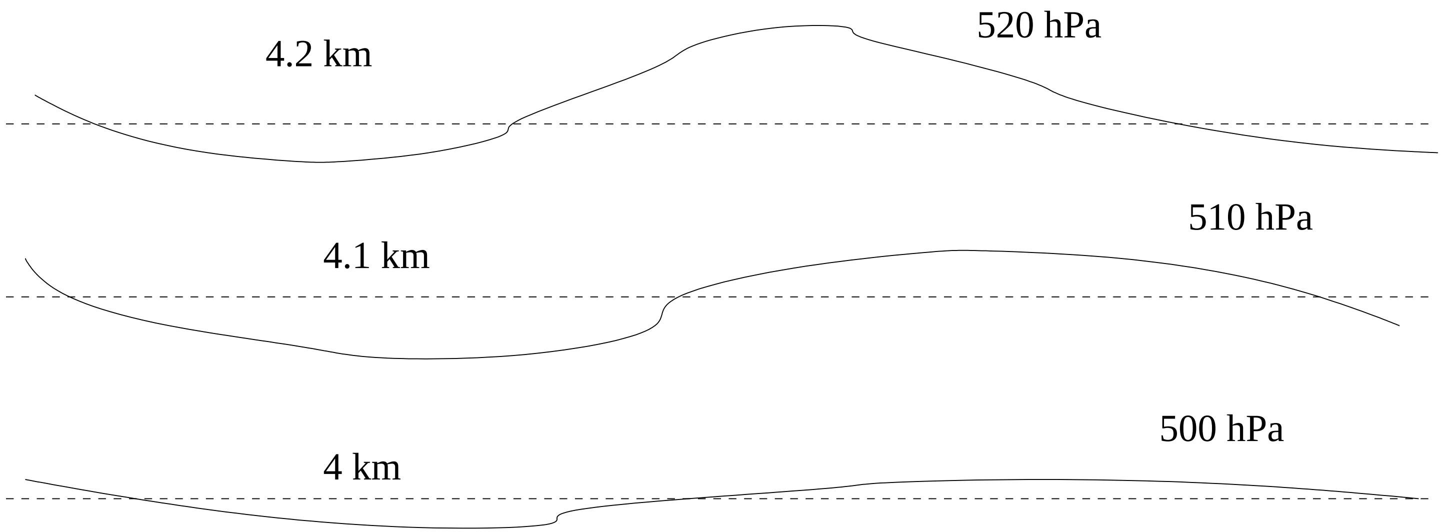
Geopotential

where Φ is the *geopotential*

This is the height of a given pressure surface

$\longrightarrow$ instead of pressure at a certain height, we think of the height of a pressure surface

Geopotential



Geostrophy

Removes density from the momentum equation!

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x}$$

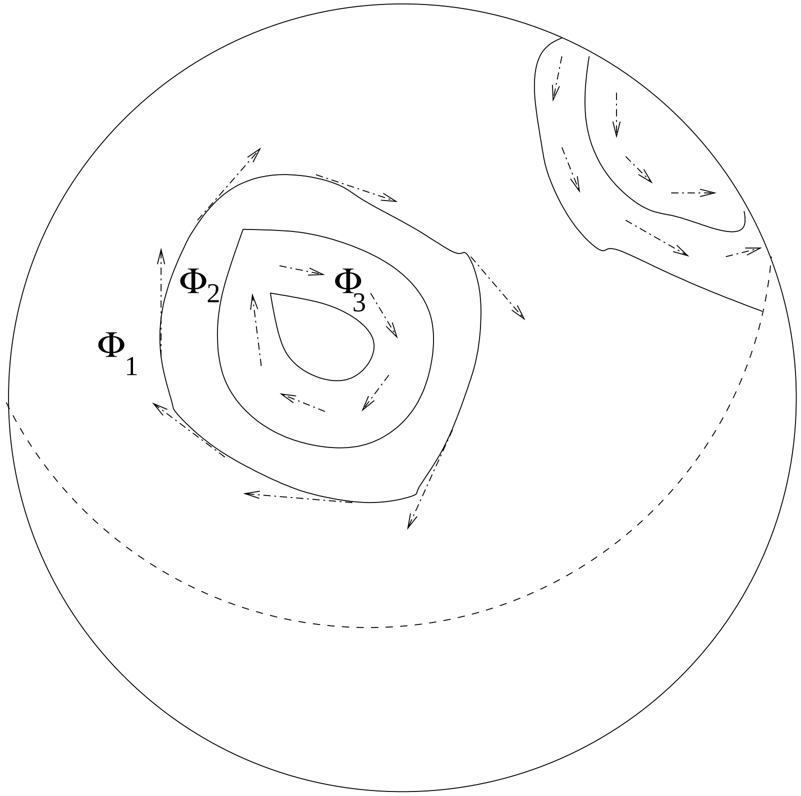
Now the geostrophic balance is:

$$fv = \frac{\partial}{\partial x}\Phi$$

$$fu = -\frac{\partial}{\partial y}\Phi$$

Geostrophy

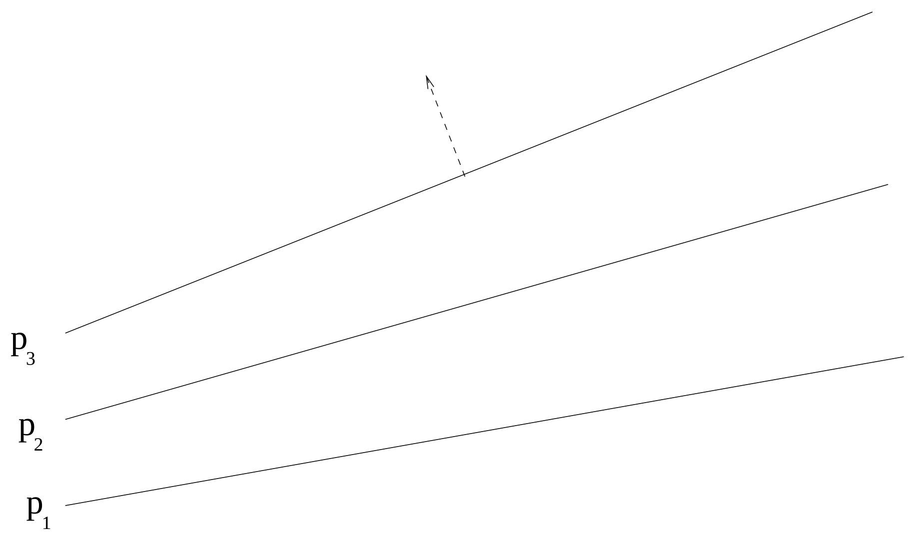
500 hPa



Vertical velocities

Different vertical velocities:

$$w = \frac{dz}{dt} \quad \rightarrow \quad \omega = \frac{dp}{dt}$$



Geopotential

Lagrangian derivative is now:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p}$$

Continuity

This changes too in pressure coordinates.

Consider a Lagrangian box:

$$V = \delta x \, \delta y \, \delta z = -\delta x \, \delta y \, \frac{\delta p}{\rho g}$$

with a mass:

$$m = \rho V = -\delta x \, \delta y \, \delta p / g$$

Continuity

Conservation of mass:

$$\frac{1}{m}\frac{d}{dt}m = \frac{g}{\delta x \delta y \delta p}\frac{d}{dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$$

Using the chain rule:

$$\frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0$$

Continuity

Let $\delta \to 0$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

The flow is *incompressible* in pressure coordinates

Much simpler to work with!

Hydrostatic balance

$$\frac{dp}{dz} = -\rho g$$

$$dp = -\rho g dz = -\rho d\Phi$$

So:

$$\frac{d\Phi}{dp} = -\frac{1}{\rho} = -\frac{RT}{p}$$

using the Ideal Gas Law

Summary: Pressure coordinates

Geostrophy:

$$fv = \frac{\partial}{\partial x}\Phi, \qquad fu = -\frac{\partial}{\partial y}\Phi$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Hydrostatic:

$$\frac{d\Phi}{dp} = -\frac{RT}{p}$$

Diagnosing vertical motion

Lost the vertical acceleration. But can find the velocity, ω, by integrating the continuity equation:

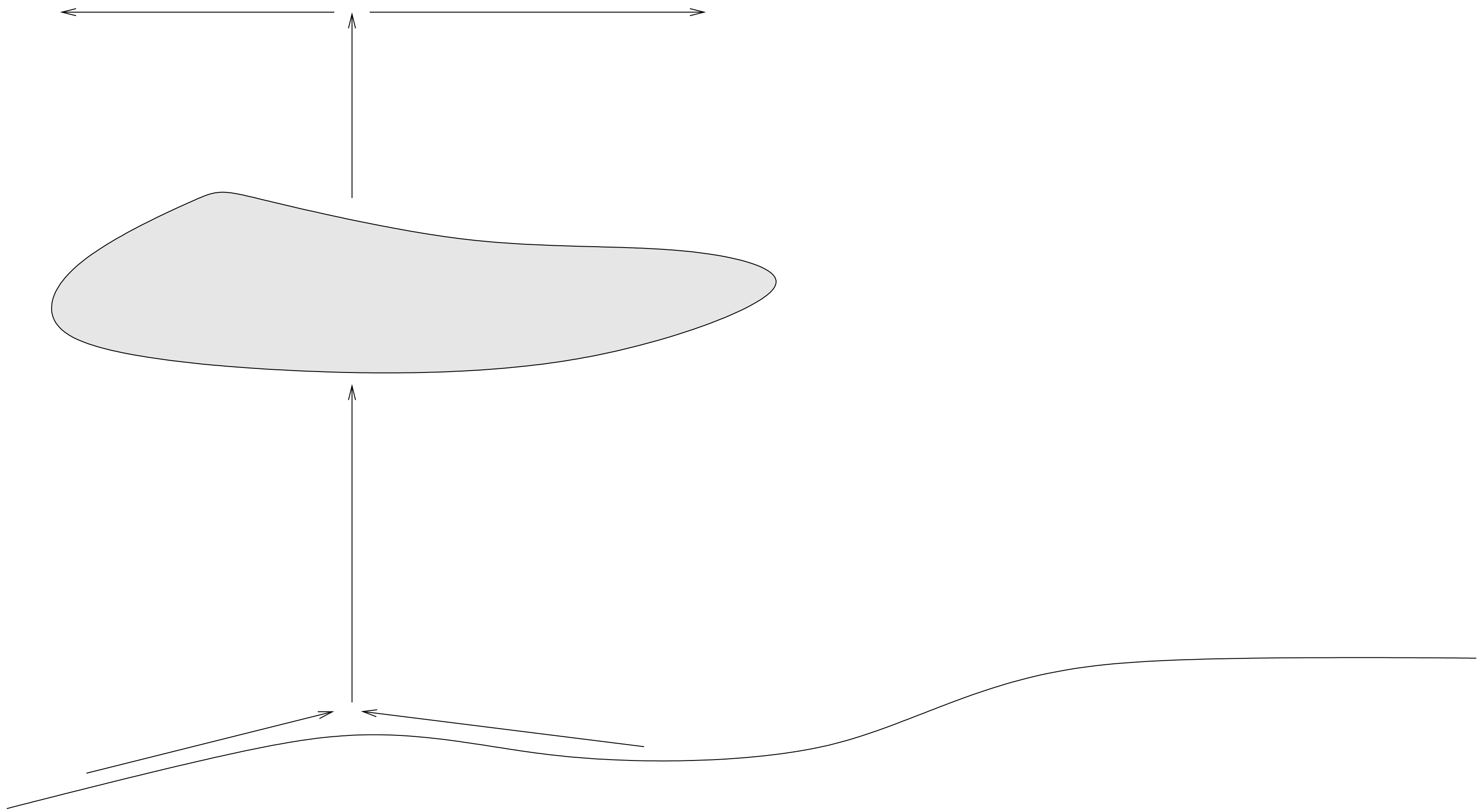
$$\omega = -\int_{p*}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\, dp$$

If the top of the atmosphere, $p* = 0$, so:

$$\omega = -\int_{0}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\, dp$$

So vertical motion occurs *when there is horizontal divergence*.

Divergence



Vertical motion

How does ω relate to the actual vertical velocity?

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p + w\frac{\partial}{\partial z}p$$

Using the hydrostatic relation:

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p - \rho g w$$

For geostrophic motion:

$$u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p = (-\frac{1}{\rho f}\frac{\partial}{\partial y}p)(\frac{\partial}{\partial x}p) + (\frac{1}{\rho f}\frac{\partial}{\partial x}p)(\frac{\partial}{\partial y}p) = 0$$

Vertical motion

So

$$\omega \approx \frac{\partial}{\partial t}p - \rho g w$$

Also:

$$\frac{\partial}{\partial t}p \approx 10hPa/day$$

$$\rho g w \approx (1.2kg/m^3)\,(9.8m/sec^2)(0.01m/sec) \approx 100hPa/day$$

Vertical motion

So:

$$\omega \approx -\rho g w$$

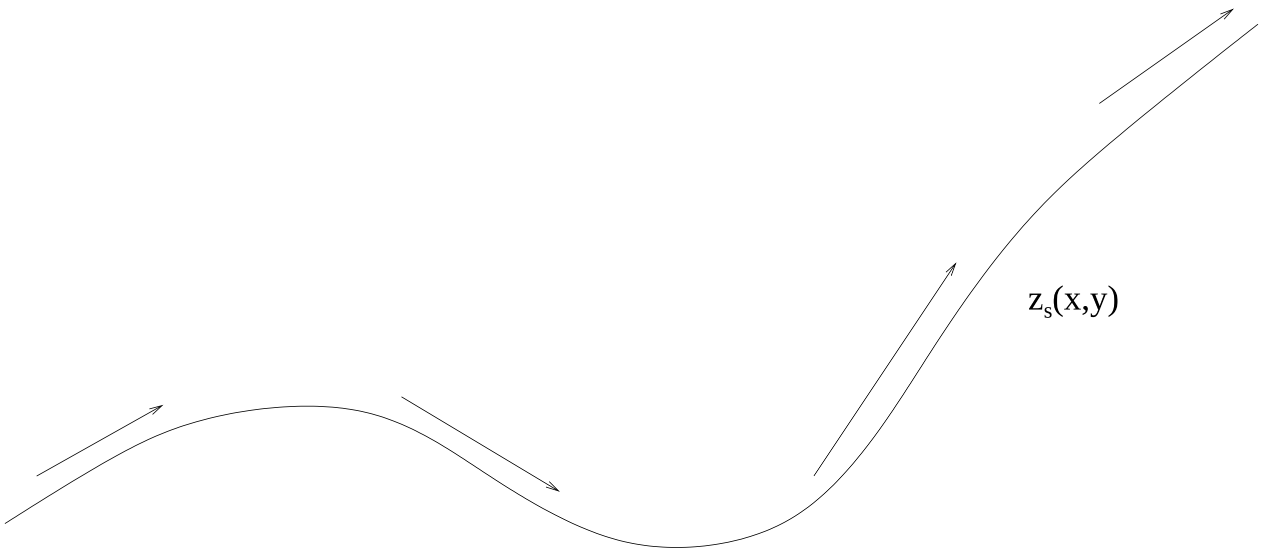
This is accurate within 10 % in the mid-troposphere

Less accurate near the ground, due to topography

At the surface:

$$w_s = u\frac{\partial}{\partial x}z_s + v\frac{\partial}{\partial y}z_s$$

Vertical motion



Topography most important for ω in the lowest 1-2 km of the troposphere

Thermal wind

Geostrophy tells us what the velocities are if we know the geopotential on a pressure surface

What about the velocities on *other* pressure surfaces?

Say we have information on the 500 hPa surface, but we wish to estimate winds on the 400 hPa surface

Requires knowing the velocity *shear*

This shear is determined by the thermal wind relation

Thermal wind

From the hydrostatic balance:

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

Now take the derivative wrt pressure of the geostrophic relation:

$$\frac{\partial}{\partial p}\left(f v_g = \frac{\partial \Phi}{\partial x}\right)$$

But:

$$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

Thermal wind

So:

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

Or:

$$\frac{\partial v_g}{\partial\, ln(p)} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

- Shear is proportional to the temperature gradient

Thermal wind

If we know the velocity at p_0, can calculate it at p_1

Integrate between two pressure levels:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} \frac{\partial T}{\partial x} \, d\, ln(p)$$

$$= -\frac{R}{f} \frac{\partial}{\partial x} \int_{p_0}^{p_1} T \, d\, ln(p)$$

Mean temperature

Define the *mean temperature* in layer between p_0 and p_1:

$$\overline{T} \equiv \frac{\int_{p_0}^{p_1} T \, d(lnp)}{\int_{p_0}^{p_1} \, d(lnp)} = \frac{\int_{p_0}^{p_1} T \, d(lnp)}{ln(\frac{p_1}{p_0})}$$

Then:

$$v_g(p_1) - v_g(p_0) = \frac{R}{f} \, ln(\frac{p_0}{p_1}) \, \frac{\partial \overline{T}}{\partial x}$$

Similarly:

$$u_g(p_1) - u_g(p_0) = -\frac{R}{f} \, ln(\frac{p_0}{p_1}) \, \frac{\partial \overline{T}}{\partial y}$$

Thermal wind

Alternately we can use geostrophy to calculate the shear between p_0 and p_1:

$$v_g(p_1) - v_g(p_0) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \equiv \frac{g}{f}\frac{\partial}{\partial x}Z_{10}$$

and:

$$u_g(p_1) - u_g(p_0) = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \equiv -\frac{g}{f}\frac{\partial}{\partial y}Z_{10}$$

Thermal wind

where:

$$Z_{10} = \frac{1}{g}\left(\Phi_1 - \Phi_0\right)$$

is the layer *thickness* between p_0 and p_1.

- Shear proportional to gradients of layer thickness

Thermal wind

Thus:

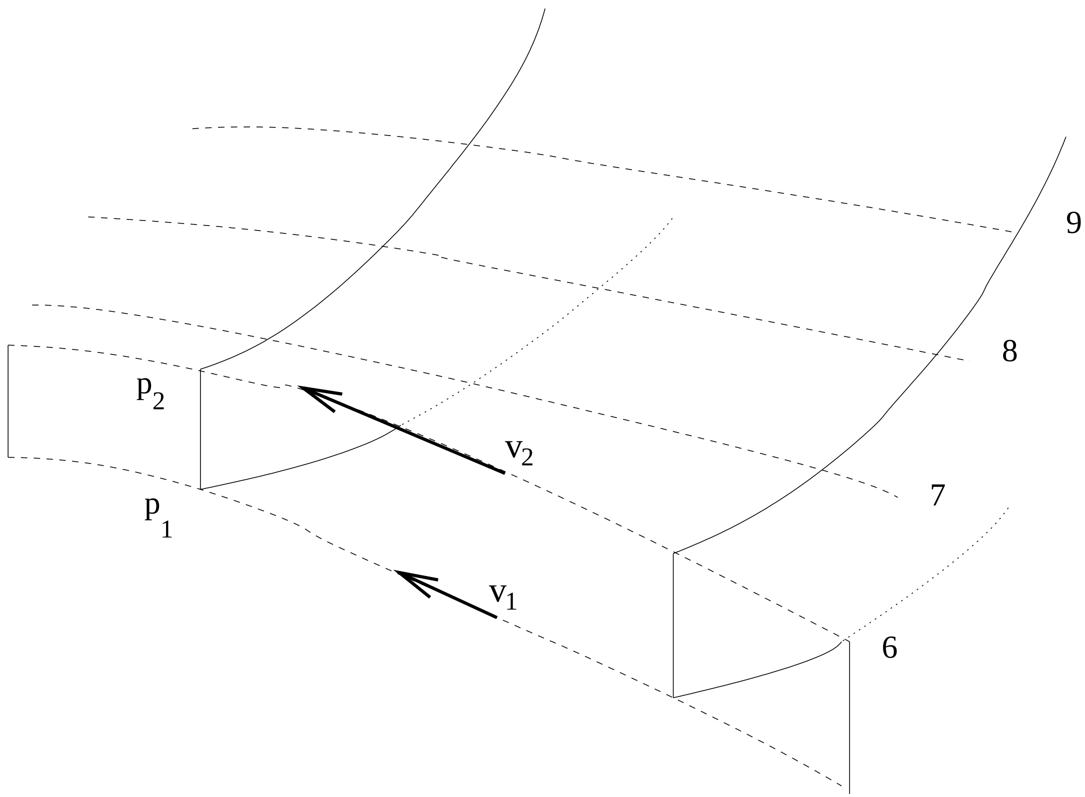
$$v_g(p_1) - v_g(p_0) = \frac{R}{f}\, ln(\frac{p_0}{p_1})\, \frac{\partial \overline{T}}{\partial x} = \frac{g}{f}\, \frac{\partial}{\partial x} Z_{10}$$

So:

$$Z_{10} = \frac{R}{g}\, ln(\frac{p_0}{p_1})\, \overline{T}$$

- Layer thickness is *proportional to the mean temperature*

Layer thickness



Barotropic atmosphere

Example 1: temperature is constant on pressure surfaces

Then $\nabla T = 0 \quad \rightarrow \quad$ *no vertical shear*

Velocities don't change with height

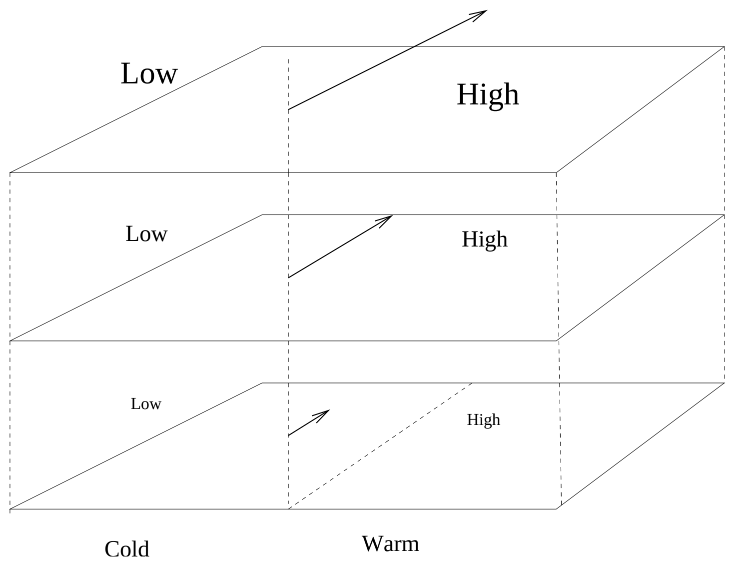
Also: all layers have *equal thickness*: stacked like pancakes

Equivalent barotropic

Example 2: temperature and geopotential contours parallel:

$$\frac{\partial}{\partial p}\vec{u}_g \parallel \vec{u}_g$$

Wind changes magnitude but *not direction* with height



Equivalent barotropic

Consider the zonal-average temperature :

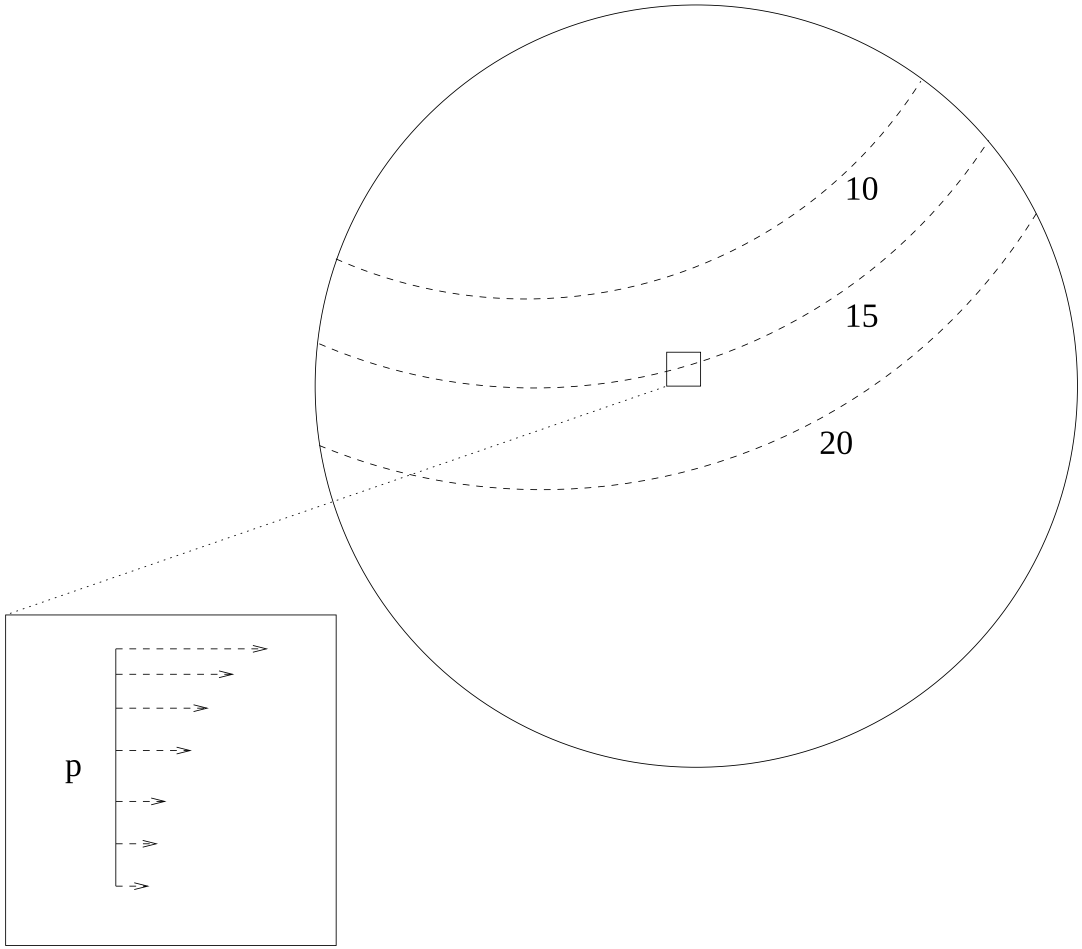
$$\frac{1}{2\pi}\int_0^{2\pi} T\,d\phi$$

Decreases from the equator to the pole

So $\frac{\partial}{\partial y}T < 0$

Thermal wind $\rightarrow$ winds increase with height

Jet Stream



Jet Stream

Example: At 30N, the zonally-averaged temperature gradient is $0.75\ K deg^{-1}$, and the average wind is zero at the earth's surface. What is the mean zonal wind at the level of the jet stream ($250\ hPa$)?

$$u_g(p_1) - u_g(p_0) = u_g(p_1) = -\frac{R}{f}\, ln(\frac{p_0}{p_1})\, \frac{\partial \overline{T}}{\partial y}$$

$$u_g(250) = -\frac{287}{2\Omega sin(30)}\, ln(\frac{1000}{250})\, (-\frac{0.75}{1.11 \times 10^5\, m}) = 36.8\ m/sec$$

Baroclinic atmosphere

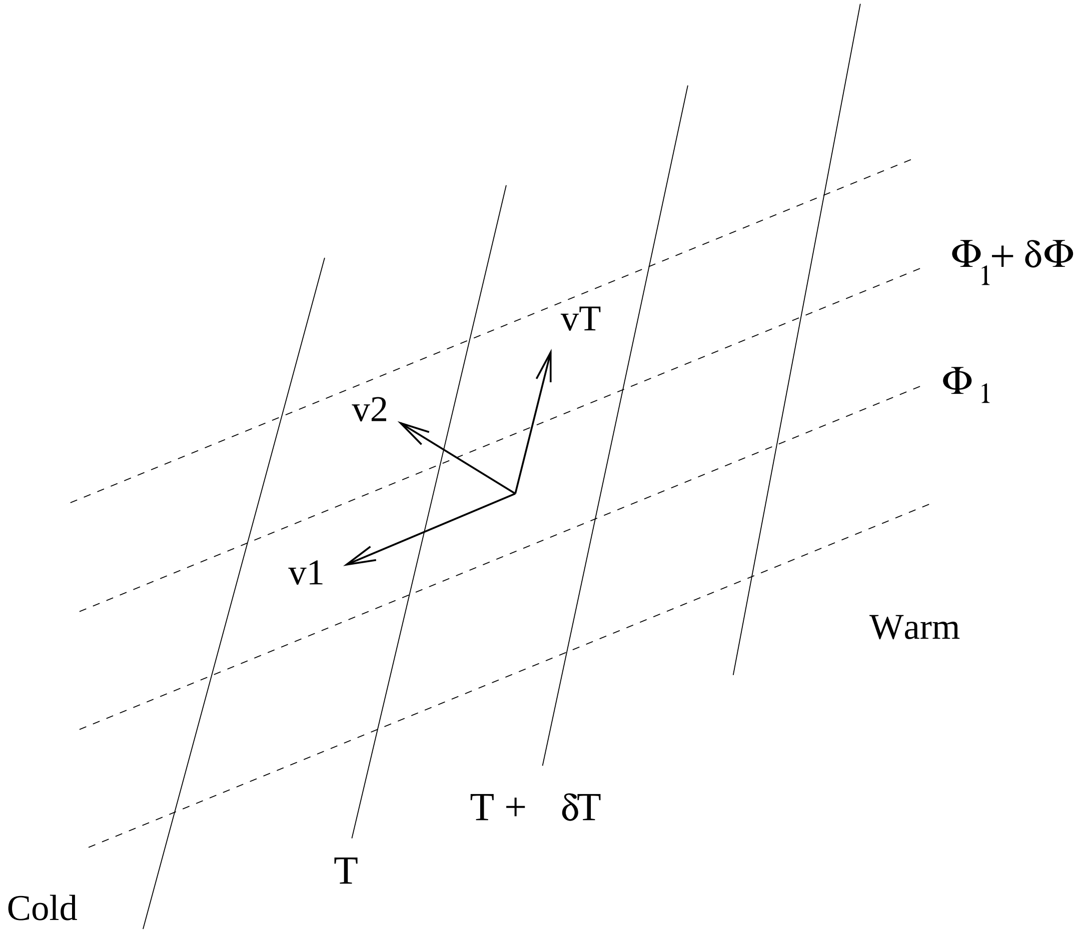
Example 3: Temperature not parallel to geopotential

Geostrophic wind has a component normal to the temperature contours (isotherms)

Produces *geostrophic temperature advection*

Winds blow from warm to cold or vice versa

Temperature advection



Temperature advection

Warm advection $\longrightarrow$ *veering*

- Anticyclonic (clockwise) rotation with height

Cold advection $\longrightarrow$ *backing*

- Cyclonic (counter-clockwise) rotation with height

Summary

Geostrophic wind parallel to geopotential contours

- high pressure to the right (North Hemisphere)

Thermal wind parallel to mean *temperature* (thickness) contours

- high thickness to the right

Divergence

Continuity equation:

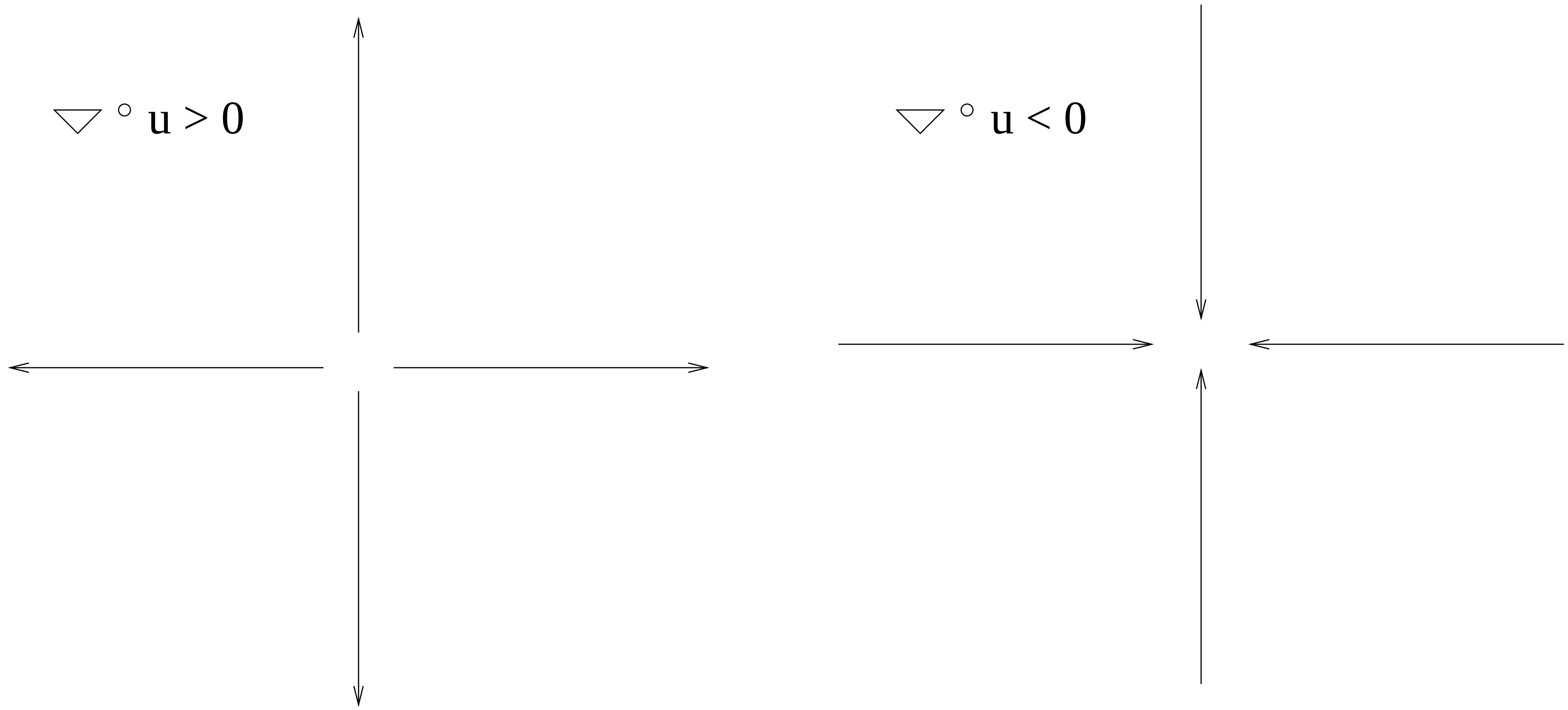
$$\frac{d\rho}{dt} + \rho \nabla \cdot \underline{u} = 0$$

or:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{u} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$$

- Density changes due to divergence

Divergence



Example

The divergence in a region is constant and positive:

$$\nabla \cdot \vec{u} = D > 0$$

What happens to the density of an air parcel?

Example

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{u} = -D$$

$$\frac{d\rho}{dt} = -\rho D$$

$$\rho(t) = \rho(0)\ e^{-Dt}$$

Density decreases exponentially in time

Vorticity

Central quantity in dynamics

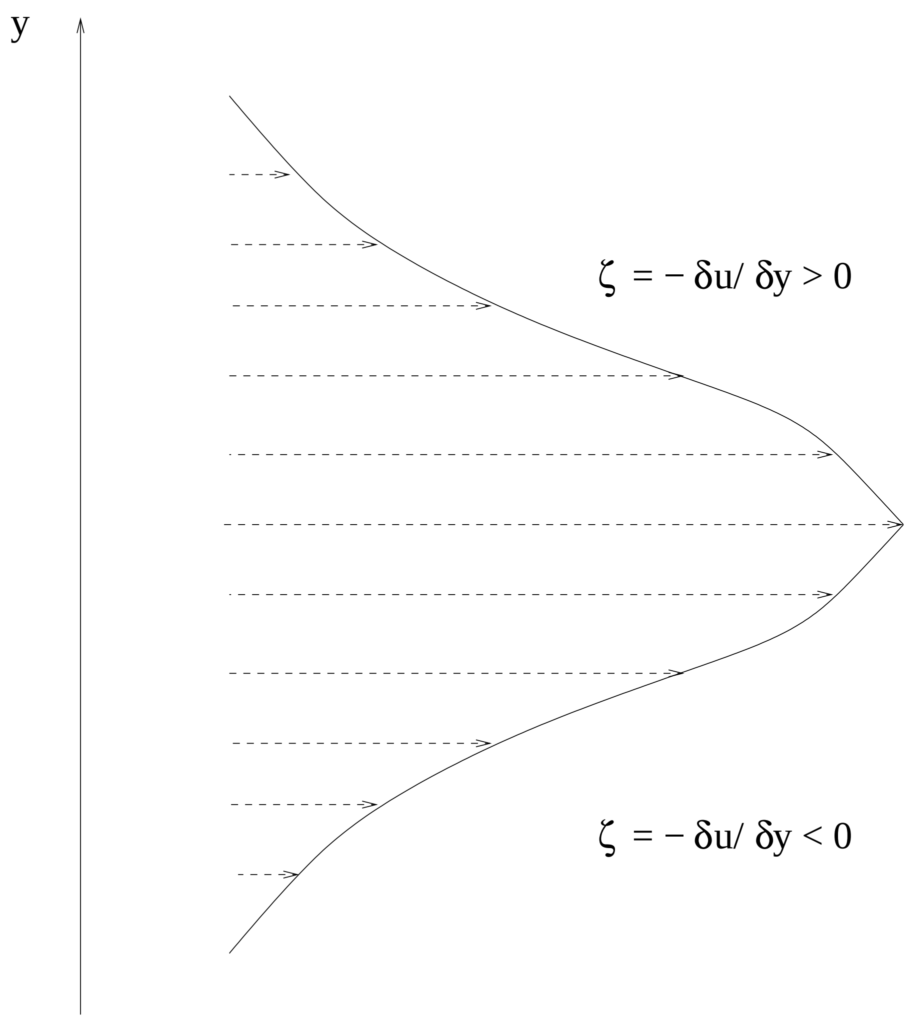
$$\vec{\zeta} \equiv \nabla \times \vec{u}$$

$$\vec{\zeta} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

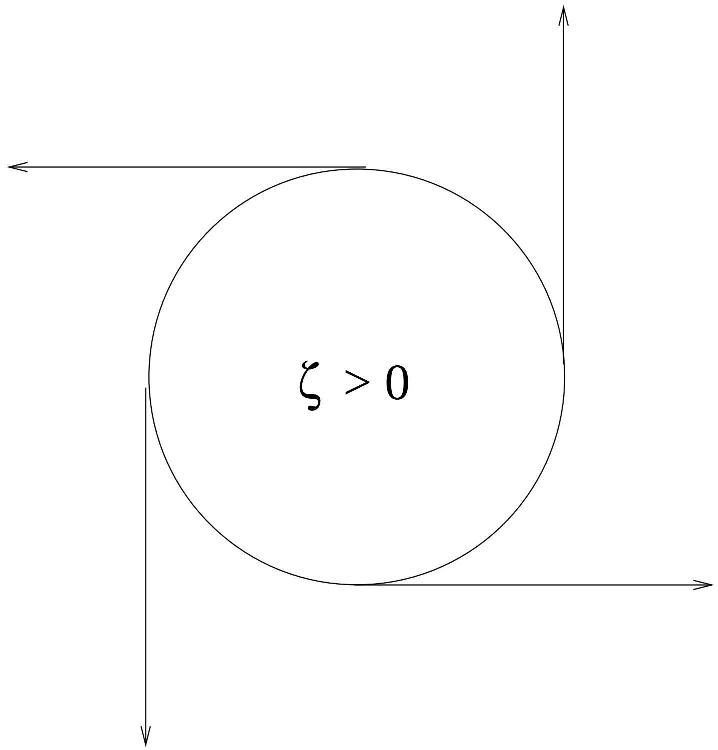
Most important at synoptic scales is *vertical component*:

$$\vec{\zeta} = \zeta \hat{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity



Vorticity



Example

What is the vorticity of a typical tornado? Assume *solid body rotation*, with a velocity of 100 m/sec, 20 m from the center.
In cylindrical coordinates, the vorticity is:

$$\zeta = \frac{1}{r}\frac{\partial\ r v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

For solid body rotation, $v_r = 0$ and

$$v_\theta = \omega r$$

with $\omega =$const.

Vorticity

So:

$$\zeta = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = \frac{1}{r}\frac{\partial \omega r^2}{\partial r} = 2\omega$$

We have $v_\theta = 100$ m/sec at $r = 20$ m:

$$\omega = \frac{v_\theta}{r} = \frac{100}{20} = 5\,rad/sec$$

So:

$$\zeta = 10\,rad/sec$$

Absolute vorticity

Now add rotation. The velocity in the fixed frame is:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

So:

$$\vec{\zeta}_a = \nabla \times (\vec{u} + \vec{\Omega} \times \vec{r}) = \vec{\zeta} + \nabla \times (\vec{\Omega} \times \vec{r})$$

$$= \vec{\zeta} + \nabla \times (z\Omega_y - y\Omega_z, x\Omega_z, -x\Omega_y)$$

$$= \vec{\zeta} + 2\vec{\Omega}$$

Absolute vorticity

Two components:

- $\nabla \times \vec{u}$ — the *relative vorticity*
- 2Ω — the *planetary vorticity*

Vertical component is the most important:

$$\zeta_a \cdot \hat{k} = (\frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u) + 2\Omega_z = \zeta + 2\Omega sin(\theta) = \zeta + f$$

ζ now refers to the vertical relative vorticity

Absolute vorticity

Scaling:

$$\zeta \propto \frac{U}{L}$$

So:

$$\frac{|\zeta|}{f} \approx \frac{U}{fL} = \epsilon$$

The Rossby number

Absolute vorticity

- $\epsilon \ll 1$

Geostrophic velocities

Planetary vorticity dominates the absolute vorticity

- $\epsilon \gg 1$

Cyclostrophic velocities

Relative vorticity dominates

Circulation

Circulation is the integral of vorticity over an area:

$$\Gamma \equiv \int\int \zeta dA$$

Due to Stoke's theorem, we can rewrite this as an integral of the velocity around the circumference:

$$\Gamma = \int\int \nabla \times \vec{u} \, dA = \oint \vec{u} \cdot \hat{n} \, dl$$

Thus we can derive an equation for the circulation by integrating the momentum equations around a closed curve.

Circulation

First write momentum equations in vector form. Turns out to be simpler using the fixed frame velocity:

$$\frac{d}{dt}\vec{u}_F = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F}$$

Integrate around a closed area:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{g} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl}$$

Circulation

Gravity vanishes because can write as the gradient of a potential:

$$\vec{g} = -g\hat{k} = \frac{\partial}{\partial z}(-gz) = \nabla \Phi_g$$

and the closed integral of a potential vanishes:

$$\oint \nabla \Phi_g \cdot \vec{dl} = \oint d\Phi_g = 0$$

Circulation

So:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

Put rotation back in. The fixed velocity is:

$$\vec{u}_F = \vec{u}_R + \Omega \times r$$

So:

$$\Gamma_F = \oint (\vec{u}_R + \Omega \times r) \cdot \vec{dl}$$

Circulation

Rewrite using Stoke's theorem:

$$\oint (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot \vec{dl} = \int\int \nabla \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot \hat{n}\, dA$$

From before:

$$\nabla \times (\vec{\Omega} \times \vec{r}) = 2\Omega$$

If the motion is quasi-horizontal, then $\hat{n} = \hat{k}$:

$$\Gamma_F = \int\int [\zeta + 2\Omega sin(\theta)] dA = \int\int (\zeta + f) dA$$

Kelvin's theorem

Thus:

$$\frac{d}{dt}\Gamma_a = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

where

$$\Gamma_a = \int\int (\zeta + f) dA$$

is the *absolute circulation*, the sum of relative and planetary circulation

Kelvin's theorem

If the atmosphere is barotropic (temperature and density constant on pressure surfaces):

$$\oint \frac{dp}{\rho} = \frac{1}{\rho} \oint dp = 0$$

If atmosphere is also frictionless ($\vec{F} = 0$), then:

$$\frac{d}{dt} \Gamma_a = 0$$

The *absolute circulation is conserved* on the parcel

Kelvin's theorem

Notice that if the area is small, so that the vorticity is approximately constant over the area, then:

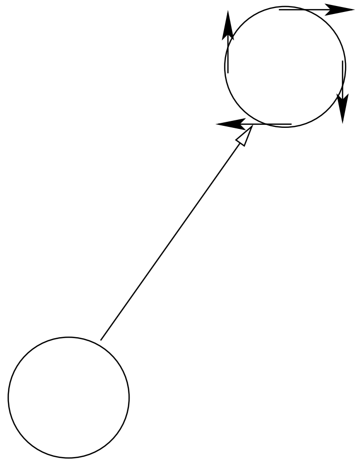
$$\frac{d}{dt}\Gamma_a \approx \frac{d}{dt}(\zeta + f)A = 0$$

which implies:

$$(\zeta + f)A = const.$$

on a parcel. Thus if a parcel's area or latitude changes, it's vorticity must change to compensate.

Kelvin's theorem



Move a parcel north, where f is larger. Either:

- Vorticity decreases
- Area decreases

Kelvin's theorem

Example: An air parcel at 30 N moves to 90 N. If its initial relative vorticity is $5 \times 10^{-5} sec^{-1}$, what is its final vorticity?

$$(\zeta_{30} + 2\Omega sin(30))A = (\zeta_{90} + 2\Omega)A$$

So:

$$\zeta_{90} = \zeta_{30} + 2\Omega(sin(30) - 1) = 5 \times 10^{-5} + 1.45 \times 10^{-4}(0.5 - 1)$$

$$= -2.25 \times 10^{-5} sec^{-1}$$

Vorticity equation

Now we will derive an equation for the vorticity.

Horizontal momentum equations (p-coords):

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,u - fv = -\frac{\partial}{\partial x}\Phi + F_x$$

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,v + fu = -\frac{\partial}{\partial y}\Phi + F_y$$

Take $\frac{\partial}{\partial x}$ of the second, subtract $\frac{\partial}{\partial y}$ of the first

Vorticity equation

Find (after some algebra):

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,\zeta + v\frac{\partial}{\partial y}f$$

$$= (\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,\zeta_a$$

$$= -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}) + (\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

where:

$$\zeta_a = \zeta + f$$

Vorticity equation

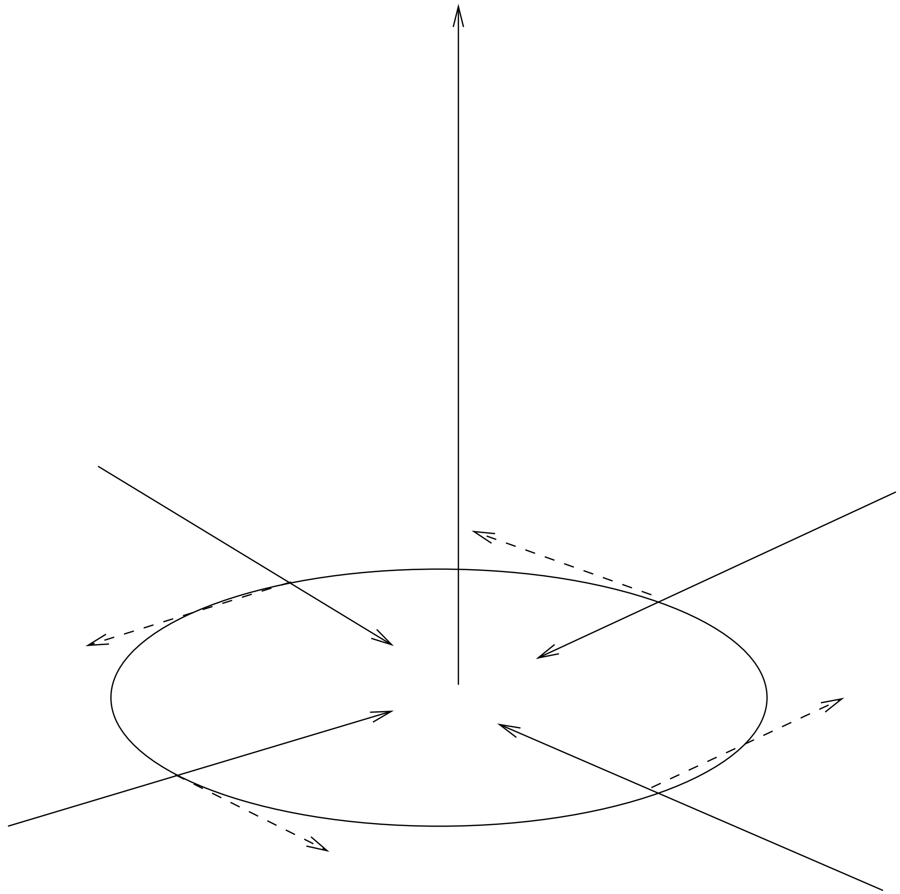
The absolute vorticity can change due to three terms

1) Divergence:

$$-\zeta_a\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Divergence changes the vorticity, just like density

Convergence



Divergence

Can absorb the divergence into the left side. Consider small area of air:

$$\delta A = \delta x \, \delta y$$

Time change in the area is:

$$\frac{\delta A}{\delta t} = \delta y \frac{\delta x}{\delta t} + \delta x \frac{\delta y}{\delta t} = \delta y \, \delta u + \delta x \, \delta v$$

Relative change is the divergence:

$$\frac{1}{\delta A} \frac{\delta A}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}$$

Divergence

So rewrite the divergence term:

$$-\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v\right)\zeta_a = -\frac{\zeta_a}{A}\frac{dA}{dt}$$

So:

$$\frac{d}{dt}\zeta_a = -\frac{\zeta_a}{A}\frac{dA}{dt} \quad \rightarrow \quad \frac{d}{dt}\zeta_a A = 0$$

This is just Kelvin's theorem again!

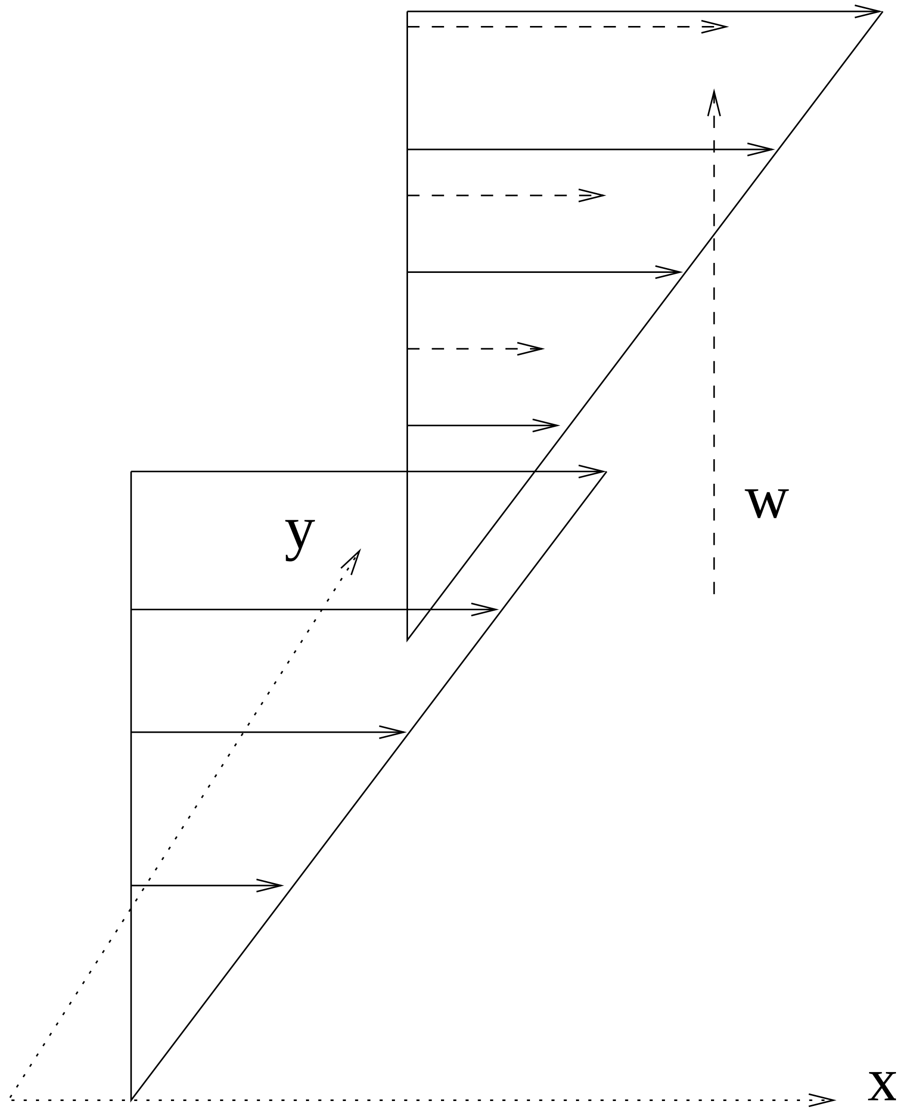
Vorticity equation

2) The *tilting* term:

$$\left(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}\right)$$

Differences in ω can affect the horizontal shear

Tilting



Vorticity equation

3) The Forcing term:

$$(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

Say frictional forcing:

$$F_x = \nu\nabla^2 u, \quad F_y = \nu\nabla^2 v$$

Friction

Then:

$$(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x) = \nu\nabla^2\,(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = \nu\nabla^2\,\zeta$$

Then:

$$\frac{d}{dt}(\zeta + f) = \nu\nabla^2\,\zeta$$

Friction

If $f \approx const.$:

$$\frac{d}{dt}\zeta = \nu \nabla^2 \zeta$$

Friction *diffuses* vorticity

Causes cyclones to spread out and weaken

Can occur due to friction in the *boundary layer*

Scaling

$$(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p})\,\zeta_a = -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x})$$

For synoptic scale motion, away from boundary layer:

$$U \approx 10m/sec \quad \omega \approx 10hPa/day \quad L \approx 10^6 m \quad \partial p \approx 100hPa$$

$$f_0 \approx 10^{-4} sec^{-1} \quad L/U \approx 10^5 sec \quad \frac{\partial f}{\partial y} \approx 10^{-11} m^{-1} sec^{-1}$$

Scaling

$$\zeta \propto \frac{U}{L} \approx 10^{-5} sec^{-1}$$

So the Rossby number is:

$$\epsilon = \frac{\zeta}{f_0} \approx 0.1$$

So:

$$(\zeta + f) \approx f$$

Scaling

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta \quad \propto \frac{U^2}{L^2} \quad \approx \quad 10^{-10}$$

$$\omega\frac{\partial}{\partial p}\zeta \quad \propto \frac{U\omega}{LP} \quad \approx \quad 10^{-11}$$

$$v\frac{\partial}{\partial y}f \quad \propto U\frac{\partial f}{\partial y} \quad \approx \quad 10^{-10}$$

$$\left(\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x}\right) \quad \propto \frac{U\omega}{LP} \quad \approx \quad 10^{-11}$$

$$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \propto \frac{fU}{L} \quad \approx \quad 10^{-9}$$

Scaling

Divergence term is unbalanced! But it's actually smaller than it appears. We can write:

$$u = u_g + u_a, \quad v = v_g + v_a$$

From the derivation of the gradient wind:

$$\frac{u_g}{u} \approx 1 + \epsilon$$

This implies:

$$\frac{|u_a|}{|u_g|} \propto \epsilon \approx 0.1$$

Ageostrophic velocities

$$u = u_g + \epsilon u_a, \quad v = v_g + \epsilon v_a$$

The vorticity is:

$$\zeta = \frac{\partial}{\partial x} v_g - \frac{\partial}{\partial y} u_g + \epsilon\left(\frac{\partial}{\partial x} v_a - \frac{\partial}{\partial y} u_a\right)$$

While the divergence is:

$$D = \frac{1}{f}\frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial y}\right) + \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x}\right) + \epsilon\left(\frac{\partial}{\partial x} u_a + \frac{\partial}{\partial y} v_a\right)$$

$$= 0 + \epsilon\left(\frac{\partial}{\partial x} u_a + \frac{\partial}{\partial y} v_a\right)$$

The divergence is order ϵ

Vertical velocities

Also:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial p}\omega = 0$$

implies:

$$\frac{\partial}{\partial p}\omega = -D = -\epsilon(\frac{\partial}{\partial x}u_a + \frac{\partial}{\partial y}v_a)$$

So the vertical velocity is also order ϵ

Planetary rotation *suppresses vertical motion*

This is why atmospheric motion is quasi-horizontal

Scaled equation

Thus the divergence estimate is smaller:

$$(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \approx f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \;\; \propto \epsilon \frac{fU}{L} \;\; \approx \;\; 10^{-10}$$

Retaining the 10^{-10} terms yields the approximate vorticity equation:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = -f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

Forecasting

Used for forecasts in the 1940's

Approach:

Assume geostrophic velocities:

$$u \approx u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}$$

$$v \approx v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

Forecasting

$$\zeta \approx \zeta_g = \frac{1}{f}\,\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f}\,(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}) = \frac{1}{f}\nabla^2 \Phi$$

The divergence vanishes identically:

$$(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}) = 0$$

Thus the vorticity equation is:

$$(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y})\,(\zeta + f) = 0$$

ζ_a is *conserved following the horizontal winds*

Remember: on a pressure surface

Forecasting

Now only *one unknown*: Φ

$$(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y})\,(\zeta + f) = 0$$

becomes:

$$(\frac{\partial}{\partial t} - \frac{1}{f}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} + \frac{1}{f}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y})\,(\frac{1}{f}\nabla^2\Phi + f) = 0$$

Forecasting

Can write equation:

$$\frac{\partial}{\partial t}\zeta + u_g \cdot \nabla\zeta + v_g \frac{\partial}{\partial y} f = 0$$

or:

$$\frac{\partial}{\partial t}\zeta = -u_g \cdot \nabla\zeta - v_g \frac{\partial}{\partial y} f$$

Can predict how ζ changes in time

Then convert $\zeta \rightarrow \Phi$ by *inversion*

Forecasting

Method:

- Obtain $\Phi(x, y, t_0)$ from measurements on p-surface
- Calculate $u_g(t_0)$, $v_g(t_0)$, $\zeta(t_0)$
- Calculate $\zeta(t_1)$
- *Invert* ζ to get $\Phi(t_1)$
- Start over
- Obtain $\Phi(t_2)$, $\Phi(t_3)$,...

Inversion

$$\zeta = \frac{1}{f} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)$$

$$\nabla^2 \Phi = f \zeta$$

Poisson's equation

Need boundary conditions to solve

Usually do this numerically

Inversion

Example: Let:

$$\zeta = sin(3x)sin(\pi y)$$

Say we have a channel:

$$x = [0, 2\pi], \quad y = [0, 1]$$

Periodic in x and solid walls at $y = 0, 1$. We have:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = sin(3x)sin(\pi y)$$

Inversion

Try a particular solution:

$$\Phi = A sin(3x) sin(\pi y)$$

This solution works in a channel, because:

$$\Phi(x = 2\pi) = \Phi(x = 0)$$

Also, at $y = 0, 1$:

$$v = \frac{1}{f_0} \frac{\partial \Phi}{\partial x} = 0$$

Inversion

Substitute into equation:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = -(9+\pi^2)Asin(3x)sin(\pi y) = sin(3x)sin(\pi y)$$

So:

$$\Phi = -\frac{1}{9+\pi^2}\, sin(3x)sin(\pi y)$$

Then we can proceed (calculate u_g, v_g, etc.)

Inversion

Inversion is a *smoothing* operation

Preferentially weights the large scale features. Say instead we had:

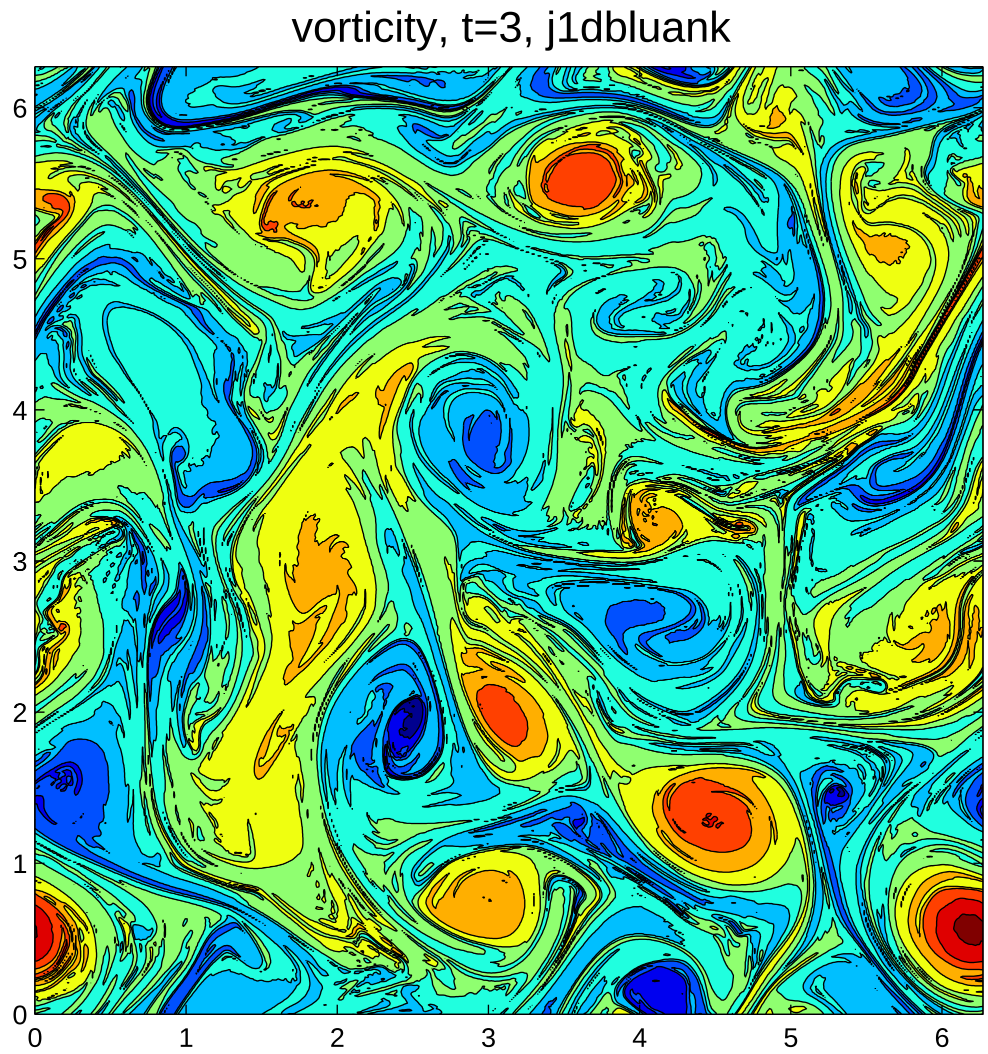
$$\zeta = sin(3x)sin(3y) + sin(x)sin(y)$$

Then:

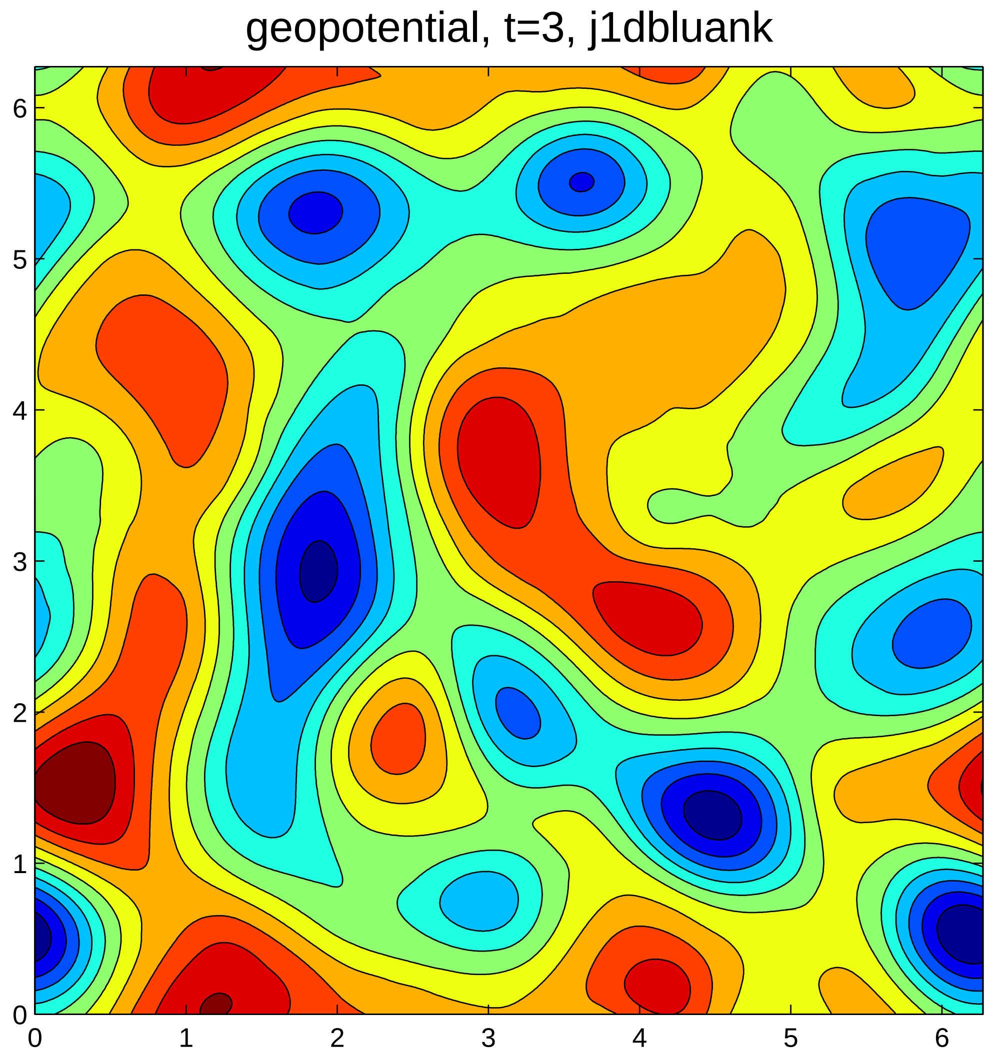
$$\Phi = \frac{1}{18} sin(3x)sin(3y) + sin(x)sin(y)$$

The smaller wave contributes less to the geopotential

Vorticity, turbulence simulation



Geopotential, turbulence simulation



Example II

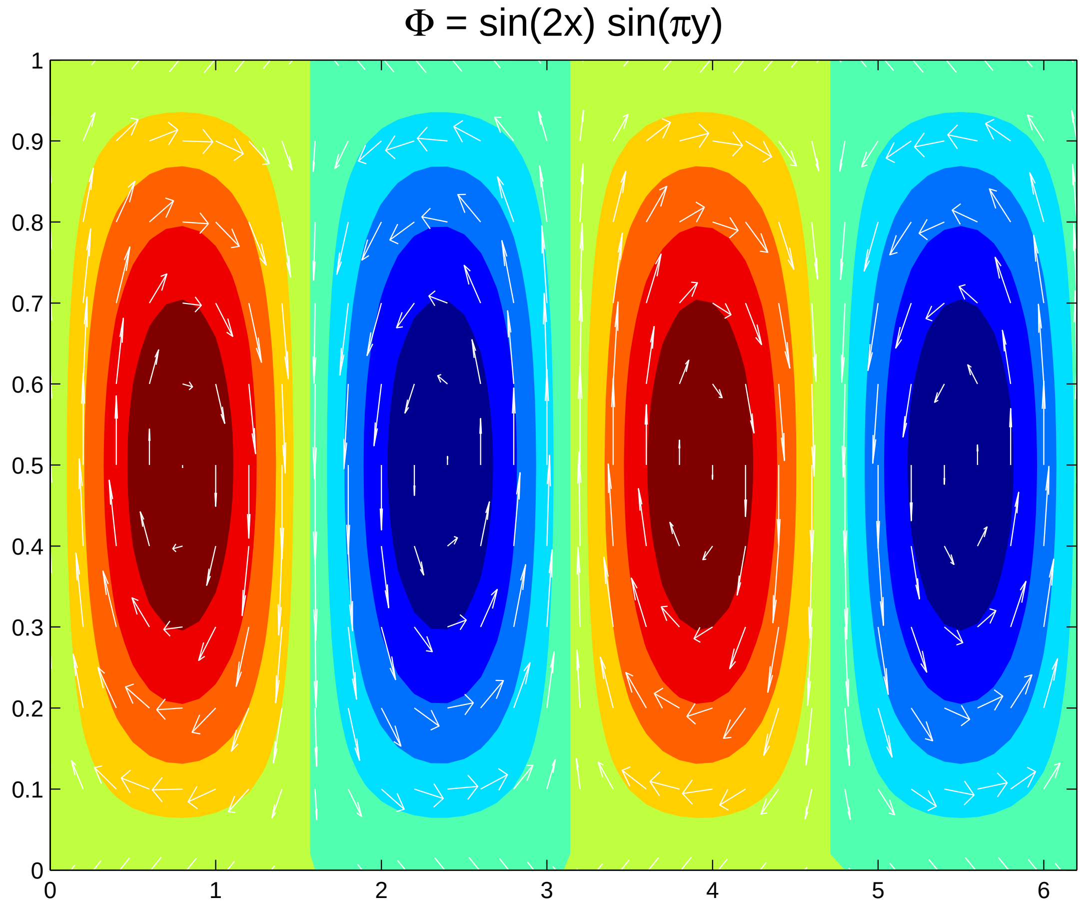
Say the geopotential is given by:

$$\Phi = f_0 A \, sin(2x - \omega t) \, sin(\pi y)$$

Describe how the field evolves in time

What is ω?

Initial geopotential



Example II

We must solve:

$$\frac{\partial}{\partial t}\zeta = -u_g \cdot \nabla\zeta - v_g \frac{\partial}{\partial y} f$$

But we have a problem—f is a function of θ, the latitude, rather than y!

We must rewrite f in terms of y

Beta-plane

If we limit the latitude range, we can expand f in a Taylor Series about the center latitude:

$$f(\theta) \approx f(\theta_0) + (\theta - \theta_0)\frac{df}{d\theta} + \frac{(\theta - \theta_0)^2}{2}\frac{d^2 f}{d\theta^2} + \ \ldots$$

We have $y = R\theta$, where R is the earth radius. Keeping the first two terms:

$$f \approx f_0 + \beta(y - y_0)$$

where:

$$f_0 = 2\Omega sin(\theta_0), \quad \beta = \frac{2\Omega}{R} cos(\theta_0)$$

Example II

So:

$$v\frac{df}{dy} = v\frac{\partial}{\partial y}(f_0 + \beta(y - y_0)) = \beta v$$

So the equation becomes:

$$\frac{\partial}{\partial t}\zeta = -u_g \cdot \nabla\zeta - \beta v_g$$

Example II

Now the velocities are:

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi = -\pi A\, sin(2x - \omega t)\, cos(\pi y)$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = 3A\, cos(2x - \omega t)\, sin(\pi y)$$

And the vorticity is:

$$\zeta = \frac{1}{f_0}\nabla^2\Phi = -(4 + \pi^2)A\, sin(2x - \omega t)\, sin(\pi y)$$

Example II

We also need the derivatives:

$$\frac{\partial}{\partial x}\zeta = -2(4+\pi^2)A\,cos(2x-\omega t)\,sin(\pi y)$$

$$\frac{\partial}{\partial y}\zeta = -\pi(4+\pi^2)A\,sin(2x-\omega t)\,cos(\pi y)$$

Example II

Collect terms:

$$-u\frac{\partial}{\partial x}\zeta - v\frac{\partial}{\partial y}\zeta = [-\pi A\, sin(2x-\omega t)\, cos(\pi y)]\times$$

$$[2(4+\pi^2)A\, cos(2x-\omega t)\, sin(\pi y)] + [2A\, cos(2x-\omega t)\, sin(\pi y)]\times$$

$$[\pi(4+\pi^2)A\, sin(2x-\omega t)\, cos(\pi y)]$$

$$= [-2\pi A^2(4+\pi^2) + 2\pi A^2(4+\pi^2)] sin(2x-\omega t) cos(2x-\omega t)$$

$$\times sin(\pi y) cos(\pi y) = 0$$

Example II

Also:

$$-\beta v = -2\beta A\, cos(2x - \omega t)\, sin(\pi y)$$

So:

$$\frac{\partial}{\partial t}\zeta = -2\beta A\, cos(2x - \omega t)\, sin(\pi y)$$

Since:

$$\zeta = -(4 + \pi^2) A\, sin(2x - \omega t)\, sin(\pi y)$$

Example II

Then:

$$\frac{\partial}{\partial t}\zeta = \omega(4+\pi^2)A\,cos(2x-\omega t)\,sin(\pi y)$$

Equate both sides:

$$\omega(4+\pi^2)A\,cos(2x-\omega t)\,sin(\pi y)$$
$$= -2\beta A\,cos(2x-\omega t)\,sin(\pi y)$$

We can cancel the $A\,cos(2x-\omega t)\,sin(\pi y)$, leaving:

$$\omega(4+\pi^2) = -2\beta$$

Example II

or:

$$\omega = -\frac{2\beta}{4+\pi^2}$$

So the solution is:

$$\Phi = A\, sin(2x + \frac{2\beta}{4+\pi^2}t)\, sin(\pi y)$$

This is a “travelling wave”

Phase speed

We can rewrite the solution:

$$\Phi = A\, cos[2(x + \frac{\beta}{4+\pi^2} t)]\, sin(\pi y)$$

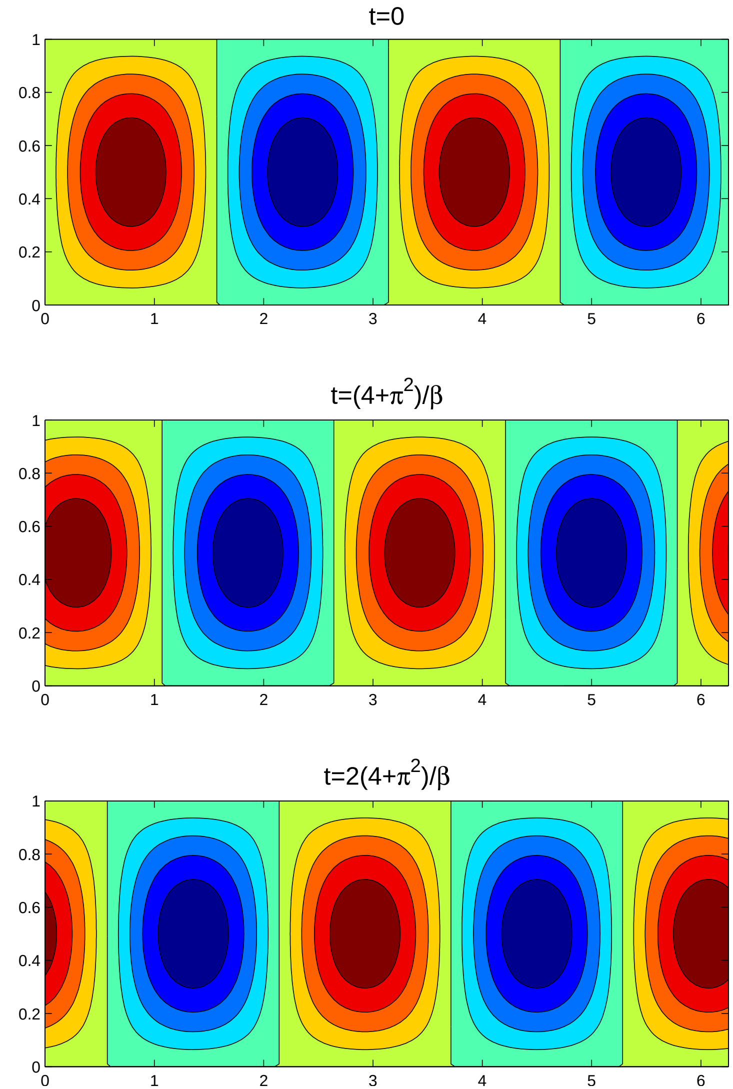
This implies that the wave has a *phase speed*:

$$c = \frac{\omega}{k} = -\frac{\beta}{4+\pi^2}$$

This is how fast the crests in the wave move

Because $c < 0$, waves move toward *negative* x (westward)

Westward



Phase speed

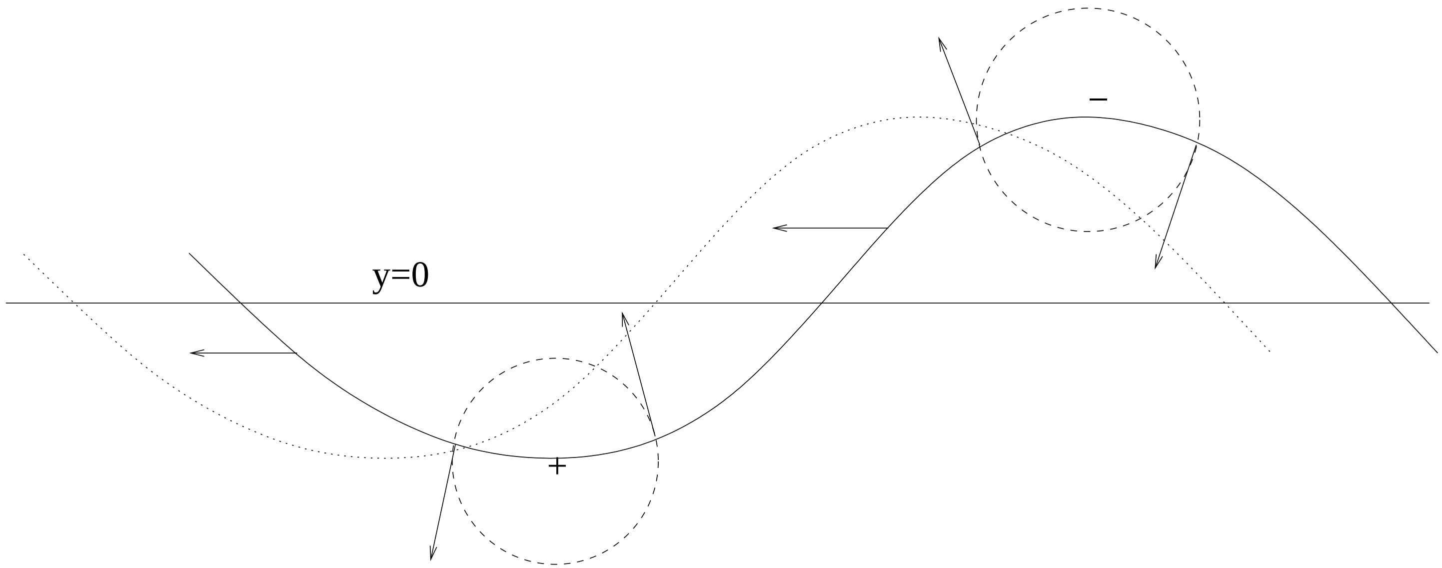
The westward propagation is actually a consequence of Kelvin's theorem

Fluid parcels advected north/south acquire relative vorticity

The parcels then advect neighboring parcels around them

Leads to a westward drift of the wave

Westward propagation



Rossby waves

Solution is known as a *Rossby wave*

Discovered by Carl Gustav Rossby (1936)

Observed in the atmosphere

Important for weather patterns

Study more later (GEF4500)

Divergence

Previously ignored divergence effects. But very important for the growth of unstable disturbances (storms)

The approximate vorticity equation is:

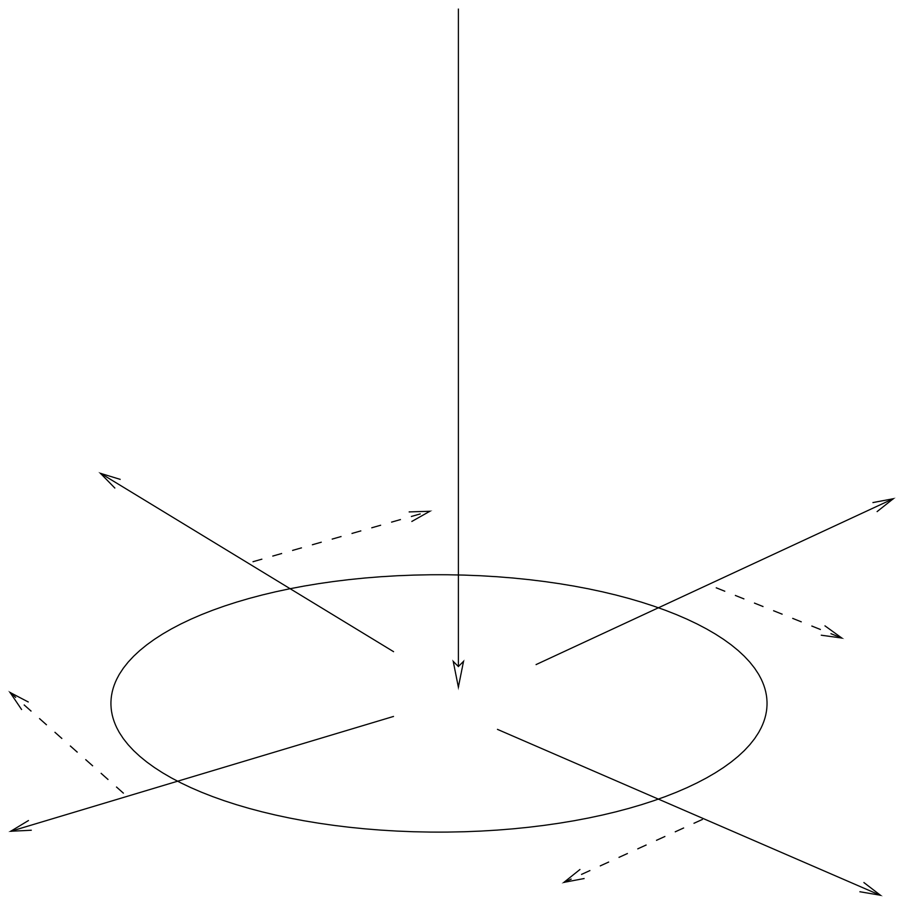
$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

where:

$$\frac{d}{dt} = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})$$

is the Lagrangian derivative following the horizontal flow

Divergence



Divergence

Consider flow with constant divergence:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D > 0$$

$$\frac{d}{dt}\zeta_a = -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -D\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\ e^{-Dt}$$

Divergence

So:

$$\zeta_a = \zeta + f \longrightarrow 0$$

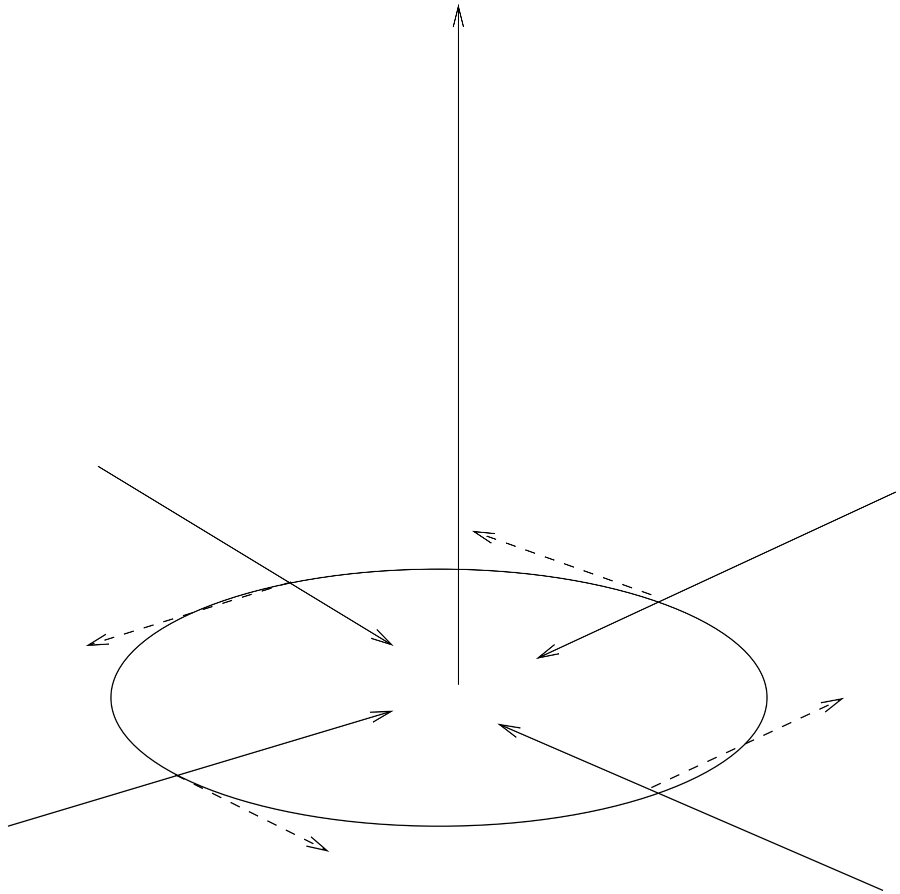
$$\zeta \longrightarrow -f$$

Divergent flow favors *anticyclonic* vorticity

Vorticity approaches $-f$, regardless of initial value

Vorticity cannot *exceed* f

Convergence



Divergence

Now say $D = -C$

$$\frac{d}{dt}\zeta_a = -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = C\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\; e^{Ct}$$

$$\zeta_a \longrightarrow \pm\infty$$

But which sign?

Divergence

If the Rossby number is small, then:

$$\zeta_a(0) = \zeta(0) + f \approx f > 0$$

So:

$$\zeta \rightarrow +\infty$$

Convergent flow favors *cyclonic* vorticity

Vorticity increases *without bound*

- Why intense storms are cyclonic

Summary

The vorticity equation is approximately:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

or:

$$\frac{d}{dt}\zeta + v\frac{df}{dy} = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

- Vorticity changes due to meridional motion
- Vorticity changes due to divergence

Barotropic potential vorticity

Consider an atmospheric layer with constant density, between two surfaces, at $z = z_1, z_2$ (e.g. the surface and the tropopause)

The continuity equation is:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0$$

If density constant, then:

$$(\nabla \cdot \vec{u}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Barotropic potential vorticity

So:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Thus the vorticity equation can be written:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\zeta + f) = (\zeta + f)\,\frac{\partial w}{\partial z}$$

Taylor-Proudman Theorem

The constant density assumption affects the shear

$$\frac{d}{dt}u - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

Taking a z-derivative:

$$\frac{d}{dt}(\frac{\partial}{\partial z}u) - f(\frac{\partial}{\partial z}v) = -\frac{1}{\rho}\frac{\partial}{\partial x}(\frac{\partial}{\partial z}p) = \frac{\rho}{\rho}\frac{\partial}{\partial x}g = 0$$

$\rightarrow$ If there is no shear initially, have no shear at any time.
With constant density:

$$\frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = 0$$

Barotropic potential vorticity

So the integral of the vorticity equation is simply:

$$\int_{z1}^{z2} (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f)dz =$$

$$h(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,[w(z_2) - w(z_1)]$$

where $h = z_2 - z_1$. Note that $w = Dz/Dt$. Thus:

$$w(z_2) - w(z_1) = \frac{d}{dt}(z_2 - z_1) = \frac{dh}{dt}$$

Barotropic potential vorticity

So:

$$h\,\frac{d}{dt}(\zeta + f) = (\zeta + f)\,\frac{dh}{dt}$$

dividing by h^2:

$$\frac{1}{h}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{dh}{dt} = 0$$

which is the same as:

$$\frac{d}{dt}\frac{\zeta + f}{h} = 0$$

Barotropic potential vorticity

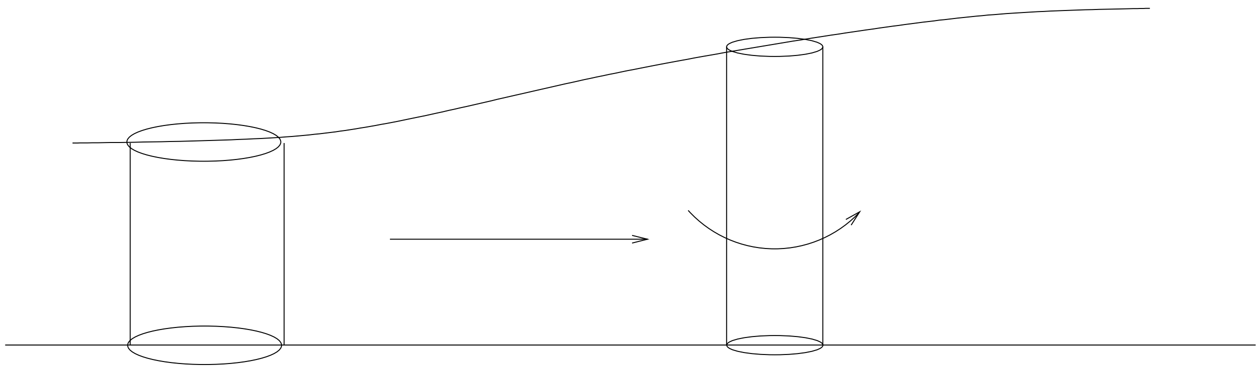
Thus the barotropic potential vorticity (PV):

$$\frac{\zeta + f}{h} = const.$$

is conserved on a fluid parcel.

If h increases, either ζ or f must also increase

Layer potential vorticity



Alternate derivation

Consider a fluid column between z_1 and z_2. As it moves, conserves mass:

$$\frac{d}{dt}(hA) = 0$$

So:

$$hA = const.$$

Because the density is constant, we can apply Kelvin's theorem:

$$\frac{d}{dt}(\zeta + f)A \propto \frac{d}{dt}\frac{\zeta + f}{h} = 0$$

Potential temperature

But the atmosphere is not constant density. What use is the potential vorticity?

As move upward in atmosphere, both temperature and pressure change—neither is absolute.

But can define the *potential temperature* which is absolute—accounts for pressure change.

The potential vorticity can then be applied in layers *between potential temperature surfaces*

Potential temperature

The thermodynamic energy equation is:

$$c_p dT - \alpha dp = dq$$

With zero heating, and using the ideal gas law:

$$c_p dT = \alpha dp = \frac{RT}{p} dp$$

Rewriting:

$$c_p \, dlnT = R \, dlnp$$

Potential temperature

Integrate up from the the surface:

$$c_p \, lnT - R \, lnp = c_p \, ln\theta - R \, lnp_0$$

where p_0 is the surface pressure:

$$p_0 = 100 \; kPa \; = 1000mb$$

Rearranging:

$$\theta = T \, (\frac{p_0}{p})^{R/c_p}$$

Potential temperature

If zero heating, a parcel conserves its potential temperature, θ

Call a surface with constant potential temperature an isentropic surface or an "adiabat"

θ is the temperature a parcel has if we move it adiabatically back to the surface

Note potential temperature depends on *both* T and p

Layer potential vorticity

Flow between two isentropic surfaces trapped if zero heating

So mass in a column between two surfaces is conserved:

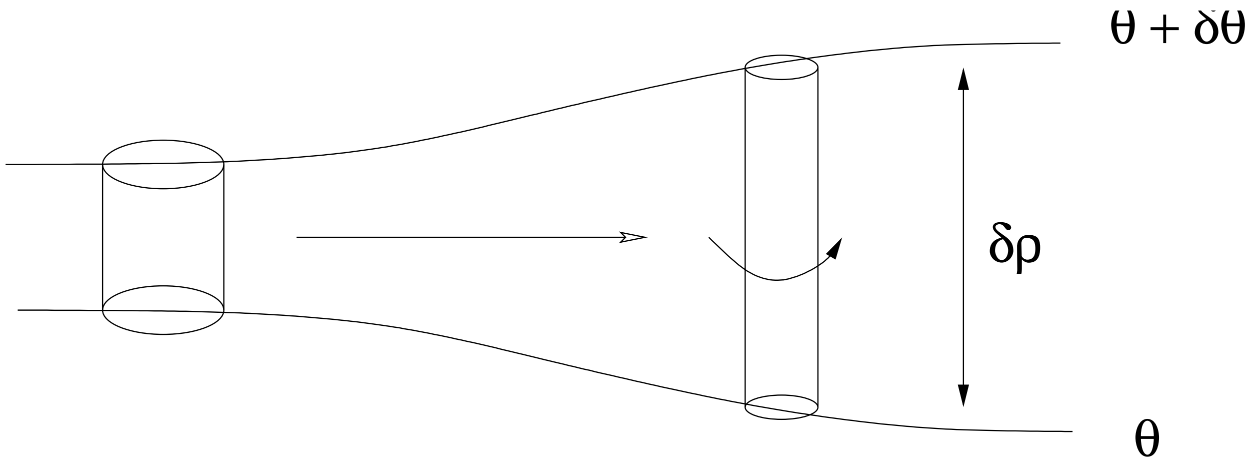
$$A\delta z = const.$$

From the hydrostatic relation:

$$-\frac{A\delta p}{\rho g} = const.$$

where δp is the spacing between surfaces

Layer potential vorticity



Layer potential vorticity

Rewrite δp thus:

$$\delta p = (\frac{\partial \theta}{\partial p})^{-1} \delta \theta$$

Here, $\frac{\partial \theta}{\partial p}$ is the *stratification*. The stronger the stratification, the smaller the pressure difference between temperature surfaces. Thus:

$$\frac{A \delta p}{\rho g} = A(\frac{\partial \theta}{\partial p})^{-1} \frac{\delta \theta}{g} = const.$$

Layer potential vorticity

From the Ideal Gas Law and the definition of potential temperature, we can write:

$$\rho = p^{c_v/c_p} (R\theta)^{-1} p_s^{R/c_p}$$

So the density is a function *only of pressure*. This means that:

$$\oint \frac{dp}{\rho} \propto \oint dp^{1-c_v/c_p} = 0$$

So Kelvin's theorem applies in the layer

Layer potential vorticity

Thus:

$$\frac{d}{dt}[(\zeta + f)A] = 0$$

implies:

$$\frac{d}{dt}[(\zeta + f)\frac{\partial \theta}{\partial p}] = 0$$

This is Ertel's (1942) "isentropic potential vorticity"

Layer potential vorticity

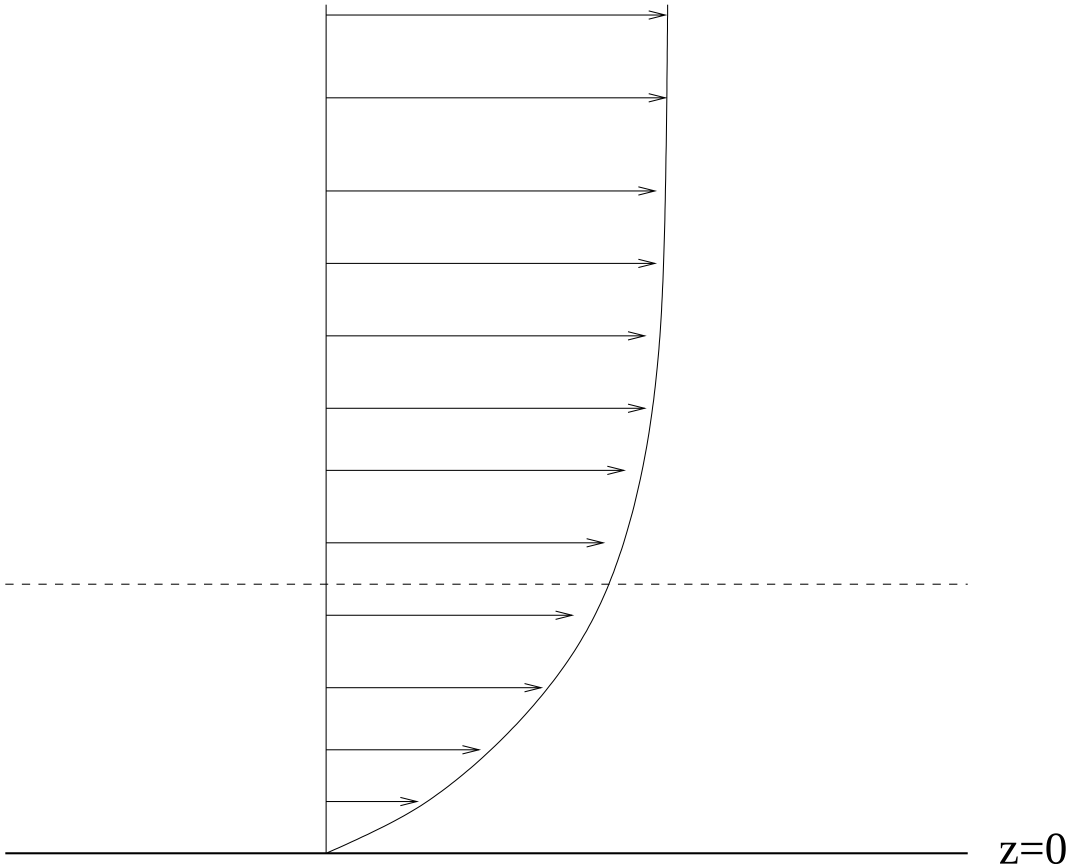
Remember: ζ evaluted on potential temperature surface

Very useful quantity: can label air by its PV

Can distinguish air in the troposphere which comes from stratosphere

Ertel's equation can also be used for prediction

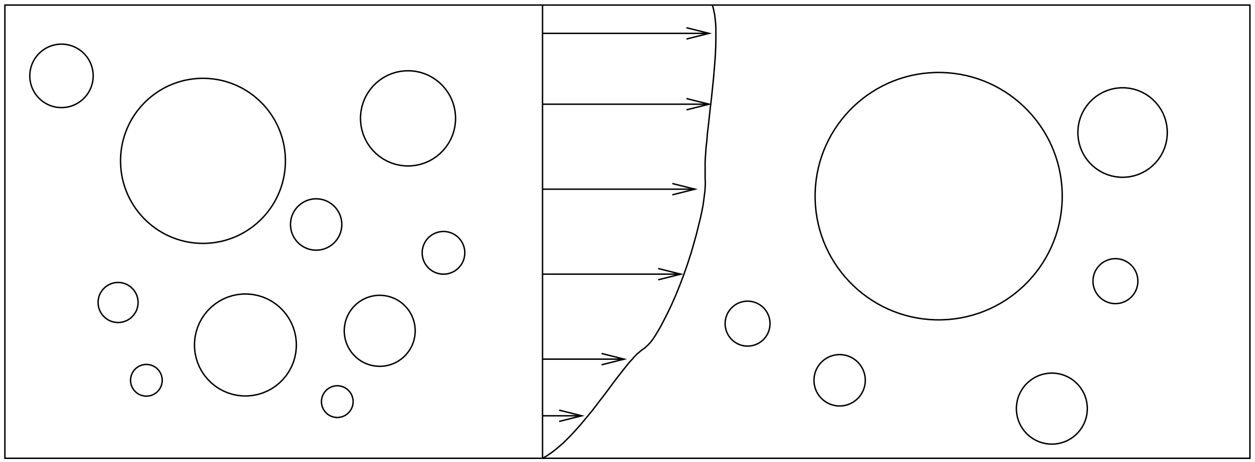
Planetary boundary layer



Turbulence

There is a *continuum* of eddy scales

Largest resolved by our models, but the smallest are not.



Boussinesq equations

Assume we can split the velocity into a time mean (over some period) and a perturbation:

$$u = \overline{u} + u'$$

Use the full momentum equations with no friction:

$$\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

Boussinesq approximation

Assume the density doesn't vary much. So we can write:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p \quad \rightarrow \quad \frac{1}{\rho_0}\frac{\partial}{\partial x}p$$

In addition, the continuity equation:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0$$

reduces to:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

So the flow is *incompressible*

Boussinesq equations

Substitute partitioned velocities into momentum equations:

$$\frac{\partial}{\partial t}(\overline{u}+u') + (\overline{u}+u')\frac{\partial}{\partial x}(\overline{u}+u') + (\overline{v}+v')\frac{\partial}{\partial y}(\overline{u}+u') - f(\overline{v}+v')$$

$$+(\overline{w}+w')\frac{\partial}{\partial z}(\overline{u}+u') = \frac{1}{\rho_0}\frac{\partial}{\partial x}(\overline{p}+p')$$

Then we average the whole equation. Note that:

$$\overline{\overline{u}+u'} = \overline{u}$$

Boussinesq equations

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{u'\frac{\partial}{\partial x}u'} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{v'\frac{\partial}{\partial y}u'} +$$

$$+\overline{w}\frac{\partial}{\partial z}\overline{u} + \overline{w'\frac{\partial}{\partial z}u'} + -f\overline{v} = \frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p}$$

Because of the continuity equation, we can write:

$$\overline{u'\frac{\partial}{\partial x}u'} + \overline{v'\frac{\partial}{\partial y}u'} + \overline{w'\frac{\partial}{\partial z}u'} = \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}$$

Boussinesq equations

So:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v}$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p} - (\frac{\partial}{\partial x}\rho_0\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'})$$

Similarly:

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u}$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial y}\overline{p} - (\frac{\partial}{\partial x}\overline{v'u'} + \frac{\partial}{\partial y}\overline{v'v'} + \frac{\partial}{\partial z}\overline{v'w'})$$

PBL equations

Prime terms on the RHS are the "eddy stresses"

Assume they don't vary horizontally in the PBL. Then:

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\overline{p} - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

If the Rossby number is small, the velocities outside the boundary layer are nearly geostrophic. So in the BL, we have:

$$-f\overline{v} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

or:

$$-f\overline{v} = -f\overline{v}_g - \frac{\partial}{\partial z}\overline{u'w'}$$

$$f\overline{u} = f\overline{u}_g - \frac{\partial}{\partial z}\overline{v'w'}$$

$\rightarrow$ The eddies *break geostrophy*

PBL equations

But we have too many unknowns! : $\overline{u}, \overline{v}, u', v', w'$

We must *parameterize* the eddy stresses, i.e. we must write the primed variables in terms of the unprimed variables.

There are two cases:

- Convective boundary layer
- Stable boundary layer

In a convective layer, heating from below causes the layer to overturn, mixing properties with height. The stable boundary layer is *stratified*.

Convective boundary layer

Due to vertical mixing, temperature and velocity do not vary with height. So we can integrate the momentum equation vertically:

$$\int_0^h -f(\overline{v} - \overline{v}_g)\, dz = -fh(\overline{v} - \overline{v}_g) =$$

$$-\int_0^h \frac{\partial}{\partial z}\overline{u'w'}\, dz = -\overline{u'w'}|_h + \overline{u'w'}|_0$$

We assume mixing vanishes at the top of the layer:

$$\overline{u'w'}|_h = 0$$

Convective boundary layer

Thus:

$$f h(\overline{v} - \overline{v}_g) = -\overline{u'w'}|_0$$

From surface measurements, can parameterize the fluxes:

$$\overline{u'w'}|_0 = -C_d \mathcal{V}\, u, \qquad \overline{v'w'}|_0 = -C_d \mathcal{V}\, v$$

where C_d is the "drag coefficient" and

$$\mathcal{V} \equiv (u^2 + v^2)^{1/2}$$

Convective boundary layer

Thus:

$$fh(\overline{v} - \overline{v}_g) = C_d \mathcal{V}\,\overline{u}$$

and:

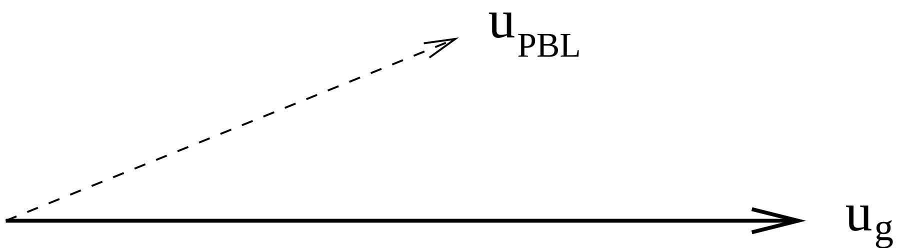
$$-fh(\overline{u} - \overline{u}_g) = C_d \mathcal{V}\,\overline{v}$$

Convective boundary layer

Say $v_g = 0$; then:

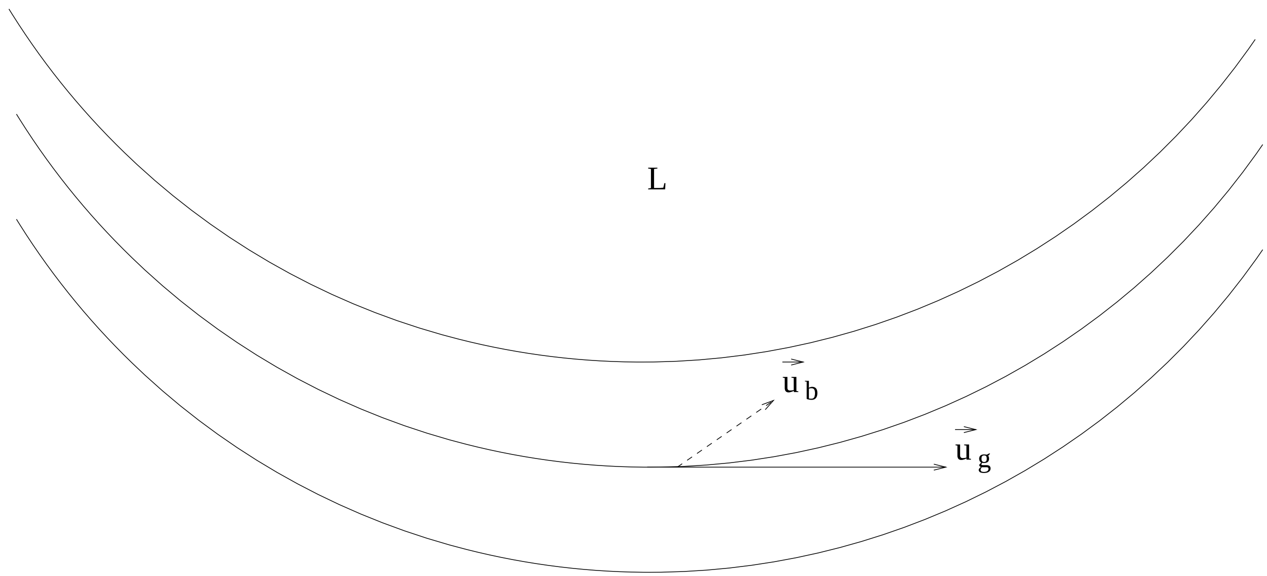
$$v = \frac{C_d}{fh} \mathcal{V} u,$$

$$u = u_g - \frac{C_d}{fh} \mathcal{V} v$$



Convective boundary layer

If $u > 0$, then $v > 0$



- Flow *down the pressure gradient*

Convective boundary layer

Solving the boundary layer equations is not so simple because $\mathcal{V} = \sqrt{u^2 + v^2}$

Coupled nonlinear equations

But we can use iterative methods

Make a first guess, then iteratively correct

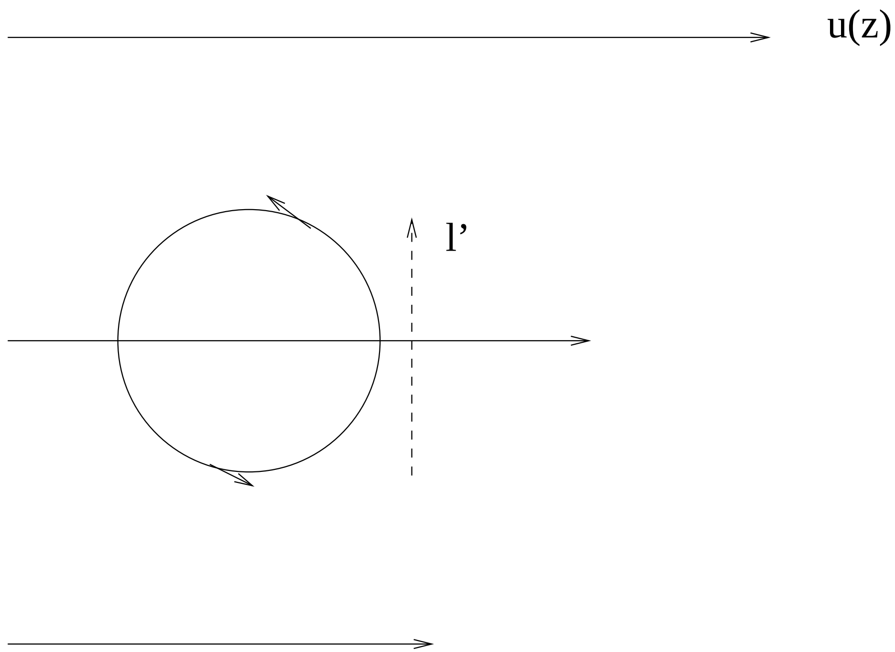
Stable boundary layer

Now assume no large scale vertical mixing

Wind speed and direction can vary with height

Specify turbulent velocities using *mixing length theory*.

Mixing length



$$u' = -l' \frac{\partial}{\partial z} \overline{u}$$

where $l' > 0$ if up.

Stable boundary layer

So:

$$-\overline{u'w'} = \overline{w'l'}\,\frac{\partial}{\partial z}\overline{u}$$

Assume same vertical and horizontal eddy scales. Write:

$$w' = l'\frac{\partial}{\partial z}\mathcal{V}$$

where again $\mathcal{V} = \sqrt{u^2 + v^2}$

Notice $w' > 0$ if $l' > 0$.

Stable boundary layer

So:

$$-\overline{u'w'} = (\overline{l'^2}\,\frac{\partial}{\partial z}\mathcal{V})\,\frac{\partial}{\partial z}\overline{u} \equiv A_z\,\frac{\partial}{\partial z}\overline{u}$$

Same argument:

$$-\overline{v'w'} = A_z\,\frac{\partial}{\partial z}\overline{v}$$

where A_z is the “eddy exchange coefficient”

Depends on the size of turbulent eddies and mean shear

Stable boundary layer

So we have:

$$f(v - v_g) = \frac{\partial}{\partial z}[A_z(z)\frac{\partial}{\partial z}u]$$

$$-f(u - u_g) = \frac{\partial}{\partial z}[A_z(z)\frac{\partial}{\partial z}v]$$

Simplest case is if $A_z(z)$ is constant

Studied by Swedish oceanographer V. W. Ekman (1905)

Consider boundary layer above a flat surface

Ekman layer

Boundary conditions: use the “no-slip condition”:

$$u = 0, v = 0 \quad at \ z = 0$$

Far from the surface, the velocities approach their geostrophic values:

$$u \longrightarrow u_g, v \longrightarrow v_g \quad z \longrightarrow \infty$$

Assume the geostrophic flow is zonal and independent of height:

$$u_g = U, \qquad v_g = 0$$

Ekman layer

Boundary layer velocities vary only in the vertical:

$$u = u(z)\ , \quad v = v(z)\ , \quad w = w(z)$$

From continuity:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = \frac{\partial}{\partial z}w = 0\ .$$

With a flat bottom, this implies:

$$w = 0$$

Ekman layer

The system is linear, so can decompose the horizontal velocities:

$$u = U + \hat{u}, \quad v = 0 + \hat{v}$$

Then:

$$-f\hat{v} = A_z \frac{\partial^2}{\partial z^2}\hat{u}$$

$$f\hat{u} = A_z \frac{\partial^2}{\partial z^2}\hat{v}$$

Ekman layer

Boundary conditions:

$$\hat{u} = -U, \hat{v} = 0 \quad at \ z = 0$$

Introduce a new variable:

$$\chi \equiv \hat{u} + i\hat{v}$$

Then:

$$\frac{\partial^2}{\partial z^2}\chi = i\frac{f}{A_z}\chi$$

Ekman layer

The solution is:

$$\chi = A\,exp(\frac{z}{\delta_E})\,exp(i\frac{z}{\delta_E}) + B\,exp(-\frac{z}{\delta_E})\,exp(-i\frac{z}{\delta_E})$$

where:

$$\delta_E = \sqrt{\frac{2A_z}{f}}$$

This is the “Ekman depth”

Corrections must decay going up, so:

$$A = 0$$

Ekman layer

Take the real part of the horizontal velocities:

$$u = Re\{\chi\} = Re\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

and

$$v = Im\{\chi\} = -Re\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

Ekman layer

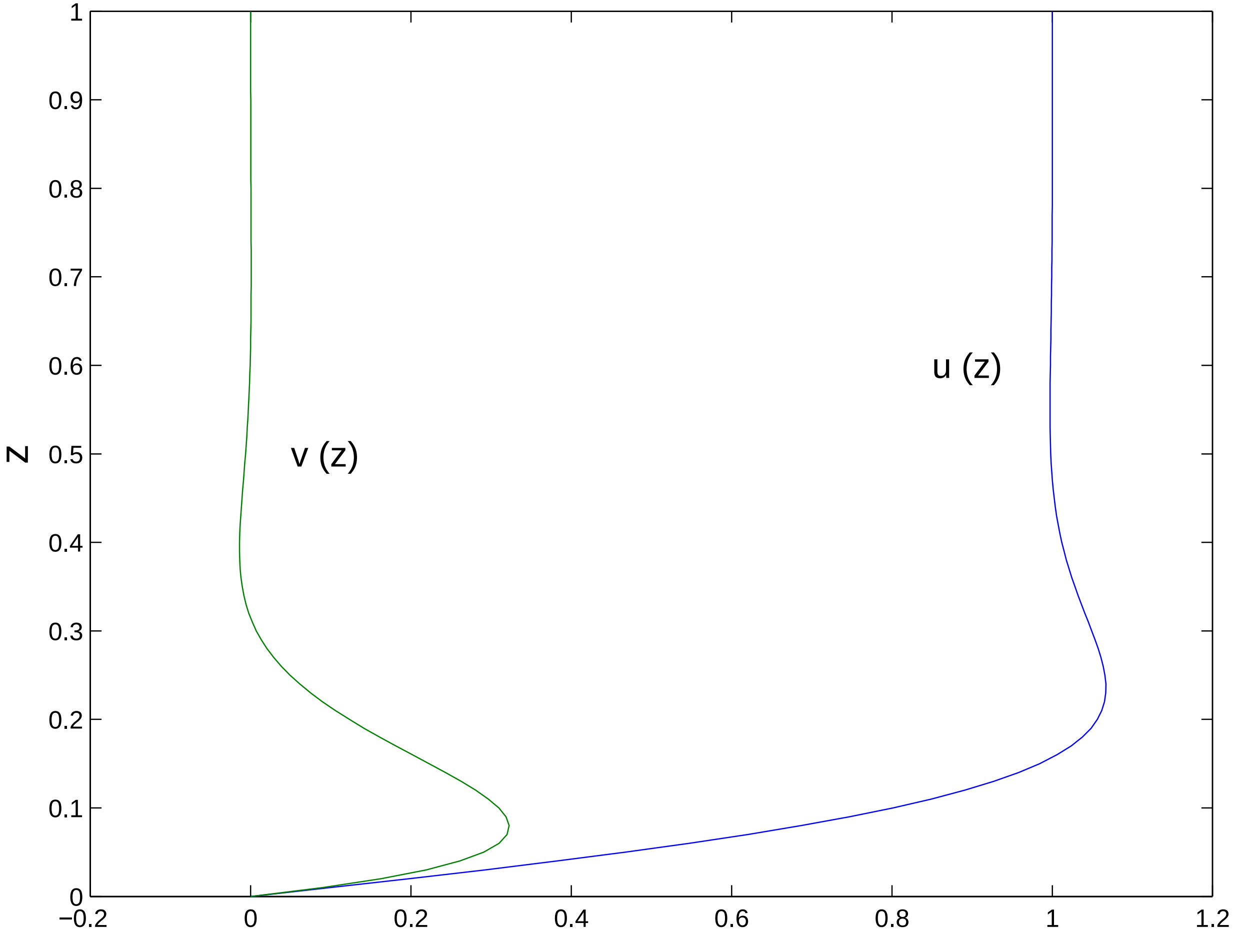
For zero flow at $z = 0$, require $Re\{B\} = -U$ and $Im\{B\} = 0$.

So:

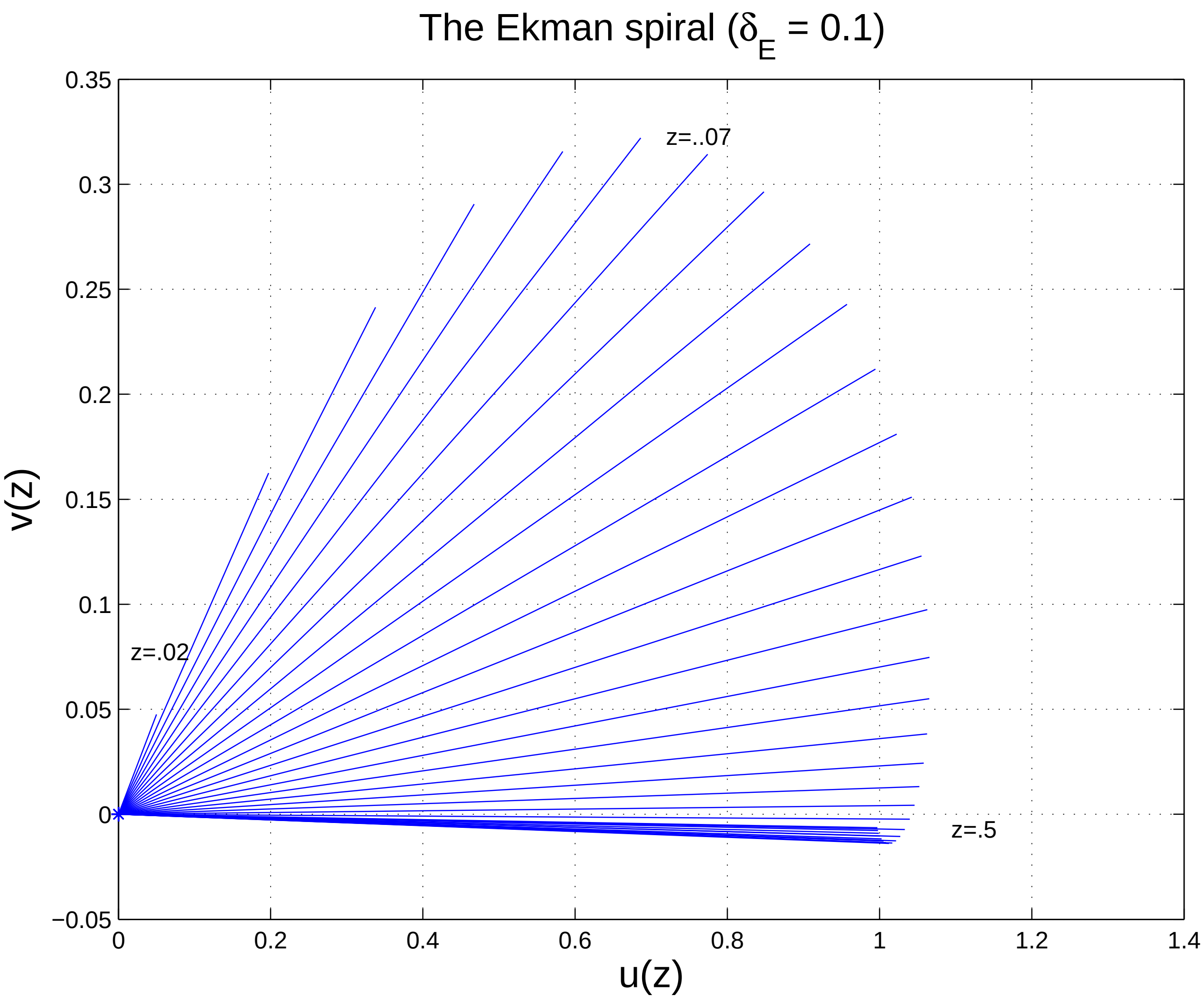
$$u = U + \hat{u} = U - U\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

$$v = \hat{v} = U\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

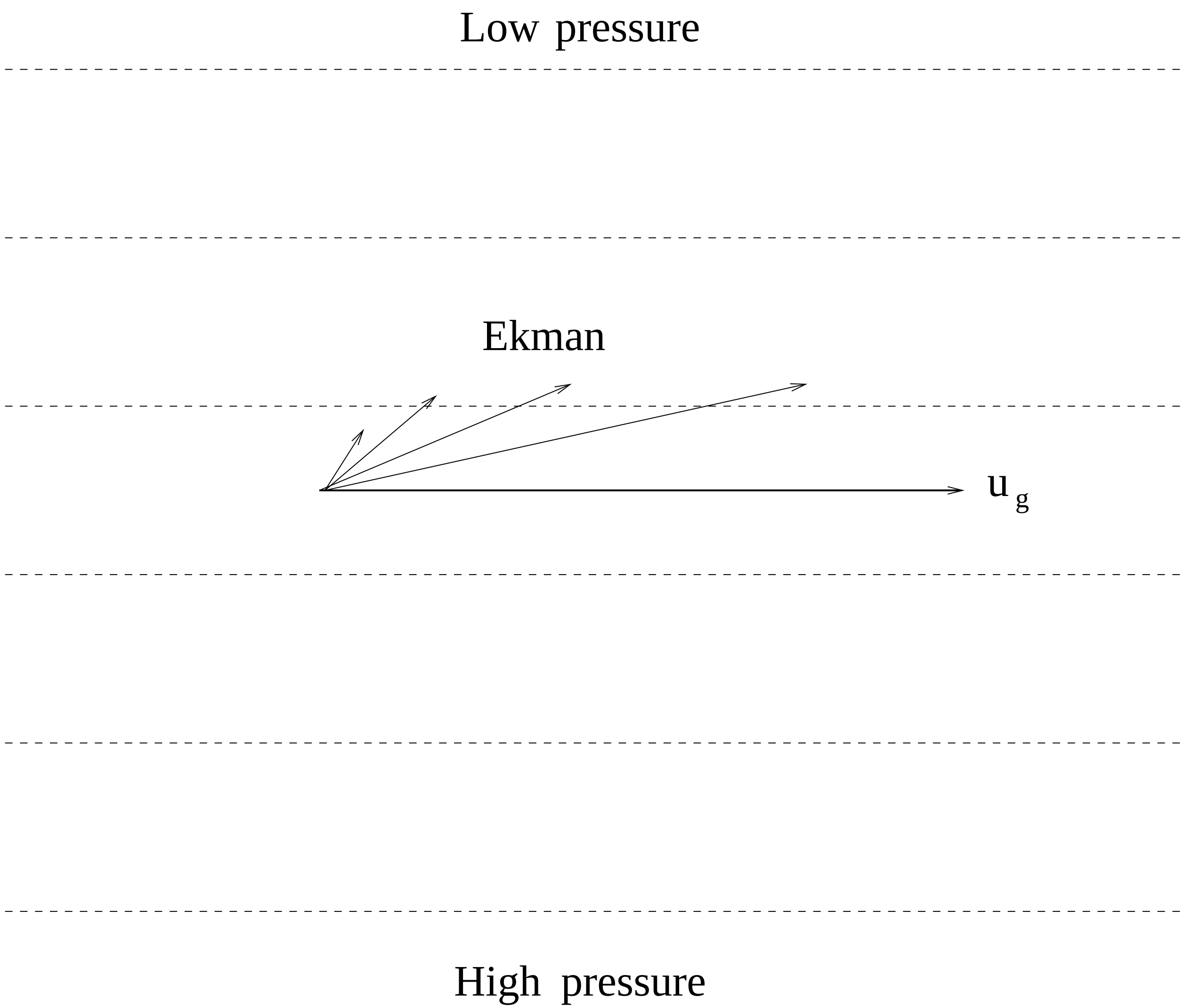
Ekman layer, $\delta_E = 0.1$



Ekman spiral, $\delta_E = 0.1$



Ekman velocities



Ekman spiral

The velocity veers to the *left*, toward low pressure

Observations suggest $u \rightarrow u_g$ at $z = 1$ km.

If $f = 10^{-4}/sec$, then $A_z \approx 5\,m^2/sec$

Typically $\frac{\partial}{\partial z}\mathcal{V}| \approx 5 \times 10^{-3}\ sec^{-1}$

So the mixing length is $l \approx 30$ m.

As in the convective boundary layer, turbulence allows flow from high pressure to low pressure.

Surface layer

Ekman layer cannot hold near surface: can't have 30 m eddies 10 m from surface. Introduce a *surface layer* where:

$$l' = kz$$

Then:

$$A_z = k^2 \, z^2 \, \frac{\partial}{\partial z} \mathcal{V}$$

So:

$$A_z \frac{\partial}{\partial z} u = k^2 z^2 |\frac{\partial}{\partial z} V| \frac{\partial}{\partial z} u \approx k^2 z^2 (\frac{\partial}{\partial z} u)^2$$

Surface layer

Measurements suggest the turbulent momentum flux is approximately constant in the surface layer:

$$\overline{u'w'} \approx u_*^2$$

where u_* is the "friction velocity". So:

$$\frac{\partial}{\partial z} u = \frac{u_*}{kz} \;\; \rightarrow \;\; u = \frac{u_*}{k} ln(\frac{z}{z_0})$$

Here:

- $k \approx 0.4$ is von Karman's constant
- z_0 is the "roughness length"

Surface layer

Match the velocity at the top of the surface layer to that at the base of the Ekman layer.

Comparisons with observations are only fair (see Fig. 5.5 of Holton)

Ekman spiral is often unstable, generating eddies that mix away the signal

Spin-down

Turbulence in both stable and convective boundary layers generates flow down the pressure gradient

Thus both should *weaken* pressure systems

Consider how an Ekman layer causes a cyclone to decay in time

Central to this is that convergence in the Ekman layer causes a vertical velocity at the top of the layer, which affects the overlying flow

Spin-down

Illustrate using the barotropic vorticity equation:

$$\frac{D}{Dt}(\zeta + f) \approx f\,\frac{\partial w}{\partial z}$$

Integrate from the top of boundary layer ($z = d$) to the tropopause:

$$(H - d)\frac{D}{Dt}(\zeta + f) = f(w(H) - w(d)) = -fw(d)$$

Spin-down

Because the boundary layer is much thinner than the troposphere, this is approximately:

$$\frac{D}{Dt}(\zeta + f) = -\frac{f}{H} w(d)$$

So vertical velocity into/out of the boundary layer changes the vorticity in the troposphere

Ekman pumping

Ekman layer. The continuity equation is:

$$\frac{\partial}{\partial z}w = -\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v$$

Integrating over the layer, we get:

$$w(d) - 0 = -\int_0^d (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\, dz \equiv -\frac{\partial}{\partial x}M_x - \frac{\partial}{\partial y}M_y$$

where M_x and M_y are the horizontal *transports*

Spin-down

Can show:

$$M_y \approx \frac{Ud}{2}$$

and:

$$M_x \approx -\frac{Vd}{2}$$

So:

$$w(d) = \frac{d}{2}\,(\frac{\partial}{\partial x}V - \frac{\partial}{\partial y}U) = \frac{d}{2}\,\zeta$$

Spin-down

Thus:

$$\frac{D}{Dt}(\zeta + f) = -\frac{fd}{2H}\zeta$$

If assume $f = const.$, then:

$$\frac{D}{Dt}\zeta = -\frac{fd}{2H}\zeta$$

So that:

$$\zeta(t) = \zeta(0)\,exp(-t/\tau_E)$$

Spin-down

where:

$$\tau_E \equiv \frac{2H}{fd}$$

is the Ekman *spin-down time*. Typical values:

$$H = 10km, \;\; f = 10^{-4} sec^{-1}, \;\; d = 0.5km$$

yield:

$$\tau_E \approx \; 5 \; days$$

Spin-down

Compare to molecular dissipation. Then:

$$\frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial z^2} u$$

where $\nu = 10^{-5}\ m^2/sec$. From scaling:

$$\frac{U}{T} \approx \frac{\nu U}{L^2} \quad \rightarrow \quad T = \frac{L^2}{\nu}$$

with $L = 10^6\ m$:

$$T \approx 10^{17}\, sec \approx 3 \times 10^9\, yr\ !$$

Spin-down

The vertical velocity is part of the *secondary circulation*

The primary flow is horizontal, (u_g, v_g)

The vertical velocities, though smaller, are extremely important nevertheless

Stratification reduces the effective H. So the geostrophic velocity over Ekman layer spins down more rapidly, leaving winds aloft alone.

Model Spin-up

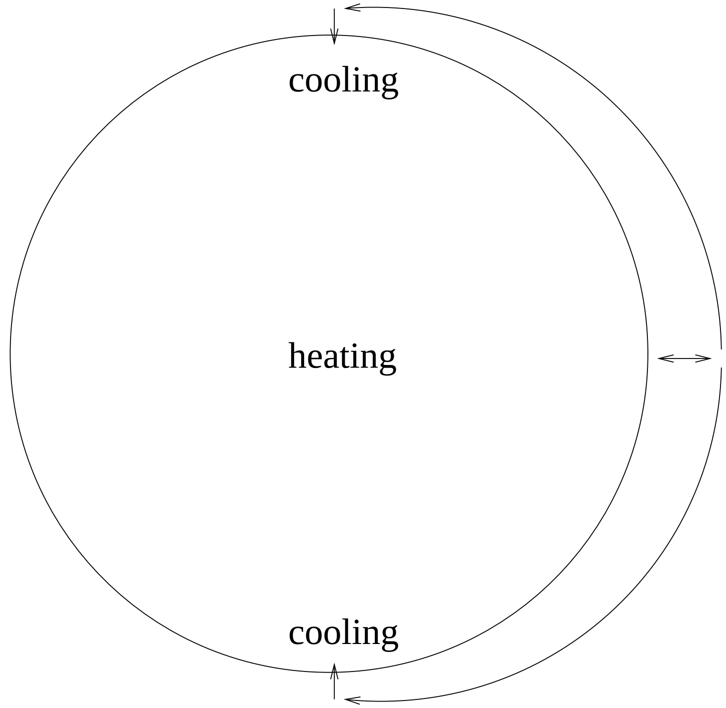
Consider an atmospheric model

Atmosphere initially *at rest*

"Turn on" solar heating

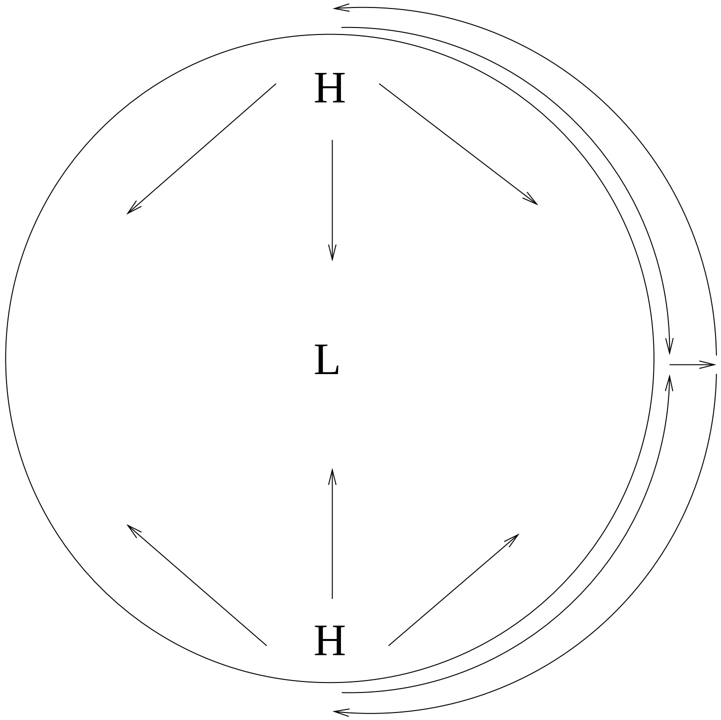
See what happens...

Spin-up



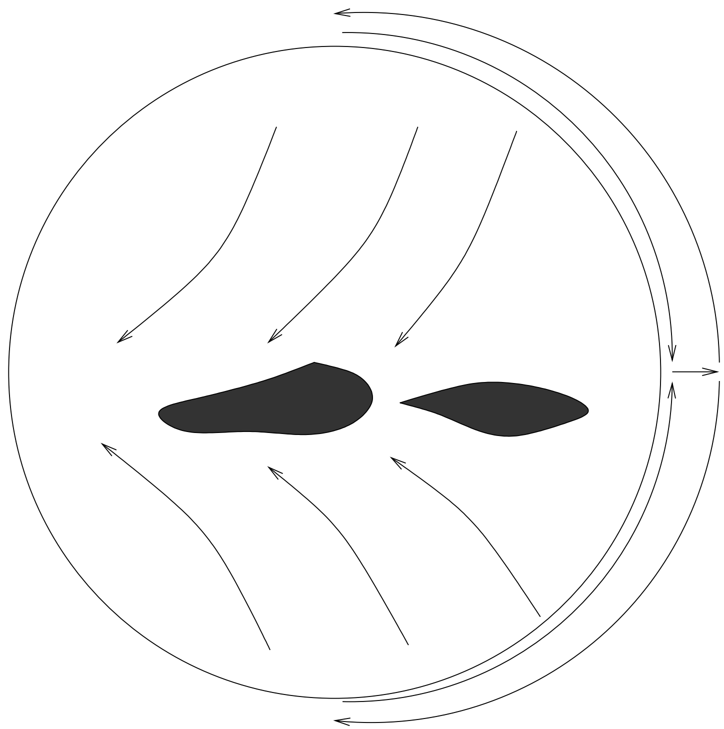
Rising motion at equator
Poleward motion aloft, equator motion near ground

Spin-up



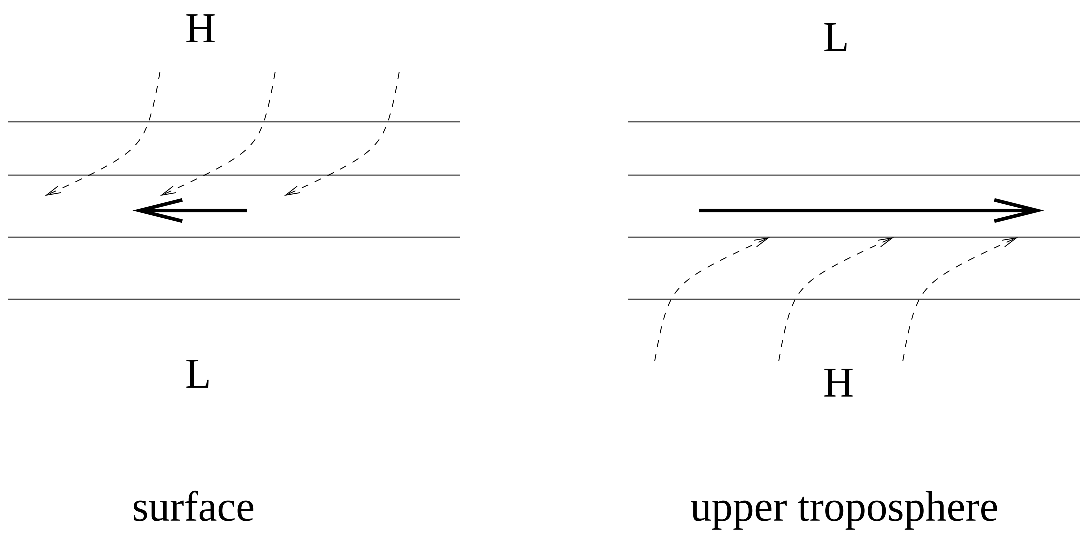
Initially high/low pressure at high/low latitudes

Spin-up

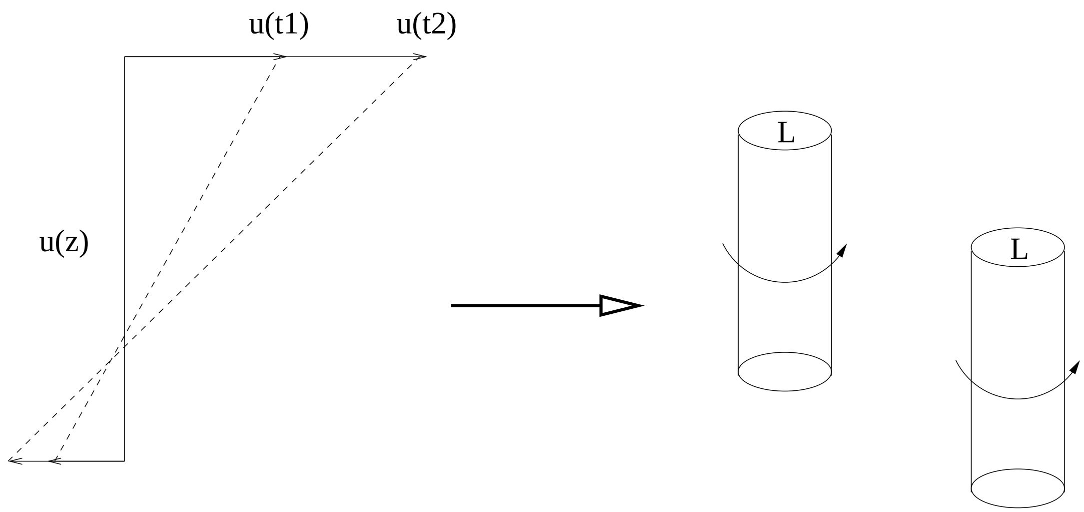


Coriolis deflects the equatorward air, westwards
Clouds formed in rising air

Spin-up

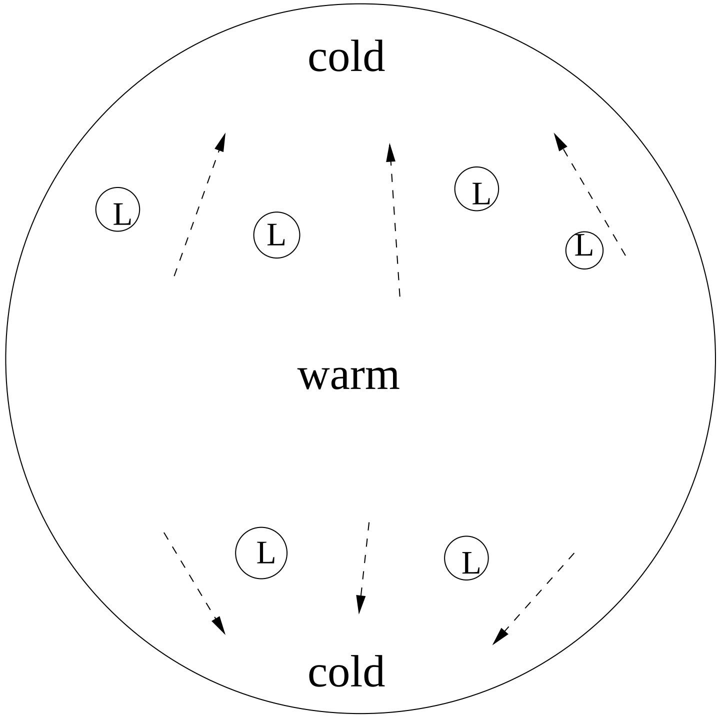


Spin-up



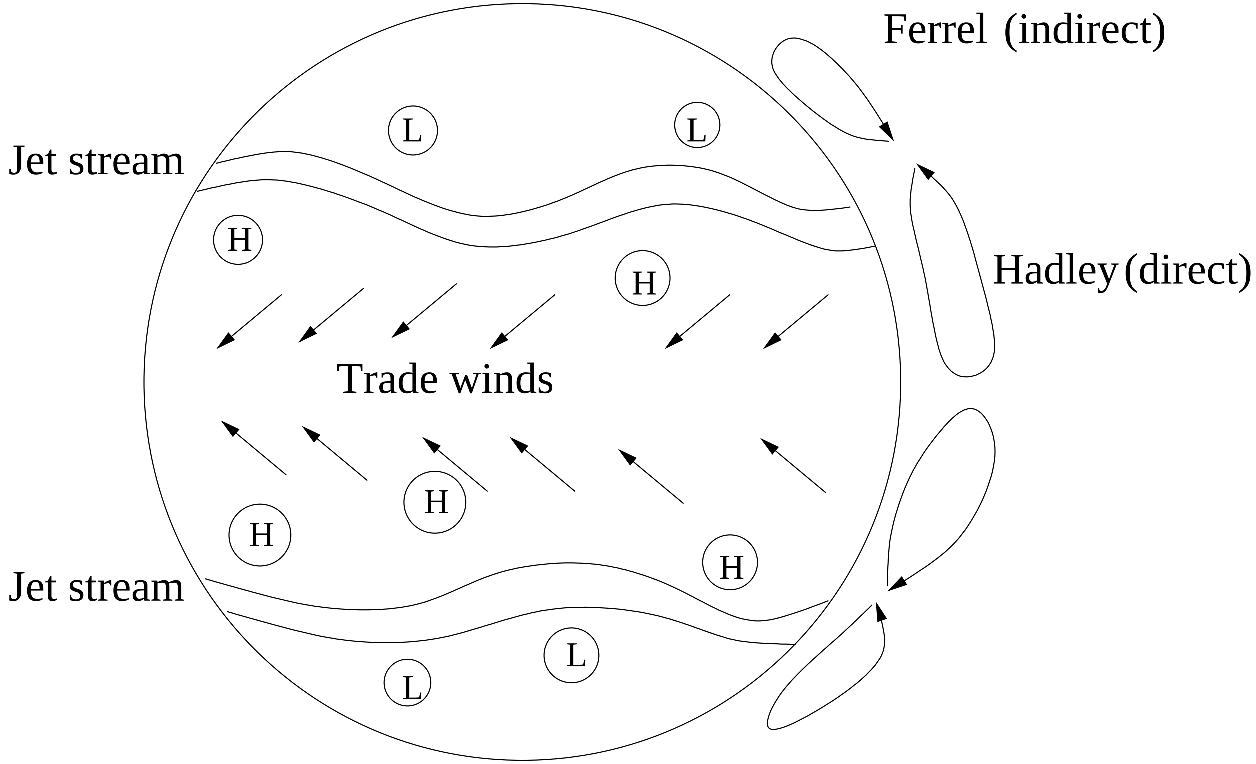
Vertical shear increases with temperature gradient
Flow becomes unstable, generating storms

Spin-up

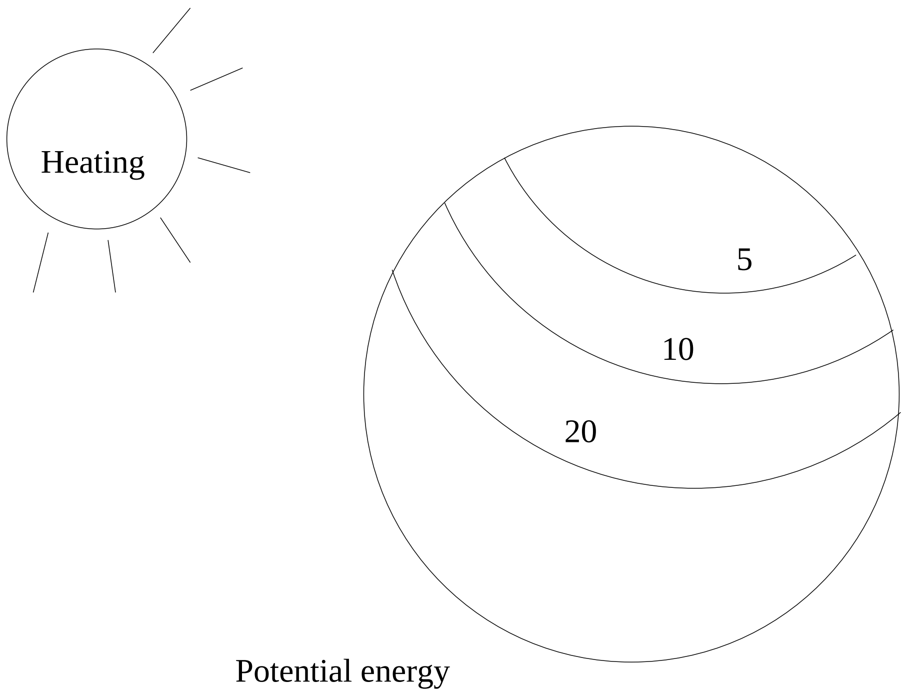


Storms transport heat toward high latitudes
Reduces the temperature gradient

General circulation

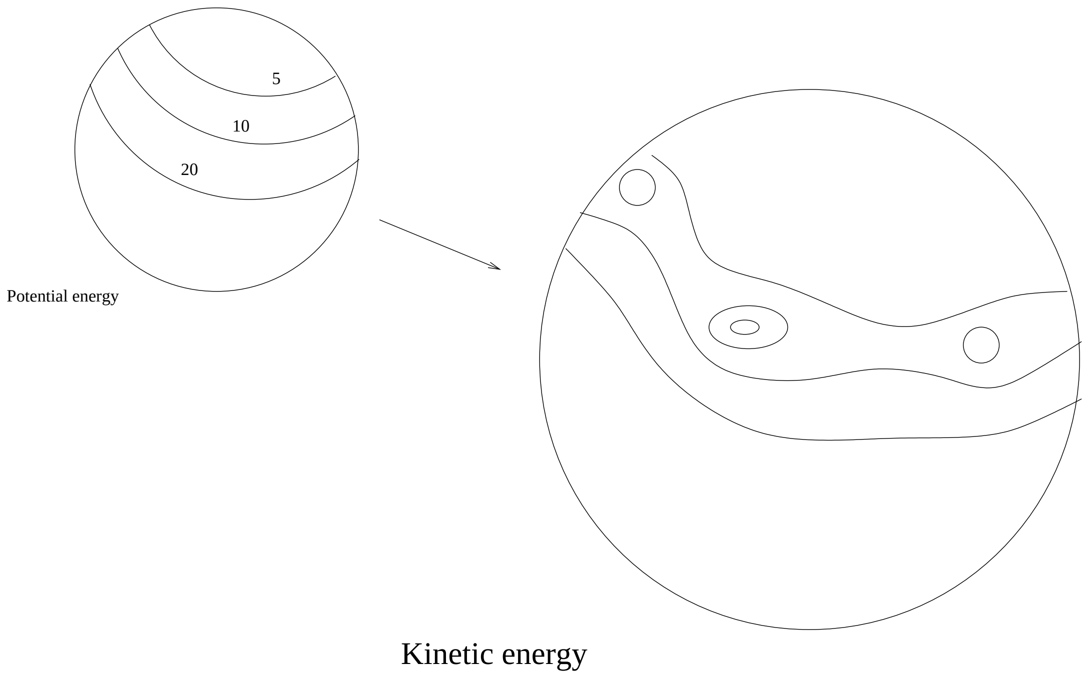


Energy cycle



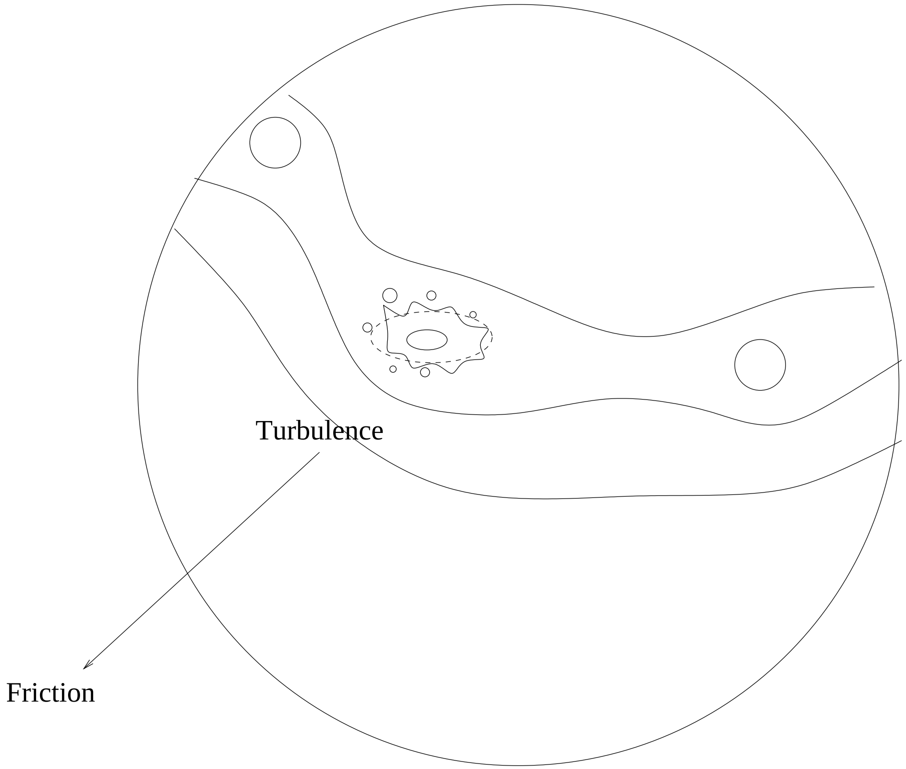
Solar heating produces the temperature gradient
The result is potential energy

Energy cycle



Instability converts potential to *kinetic* energy

Energy cycle



Energy is ultimately dissipated at small scales, via *turbulence*