GEF 2220: Dynamics

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Course

Part 1: Dynamics: *LaCasce*

Chapter 7, Wallace and Hobbs + my notes

Part 2: Weather systems: *Røsting*

Chapter 8, Wallace and Hobbs + extra articles + DIANA

Dynamics

1) Derive the equations which describe atmospheric motion

2) Derive approximate balances

3) Understand pressure systems, temperature gradients

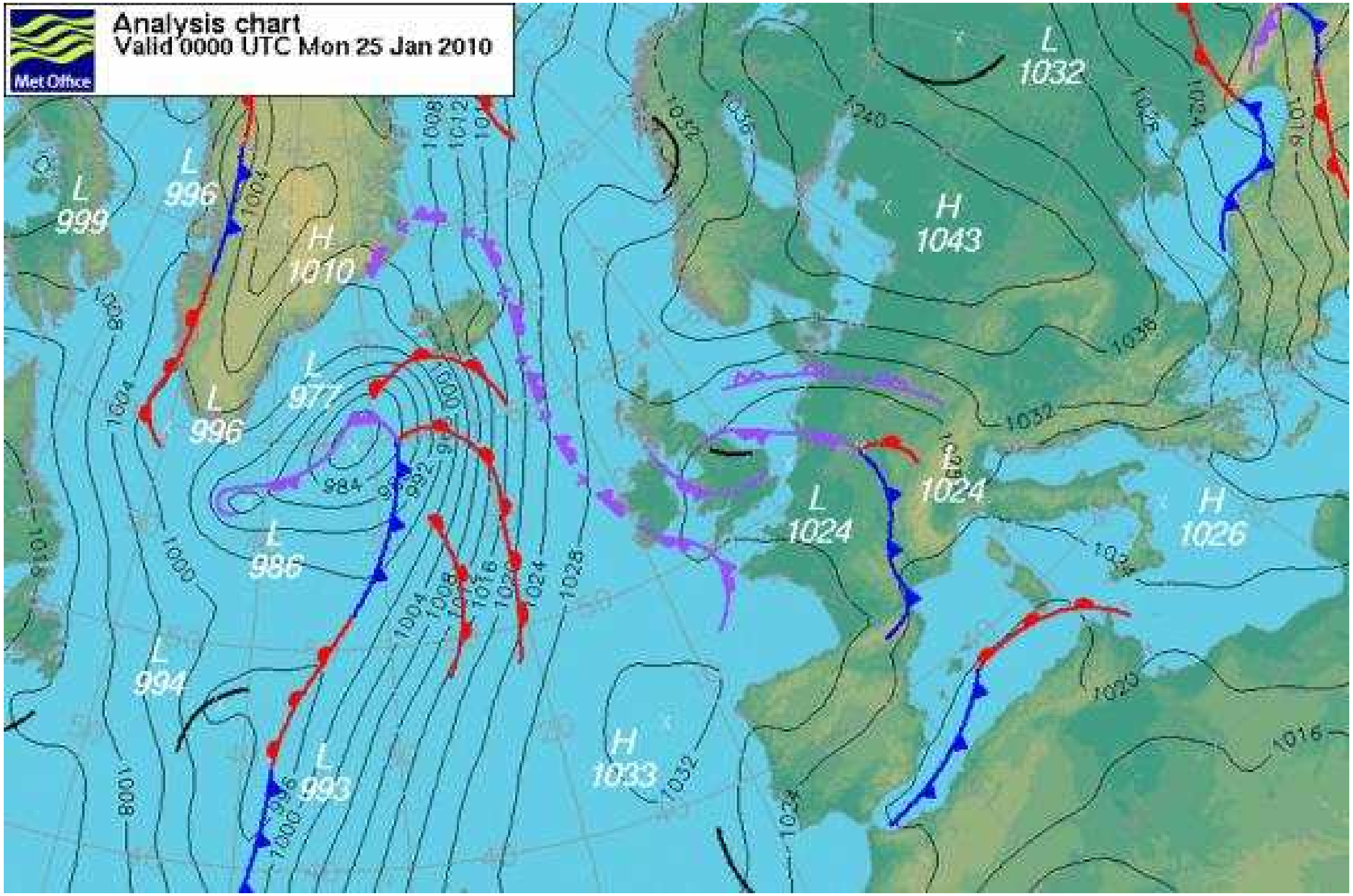
4) Introduce the general circulation

Variables

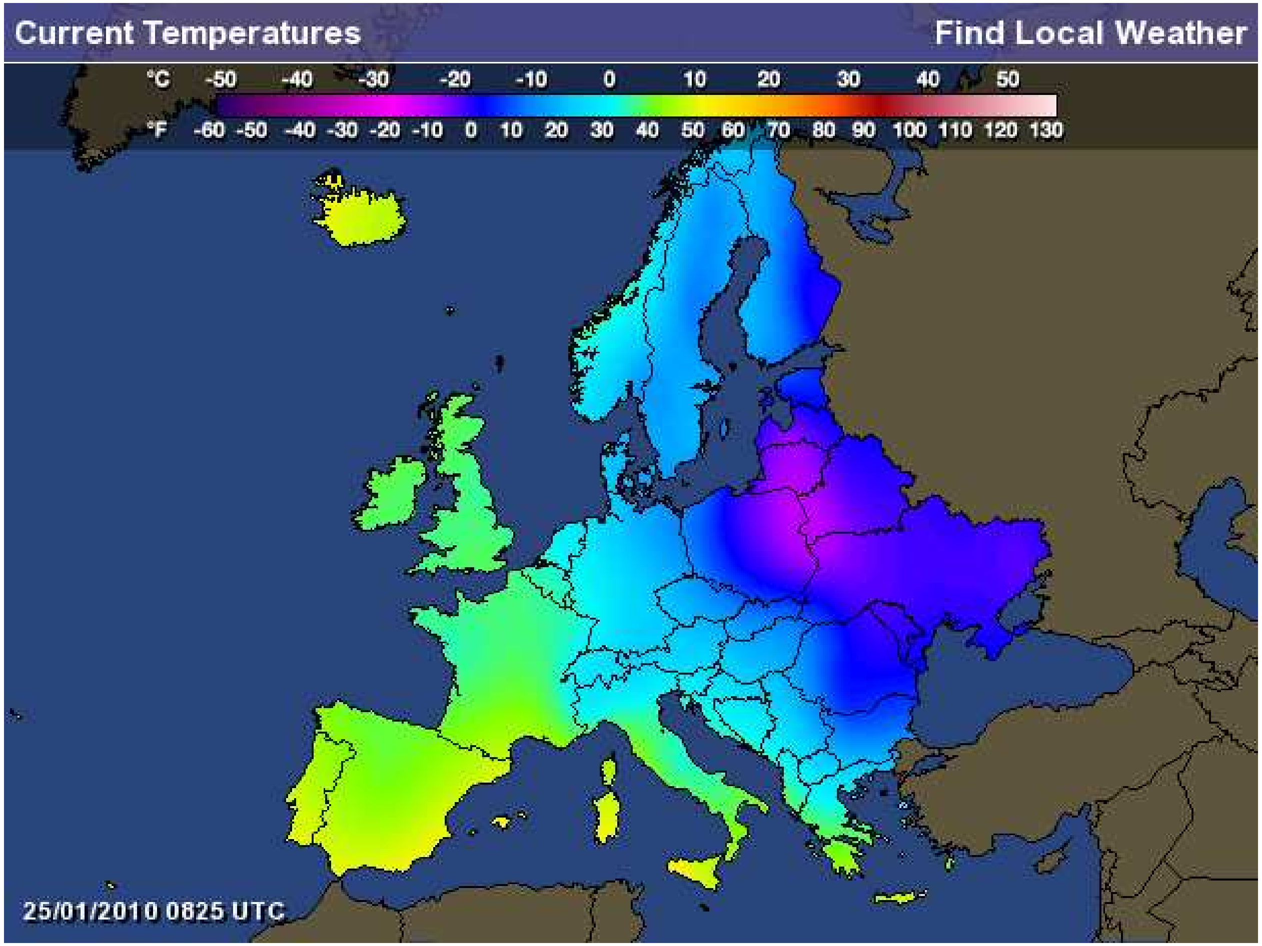
Six unknowns:

- (u, v, w) — Wind velocities
- p — Pressure
- T — Temperature
- ρ — Density

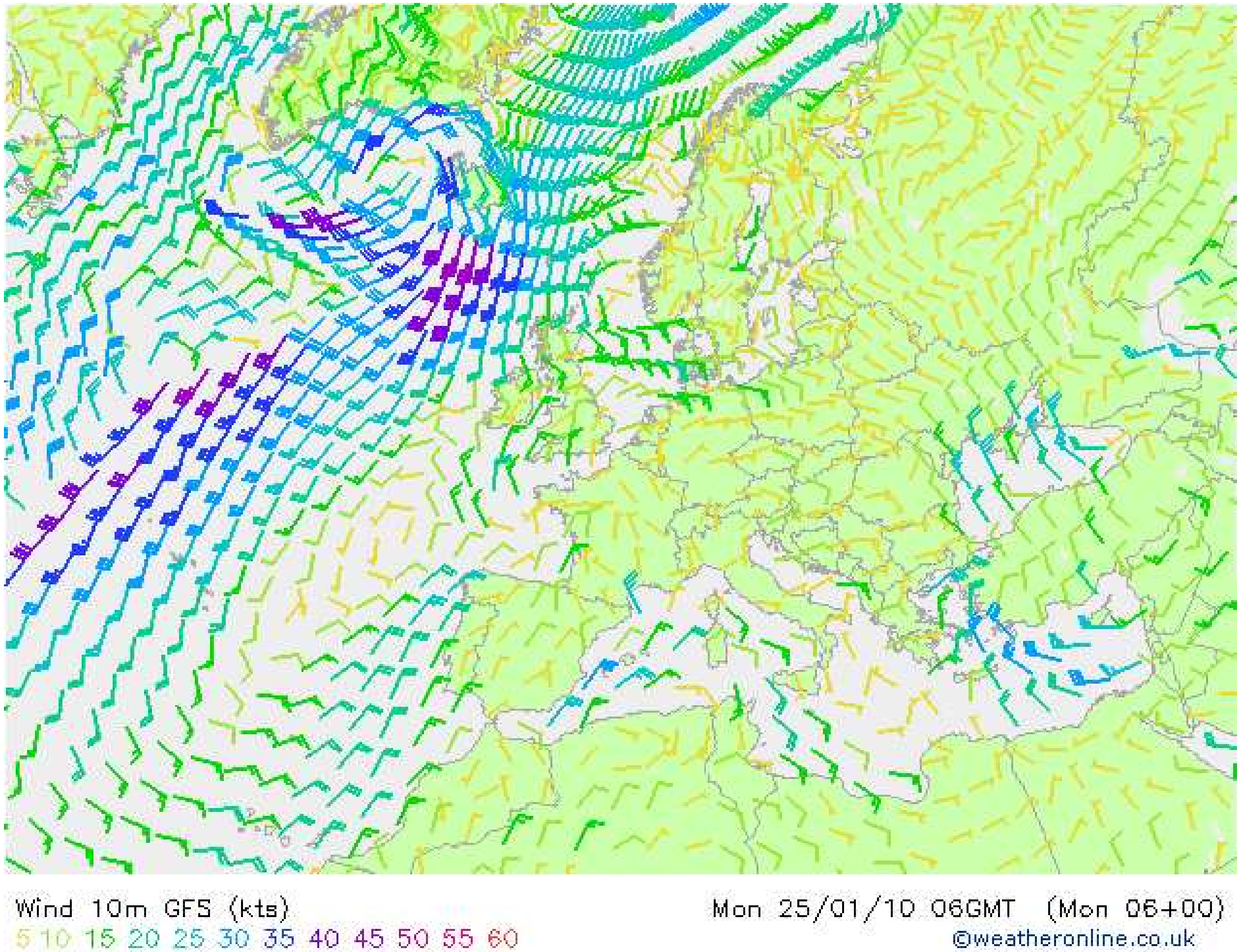
Pressure



Temperature



Winds


Wind 10m GFS (kts)
5 10 15 20 25 30 35 40 45 50 55 60
Mon 25/01/10 06GMT (Mon 06+00)
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Primitive equations

Momentum equations $\leftarrow$ $F = ma$

Thermodynamic energy equation $\leftrightarrow$ T

Continuity $\leftrightarrow$ ρ

Ideal gas law

Primitive equations

Momentum:

$$\frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \frac{dq}{dt}$$

Prediction

Solve the equations numerically with weather models

Issues:

- Numerical resolution
- Vertical coordinate
- Small scale mixing
- Convection
- Clouds

Goal: forecasting

Dynamics

Solve a simplified set of equations

- Identify dominant balances
- Simplify the equations
- Obtain solutions (analytical, numerical)
- Look for similiarities with observations

Goal: *understanding* the atmosphere

Derivatives

Consider an air parcel, with temperature $T = T(x, y, z, t)$

The change in temperature, from the chain rule:

$$dT = \frac{\partial T}{\partial t}\,dt + \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz$$

So:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T$$

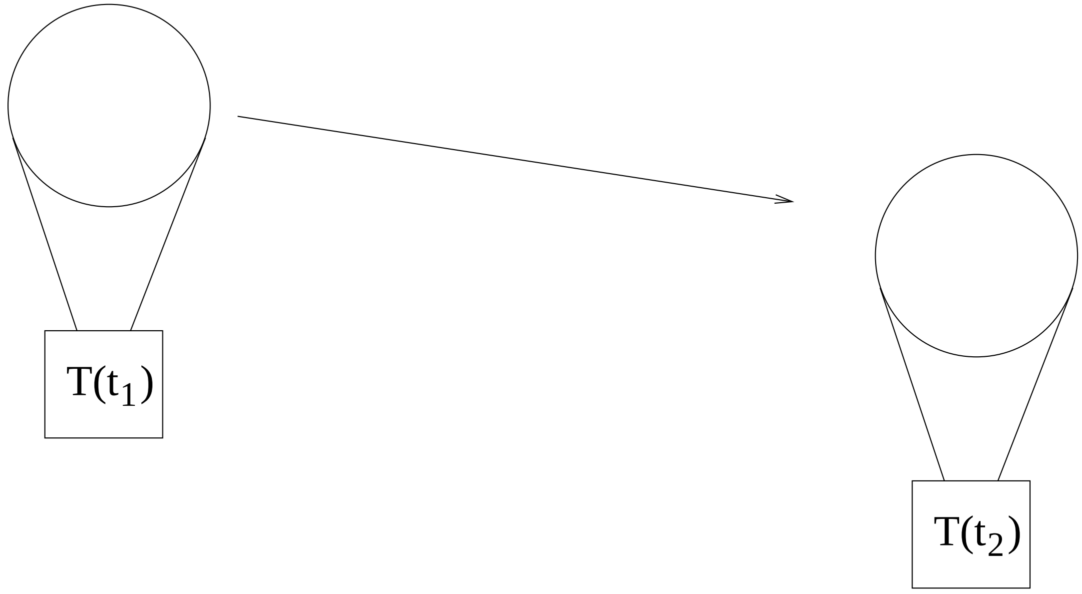
Derivatives

(u, v, w) are the wind velocities in the (x, y, z) directions

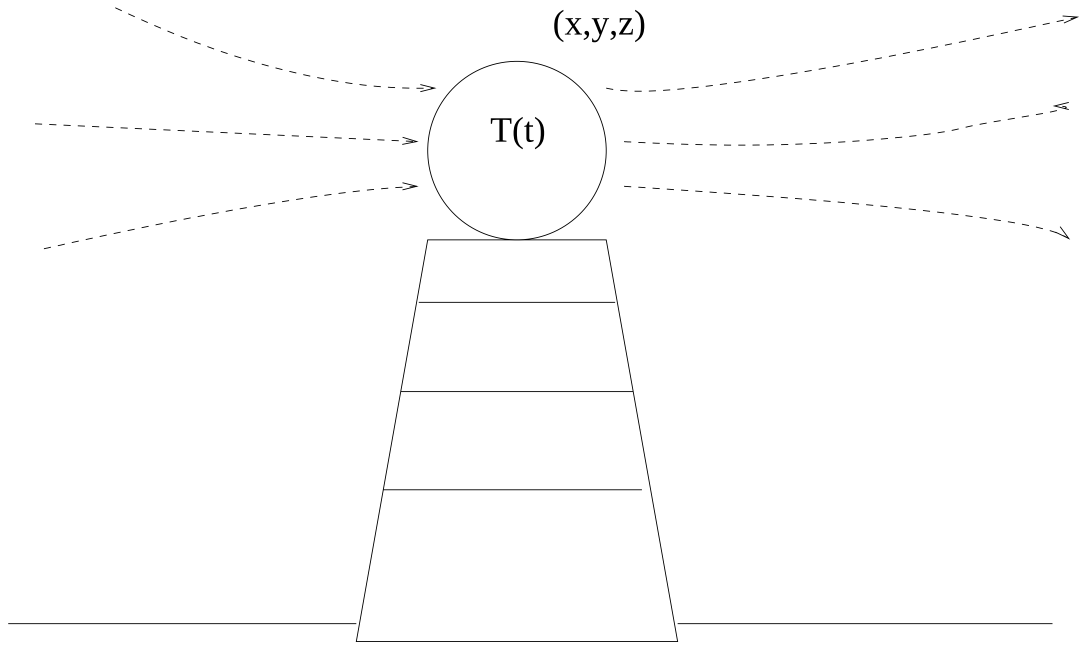
$\frac{d}{dt}$ is the "Lagrangian" derivative

$\frac{\partial}{\partial t} + \vec{u} \cdot \nabla$ is the "Eulerian" derivative

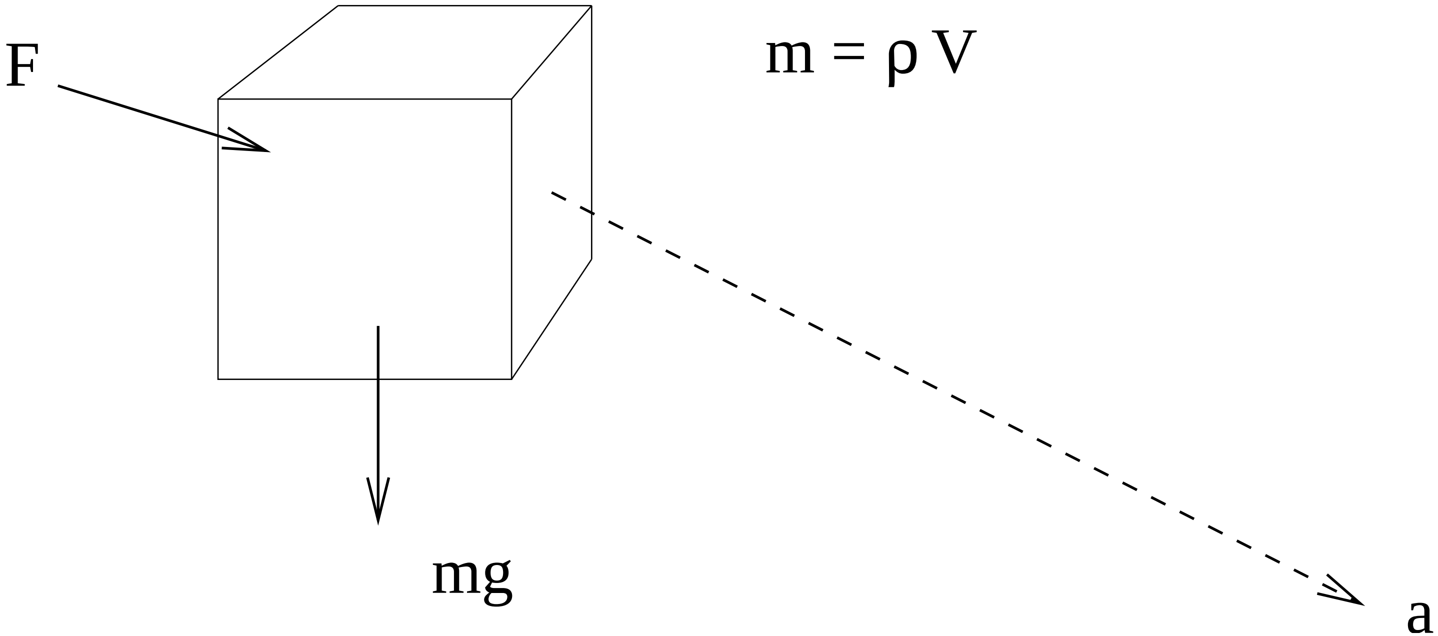
Lagrangian



Eulerian



Momentum equations



Momentum equations

The acceleration in the x-direction is:

$$a_x = \frac{1}{m} \sum_i F_i$$

Two types of force:

- Real
- Apparent

Forces

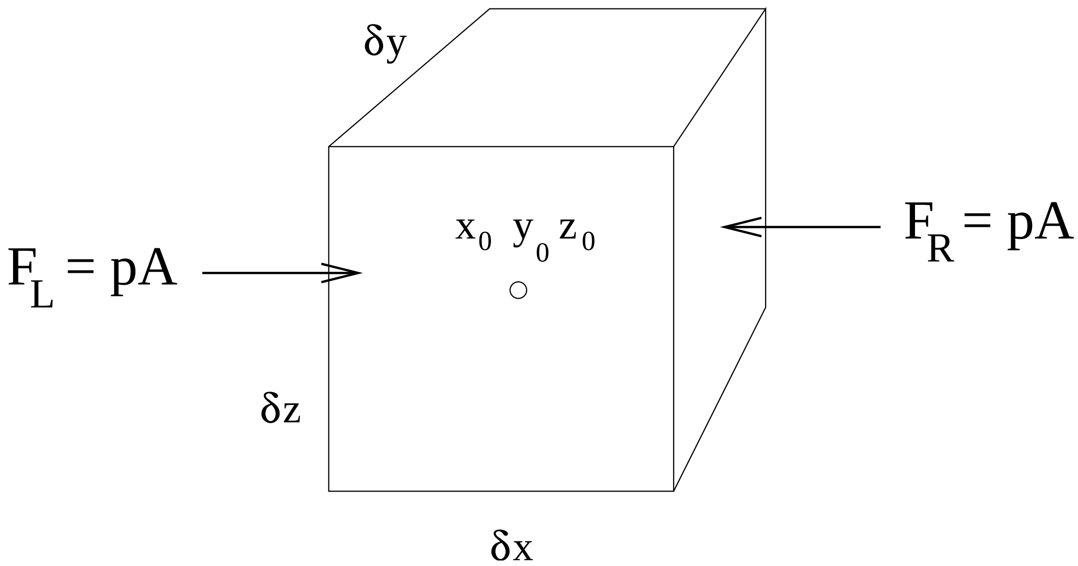
Real forces

- Pressure gradient
- Gravity
- Friction

Apparent forces

- Coriolis
- Centrifugal

Pressure gradient



$$\delta V = \delta x \ \delta y \ \delta z$$

Pressure gradient

Using a *Taylor series*, we can write the pressure on the right side of the box:

$$p_R = p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2} + ...$$

Similarly, the pressure on left side of the box is:

$$p_L = p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2} + ...$$

Pressure gradient

The force on the right hand side (directed inwards):

$$F_R = -p_R A = [p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

On left side:

$$F_L = p_L A = [p(x_0, y_0, z_0) - \frac{\partial p}{\partial x}\frac{\delta x}{2}]\,\delta y \delta z$$

So the net force is:

$$F_x = F_L + F_R = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

Pressure gradient

The volume weighs:

$$m = \rho \, \delta x \, \delta y \, \delta z$$

So:

$$a_x \equiv \frac{du}{dt} = \frac{F_x}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

Same derivation for the y and z directions.

Note this is a *Lagrangian* derivative

Momentum equations

Momentum with pressure gradients:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p$$

Gravity

Acts downward (toward the center of the earth):

So only affects the vertical acceleration

$$a_z = \frac{F_z}{m} = -g$$

Momentum equations

Momentum with pressure gradients and gravity:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

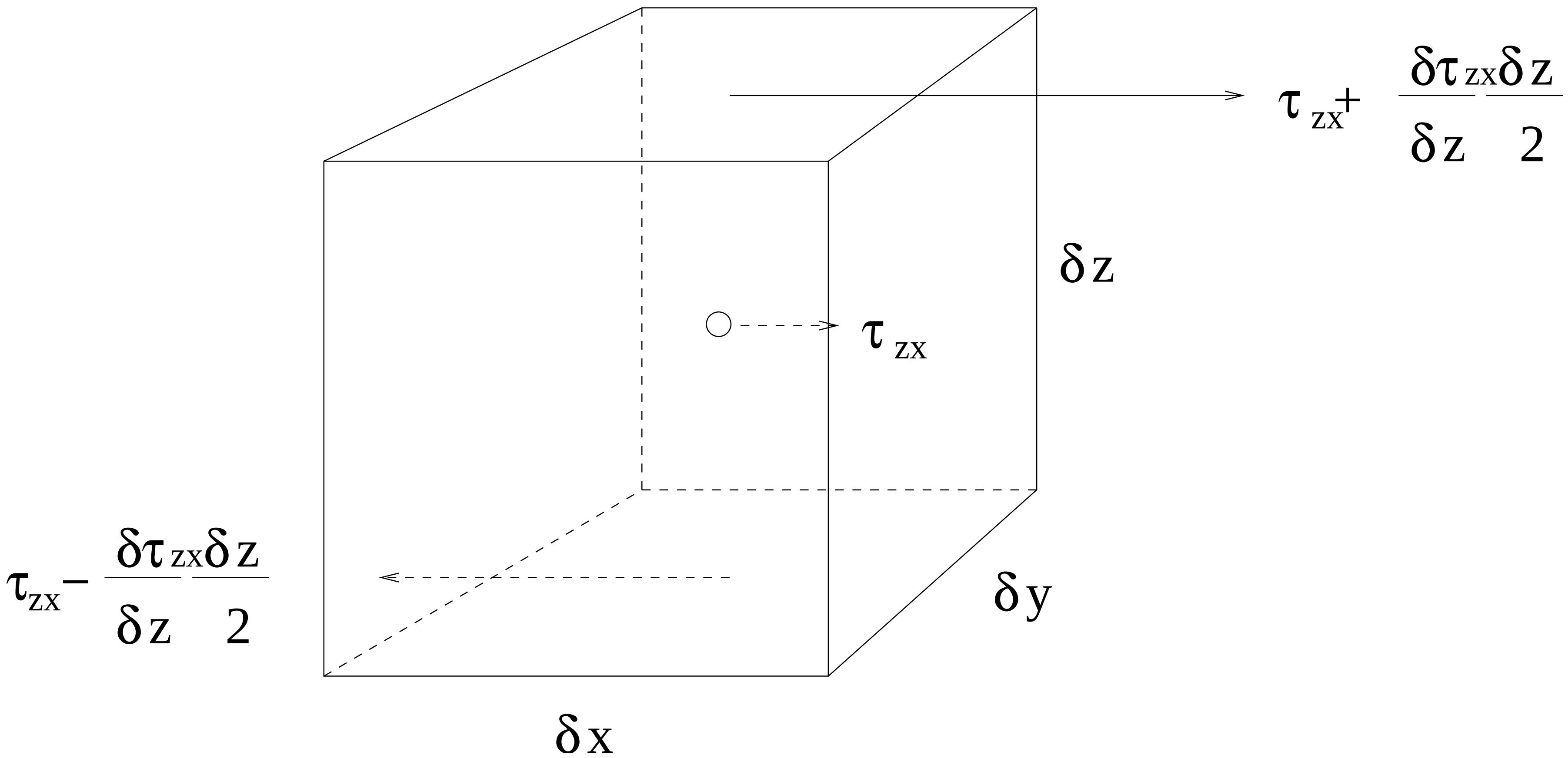
Friction

Frictional stress represented by a 3x3 matrix, $\vec{\vec{\tau}}$

$$\vec{\vec{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

So τ_{zx} is the stress which acts in the x direction and varies with height

Friction



Friction

A stress gradient causes an accerleration:

$$\frac{du}{dt} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$$

We don't know the stress. So we *parametrize* it in terms of the velocity:

$$\frac{1}{\rho}\tau_{zx} = \nu \frac{\partial}{\partial z} u$$

where ν is a molecular mixing coefficient. So:

$$\frac{du}{dt} = \nu \frac{\partial^2}{\partial z^2} u$$

Friction

In 3 dimensions:

$$\frac{du}{dt} = \nu\,(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u = \nu\,\nabla^2 u$$

- Friction *diffuses momentum*, reducing velocity gradients

But ν is typically small ($\nu \approx 10^{-5}\ m^2/sec$), so friction is generally not important for large scale motion

Momenutum equations

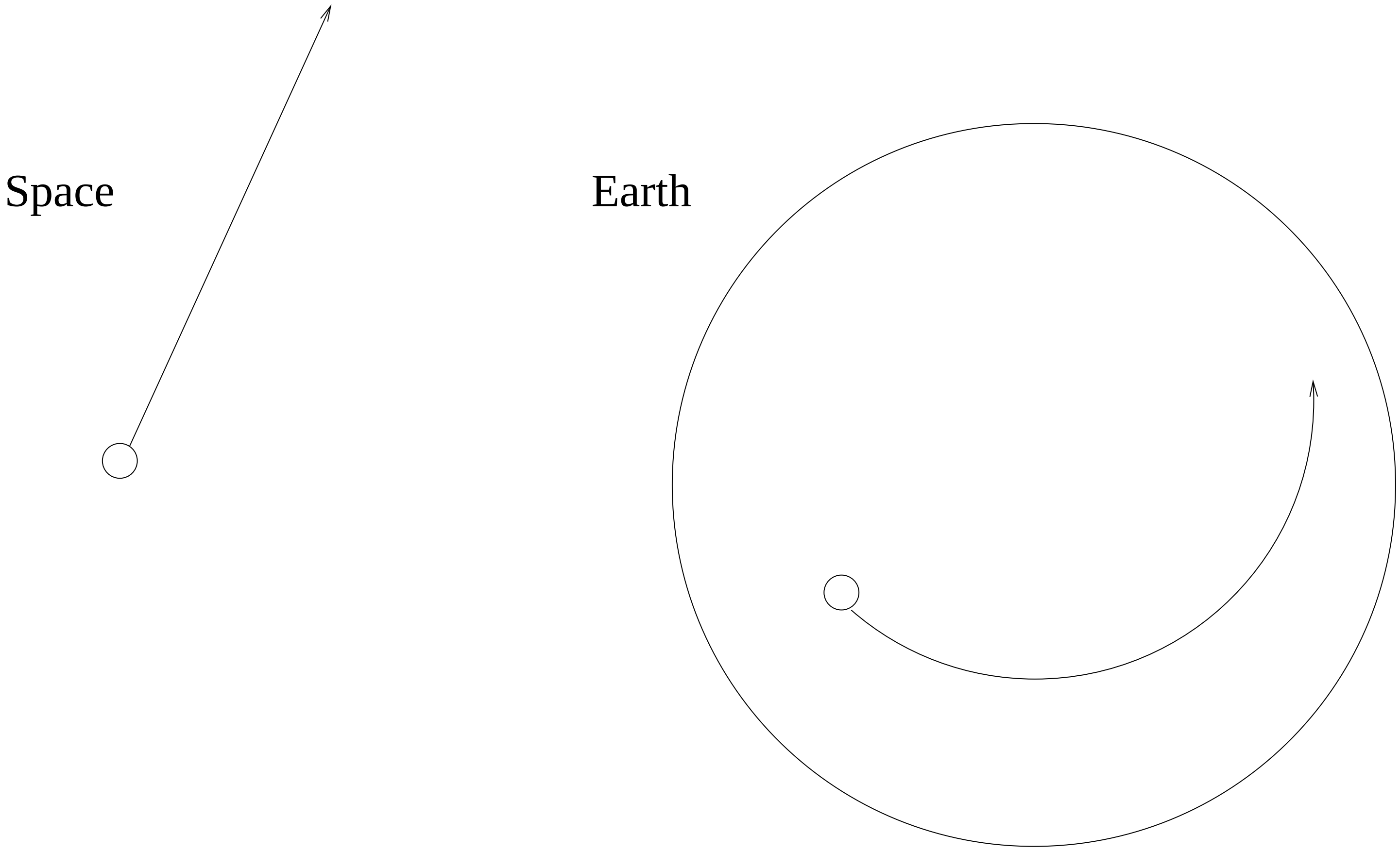
With all the real forces:

$$\frac{du}{dt} = \frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

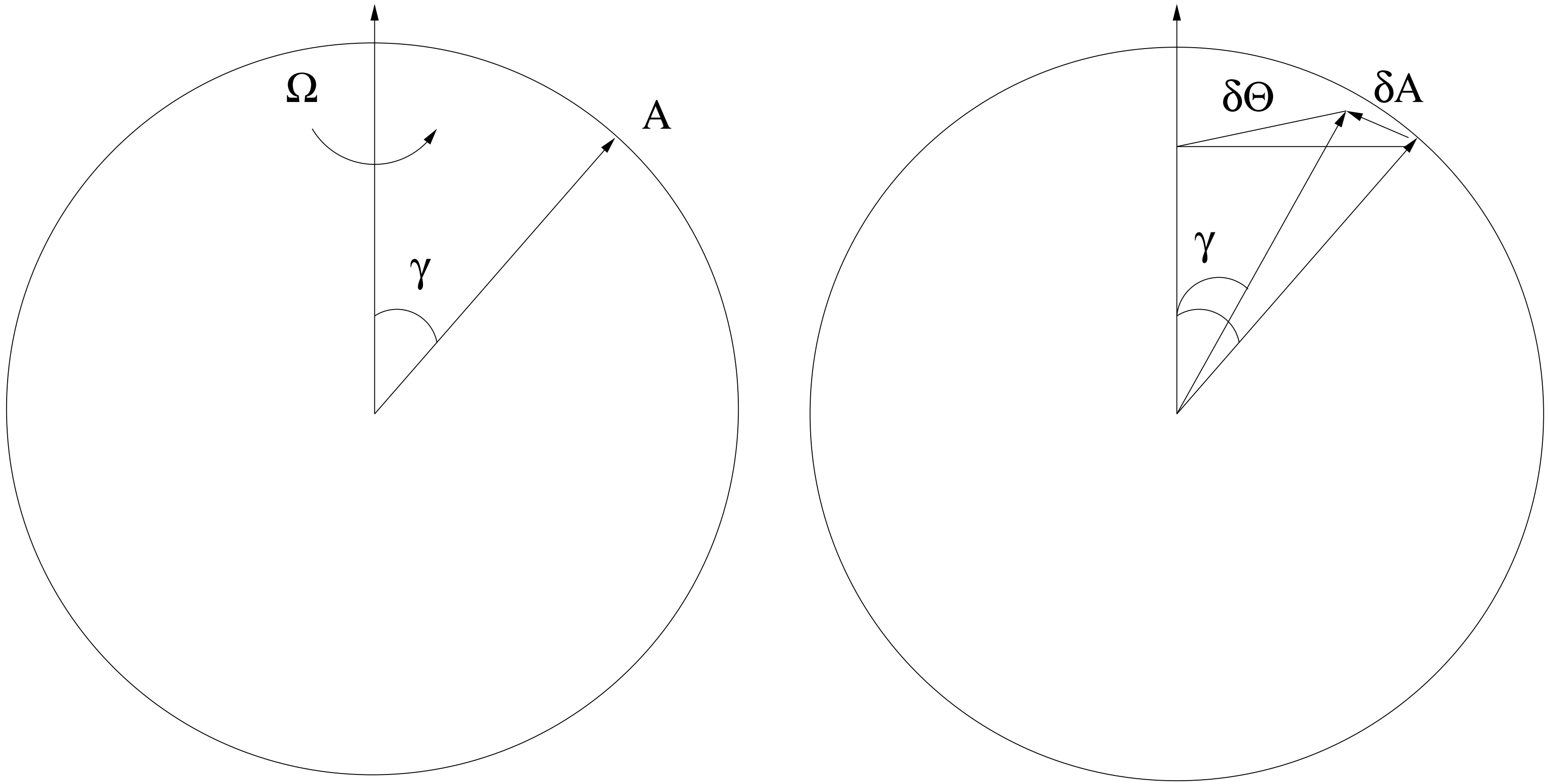
$$\frac{dv}{dt} = \frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{dw}{dt} = \frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Apparent forces



Rotation



Rotation

$$\delta\Theta = \Omega\delta t$$

Assume $\Omega = const.$ (reasonable for the earth)

Change in A is δA, the arc-length:

$$\delta\vec{A} = |\vec{A}|sin(\gamma)\delta\Theta = \Omega|\vec{A}|sin(\gamma)\delta t = (\vec{\Omega} \times \vec{A})\,\delta t$$

Rotation

So:

$$\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A}$$

This is the motion of a *fixed* vector. For a moving vector:

$$(\frac{d\vec{A}}{dt})_F = (\frac{d\vec{A}}{dt})_R + \vec{\Omega} \times \vec{A}$$

So the velocity in the fixed frame is equal to that in the rotating frame plus the rotational movement

Rotation

If $\vec{A} = \vec{r}$, the position vector, then:

$$(\frac{d\vec{r}}{dt})_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

If $\vec{A} = \vec{u}$, we get the acceleration:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_F}{dt})_R + \vec{\Omega} \times \vec{u}_F = [\frac{d}{dt}(u_R + \vec{\Omega} \times \vec{r})]_R$$

$$+\vec{\Omega} \times (\vec{u}_R + \vec{\Omega} \times \vec{r})$$

$$= (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Rotation

Rearranging:

$$\left(\frac{d\vec{u}_R}{dt}\right)_R = \left(\frac{d\vec{u}_F}{dt}\right)_F - 2\vec{\Omega} \times \vec{u}_R - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Two additional terms:

- Coriolis acceleration $\rightarrow \quad -2\vec{\Omega} \times \vec{u}_R$
- Centrifugal acceleration $\rightarrow \quad -\vec{\Omega} \times \vec{\Omega} \times \vec{r}$

Centrifugal acceleration

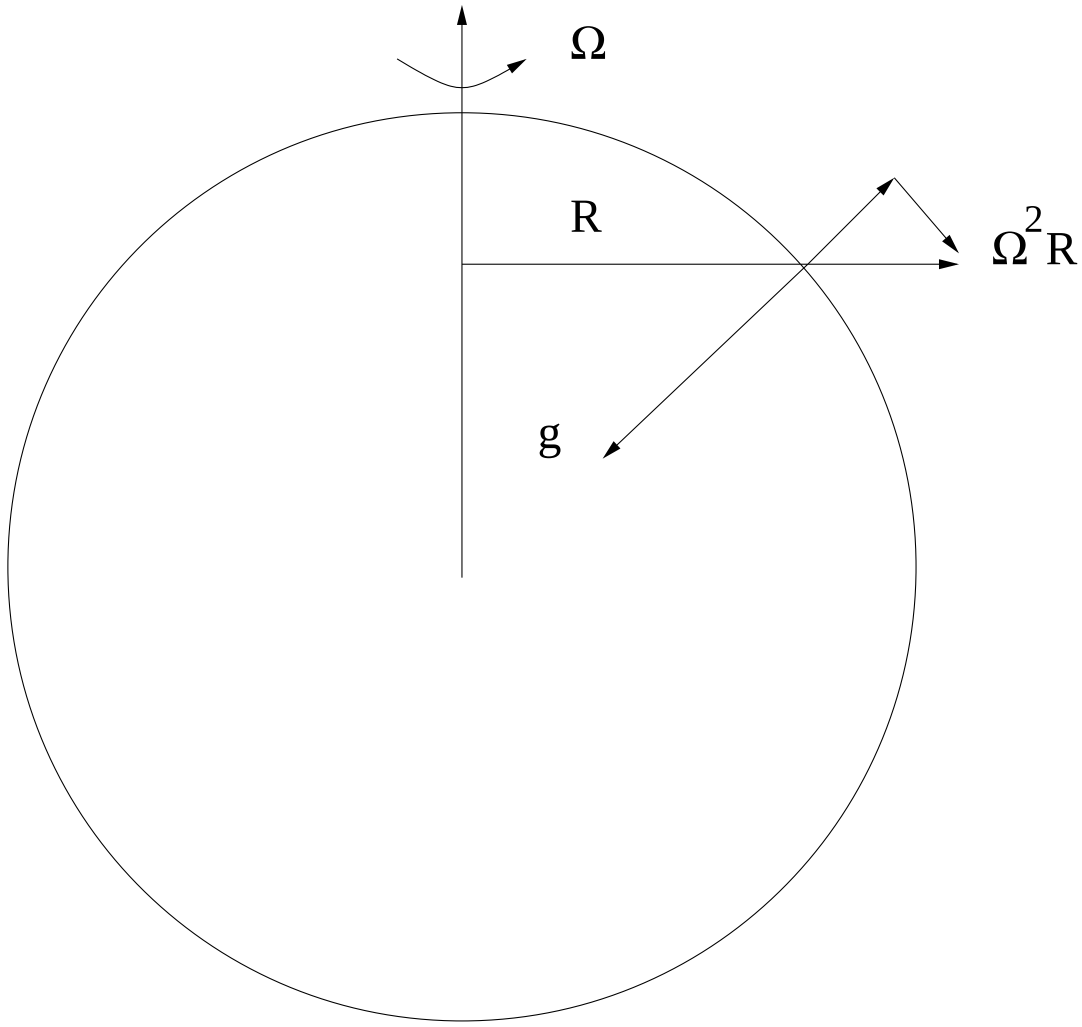
Rotation requires a force towards the center of rotation—the *centripetal acceleration*

From the rotating frame, the sign is opposite—the *centrifugal acceleration*

Acceleration points out from the earth's radius of rotation

So has components in the radial and N-S directions

Centrifugal



Centrifugal

If the earth were perfectly spherical, the N-S component of the force would be *unbalanced*

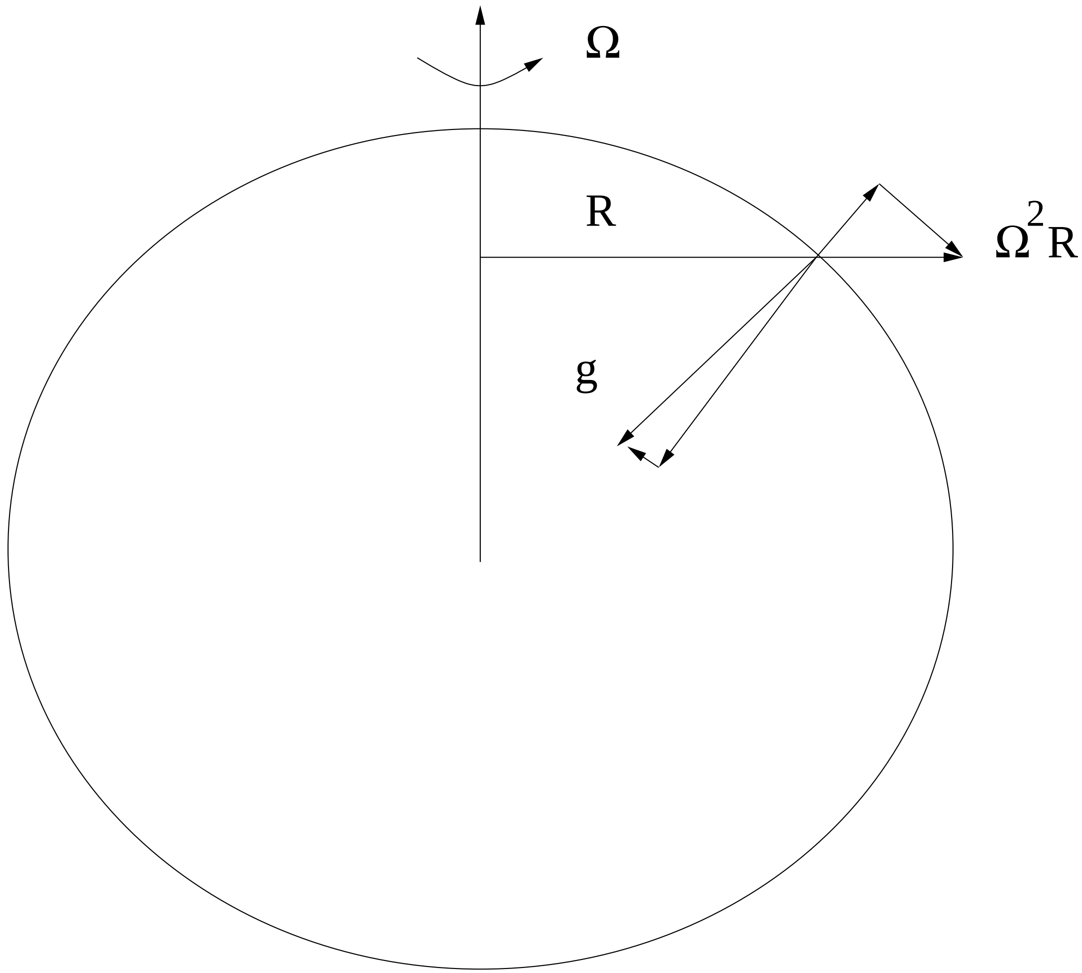
This would cause material (rocks, etc.) to move toward the equator

In fact, this has already happened

The result is that the earth is not spherical, but an *oblate spheroid*

With the redistributed mass, there is a component of gravity which exactly balances the centrifugal force in the N-S direction

Centrifugal



Centrifugal

What's left are the vertical components of gravity and of the centrifugal force

The vertical centrifugal acceleration can be absorbed into gravity:

$$g' = g - (\vec{\Omega} \times \vec{\Omega} \times \vec{r}) \cdot \hat{k}$$

This is still negative (pointed downward)

If not, the atmosphere would fly off the spinning earth!

Centrifugal

Example: What is the centrifugal acceleration for a parcel of air at the Equator?

$$-\vec{\Omega} \times \vec{\Omega} \times \vec{r} = \Omega^2 r$$

with:

$$r_e = 6.378 \times 10^6 \ m$$

and:

$$\Omega = \frac{2\pi}{3600(24)} \ sec^{-1}$$

Centrifugal

So:

$$\Omega^2 r_e = 0.034\ m/sec^2$$

This is much smaller than $g = 9.8\ m^2/sec$

- Only a minor change to absorb into g'

Cartesian coordinates

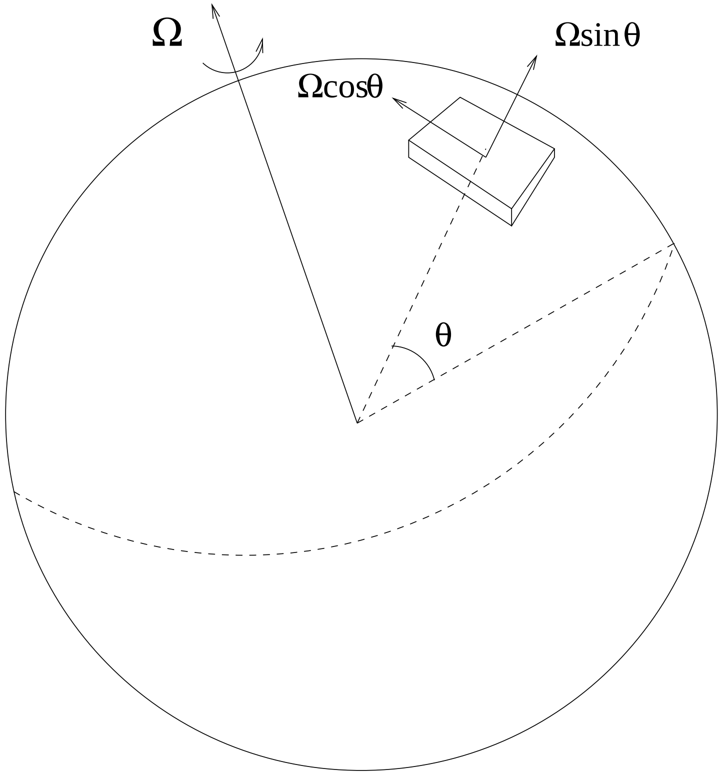
Equatorial radius is only 21 km larger than at poles

So can use spherical coordinates

However, we will use *Cartesian* coordinates

- Simplifies the math
- Neglected terms are *unimportant* at weather scales

Cartesian coordinates



Coriolis force

Rotation vector projects onto local vertical and meridional directions:

$$2\vec{\Omega} = 2\Omega cos\theta\ \hat{j} + 2\Omega sin\theta\ \hat{k} \equiv f_y\ \hat{j} + f_z\ \hat{k}$$

So the Coriolis force is:

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, v, w)$$

$$= -(f_y w - f_z v, f_z u, -f_y u)$$

Coriolis force

Example: What is the Coriolis acceleration on a parcel moving eastward at $10\ m/sec$ at 45 N?

We have:

$$f_y = 2\Omega cos(45) = 1.03 \times 10^{-4}\ sec^{-1}$$

$$f_z = 2\Omega sin(45) = 1.03 \times 10^{-4}\ sec^{-1}$$

$$-2\vec{\Omega} \times \vec{u} = -(0, f_y, f_z) \times (u, 0, 0) = -f_z u\ \hat{j} + f_y u\ \hat{k}$$

$$= (0, -1.03 \times 10^{-3}, 1.03 \times 10^{-3})\ m/sec^2$$

Coriolis force

Vertical acceleration is negligible compared to gravity ($g = 9.8\ m/sec^2$), so has little effect in z

But unbalanced in the horizontal direction

Note the acceleration is to the *south*

- Coriolis acceleration is most important in the *horizontal*
- Acts *to the right* in the Northern Hemisphere

Coriolis force

In the Southern hemisphere, $\theta < 0$. Same problem, at 45 S:

$$f_y = 2\Omega cos(-45) = 1.03 \times 10^{-4}\ sec^{-1}$$

$$f_z = 2\Omega sin(-45) = -1.03 \times 10^{-4}\ sec^{-1}$$

$$-2\vec{\Omega} \times \vec{u} = -f_z u\ \hat{j} + f_y u\ \hat{k}$$

$$= (0, +1.03 \times 10^{-3}, 1.03 \times 10^{-3})\ m/sec^2$$

Acceleration to the north, to the *left* of the parcel velocity.

Momentum equations

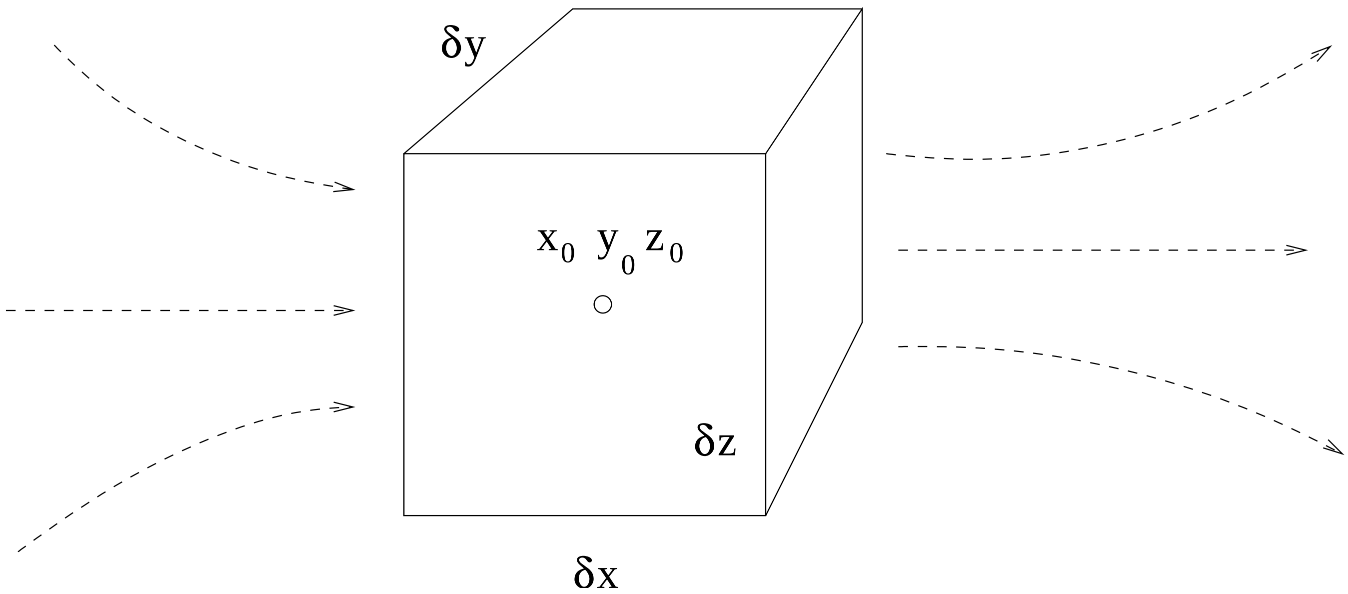
Move Coriolis terms to the LHS:

$$\frac{\partial}{\partial t}u + \vec{u} \cdot \nabla u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu \nabla^2 u$$

$$\frac{\partial}{\partial t}v + \vec{u} \cdot \nabla v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu \nabla^2 v$$

$$\frac{\partial}{\partial t}w + \vec{u} \cdot \nabla w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu \nabla^2 w$$

Continuity



Continuity

Consider a fixed volume

Density flux through the left side:

$$[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Through the right side:

$$[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\delta y\,\delta z$$

Continuity

So the net rate of change in mass is:

$$\frac{\partial}{\partial t}m = \frac{\partial}{\partial t}(\rho\,\partial x\,\partial y\,\partial z) = [\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\partial y\,\partial z$$

$$-[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}]\,\partial y\,\partial z = -\frac{\partial}{\partial x}(\rho u)\partial x\,\partial y\,\partial z$$

The volume δV is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u)$$

Continuity

Taking the other sides of the box:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = -\nabla \cdot (\rho \vec{u})$$

Can rewrite:

$$\nabla \cdot (\rho \vec{u}) = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho \ .$$

So:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = 0$$

Continuity

Can also derive using a *Lagrangian* box

As the box moves, it conserves it mass. So:

$$\frac{1}{m}\frac{d}{dt}(\partial m) = \frac{1}{\rho\delta V}\frac{d}{dt}(\rho\delta V) = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d\delta V}{dt} = 0$$

Expand the volume term:

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{1}{\delta x}\frac{d}{dt}\delta x + \frac{1}{\delta y}\frac{d}{dt}\delta y + \frac{1}{\delta z}\frac{d}{dt}\delta z$$

$$= \frac{1}{\delta x}\delta\frac{dx}{dt} + \frac{1}{\delta y}\delta\frac{dy}{dt} + \frac{1}{\delta z}\delta\frac{dz}{dt} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}$$

Continuity

As $\delta \to 0$:

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} \longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

So:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{u} = 0$$

Change in density proportional to the velocity *divergence*.

If volume changes, density changes to keep mass constant.

Ideal Gas Law

Five of the equations are *prognostic*: they describe the time evolution of fields.

But we have one *diagnostic* relation.

This relates the density, pressure and temperature

Ideal Gas Law

For dry air:

$$p = \rho R T$$

where

$$R = 287 \; Jkg^{-1}K^{-1}$$

Moist air

Law moist air, can write (Chp. 3):

$$p = \rho R T_v$$

where the *virtual temperature* is:

$$T_v \equiv \frac{T}{1 - e/p(1-\epsilon)}$$

$$\epsilon \equiv \frac{R_d}{R_v} = 0.622$$

We will ignore moisture. But remember that we *can* take it into account in this way.

Primitive equations

Continuity:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

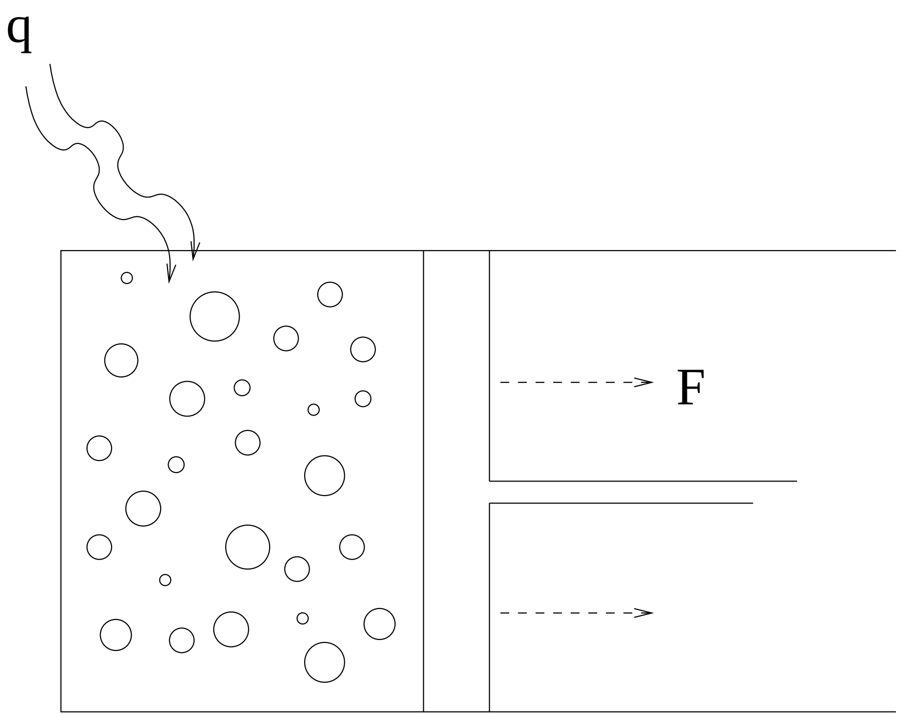
Ideal gas:

$$p = \rho R T$$

Thermodynamic energy:

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \frac{dq}{dt}$$

Thermodynamic equation



First law of thermodynamics

Change in internal energy = heat added - work done:

$$de = dq - dw$$

Work is done by expanding against external forces:

$$dw = Fdx = pAdx = pdV$$

If $dV > 0$, the volume is doing the work

First law of thermodynamics

Assume the volume has a unit mass, so that:

$$\rho V = 1$$

Then:

$$dV = d(\frac{1}{\rho}) = d\alpha$$

where α is the *specific volume*. So:

$$de = dq - p\, d\alpha$$

First law of thermodynamics

Add heat to the volume, the temperature rises. The *specific heat* (c_v) determines how much. If the volume is held constant:

$$dq_v = c_v dT$$

With $dV = 0$, equals the change in internal energy:

$$dq_v = de_v = c_v dT$$

First Law of thermodynamics

Joule's Law: e only depends on temperature for an ideal gas. So even if V changes:

$$de = c_v dT$$

So:

$$dq = c_v dT + p\, d\alpha$$

Divide by dt to find the theromdynamic energy equation:

$$\frac{dq}{dt} = c_v \frac{dT}{dt} + p\, \frac{d\alpha}{dt}$$

First law of thermodynamics

Now imagine we keep the pressure constant:

$$dq_p = c_p dT$$

We let the volume expand while keeping p constant. This requires more heat to raise the temperature. Rewrite the work term:

$$p\,d\alpha = d(p\alpha) - \alpha dp$$

So:

$$dq = c_v dT + d(p\alpha) - \alpha dp$$

First law of thermodynamics

The ideal gas law is:

$$p = \rho RT = \alpha^{-1} RT$$

So:

$$d(p\alpha) = RdT$$

Thus:

$$dq = (c_v + R)dT - \alpha dp$$

First law of thermodynamics

At constant pressure, $dp = 0$, so:

$$dq_p = (c_v + R)dT = c_p dT$$

So the specific heat at constant pressure is *greater* than at constant volume. For dry air, measurements yield:

$$c_v = 717 Jkg^{-1}K^{-1}, \qquad c_p = 1004 Jkg^{-1}K^{-1}$$

so:

$$R = 287\ Jkg^{-1}K^{-1}$$

First law of thermodynamics

So we can also write:

$$dq = c_p dT - \alpha dp$$

Dividing by dt, we have:

$$\frac{dq}{dt} = c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}$$

Basic balances

Not all terms in the momentum equations are equally important for weather systems.

Will simplify the equations by identifying primary balances (throw out as many terms as possible).

Begin with horizontal momentum equations.

Scaling

General technique: *scale* equations using estimates of the various parameters. Take the x-momentum equation:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad f_y W \qquad f_z U \qquad \frac{\triangle_H P}{\rho L} \qquad \frac{\nu U}{L^2}$$

Scaling

Now use typical values. Length scales:

$$L \approx 10^6 m, \quad D \approx 10^4 m$$

Horizontal scale is 1000 km, the *synoptic scale* (of weather systems).

Velocities:

$$U \approx V \approx 10\, m/sec, \quad W \approx 1\, cm/sec$$

Notice the winds are *quasi-horizontal*

Scaling

Pressure term, from measurements:

$$\triangle_H P/\rho \approx 10^3 m^2/sec^2$$

Time scale:

$$T = L/U \approx 10^5 sec$$

Called an "advective time scale" (≈ 1 day).

Also $\nu \approx 10^{-5} m^2 \,\mathrm{sec}$ for the friction term

Scaling

Coriolis terms:

$$(f_y, f_z) = 2\Omega(cos\theta, sin\theta)$$

with

$$\Omega = 2\pi(86400)^{-1} sec^{-1}$$

Assume at mid-latitudes:

$$f_y \approx f_z \approx 10^{-4} sec^{-1}$$

Scaling

Plug in:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad fW \qquad fU \qquad \frac{\triangle_H P}{\rho L} \qquad \frac{\nu U}{L^2}$$

$$10^{-4} \qquad 10^{-4} \qquad 10^{-4} \qquad 10^{-5} \qquad 10^{-6} \qquad 10^{-3} \qquad 10^{-3} \qquad 10^{-16}$$

Geostrophy

Keeping only the 10^{-3} terms:

$$f_z v = \frac{1}{\rho} \frac{\partial}{\partial x} p$$

$$f_z u = -\frac{1}{\rho} \frac{\partial}{\partial y} p$$

These are the *geostrophic* relations.

Balance between the pressure gradient and Coriolis force.

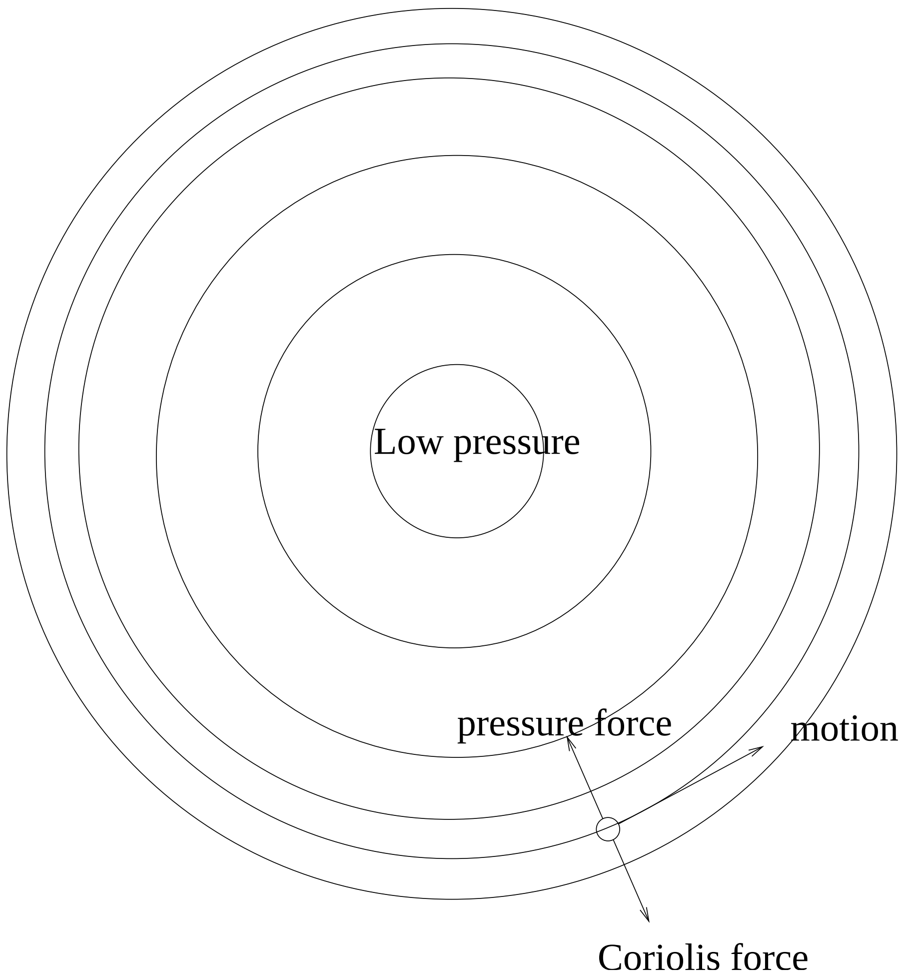
Geostrophy

Fundamental momentum balance at synoptic scales

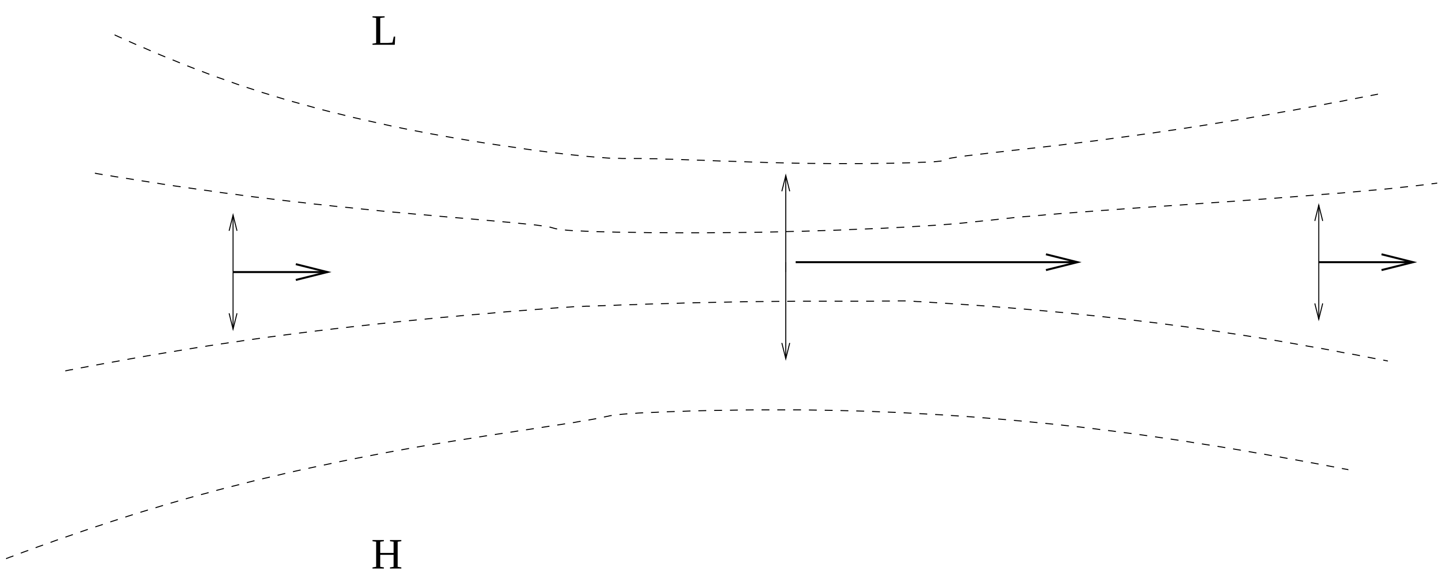
- Low pressure to left of the wind in Northern Hemisphere
- Low pressure to *right* in Southern Hemisphere

But balance *fails* at equator, because $f_z = 2\Omega sin(0) = 0$. There we must keep other terms.

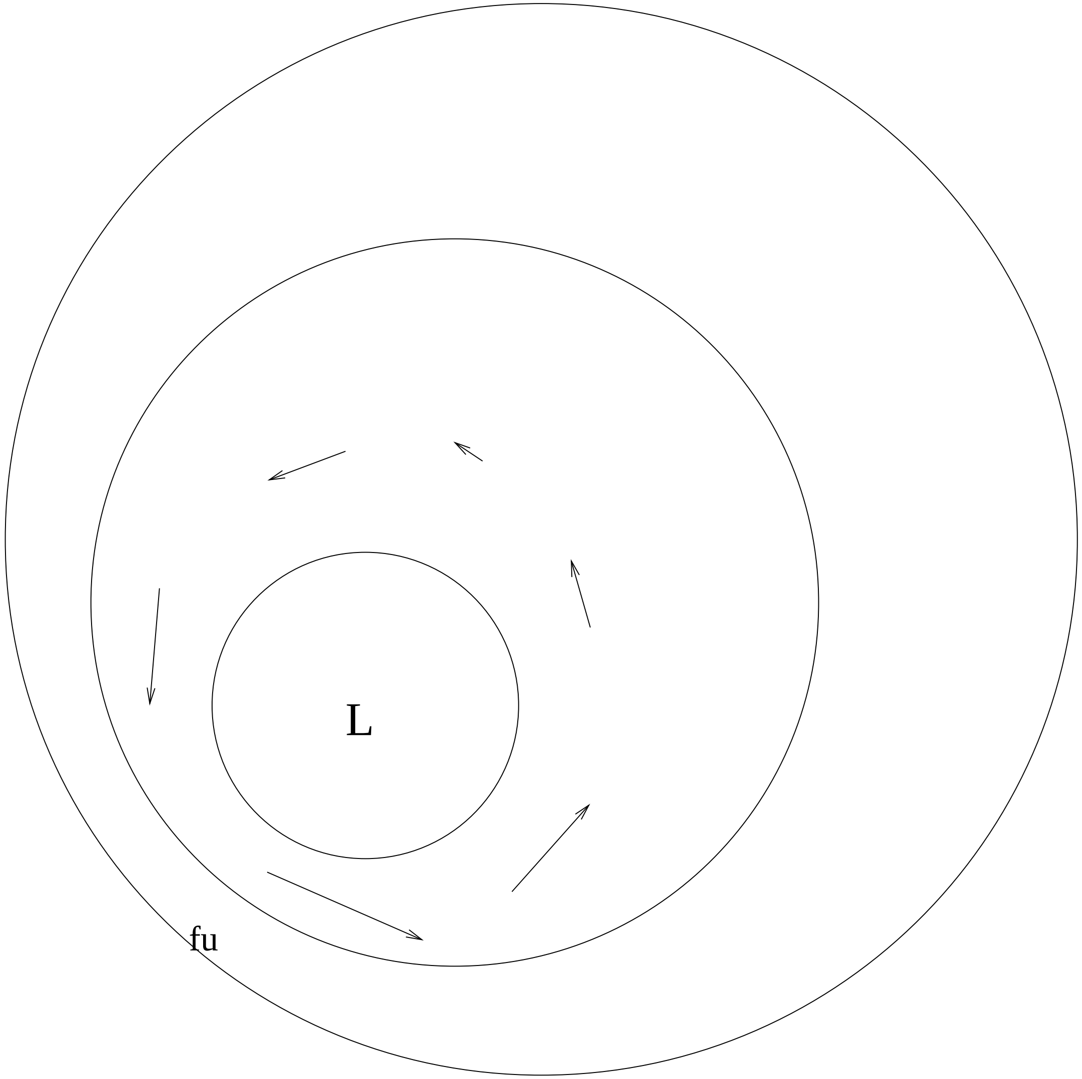
Geostrophy



Geostrophy



Geostrophy



Geostrophy

Example: The pressure difference is 0.37 kPa over 100 km at 45 N. How strong are the winds?

$$f_z = 2\Omega sin(45) = (1.45\times10^{-4})(.7071)\ sec^{-1} = 1.03\times10^{-4}\ sec^{-1}$$

$$\frac{\partial p}{\partial l} = \frac{0.37 \times 10^3\ N/m^2}{10^5\ m} = 3.7 \times 10^{-3}\ N/m^3$$

So:

$$u = \frac{1}{\rho_0 f_z}\frac{\partial p}{\partial l} = \frac{1}{(1.2\ kg/m^3)(1.03 \times 10^{-4} sec^{-1})}(3.7\times10^{-3}\ N/m^3)$$

= 29.9 m/sec (Strong!)

Geostrophy

Is a *diagnostic relation*

- Given the pressure, can calculate the horizontal velocities

But geostrophy cannot be used for *prediction*

Means that we must also retain the 10^{-4} terms in the scaling

Approximate horizontal momentum

So:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

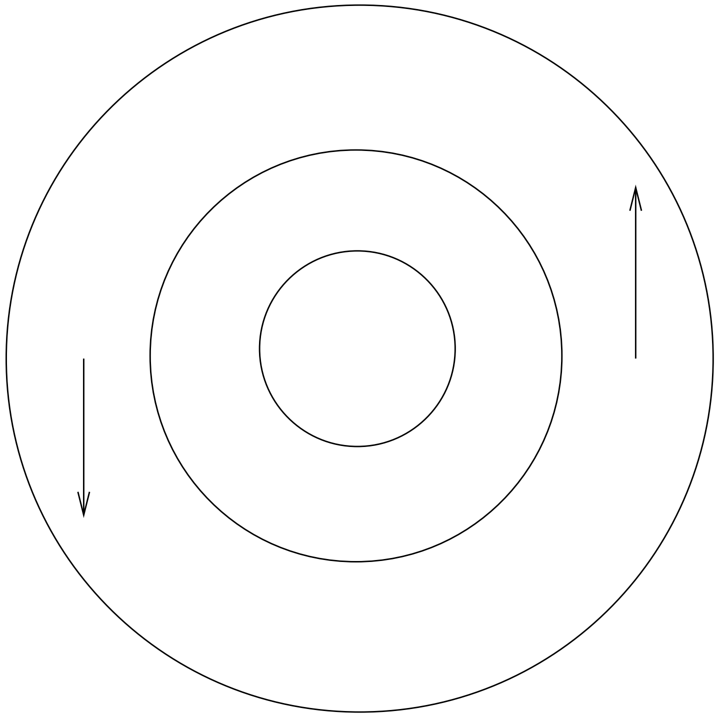
$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

These equations are *quasi-horizontal*: neglect vertical motion

Explains why the horizontal winds are so much larger than in the vertical

Other momentum balances

Geostrophy most important balance at synoptic scales. But other balances possible. Consider purely circular flow:



Other momentum balances

Must use cylindrical coordinates. From standard text books, can find that the acceleration in the radial direction is:

$$\frac{d}{dt}u_r - \frac{u_\theta^2}{r} - fu_\theta = -\frac{1}{\rho}\frac{\partial}{\partial r}p$$

u_θ^2/r is the *cyclostrophic* term

This is related to centripetal acceleration.

Other momentum balances

Assume no radial motion: $u_r = 0$. Then:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r} p$$

Scaling we get:

$$\frac{U^2}{R} \qquad fU \qquad \frac{\triangle_H P}{\rho R}$$

Or:

$$\frac{U}{fR} \qquad 1 \qquad \frac{\triangle_H P}{\rho f U R}$$

Other momentum balances

The ratio:

$$\frac{U}{fR} \equiv \epsilon$$

is called the *Rossby number*. If $\epsilon \ll 1$, the first term is very small. So we have:

$$fu_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

The geostrophic relation.

Other momentum balances

If $\epsilon \gg 1$, the first term dominates.

A tornado at mid-latitudes has:

$$U \approx 30m/s, \ f = 10^{-4} sec^{-1}, \ R \approx 300m \rightarrow \epsilon \approx 1000$$

Cyclostrophic wind balance

Then we have:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

or:

$$u_\theta = \pm\left(\frac{r}{\rho}\frac{\partial}{\partial r}p\right)^{1/2}$$

- Rotation does not enter.
- Winds can go *either way*.

Inertial oscillations

Third possibility: there is no radial pressure gradient:

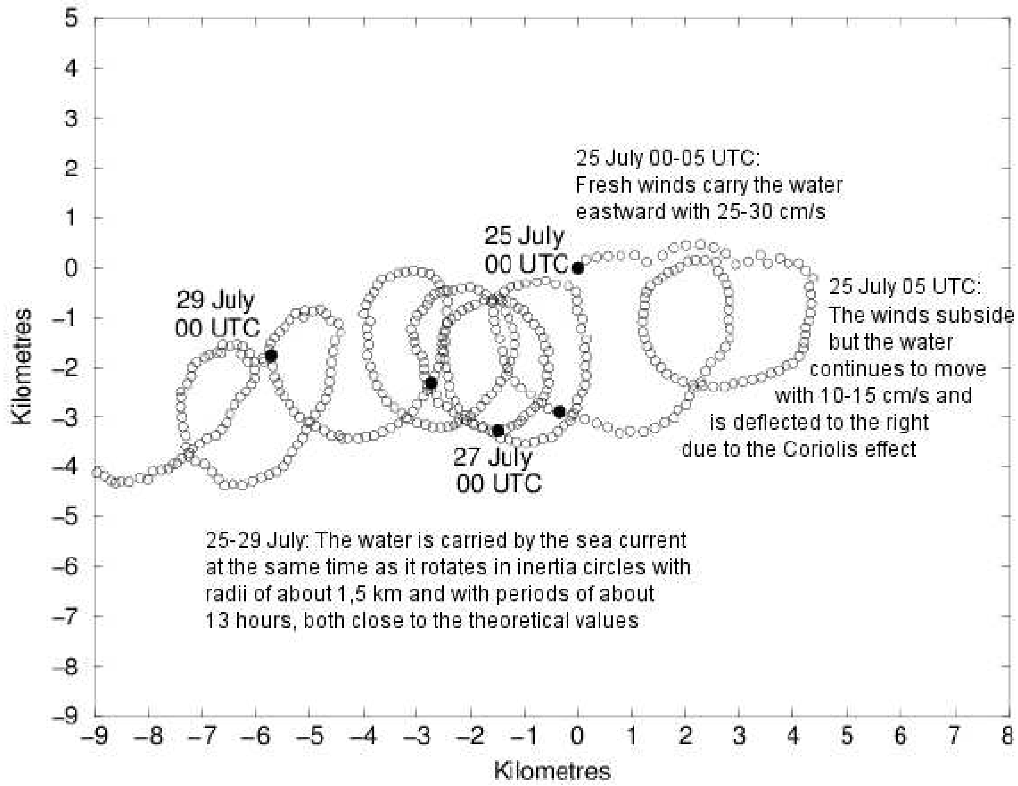
$$\frac{u_\theta^2}{r} + f u_\theta = 0$$

then:

$$u_\theta = -fr$$

Rotation is clockwise (*anticyclonic*) in the Northern Hemisphere.

Inertial oscillations



A drifting buoy in the Baltic Sea, July 1969. Courtesy Persson and Broman.

Inertial oscillations

The time for a fluid parcel to complete a loop is:

$$\frac{2\pi r}{u_\theta} = \frac{2\pi}{f} = \frac{0.5\ day}{|sin\theta|}$$

Called the “inertial period”

Strong effect in the surface ocean

Less frequently observed in the atmosphere

Gradient wind balance

Fourth possibility: all terms are important ($\epsilon \approx 1$)

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p$$

Solve using the quadratic formula:

$$u_\theta = -\frac{1}{2}fr \pm \frac{1}{2}(f^2r^2 + \frac{4r}{\rho}\frac{\partial}{\partial r}p)^{1/2}$$

$$= -\frac{1}{2}fr \pm \frac{1}{2}(f^2r^2 + 4\,r\,f\,u_g)^{1/2}$$

Gradient wind balance

If $u_g < 0$ (anticyclone), we require:

$$|u_g| < \frac{fr}{4}$$

If $u_g > 0$ (cyclone), there is *no limit*

Wind gradients can be *much stronger* in cyclones than in anticyclones

Gradient wind balance

Alternately can write:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p = f u_g$$

Divide through by $f u_\theta$:

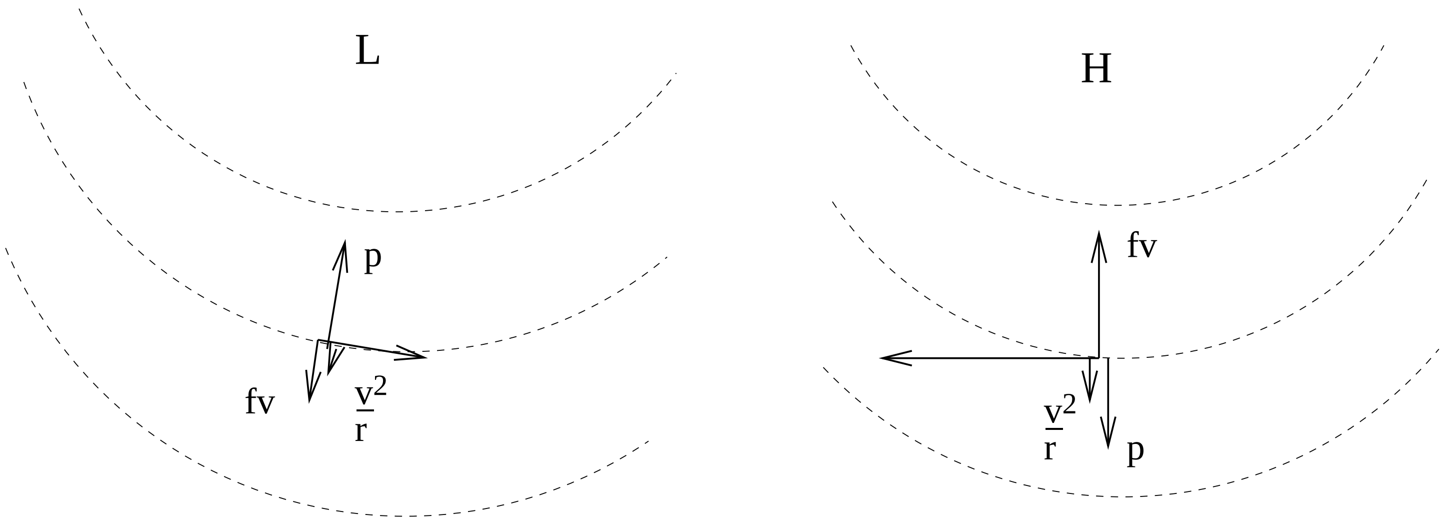
$$\frac{u_\theta}{fr} + 1 = \epsilon + 1 = \frac{u_g}{u_\theta}$$

So if $\epsilon = 0.1$, the gradient wind estimate differs by 10 %

Gradient wind balance

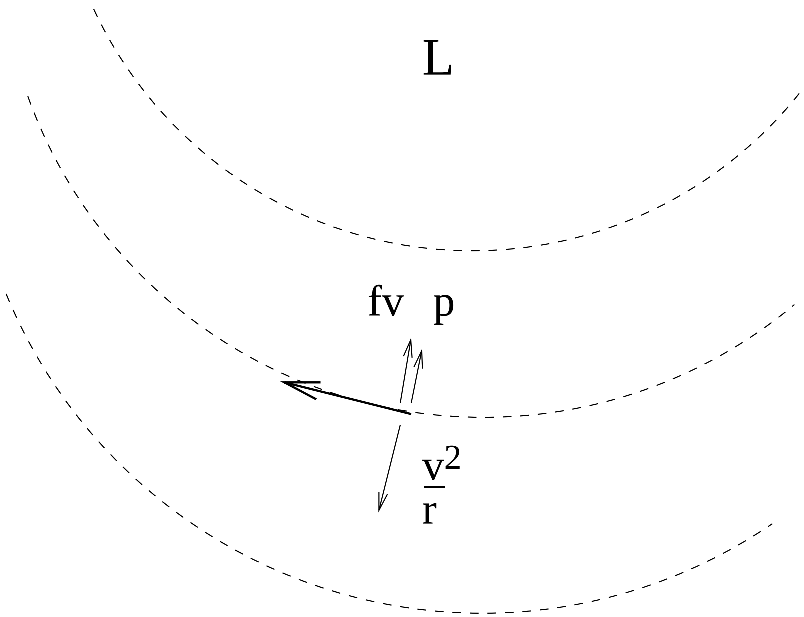
- At low latitudes, ϵ can be 1-10. Then the gradient wind estimate is more accurate.
- Geostrophy is *symmetric to sign changes*: no difference between cyclones and anticyclones
- The gradient wind balance is *not* symmetric to sign change. Cyclones can be stronger.

Gradient wind balance



Winds weaker than geostrophic for a low pressure system; they are stronger for a high pressure system.

Gradient wind balance



An *anomalous low*: low pressure with clockwise flow

Usually only occurs at low latitudes, where Coriolis weak

Hydrostatic balance

Now scale the vertical momentum equation

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad fU \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

Hydrostatic balance

We must scale:

$$\frac{1}{\rho}\frac{\partial}{\partial z}p$$

The vertical variation of pressure much greater than the horizontal variation:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Hydrostatic balance

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g$$

$$\frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad fU \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

$$10^{-7} \qquad 10^{-7} \qquad 10^{-7} \qquad 10^{-10} \qquad 10^{-3} \qquad 10 \qquad 10$$

Static atmosphere

Dominant balance is between the vertical pressure gradient and gravity

However, same balance *if there no motion at all* !

Setting $(u, v, w) = 0$ in the equations of motion yields:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p = \frac{1}{\rho}\frac{\partial}{\partial y}p = \frac{\partial}{\partial t}\rho = \frac{dT}{dt} = 0$$

Which implies:

$$\rho = \rho(z), \quad p = p(z), \quad T = T(z)$$

Static atmosphere

Two equations left:

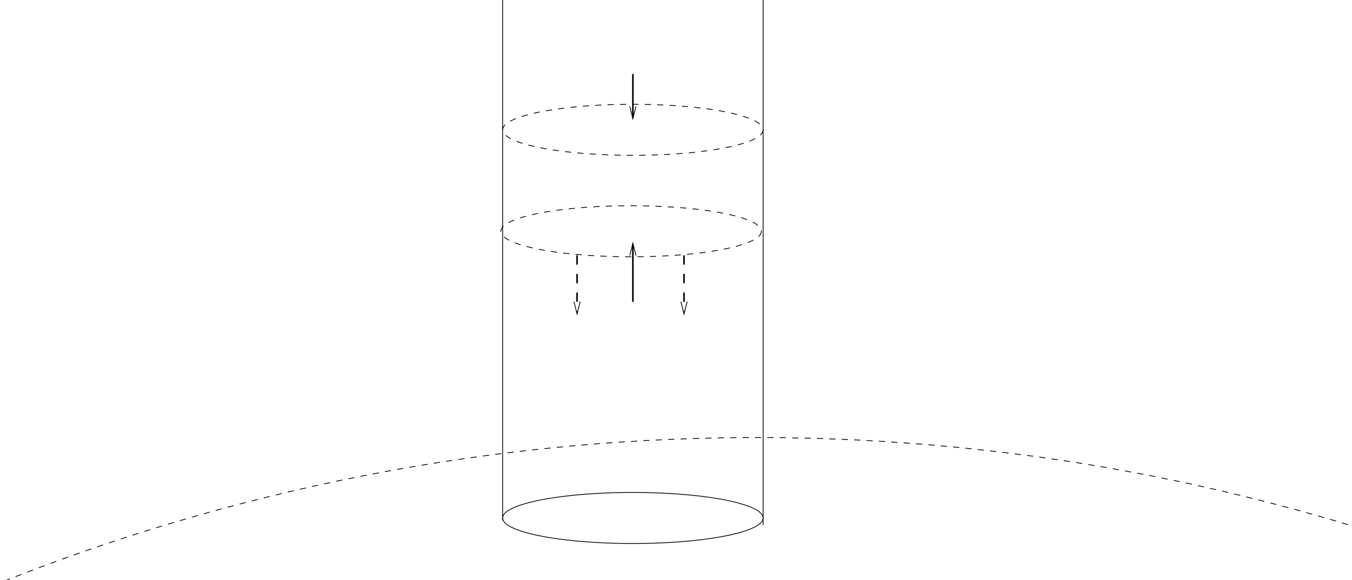
$$\frac{\partial}{\partial z} p = -\rho g$$

the *hydrostatic balance* and

$$p = \rho R T$$

Equations describe a non-moving atmosphere

Hydrostatic balance



Static atmosphere

Aside: what is “sea level pressure”? Integrate the hydrostatic relation:

$$p(z) = \int_z^{\infty} \rho g \, dz \ .$$

The pressure at any point is equal to the weight of air above it. Sea level pressure is:

$$p(0) = 101.325 \ kPa \ (1013.25 mb)$$

The average weight per square meter of the *entire* atmospheric column

Scaling

Static hydrostatic balance not interesting for weather. Separate the pressure and density into static and non-static (moving) components:

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t)$$

$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$$

Assume:

$$|p'| \ll |p_0|, \quad |\rho'| \ll |\rho_0|$$

Scaling

Then:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(p_0 + p') - g$$

$$\approx -\frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}(p_0 + p') - g$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + \left(\frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$\rightarrow$ Neglect $(\rho' p')$

Scaling

Use these terms in the vertical momentum equation

But how to scale?

Vertical variation of the perturbation pressure comparable to the horizontal perturbation:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2$$

Scaling

Also:

$$|\rho'| \approx 0.01|\rho_0|$$

So:

$$\frac{\rho'}{\rho_0} g \approx 10^{-1} m/sec^2$$

Scaling

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - f_y u = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g$$

$$10^{-7} \quad 10^{-7} \quad 10^{-7} \quad 10^{-10} \quad 10^{-3} \qquad 10^{-1} \qquad 10^{-1}$$

Hydrostatic perturbations

Dominant balance still hydrostatic, but with perturbations:

$$\frac{\partial}{\partial z} p' = -\rho' g$$

thus vertical acceleration unimportant at synoptic scales

But we lost the vertical velocity! Deal with this later.

Coriolis parameter

So *all* terms with f_y are unimportant

From now on, neglect f_y and write f_z simply as f:

$$f \equiv 2\Omega sin(\theta)$$

f_y only important near the equator

Static atmosphere

One result of the hydrostatic relation is that we can estimate how density changes with height.

Say $T = const.$ (an *isothermal* atmosphere):

$$\frac{\partial}{\partial z} p_0 = -\rho_0 g$$

Substituting from the ideal gas law, we have:

$$RT \frac{\partial}{\partial z} \rho_0 = -\rho_0 g$$

Static atmosphere

The solution to this is:

$$\rho_0(z) = \rho_0(0)\, e^{-z/H}$$

So the density decays exponentially with height. The e-folding scale is the “scale height”:

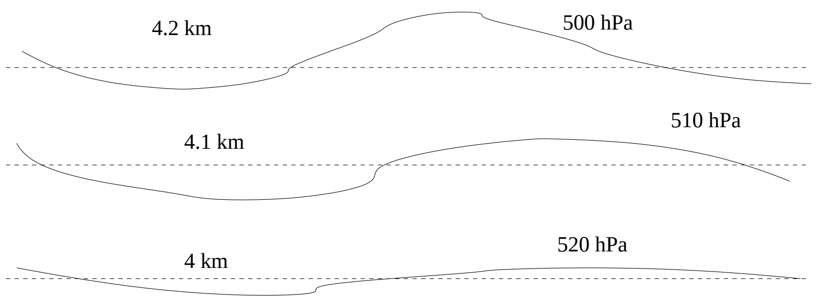
$$H \equiv \frac{RT}{g}$$

Pressure coordinates

The hydrostatic balance implies an equivalence between changes in pressure and z

Can use it to change vertical coordinates

Consider constant pressure surfaces (here in two dimensions):



Pressure coordinates

On a pressure surface:

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial z}\,dz = 0$$

Substitute hydrostatic relation:

$$dp = \frac{\partial p}{\partial x}\,dx - \rho g\,dz = 0$$

So:

$$\frac{\partial p}{\partial x} = \rho g \frac{dz}{dx} \equiv \rho\,\frac{\partial \Phi}{\partial x}$$

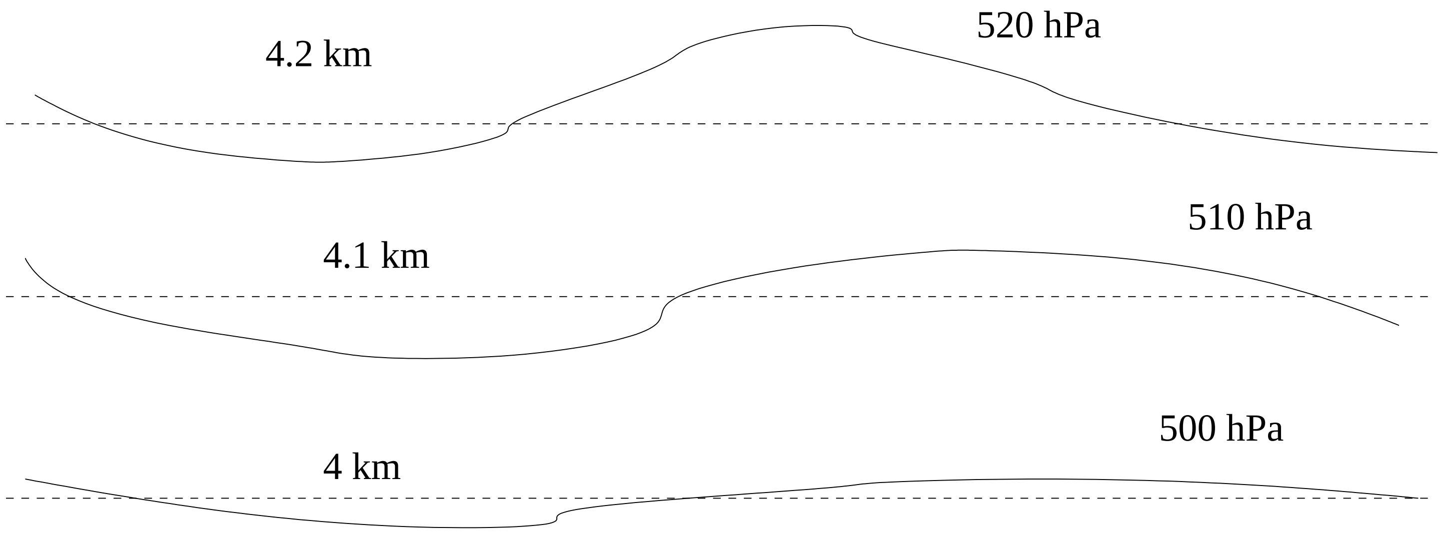
Geopotential

where Φ is the *geopotential*

This is proportional to the height of a given pressure surface

$\longrightarrow$ instead of pressure at a certain height, we think of the height of a pressure surface

Geopotential



Geostrophy

Removes density from the momentum equation!

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x}$$

Now the geostrophic balance is:

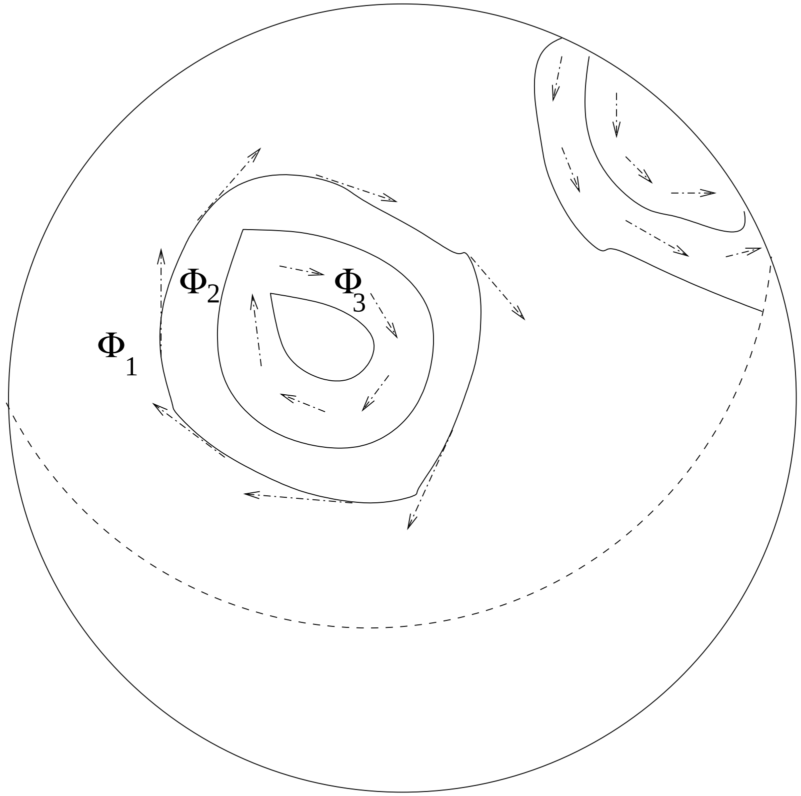
$$fv = \frac{\partial}{\partial x}\Phi$$

$$fu = -\frac{\partial}{\partial y}\Phi$$

These are *linear* relations

Geostrophy

500 hPa



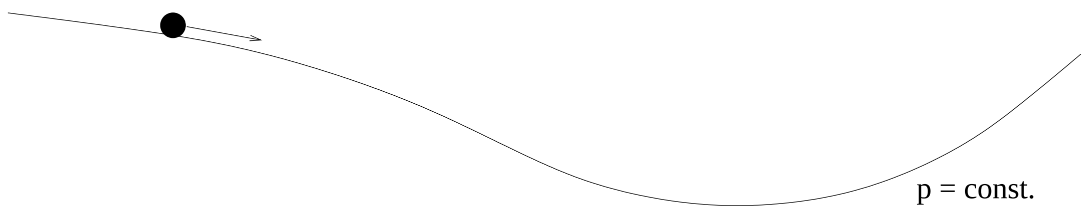
Geopotential

How do we understand the geopotential?

$$\Phi \approx gz$$

So geopotential is proportional to potential energy ($\rho g z$)

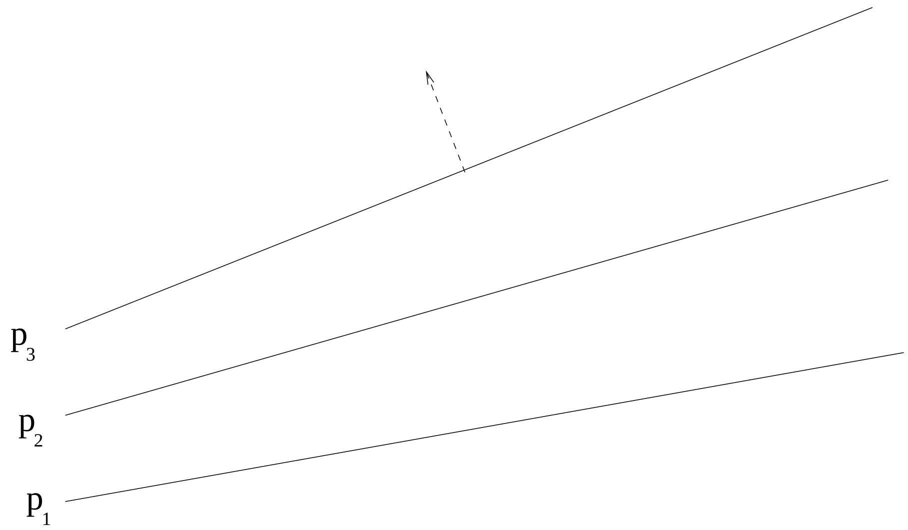
Particles accelerate toward minima in potential energy



Vertical velocities

Different vertical velocities:

$$w = \frac{dz}{dt} \quad \rightarrow \quad \omega = \frac{dp}{dt}$$



Lagrangian derivative

Lagrangian derivative is now:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + \omega\,\frac{\partial}{\partial p}$$

Vertical motion

How does ω relate to the actual vertical velocity?

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p + w\frac{\partial}{\partial z}p$$

Using the hydrostatic relation:

$$\omega = \frac{dp}{dt} = \frac{\partial}{\partial t}p + u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p - \rho g w$$

For geostrophic motion:

$$u\frac{\partial}{\partial x}p + v\frac{\partial}{\partial y}p = (-\frac{1}{\rho f}\frac{\partial}{\partial y}p)(\frac{\partial}{\partial x}p) + (\frac{1}{\rho f}\frac{\partial}{\partial x}p)(\frac{\partial}{\partial y}p) = 0$$

Vertical motion

So

$$\omega \approx \frac{\partial}{\partial t} p - \rho g w$$

Also:

$$\frac{\partial}{\partial t} p \approx 10 hPa/day$$

$$\rho g w \approx (1.2 kg/m^3)\,(9.8 m/sec^2)(0.01 m/sec) \approx 100 hPa/day$$

Vertical motion

So:

$$\omega \approx -\rho g w$$

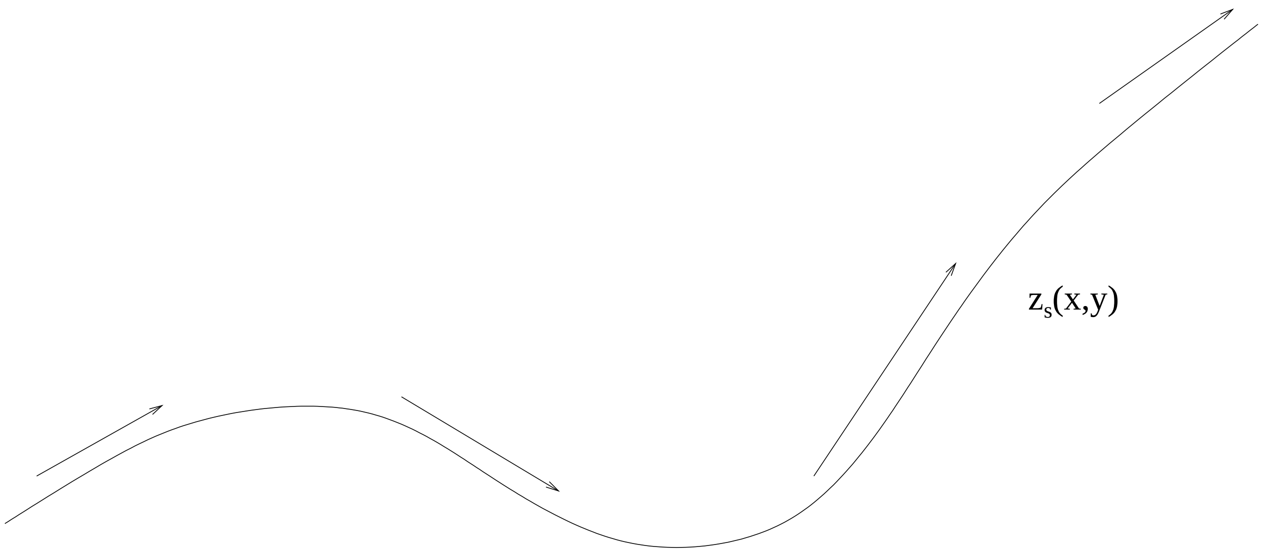
This is accurate within 10 % in the mid-troposphere

Less accurate near the ground, due to topography

At the surface:

$$w_s = u\frac{\partial}{\partial x}z_s + v\frac{\partial}{\partial y}z_s$$

Vertical motion



Topography most important for ω in the lowest 1-2 km of the troposphere

Continuity

Continuity equation also changes in pressure coordinates

Consider a Lagrangian box:

$$V = \delta x \, \delta y \, \delta z = -\delta x \, \delta y \, \frac{\delta p}{\rho g}$$

with a mass:

$$m = \rho V = -\delta x \, \delta y \, \delta p / g$$

Continuity

Conservation of mass:

$$\frac{1}{m}\frac{d}{dt}m = \frac{g}{\delta x \delta y \delta p}\frac{d}{dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$$

Using the chain rule:

$$\frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0$$

Continuity

Let $\delta \to 0$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

The flow is *incompressible* in pressure coordinates

This equation is also linear

Much simpler to work with than continuity equation in z-coordinates (which is nonlinear)

Hydrostatic balance

$$\frac{dp}{dz} = -\rho g$$

$$dp = -\rho g dz = -\rho d\Phi$$

So:

$$\frac{d\Phi}{dp} = -\frac{1}{\rho} = -\frac{RT}{p}$$

using the Ideal Gas Law

Summary: Pressure coordinates

Geostrophy:

$$fv = \frac{\partial}{\partial x}\Phi, \qquad fu = -\frac{\partial}{\partial y}\Phi$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Hydrostatic:

$$\frac{d\Phi}{dp} = -\frac{RT}{p}$$

Thermal wind

Geostrophy tells us what the velocities are if we know the geopotential on a pressure surface

What about the velocities on *other* pressure surfaces?

Say we have information on the 500 hPa surface, but we wish to estimate winds on the 400 hPa surface

Requires knowing the velocity *shear*

This shear is determined by the thermal wind relation

Thermal wind

From the hydrostatic balance:

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

Now take the derivative wrt pressure of the geostrophic relation:

$$\frac{\partial}{\partial p}\,(f v_g = \frac{\partial \Phi}{\partial x})$$

But:

$$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

Thermal wind

So:

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

Or:

$$\frac{\partial v_g}{\partial\, ln(p)} = -\frac{R}{f}\frac{\partial T}{\partial x}$$

- Shear is proportional to the temperature gradient

Thermal wind

If we know the velocity at p_0, can calculate it at p_1

Integrate between two pressure levels:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} \frac{\partial T}{\partial x} \, d\, ln(p)$$

$$= -\frac{R}{f} \frac{\partial}{\partial x} \int_{p_0}^{p_1} T \, d\, ln(p)$$

Mean temperature

Define the *mean temperature* in layer between p_0 and p_1:

$$\overline{T} \equiv \frac{\int_{p_0}^{p_1} T \, d(lnp)}{\int_{p_0}^{p_1} \, d(lnp)} = \frac{\int_{p_0}^{p_1} T \, d(lnp)}{ln(\frac{p_1}{p_0})}$$

Then:

$$v_g(p_1) - v_g(p_0) = \frac{R}{f} \, ln(\frac{p_0}{p_1}) \, \frac{\partial \overline{T}}{\partial x}$$

Similarly:

$$u_g(p_1) - u_g(p_0) = -\frac{R}{f} \, ln(\frac{p_0}{p_1}) \, \frac{\partial \overline{T}}{\partial y}$$

Thermal wind

Alternately we can use geostrophy to calculate the shear between p_0 and p_1:

$$v_g(p_1) - v_g(p_0) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \equiv \frac{g}{f}\frac{\partial}{\partial x}Z_{10}$$

and:

$$u_g(p_1) - u_g(p_0) = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \equiv -\frac{g}{f}\frac{\partial}{\partial y}Z_{10}$$

Thermal wind

where:

$$Z_{10} = \frac{1}{g}\left(\Phi_1 - \Phi_0\right)$$

is the layer *thickness* between p_0 and p_1.

- Shear proportional to gradients of layer thickness

Thermal wind

Thus:

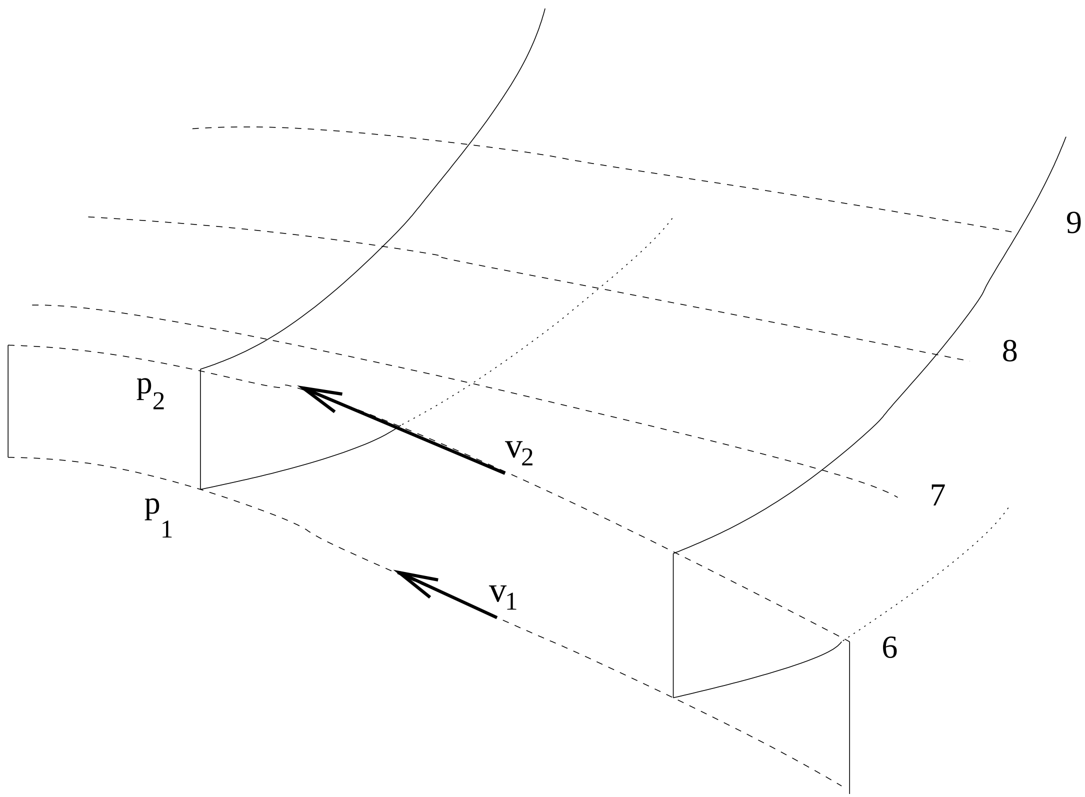
$$v_g(p_1) - v_g(p_0) = \frac{R}{f}\, ln(\frac{p_0}{p_1})\, \frac{\partial \overline{T}}{\partial x} = \frac{g}{f}\, \frac{\partial}{\partial x} Z_{10}$$

So:

$$Z_{10} = \frac{R}{g}\, ln(\frac{p_0}{p_1})\, \overline{T}$$

- Layer thickness is *proportional to the mean temperature*

Layer thickness



Barotropic atmosphere

Example 1: temperature is constant on pressure surfaces

Then $\nabla T = 0 \quad \rightarrow$ *no vertical shear*

Velocities don't change with height

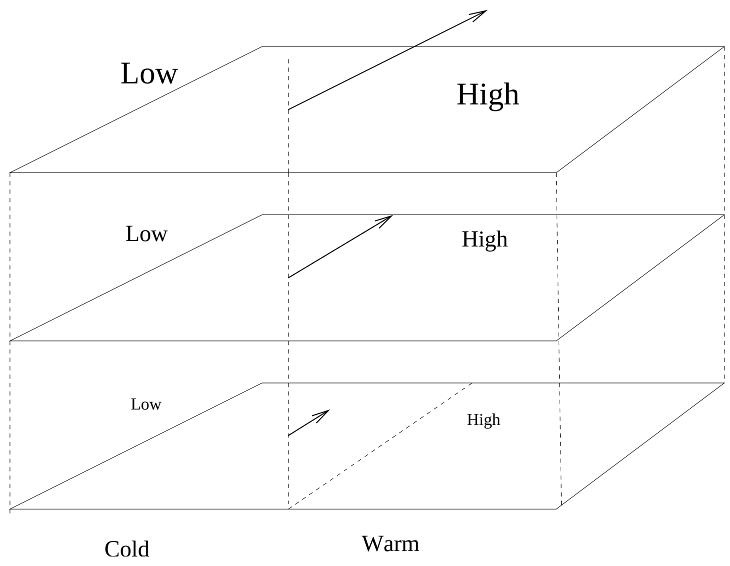
Also: all layers have *equal thickness*: stacked like pancakes

Equivalent barotropic

Example 2: temperature and geopotential contours parallel:

$$\frac{\partial}{\partial p}\vec{u}_g \parallel \vec{u}_g$$

Wind changes magnitude but *not direction* with height



Equivalent barotropic

Consider the zonal-average temperature :

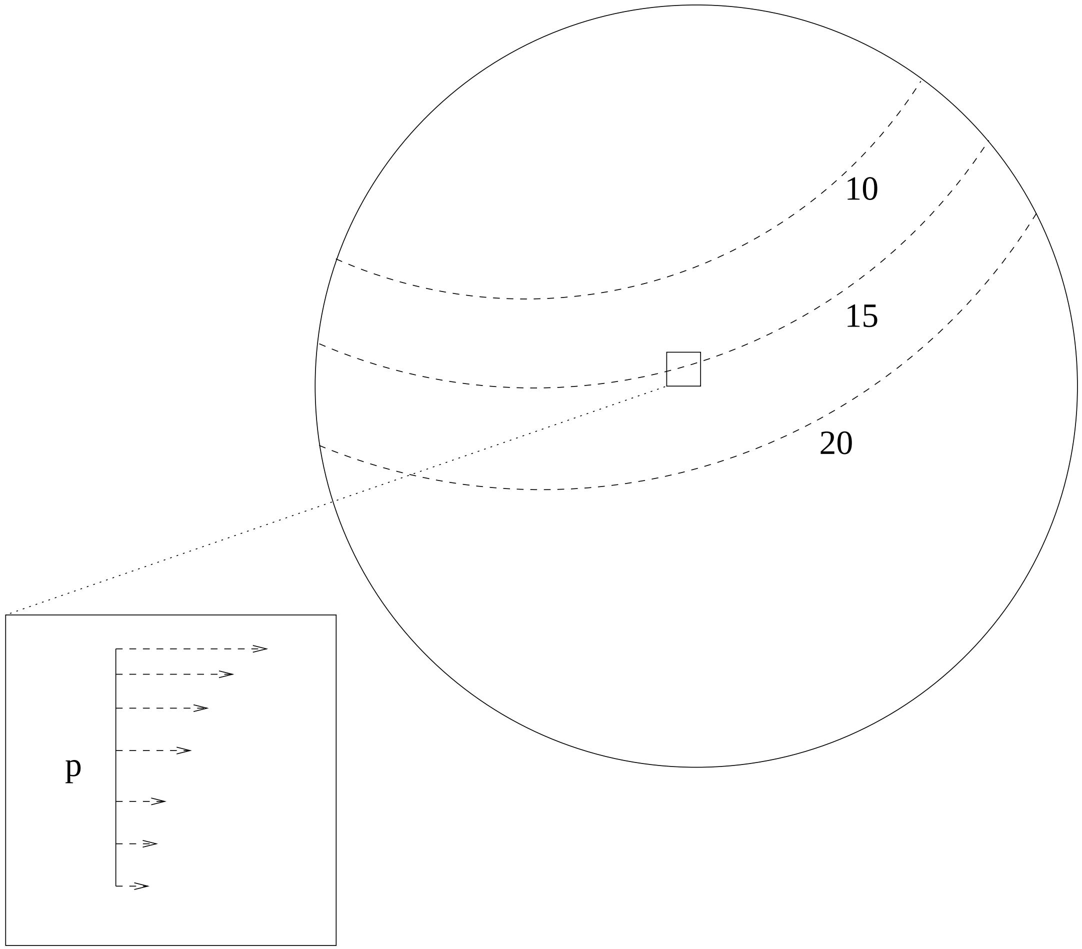
$$\frac{1}{2\pi}\int_0^{2\pi} T\, d\phi$$

Decreases from the equator to the pole

So $\frac{\partial}{\partial y}T < 0$

Thermal wind $\rightarrow$ winds increase with height

Jet Stream



Jet Stream

Example: At 30N, the zonally-averaged temperature gradient is $0.75\ Kdeg^{-1}$, and the average wind is zero at the earth's surface. What is the mean zonal wind at the level of the jet stream ($250\ hPa$)?

$$u_g(p_1) - u_g(p_0) = u_g(p_1) = -\frac{R}{f}\, ln(\frac{p_0}{p_1})\, \frac{\partial \overline{T}}{\partial y}$$

$$u_g(250) = -\frac{287}{2\Omega sin(30)}\, ln(\frac{1000}{250})\, (-\frac{0.75}{1.11 \times 10^5\, m}) = 36.8\ m/sec$$

Baroclinic atmosphere

Example 3: Temperature not parallel to geopotential

Geostrophic wind has a component normal to the temperature contours (isotherms)

Produces *geostrophic temperature advection*

Winds blow from warm to cold or vice versa

Vertical shear

We have derived the vertical shear in pressure coordinates

Recall that pressure increases going down

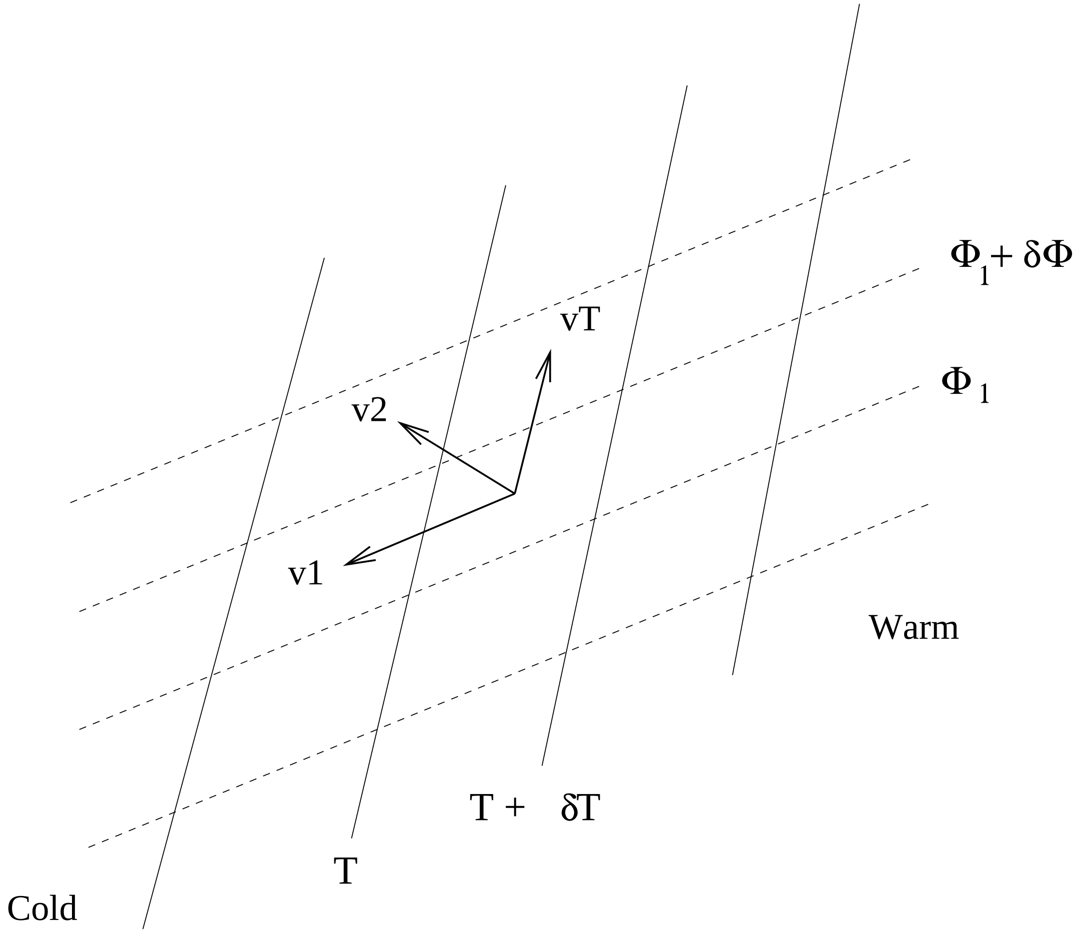
The shear in z-coordinates has the opposite sign

For example:

$$\frac{\partial T}{\partial y} < 0 \quad \rightarrow \quad \frac{\partial u}{\partial p} < 0 \quad \rightarrow \quad \frac{\partial u}{\partial z} > 0$$

So the Jet Stream gets stronger going upward

Temperature advection



Temperature advection

Warm advection $\longrightarrow$ *veering*

- Anticyclonic (clockwise) rotation with height

Cold advection $\longrightarrow$ *backing*

- Cyclonic (counter-clockwise) rotation with height

Summary

Geostrophic wind parallel to geopotential contours

- high pressure to the right (North Hemisphere)

Thermal wind parallel to mean *temperature* (thickness) contours

- high temperature (thickness) to the right

Divergence and vorticity

Two important quantities in dynamical meteorology:

- **Divergence**

$$\chi = \nabla \cdot \vec{u}$$

- **Vorticity**

$$\vec{\zeta} = \nabla \times \vec{u}$$

Divergence and density

Continuity equation:

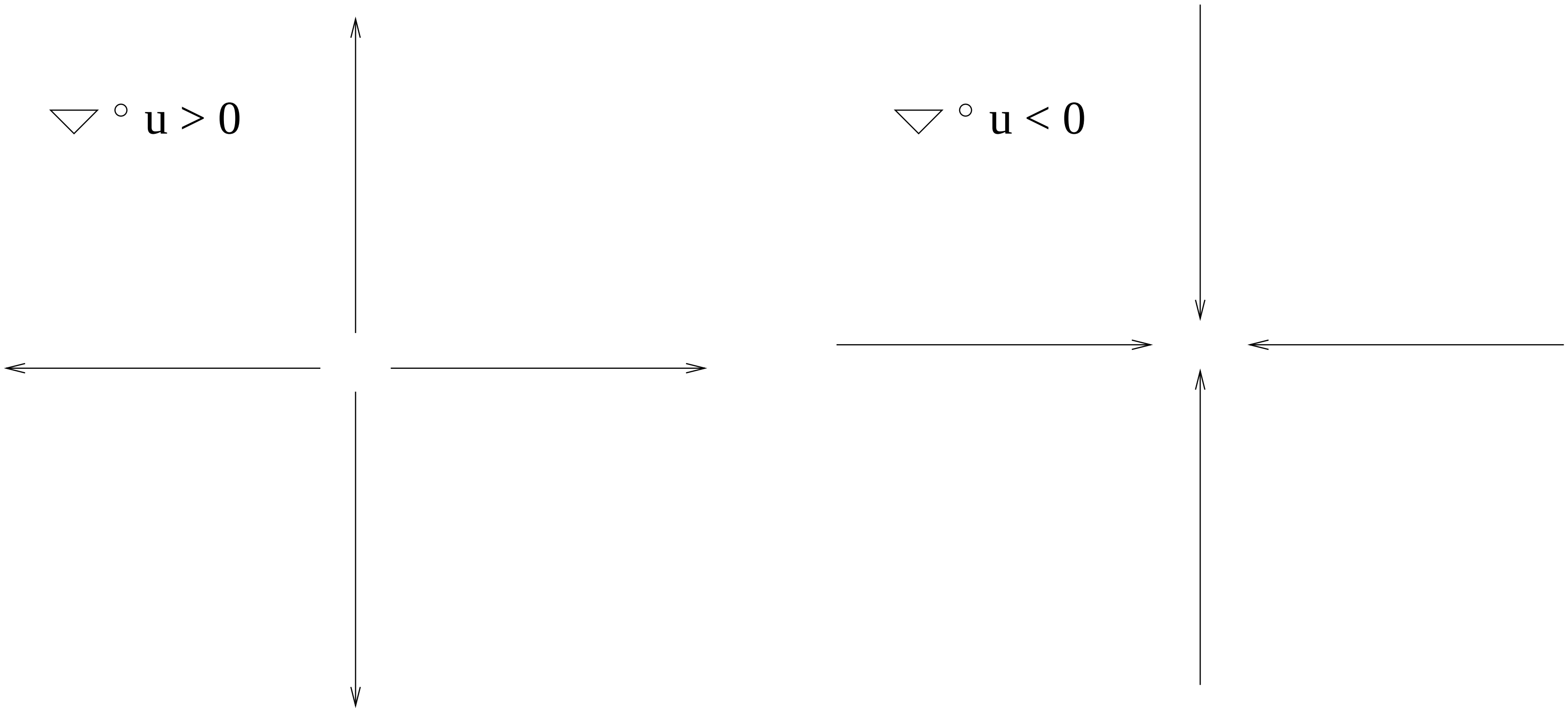
$$\frac{d\rho}{dt} + \rho \nabla \cdot \underline{u} = 0$$

So:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\chi$$

- Density changes due to divergence

Divergence



Example

The divergence in a region is constant and positive:

$$\chi = D > 0$$

What happens to the density of an air parcel?

Example

$$\frac{1}{\rho}\frac{d\rho}{dt} = -D$$

$$\frac{d\rho}{dt} = -\rho D$$

$$\rho(t) = \rho(0)\ e^{-Dt}$$

Density decreases exponentially in time

Divergence and vertical motion

In pressure coordinates, the total divergence is zero:

$$\chi = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial p}\omega = 0$$

This gives us a way of estimating the vertical velocity, ω, from the horizontal velocities:

$$\frac{\partial}{\partial p}\omega = -(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)$$

Divergence and vertical motion

Integrating with respect to pressure, we get:

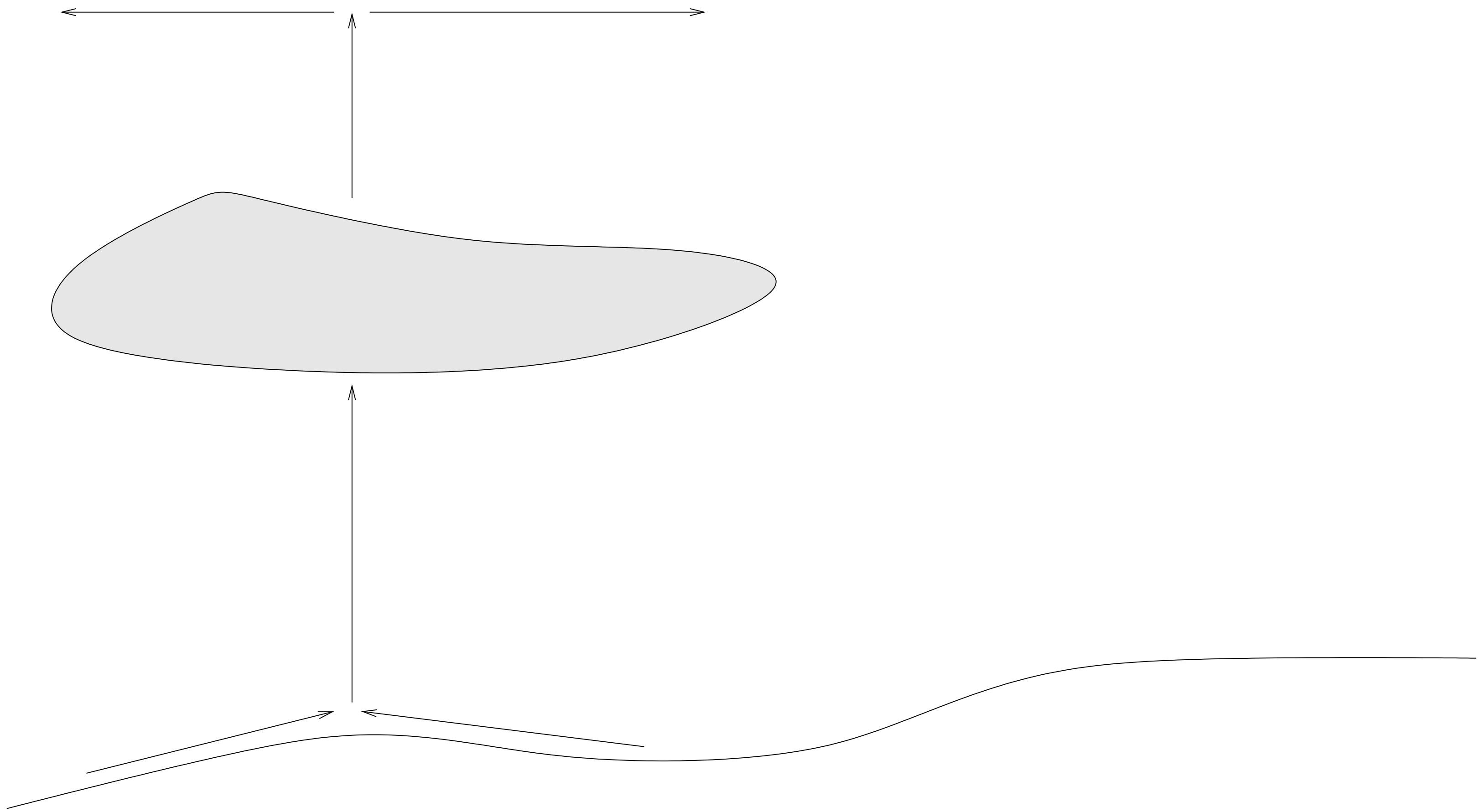
$$\omega(p) - \omega(p^*) = -\int_{p*}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\, dp$$

If we take $p* = 0$ (the top of the atmosphere):

$$\omega(p) = -\int_{0}^{p} (\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\, dp$$

Where there is *horizontal* divergence, there is vertical motion

Divergence



Vorticity

The full vorticity is a vector:

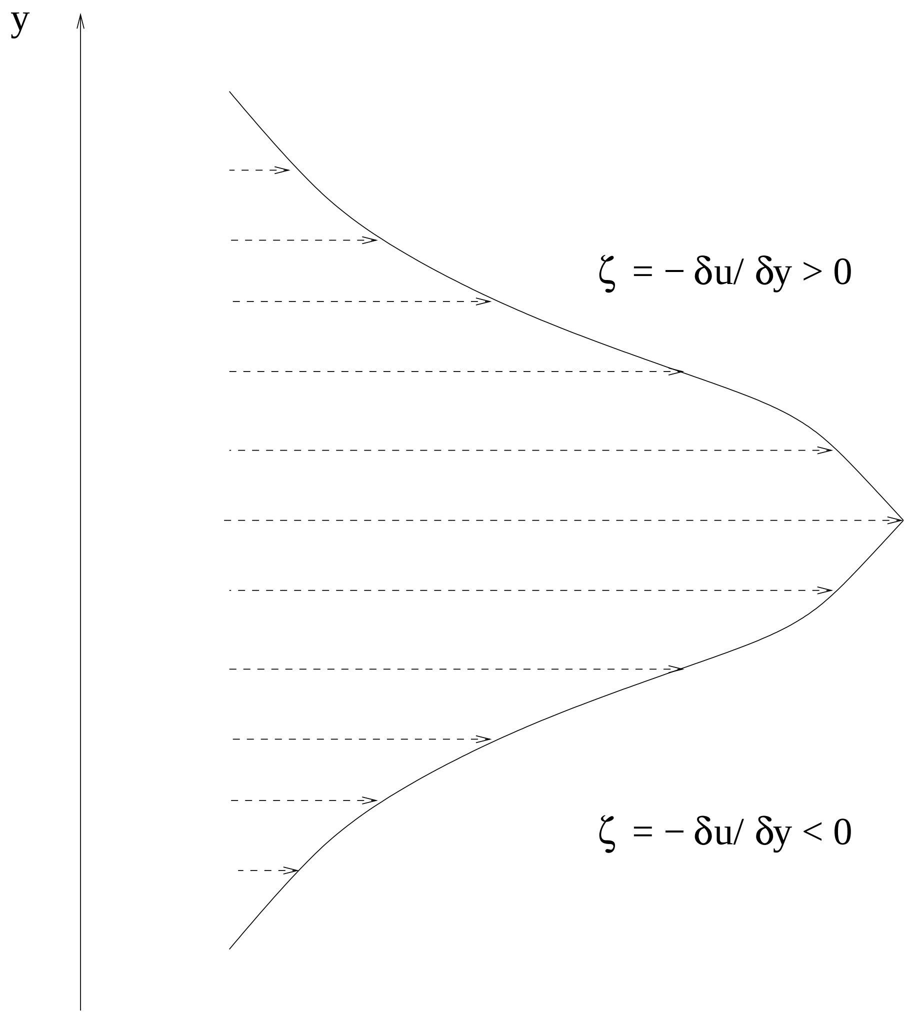
$$\vec{\zeta} \equiv \nabla \times \vec{u}$$

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

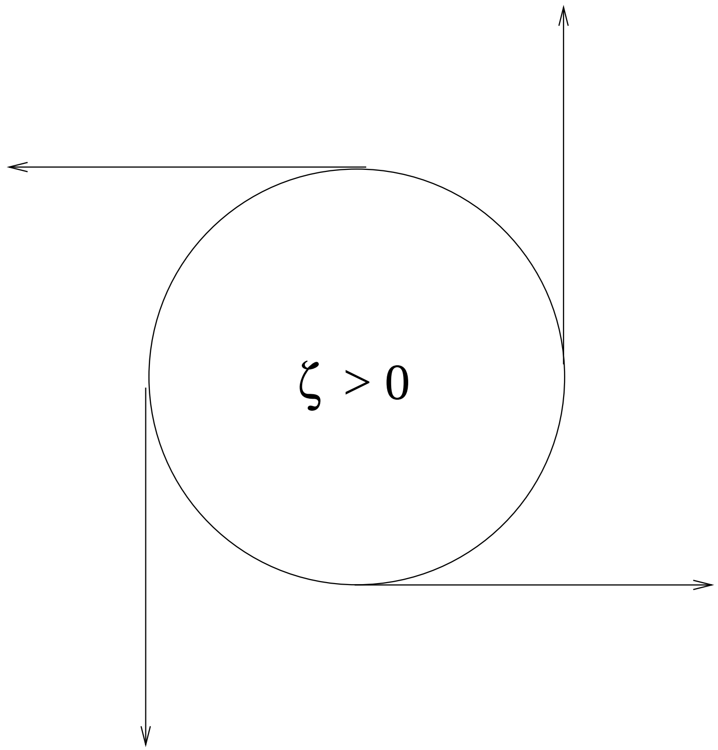
Because the winds are quasi-two dimensional, the most important is the *vertical component*:

$$\vec{\zeta} = \zeta \hat{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity



Vorticity



Example

What is the vorticity of a typical tornado? Assume *solid body rotation*, with a velocity of 100 m/sec, 20 m from the center.

In cylindrical coordinates, the vorticity is:

$$\zeta = \frac{1}{r}\frac{\partial\ r v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

For solid body rotation, $v_r = 0$ and

$$v_\theta = \omega r$$

with $\omega =$const.

Vorticity

So:

$$\zeta = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = \frac{1}{r}\frac{\partial \omega r^2}{\partial r} = 2\omega$$

We have $v_\theta = 100$ m/sec at $r = 20$ m:

$$\omega = \frac{v_\theta}{r} = \frac{100}{20} = 5\, rad/sec$$

So:

$$\zeta = 10\, rad/sec$$

Absolute vorticity

Now add rotation. Recall the velocity in the fixed frame:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

So:

$$\nabla \times \vec{u}_F = \nabla \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) = \vec{\zeta} + \nabla \times (\vec{\Omega} \times \vec{r})$$

$$= \vec{\zeta} + \nabla \times (z\Omega_y - y\Omega_z, x\Omega_z, -x\Omega_y)$$

$$= \vec{\zeta} + 2\vec{\Omega}$$

- Like the tornado, the earth is in solid body rotation—its vorticity is *twice the rotation rate*

Absolute vorticity

Two components:

- $\nabla \times \vec{u}$ — the *relative vorticity*
- 2Ω — the *planetary vorticity*

Vertical component is the most important:

$$\zeta_a \cdot \hat{k} = (\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u) + 2\Omega_z = \zeta + 2\Omega sin(\theta) = \zeta + f$$

Hereafter, ζ refers only to the vertical relative vorticity

Absolute vorticity

Scaling:

$$\zeta \propto \frac{U}{L}$$

So:

$$\frac{|\zeta|}{f} \approx \frac{U}{fL} = \epsilon$$

The Rossby number

Absolute vorticity

- $\epsilon \ll 1$

Geostrophic velocities

Planetary vorticity dominates the absolute vorticity

- $\epsilon \gg 1$

Cyclostrophic velocities

Relative vorticity dominates

Circulation

Circulation is the integral of vorticity over an area:

$$\Gamma \equiv \iint \zeta \, dA$$

Due to Stoke's theorem, we can rewrite this as an integral of the velocity around the circumference:

$$\Gamma = \iint \nabla \times \vec{u} \, dA = \oint \vec{u} \cdot \hat{n} \, dl$$

Thus we can derive an equation for the circulation by integrating the momentum equations around a closed curve.

Circulation

Consider the full momentum equations in z-coordinates:

$$\frac{d}{dt}u + f_y w - f_z v = -\frac{1}{\rho}\frac{\partial}{\partial x}p + F^x$$

$$\frac{d}{dt}v + f_z u = -\frac{1}{\rho}\frac{\partial}{\partial y}p + F^y$$

$$\frac{d}{dt}w - f_y u = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + F^z$$

We can write these in vector form as:

$$\frac{d}{dt}\vec{u}_F = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F}$$

Circulation

Define the *absolute circulation* as:

$$\Gamma_F = \oint \vec{u}_F \cdot \vec{dl}$$

Notice that:

$$\frac{d}{dt}\Gamma_F = \oint \frac{d}{dt}\vec{u}_F \cdot \vec{dl} + \oint \vec{u}_F \cdot \frac{d}{dt}\vec{dl}$$

Rewrite the last term as:

$$\oint \vec{u}_F \cdot \frac{d}{dt}\vec{dl} = \oint \vec{u}_F \cdot \vec{du} = \frac{1}{2}\oint d|\vec{u}^2| = 0$$

This vanishes integrated around a closed circuit

Circulation

So we have:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{g} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl}$$

Gravity vanishes because can write as the gradient of a potential:

$$\vec{g} = -g\hat{k} = \frac{\partial}{\partial z}(-gz) = \nabla \Phi_g$$

and the closed integral of a gradient vanishes:

$$\oint \nabla \Phi_g \cdot \vec{dl} = \oint d\Phi_g = 0$$

Circulation

So:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

Put rotation back in:

$$\Gamma_F = \oint (\vec{u}_R + \Omega \times r) \cdot \vec{dl}$$

Using Stoke's theorem:

$$\oint (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot \vec{dl} = \iint \nabla \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot \hat{n}\, dA$$

Circulation

From before:

$$\nabla \times (\vec{\Omega} \times \vec{r}) = 2\vec{\Omega}$$

So:

$$\frac{d}{dt}\iint (\vec{\zeta} + 2\vec{\Omega}) \cdot \hat{n}\, dA = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

Notice we have not made *any* approximations up to now. If the motion is quasi-horizontal, then $\hat{n} = \hat{k}$, so:

$$\frac{d}{dt}\iint (\zeta + f)\, dA = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl}$$

Kelvin's theorem

Consider the pressure term. If the atmosphere is *barotropic* (the temperature and density are constant on pressure surfaces):

$$\oint \frac{dp}{\rho} = \frac{1}{\rho} \oint dp = 0$$

Also, if we use pressure surfaces, then:

$$dp = \rho d\Phi$$

so:

$$\oint \frac{dp}{\rho} = \oint d\Phi = 0$$

Kelvin's theorem

In either case, the pressure term vanishes

So if friction is also unimportant ($\vec{F} = 0$), then:

$$\frac{d}{dt}\Gamma_a = 0$$

The *absolute circulation* (the sum of relative and planetary circulations) is conserved on the parcel

Kelvin's theorem

If the area is small, the vorticity is approximately constant over the area, so:

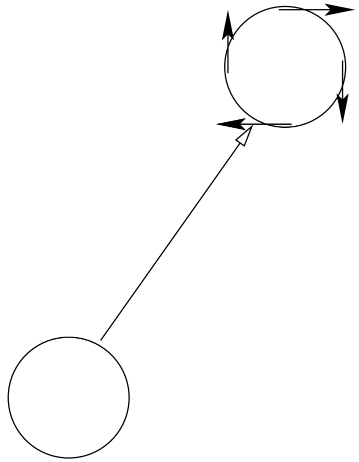
$$\frac{d}{dt}\Gamma_a \approx \frac{d}{dt}(\zeta + f)A = 0$$

which implies:

$$(\zeta + f)A = const.$$

on a parcel. Thus if a parcel's area or latitude changes, it's vorticity must change to compensate.

Kelvin's theorem



Move a parcel north, where f is larger. Either:

- Vorticity decreases
- Area decreases

Vorticity equation

Kelvin's theorem applies to a region of air

Derive an equation which works pointwise

First expand for a small area:

$$\frac{d}{dt}(\zeta + f)A = A\frac{d}{dt}(\zeta + f) + (\zeta + f)\frac{dA}{dt} = 0$$

or:

$$\frac{d}{dt}(\zeta + f) = -\frac{1}{A}(\zeta + f)\frac{dA}{dt}$$

Vorticity equation

Let the small region have an area:

$$A = \delta x \, \delta y$$

Then:

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{\delta x \, \delta y}(\delta y \frac{d}{dt}\delta x + \delta x \frac{d}{dt}\delta y)$$

$$= \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}$$

As $\delta \rightarrow 0$, have:

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Vorticity equation

So we have the *vorticity equation*:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

This is a *very* useful equation. It applies at all points, so it tells us how vorticity is changing everywhere.

But notice: the equation is in Cartesian coordinates, but $f = 2\Omega sin(\theta)$ is in polar coordinates!

Beta-plane approximation

If we limit the latitude range, we can expand f in a Taylor Series about the center latitude:

$$f(\theta) \approx f(\theta_0) + (\theta - \theta_0)\frac{df}{d\theta} + \frac{(\theta - \theta_0)^2}{2}\frac{d^2 f}{d\theta^2} + \ \ldots$$

We have $y = R\theta$, where R is the earth radius. Keeping the first two terms:

$$f \approx f_0 + \beta(y - y_0)$$

where:

$$f_0 = 2\Omega sin(\theta_0), \quad \beta = \frac{2\Omega}{R} cos(\theta_0)$$

Beta-plane

In order for this to work, we require that the range of latitudes be small. In particular, we require the second term to be small compared to the first:

$$|f_0| \gg |\beta(y - y_0)|$$

If the range of y is equal to L, then we require:

$$L \ll \frac{f_0}{\beta} = \frac{2\Omega\, sin(\theta)}{2\Omega\, cos(\theta)/R} = R\, tan(\theta) \approx R$$

So the domain scale in y must be small compared to the earth's radius (6400 km)

Beta-plane

With the β-plane approximation, we have:

$$\frac{d}{dt}f = v\frac{df}{dy} = \beta v$$

So the vorticity equation is just:

$$\frac{d}{dt}\zeta + \beta v = -(f + \zeta)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

Non-divergence

Another benefit of the β-plane approximation is that the geostrophic velocities are *non-divergent*

$$u_g = -\frac{1}{f}\,\frac{\partial \Phi}{\partial y} \approx -\frac{1}{f_0}\,\frac{\partial \Phi}{\partial y}$$

$$v_g \approx \frac{1}{f_0}\,\frac{\partial \Phi}{\partial x}$$

So:

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = -\frac{1}{f_0}\,\frac{\partial^2 \Phi}{\partial y \partial x} + \frac{1}{f_0}\,\frac{\partial^2 \Phi}{\partial x \partial y} = 0$$

Beta-plane

So with geostrophic velocities, on the β-plane, the vorticity equation is approximately:

$$\frac{d}{dt}\zeta_g + \beta v_g = 0$$

Here:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi$$

So:

$$\frac{d}{dt}\nabla^2 \Phi + \beta \frac{\partial \Phi}{\partial x} = 0$$

Example

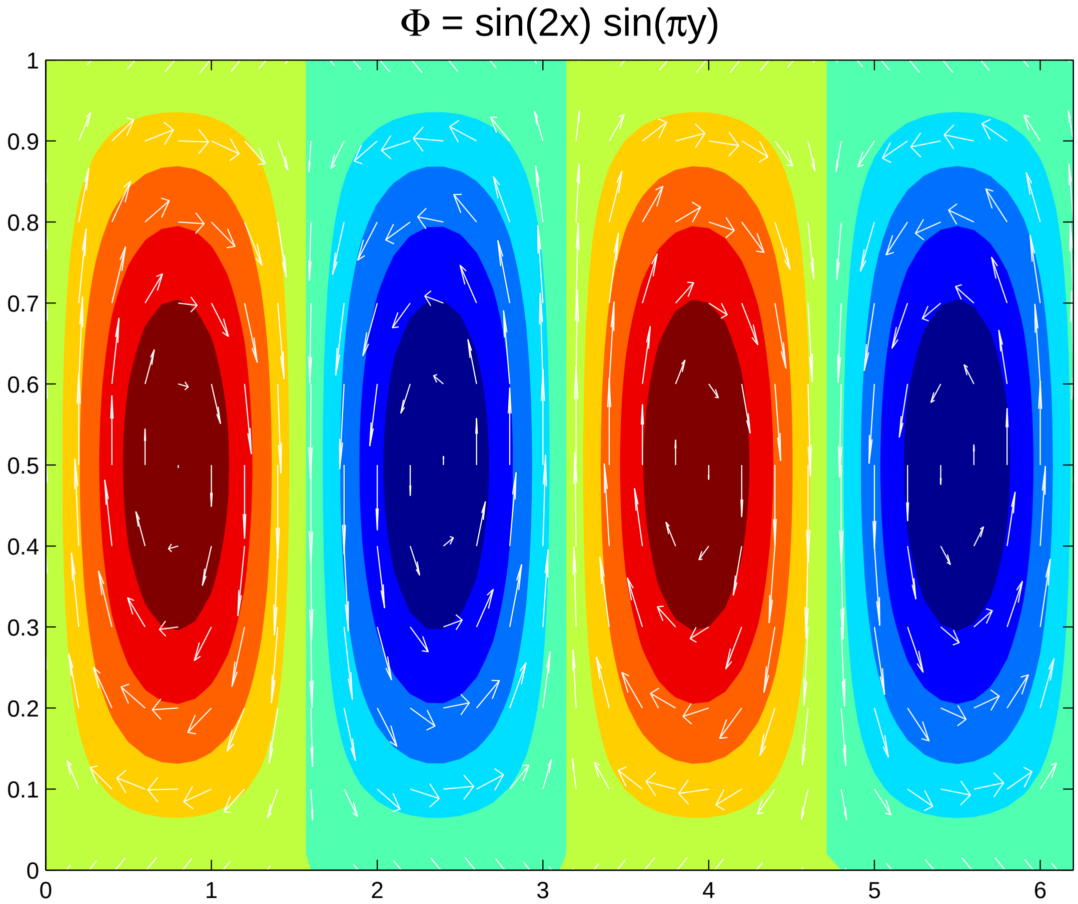
Say the geopotential is initially:

$$\Phi = f_0 A\, sin(2x - \omega t)\, sin(\pi y)$$

Describe how the field evolves in time

What is ω?

Initial geopotential



Example I

We must solve:

$$\frac{d}{dt}\nabla^2\Phi + \beta\frac{\partial\Phi}{\partial x} = 0$$

or:

$$\frac{\partial}{\partial t}\nabla^2\Phi + u_g \cdot \nabla(\nabla^2\Phi) + \beta\frac{\partial\Phi}{\partial x} = 0$$

Evaluate each of the terms.

Example I

The velocities are:

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi = -\pi A\, sin(2x - \omega t)\, cos(\pi y)$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = 3A\, cos(2x - \omega t)\, sin(\pi y)$$

and:

$$\nabla^2\Phi = -(4 + \pi^2) f_0 A\, sin(2x - \omega t)\, sin(\pi y)$$

Example I

We also need the derivatives:

$$\frac{\partial}{\partial x}\nabla^2\Phi = -2(4+\pi^2)f_0 A\, cos(2x-\omega t)\, sin(\pi y)$$

$$\frac{\partial}{\partial y}\nabla^2\Phi = -\pi(4+\pi^2)f_0 A\, sin(2x-\omega t)\, cos(\pi y)$$

Example I

Collect terms:

$$u_g \frac{\partial}{\partial x} \nabla^2 \Phi + v_g \frac{\partial}{\partial y} \nabla^2 \Phi = [\pi A\, sin(2x - \omega t)\, cos(\pi y)] \times$$

$$[2(4 + \pi^2) f_0 A\, cos(2x - \omega t)\, sin(\pi y)] - [2A\, cos(2x - \omega t)\, sin(\pi y)] \times$$

$$[\pi (4 + \pi^2) f_0 A\, sin(2x - \omega t)\, cos(\pi y)]$$

$$= [2\pi f_0 A^2 (4 + \pi^2) - 2\pi f_0 A^2 (4 + \pi^2)] sin(2x - \omega t) cos(2x - \omega t)$$

$$\times sin(\pi y) cos(\pi y) = 0$$

Example I

Also:

$$\beta \frac{\partial}{\partial x} \Phi = 2\beta f_0 A\, cos(2x - \omega t)\, sin(\pi y)$$

And:

$$\frac{\partial}{\partial t} \nabla^2 \Phi = \omega(4 + \pi^2) f_0 A\, cos(2x - \omega t)\, sin(\pi y)$$

So:

$$\omega(4 + \pi^2) f_0 A\, cos(2x - \omega t)\, sin(\pi y)$$

$$+2\beta f_0 A\, cos(2x - \omega t)\, sin(\pi y) = 0$$

Example I

This simplifies to:

$$\omega = -\frac{2\beta}{4+\pi^2}$$

So the solution is:

$$\Phi = A\,sin(2x + \frac{2\beta}{4+\pi^2}t)\,sin(\pi y)$$

This is a “travelling wave”

Note we have obtained a “prediction” of the field valid at *all times* in the future

Phase speed

We can rewrite the solution:

$$\Phi = A\,cos[2(x + \frac{\beta t}{4+\pi^2})]\,sin(\pi y)$$

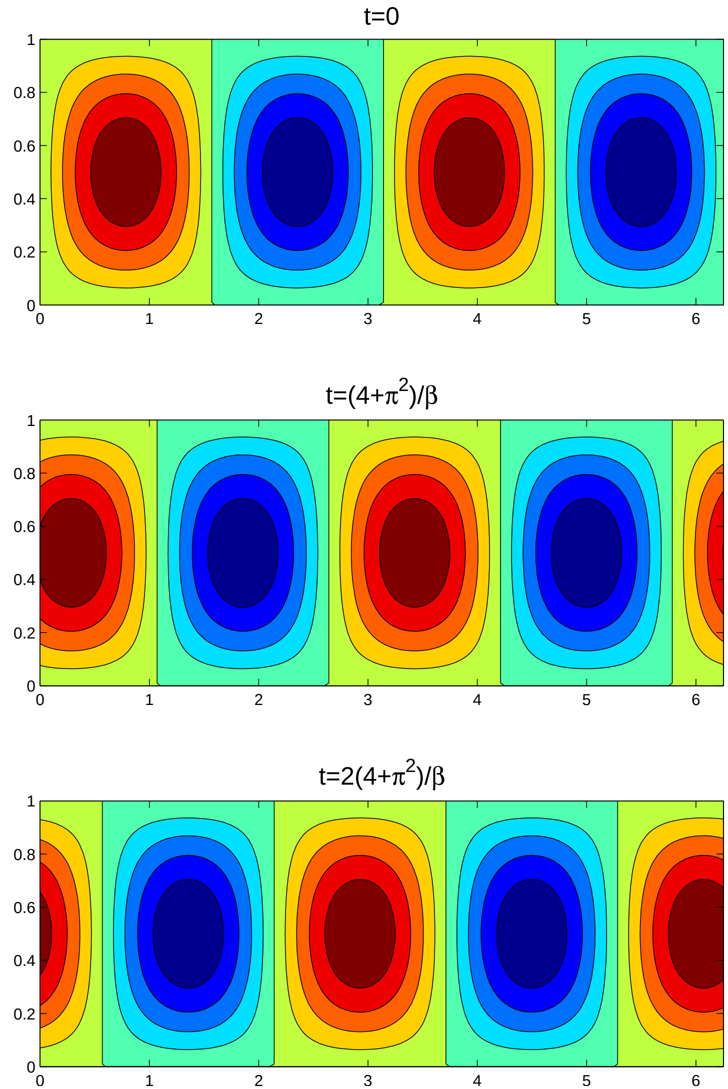
Thus the wave preserves it shape and tranlates with a *phase speed*:

$$c = \frac{\omega}{k} = -\frac{\beta}{4+\pi^2}$$

This is how fast the crests in the wave move

Because $c < 0$, waves move toward *negative* x (westward)

Westward



More general solution

In fact, the solution holds for any type of sinusoidal wave. Consider:

$$\Phi = Aexp(ikx + ily - i\omega t)$$

Here (k, l) are the *wavenumbers* in the x- and y-directions:

$$k = \frac{2\pi}{\lambda_x} \quad , \quad l = \frac{2\pi}{\lambda_y}$$

In the previous example, $k = 2$ and $l = \pi$

More general solution

With this exponential form, it's easy to derive all the terms in the vorticity equation:

$$u_g = -\frac{1}{f_0} il\Phi, \quad v_g = \frac{1}{f_0} ik\Phi, \quad \zeta = -\frac{1}{f_0}(k^2 + l^2)\Phi$$

The advective term is:

$$u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta = -\frac{1}{f_0^2} kl(k^2 + l^2)\Phi^2 + \frac{1}{f_0^2} kl(k^2 + l^2)\Phi^2 = 0$$

So this vanishes for *any single wave*

$\rightarrow$ The wave does not advect its own vorticity

More general solution

So the vorticity equation is:

$$\frac{\partial}{\partial t}\nabla^2\Phi + \beta\frac{\partial}{\partial x}\Phi = 0$$

or:

$$i\omega(k^2 + l^2)\Phi + ik\beta\Phi = 0$$

Solving for ω:

$$\omega = -\frac{\beta k}{k^2 + l^2}$$

This is known as the *dispersion relation* for the wave

More general solution

So the solution is:

$$\Phi = Aexp(ikx + ily + i\frac{\beta kt}{k^2 + l^2})$$

This has a phase speed in the x-direction of:

$$c = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2}$$

This is *always negative*, so all waves go west

Also, big waves (with small (k, l)) move fastest

Westward propagation

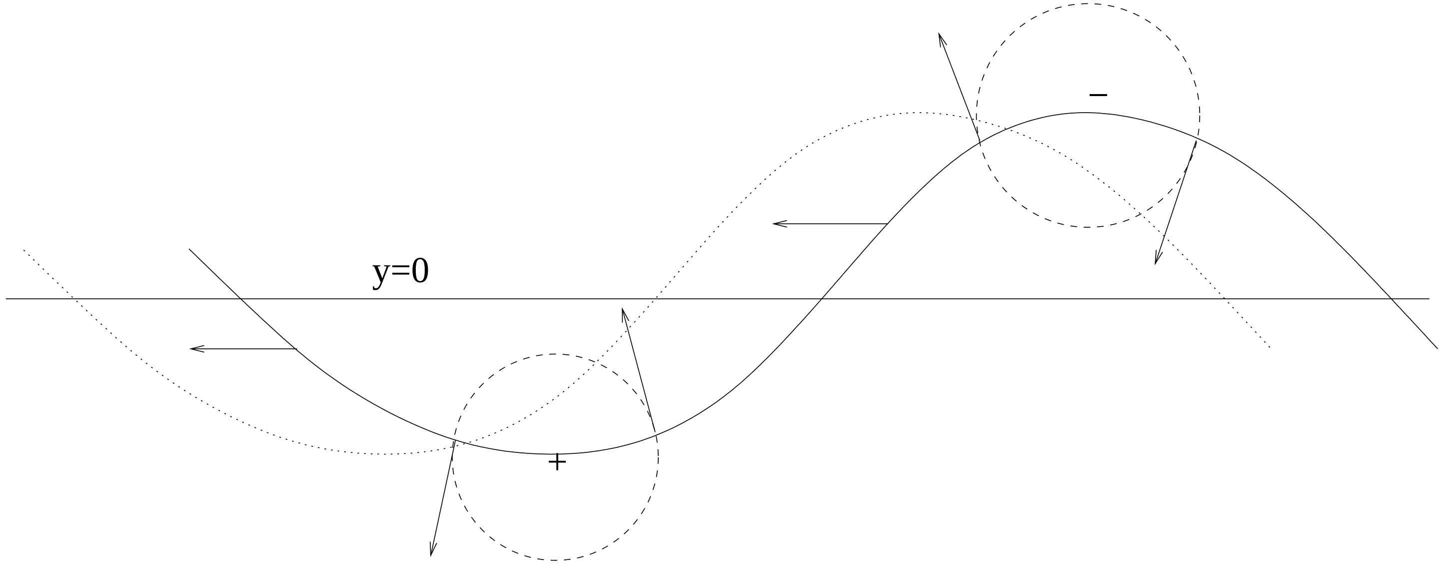
The westward propagation is actually a consequence of Kelvin's theorem

Fluid parcels advected north/south acquire relative vorticity

The parcels then advect neighboring parcels around them

Leads to a westward drift of the wave

Westward propagation



Rossby waves

Solution is known as a *Rossby wave*

Discovered by Carl Gustav Rossby (1936)

Observed in the atmosphere

Important for weather patterns

Study more later (GEF4500)

Divergence

In previous example, we ignored the divergence. But very important for the growth of unstable disturbances (storms)

The vorticity equation is:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

We can write this as:

$$\frac{d}{dt}\zeta_a = -\zeta_a\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

where $\zeta_a = \zeta + f$ is the absolute vorticity

Divergence

Consider flow with constant divergence:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D > 0$$

$$\frac{d}{dt}\zeta_a = -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -D\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\ e^{-Dt}$$

Divergence

So:

$$\zeta_a = \zeta + f \rightarrow 0$$

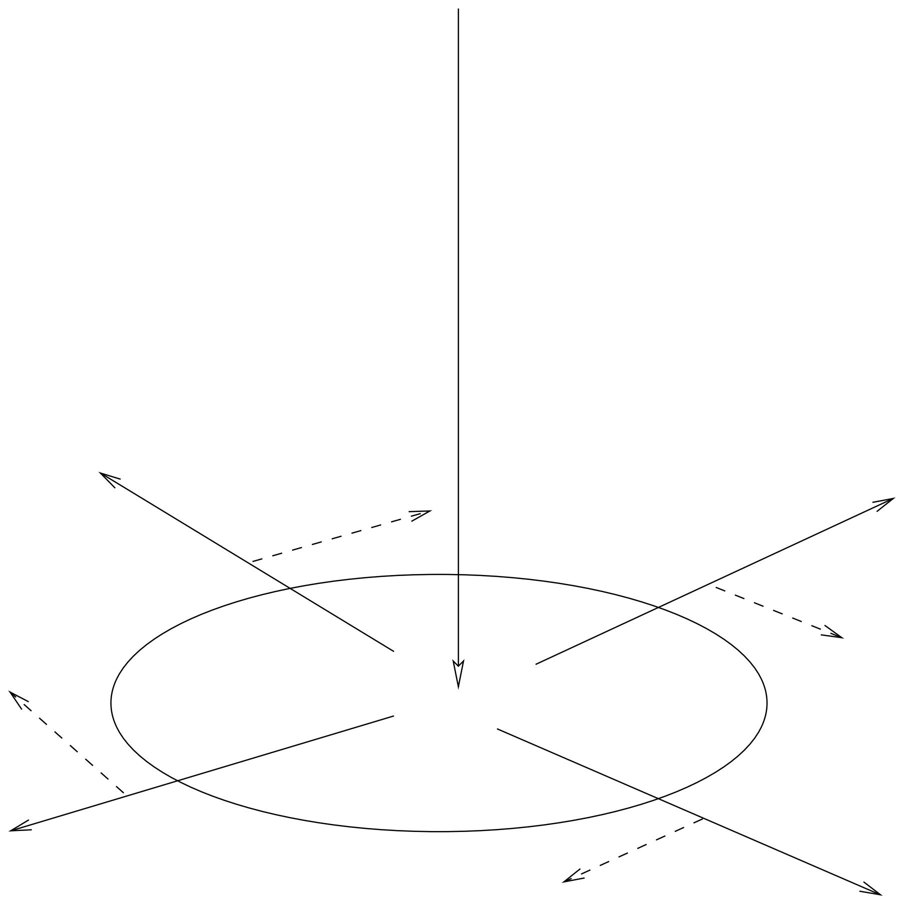
$$\zeta \rightarrow -f$$

Divergent flow favors *anticyclonic* vorticity

Vorticity approaches $-f$, regardless of initial value

Vorticity cannot *exceed* f

Divergence



Divergence

Now say $D = -C$

$$\frac{d}{dt}\zeta_a = -\zeta_a(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = C\zeta_a$$

$$\zeta_a(t) = \zeta_a(0)\ e^{Ct}$$

$$\zeta_a \longrightarrow \pm\infty$$

But which sign?

Divergence

If the Rossby number is small, then:

$$\zeta_a(0) = \zeta(0) + f \approx f > 0$$

So:

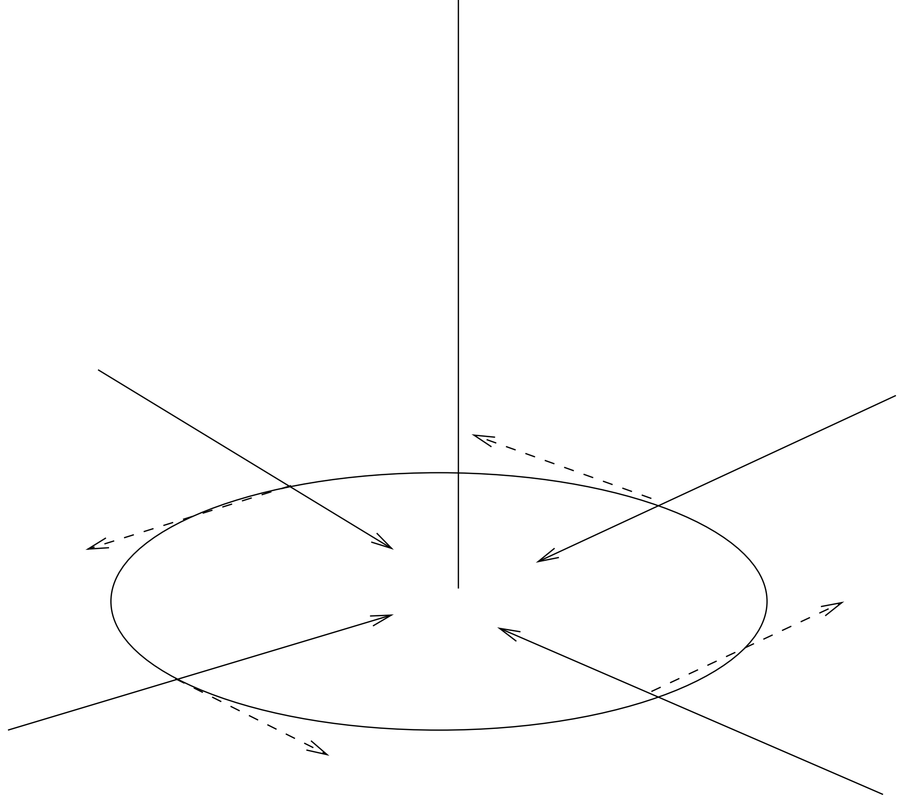
$$\zeta \rightarrow +\infty$$

Convergent flow favors *cyclonic* vorticity

Vorticity increases *without bound*

- Why intense storms are cyclonic

Convergence



Forecasting

In realistic cases though, require numerical solutions of the vorticity equation

This was done for weather forecasts in the 1940's

Approach:

Assume geostrophic velocities:

$$u \approx u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y}$$

$$v \approx v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

Forecasting

$$\zeta = \zeta_g = \frac{1}{f_0}\,\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f}\,(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}) = \frac{1}{f}\nabla^2 \Phi$$

Again, the divergence vanishes identically:

$$(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}) = 0$$

so the vorticity equation is:

$$(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y})\zeta_g + \beta v_g = 0$$

(Notice we neglect the vertical advection on the LHS)

Forecasting

Now only *one unknown*: Φ

$$(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y})\zeta_g + \beta v_g = 0$$

becomes:

$$(\frac{\partial}{\partial t} - \frac{1}{f_0}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} + \frac{1}{f_0}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y})\,(\frac{1}{f_0}\nabla^2\Phi + \frac{\beta}{f_0}\frac{\partial \Phi}{\partial x} = 0$$

Forecasting

Or:

$$\frac{\partial}{\partial t}\nabla^2\Phi = \frac{1}{f_0}\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial x}\nabla^2\Phi - \frac{1}{f_0}\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial y}\nabla^2\Phi - \beta\frac{\partial\Phi}{\partial x}$$

Predict how $\nabla^2\Phi$ changes in time

Then we obtain Φ by *inversion*

Forecasting

Method:

- Obtain $\Phi(x, y, t_0)$ from measurements on p-surface
- Calculate $u_g(t_0)$, $v_g(t_0)$, $\nabla^2\Phi(t_0)$
- Calculate $\nabla^2\Phi(t_1)$
- Invert to get $\Phi(t_1)$
- Start over
- Obtain $\Phi(t_2)$, $\Phi(t_3)$,...

Inversion

$$\nabla^2 \Phi = f_0 \zeta$$

Poisson's equation

Need boundary conditions to solve

Usually do this numerically

Inversion

Example: Let:

$$\zeta = sin(3x)sin(\pi y)$$

Say we have a channel:

$$x = [0, 2\pi], \quad y = [0, 1]$$

Periodic in x. Also solid walls at $y = 0, 1$ so that:

$$v = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} = 0$$

Inversion

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = sin(3x)sin(\pi y)$$

Try a particular solution:

$$\Phi = Asin(3x)sin(\pi y)$$

This solution works in a channel, because:

$$\Phi(x = 2\pi) = \Phi(x = 0)$$

and $\frac{\partial}{\partial x}\Phi = 0$ at $y = 0, 1$:

Inversion

Substitute into equation:

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = -(9+\pi^2)Asin(3x)sin(\pi y) = sin(3x)sin(\pi y)$$

So:

$$\Phi = -\frac{1}{9+\pi^2}\, sin(3x)sin(\pi y)$$

Then we can proceed (calculate u_g, v_g, etc.)

Inversion

Inversion is a *smoothing* operation

Preferentially weights the large scale features. Say instead we had:

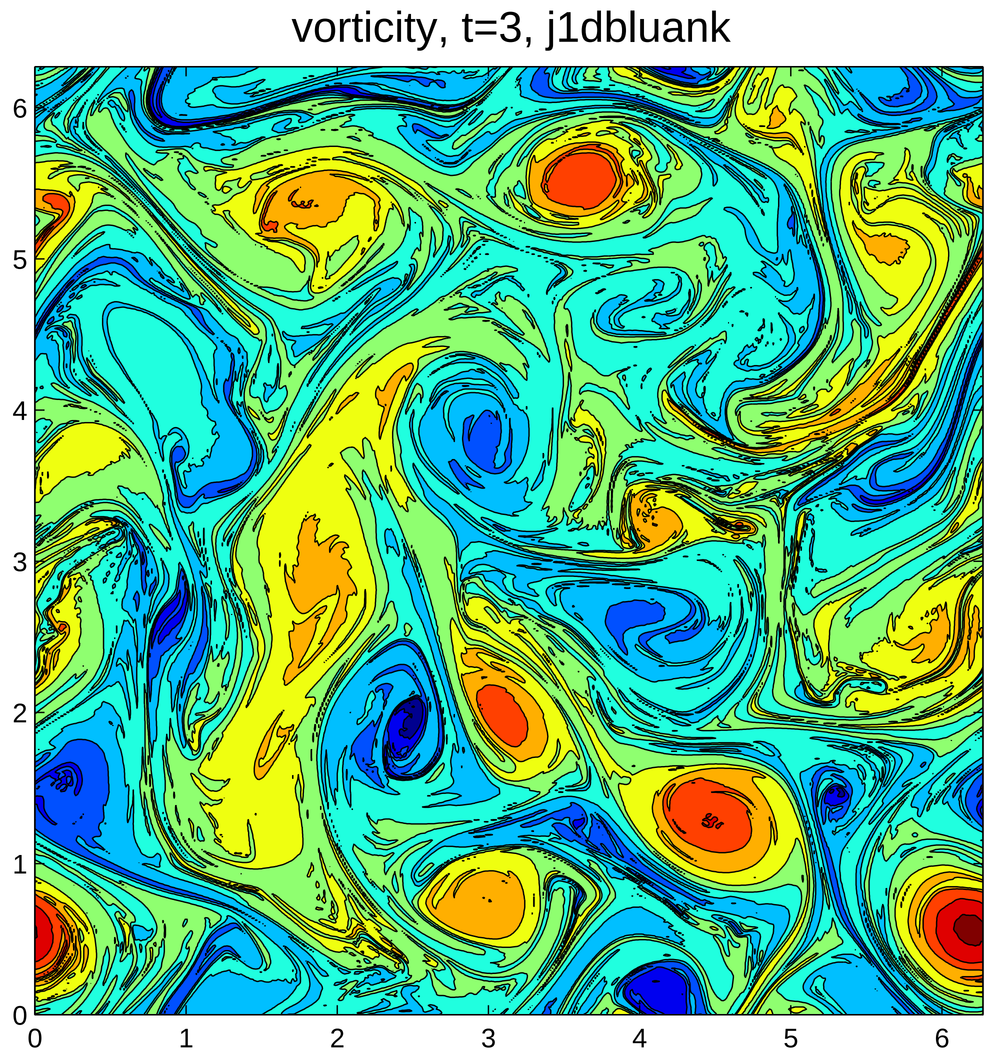
$$\nabla^2 \Phi = sin(3x)sin(3y) + sin(x)sin(y)$$

Then:

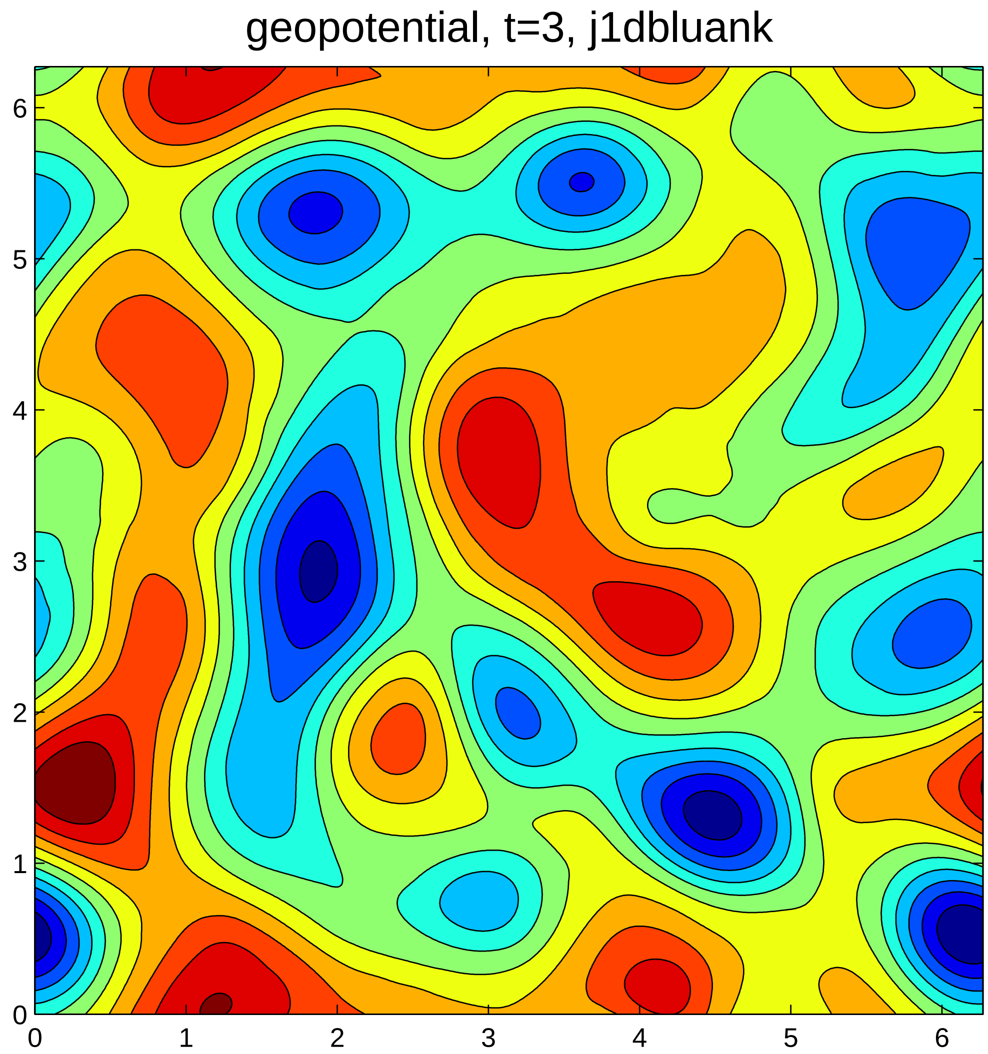
$$\Phi = \frac{1}{18} sin(3x)sin(3y) + sin(x)sin(y)$$

The smaller wave contributes less to the geopotential

Vorticity, turbulence simulation



Geopotential, turbulence simulation



Frictional effect

Friction is weak in most of the troposphere, but acts near the ground. How does this affect the vorticity? From Kelvin's theorem:

$$\frac{d}{dt}\iint (\vec{\zeta} + 2\vec{\Omega}) \cdot \hat{k}\, dA = \oint \vec{F} \cdot \vec{dl}$$

From Stoke's theorem:

$$\oint \vec{F} \cdot \vec{dl} = \iint (\nabla \times \vec{F}) \cdot \hat{k}\, dA$$

For a small area, we have:

$$\frac{d}{dt}(\zeta + f)A = (\nabla \times \vec{F}) \cdot \hat{k}\, A$$

Frictional effect

or:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (\nabla \times \vec{F}) \cdot \hat{k}$$

Say the frictional forcing is:

$$F_x = \nu \nabla^2 u, \quad F_y = \nu \nabla^2 v$$

Then:

$$(\nabla \times \vec{F}) \cdot \hat{k} = (\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x) = \nu \nabla^2 (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = \nu \nabla^2 \zeta$$

Friction

Let's ignore the divergence term for a moment. Then:

$$\frac{d}{dt}(\zeta + f) = \nu \nabla^2 \zeta$$

If, in addition, $f \approx const.$:

$$\frac{d}{dt}\zeta = \nu \nabla^2 \zeta$$

So friction *diffuses* vorticity. It causes cyclones to spread out and weaken

Friction

But how long does this take?

Scaling the vorticity equation, we find:

$$\frac{\zeta}{T} = \frac{\nu\zeta}{L^2} \quad \rightarrow \quad T = \frac{L^2}{\nu}$$

If $\nu = 10^{-5}\, m^2/sec$ and $L = 10^6\, m$, then:

$$T = 10^{17}\, sec \approx 3.17 \times 10^9\, yr\ !$$

- Molecular friction ineffective at damping storms

Summary

The vorticity equation is approximately:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

On the β-plane, we can write:

$$\frac{d}{dt}\zeta + \beta v = -(\zeta + f)\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

- Vorticity changes due to meridional motion
- Vorticity changes due to divergence
- Including friction diffuses vorticity

Barotropic potential vorticity

Consider an atmospheric layer with *constant density*, between two surfaces, at $z = z_1, z_2$ (e.g. the surface and the tropopause)

The continuity equation is:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0$$

If density constant, then:

$$(\nabla \cdot \vec{u}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Barotropic potential vorticity

So:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Thus the vorticity equation can be written:

$$\frac{d}{dt}(\zeta + f) = (\zeta + f)\frac{\partial w}{\partial z}$$

Barotropic fluid

Recall a barotropic fluid has no vertical shear

The thermal wind relation in z-coordinates is:

$$f\frac{\partial}{\partial z}v = \frac{1}{\rho}\frac{\partial}{\partial x}(\frac{\partial}{\partial z}p) = \frac{1}{\rho}\frac{\partial}{\partial x}(-\rho g) = 0$$

and

$$f\frac{\partial}{\partial z}u = -\frac{1}{\rho}\frac{\partial}{\partial y}(\frac{\partial}{\partial z}p) = -\frac{1}{\rho}\frac{\partial}{\partial y}(-\rho g) = 0$$

- $(\zeta + f)$ does not vary with z!

(This is an example of the “Taylor-Proudman theorem”)

Barotropic potential vorticity

So the integral of the vorticity equation is simply:

$$\int_{z1}^{z2} (\frac{\partial}{\partial t} + \vec{u} \cdot \nabla)\,(\zeta + f)dz =$$

$$h(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla)\,(\zeta + f) = (\zeta + f)\,[w(z_2) - w(z_1)]$$

where $h = z_2 - z_1$. Note that $w = Dz/Dt$. Thus:

$$w(z_2) - w(z_1) = \frac{d}{dt}(z_2 - z_1) = \frac{dh}{dt}$$

Barotropic potential vorticity

So:

$$h \frac{d}{dt}(\zeta + f) = (\zeta + f)\frac{dh}{dt}$$

dividing by h^2:

$$\frac{1}{h}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{dh}{dt} = 0$$

which is the same as:

$$\frac{d}{dt}\frac{\zeta + f}{h} = 0$$

Barotropic potential vorticity

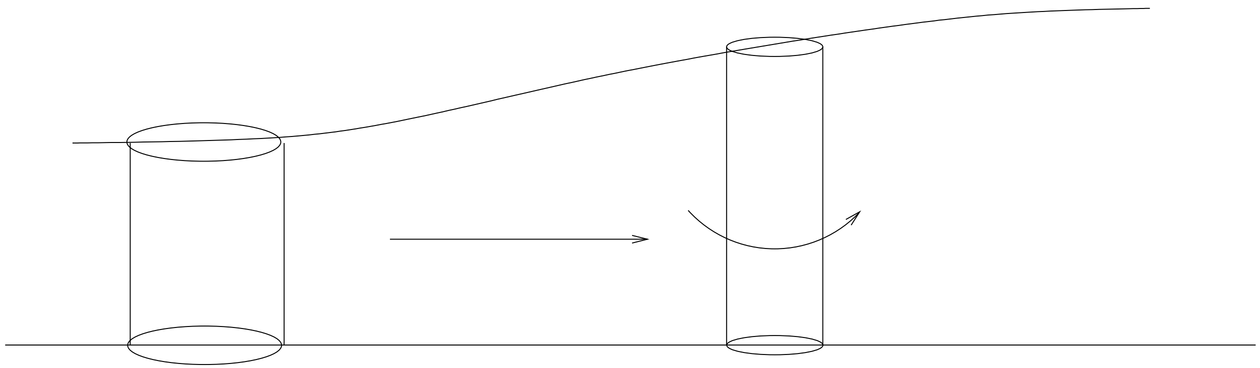
Thus the barotropic potential vorticity (PV):

$$\frac{\zeta + f}{h} = const.$$

is conserved on a fluid parcel in the layer.

If h increases, either ζ or f must also increase

Layer potential vorticity



Alternate derivation

Consider a fluid column between z_1 and z_2. As it moves, conserves mass:

$$\frac{d}{dt}(hA) = 0$$

So:

$$hA = const.$$

Because the density is constant, we can apply Kelvin's theorem:

$$\frac{d}{dt}(\zeta + f)A \propto \frac{d}{dt}\frac{\zeta + f}{h} = 0$$

Potential temperature

But the atmosphere is not constant density. What use is the potential vorticity?

As move upward in atmosphere, both temperature and pressure change—neither is absolute.

But can define the *potential temperature* which is absolute—accounts for pressure change.

The potential vorticity can then be applied in layers *between potential temperature surfaces*

Potential temperature

We can write the thermodynamic energy equation as:

$$c_p dT - \alpha dp = dq$$

With zero heating, $dq = 0$, so:

$$c_p dT - \alpha dp = c_p dT - \frac{RT}{p} dp = 0$$

after using the ideal gas law. Rewriting:

$$c_p \, dlnT - R \, dlnp = 0$$

Potential temperature

Thus:

$$c_p \, lnT - R \, lnp = const.$$

So:

$$c_p \, lnT - R \, lnp = c_p \, ln\theta - R \, lnp_0$$

Here θ and p_0 are the temperature and pressure at a reference level, usually the surface. Rearranging:

$$\theta = T \, (\frac{p_0}{p})^{R/c_p}$$

Potential temperature

The *potential temperature*, θ, is the temperature a parcel would have if moved back to the earth's surface without any heating.

Alternately, if there is no heating, an air parcel conserves its potential temperature, θ

We call a surface with constant potential temperature an isentropic surface or an "adiabat"

Note the potential temperature depends on *both* T and p

Layer potential vorticity

Flow between two isentropic surfaces is trapped if there is zero heating

So mass in a column between two surfaces is conserved:

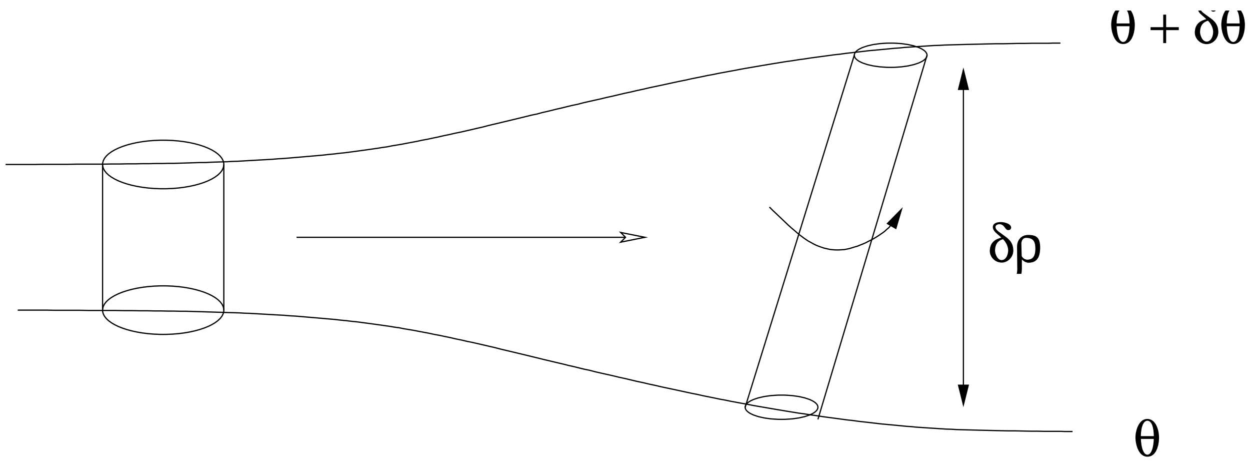
$$\rho A \delta z = const.$$

We want to express this in terms of the separation between potential temperature surfaces. From the hydrostatic relation, we can write:

$$-\rho A \frac{\delta p}{\rho g} = -A \frac{\delta p}{g} = const.$$

where δp is the spacing in pressure

Layer potential vorticity



Layer potential vorticity

Rewrite δp thus:

$$\delta p = (\frac{\partial \theta}{\partial p})^{-1} \delta \theta$$

Here, $\frac{\partial \theta}{\partial p}$ is the *stratification*. The stronger the stratification, the smaller the pressure difference between temperature surfaces. Thus:

$$-\frac{A\delta p}{g} = A(\frac{\partial \theta}{\partial p})^{-1} \frac{\delta \theta}{g} = const. \quad \rightarrow \quad A(\frac{\partial \theta}{\partial p})^{-1} = const.$$

Layer potential vorticity

Mass is conserved between the adibatic surfaces. But the flow is generally *not* barotropic (the flow will generally have vertical shear)

- Can we still apply Kelvin's theorem?

Assuming no friction, we have:

$$\frac{d}{dt}\iint(\zeta + f)\,dA = \oint \frac{dp}{\rho}$$

Layer potential vorticity

To evaluate the RHS, we write:

$$T = \theta(\frac{p}{p_0})^{R/c_p}$$

from the definition of potential temperature. With the Ideal Gas Law, we get:

$$p = \rho R\theta(\frac{p}{p_0})^{R/c_p}$$

or:

$$\rho = p^{c_v/c_p}\frac{1}{R\theta}p_0^{R/c_p} = Cp^{c_v/c_p}$$

Layer potential vorticity

The density is a function *only of pressure*. So:

$$\oint \frac{dp}{\rho} \propto \oint dp^{1-c_v/c_p} = 0$$

$\rightarrow$ Kelvin's theorem applies in the layer. Thus:

$$\frac{d}{dt}[(\zeta + f)A] = 0$$

This implies:

$$\frac{d}{dt}[(\zeta + f)\frac{\partial\theta}{\partial p}] = 0$$

Layer potential vorticity

This is Ertel's (1942) "isentropic potential vorticity"

It is conserved for adiabatic flows

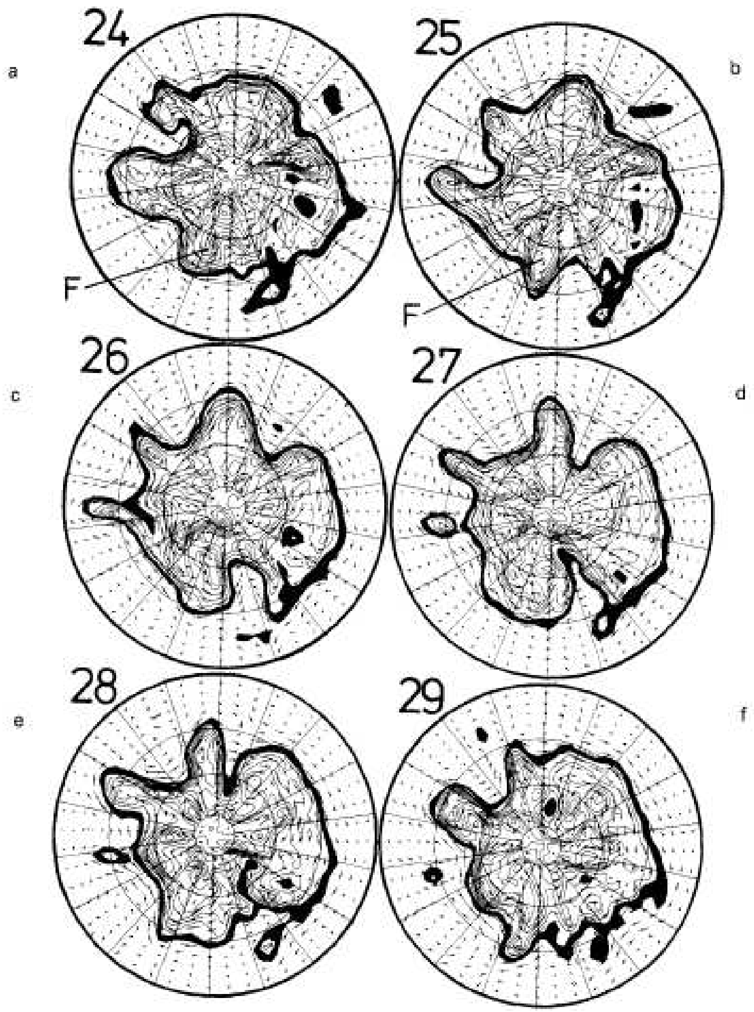
A very useful quantity: can label air by its PV

Can distinguish air in the troposphere (which has high background PV) which comes from stratosphere (which has low PV)

Visible in satellite images because stratospheric air is also dry (less cloudy)

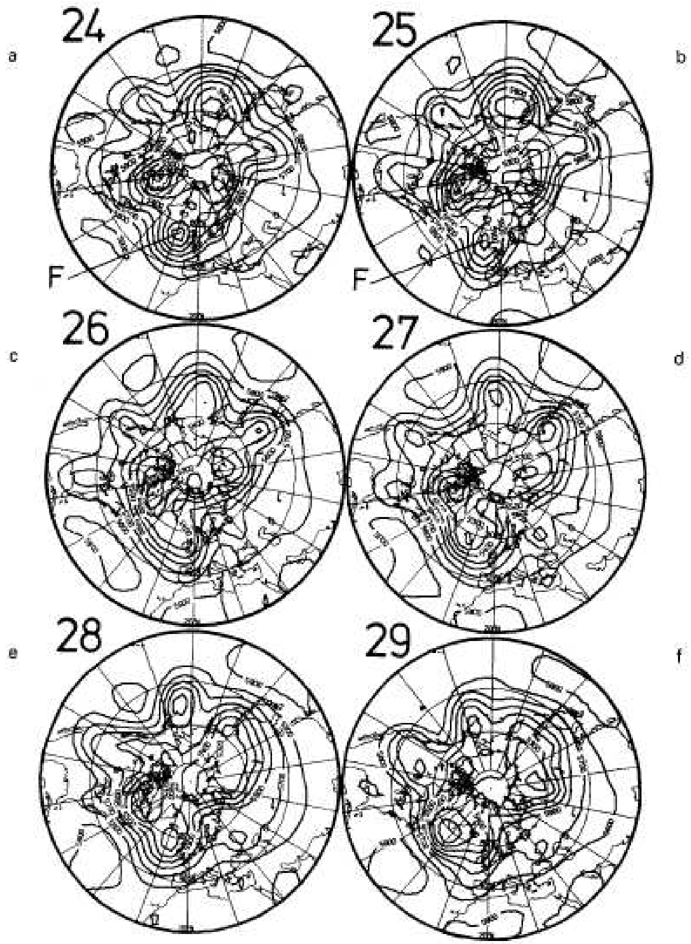
Isentropic PV

PV on the 330 K surface (24-29/9/82); Hoskins et al. (1985)

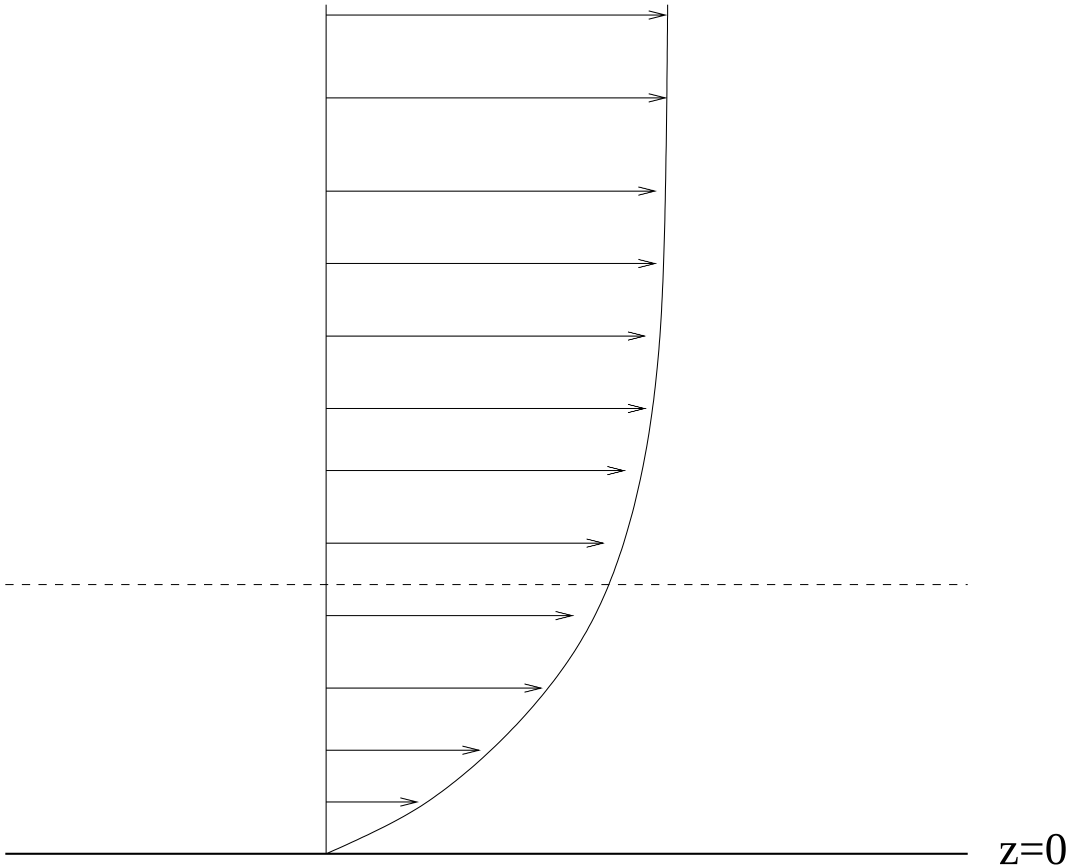


Isentropic PV

$\Phi(500)$ (24-29/9/82); Hoskins et al. (1985)



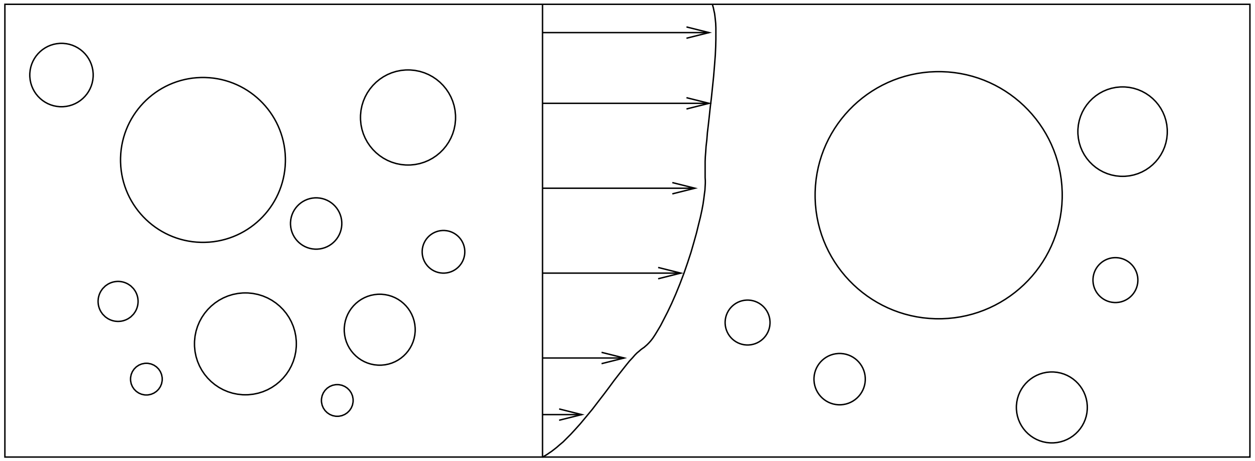
Planetary boundary layer



Turbulence

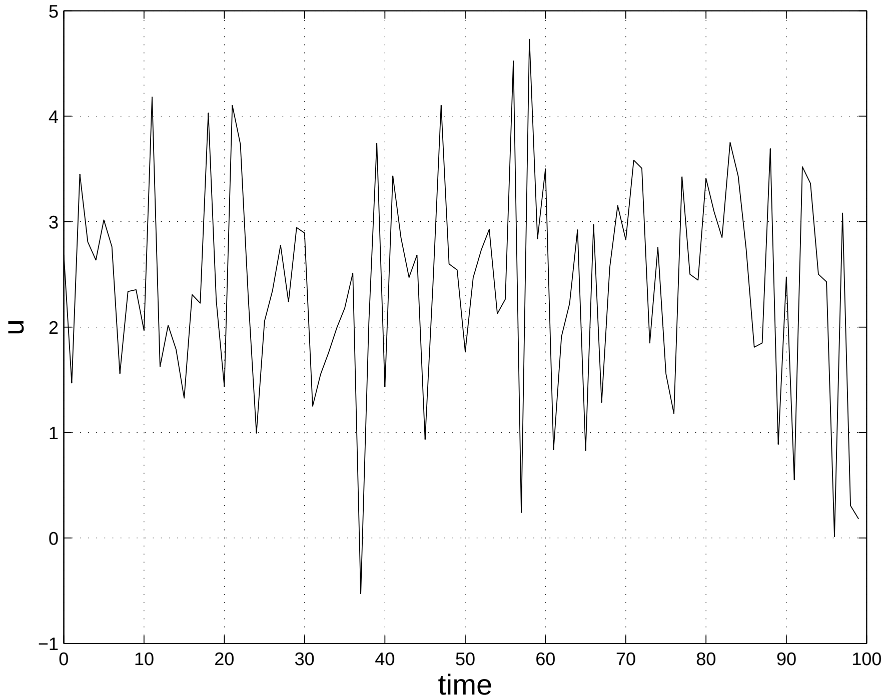
There is a *continuum* of eddy scales

Largest resolved by our models, but the smallest are not.



Turbulence

Turbulence causes flucutations with short time scales



Distinguish the time mean velocity and fluctuations

Turbulence

Define averaging procedure as:

$$\overline{a} \equiv \frac{1}{T}\int_0^T a\,dt$$

So if we write:

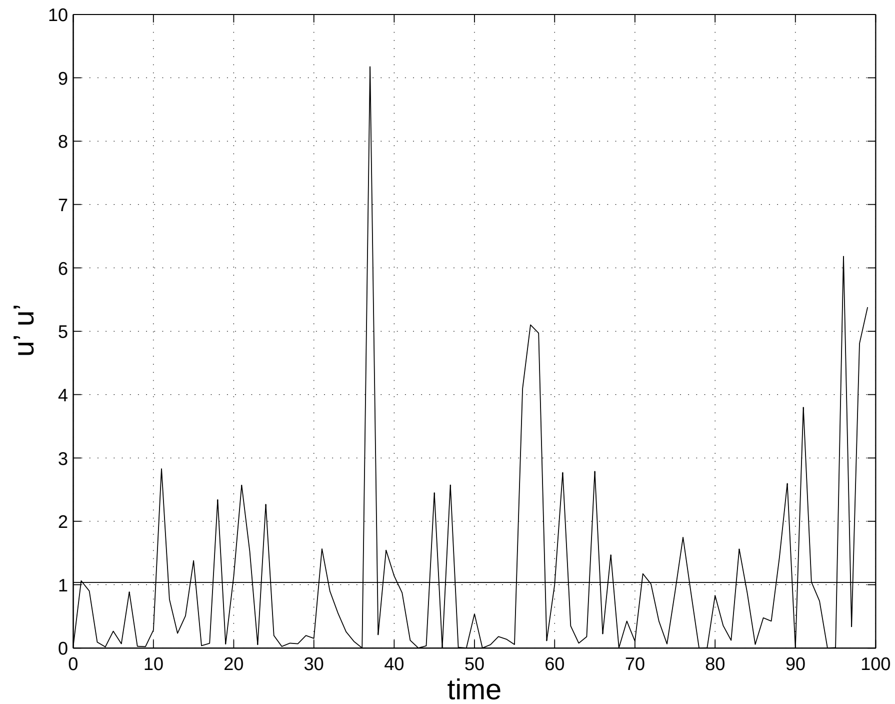
$$u = \overline{u} + u'$$

then:

$$\overline{u'} = 0$$

Turbulence

But note that $\overline{u'u'} \neq 0$



So products of primed variables don't vanish on average

Reynolds decomposition

Because the flow in the boundary layer is fully three dimensional and near the groud, we use the full momentum equations:

$$\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

Again neglect molecular friction.

Boussinesq approximation

Also, the density doesn't vary much. So:

$$\rho \approx \rho_0 + \rho'(x, y, z, t)$$

where ρ_0=const. and:

$$|\rho'| \ll |\rho_0|$$

This allows us to write:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p \quad \rightarrow \quad \frac{1}{\rho_0}\frac{\partial}{\partial x}p$$

This is called the *Boussinesq approximation*. It is valid in the boundary layer (and in the ocean).

Boussinesq approximation

The Boussinesq approximation also simplifies the continuity equation:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0$$

Assuming $rho \approx$ const., this is simply:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

So the boundary layer flow is *incompressible*

Boussinesq approximation

Incompressibility allows us to write the advection terms in a more usable form:

$$\vec{u} \cdot \nabla u = \nabla \cdot (\vec{u} u) - u(\nabla \cdot \vec{u}) = \nabla \cdot (\vec{u} u)$$

So the x-momentum equation can be written:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x} p$$

PBL equations

Substitute the partitioned variables into the x-momentum equation:

$$\frac{\partial}{\partial t}(\overline{u}+u')+\frac{\partial}{\partial x}(\overline{u}+u')(\overline{u}+u')+\frac{\partial}{\partial y}(\overline{v}+v')(\overline{u}+u')+\frac{\partial}{\partial z}(\overline{w}+w')(\overline{u}+u')$$

$$-f(\overline{v}+v') = \frac{1}{\rho_0}\frac{\partial}{\partial x}(\overline{p}+p')$$

Then we average the whole equation.

PBL equations

$$\frac{\partial}{\partial t}\overline{u} + \frac{\partial}{\partial x}(\overline{uu}) + \frac{\partial}{\partial y}(\overline{vu}) + \frac{\partial}{\partial z}(\overline{wu}) - f\overline{v}$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p} - (\frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'})$$

Notice we moved the eddy terms to the RHS. Similarly:

$$\frac{\partial}{\partial t}\overline{v} + \frac{\partial}{\partial x}(\overline{uv}) + \frac{\partial}{\partial y}(\overline{vv}) + \frac{\partial}{\partial z}(\overline{wv}) + f\overline{u}$$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial y}\overline{p} - (\frac{\partial}{\partial x}\overline{v'u'} + \frac{\partial}{\partial y}\overline{v'v'} + \frac{\partial}{\partial z}\overline{v'w'})$$

PBL equations

Prime terms on the RHS are the "eddy stresses". Because the aspect ratio of the BL is small, we can focus on the vertical terms:

$$\frac{\partial}{\partial t}\overline{u} + \frac{\partial}{\partial x}(\overline{uu}) + \frac{\partial}{\partial y}(\overline{vu}) + \frac{\partial}{\partial z}(\overline{wu}) - f\overline{v} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

$$\frac{\partial}{\partial t}\overline{v} + \frac{\partial}{\partial x}(\overline{uv}) + \frac{\partial}{\partial y}(\overline{vv}) + \frac{\partial}{\partial z}(\overline{wv}) + f\overline{u} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\overline{p} - \frac{\partial}{\partial z}\overline{v'w'}$$

PBL equations

If the Rossby number is small, the velocities outside the boundary layer are nearly geostrophic. So:

$$-f\overline{v} \approx -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{p} - \frac{\partial}{\partial z}\overline{u'w'}$$

or:

$$-f\overline{v} \approx -f\overline{v}_g - \frac{\partial}{\partial z}\overline{u'w'}$$

$$f\overline{u} \approx f\overline{u}_g - \frac{\partial}{\partial z}\overline{v'w'}$$

$\rightarrow$ The eddies *break geostrophy*

PBL equations

But we have too many unknowns! : $\overline{u}, \overline{v}, u', v', w'$

We must *parametrize* the eddy terms, i.e. we must write the primed variables in terms of the unprimed variables.

There are two cases:

- Convective boundary layer
- Stable boundary layer

In a convective layer, heating from below causes the layer to overturn, mixing properties with height. The stable boundary layer is *stratified*.

Convective boundary layer

Due to vertical mixing, temperature and velocity do not vary with height. So we can integrate the momentum equation vertically:

$$\int_0^h -f(\overline{v} - \overline{v}_g)\, dz = -fh(\overline{v} - \overline{v}_g) =$$

$$-\int_0^h \frac{\partial}{\partial z}\overline{u'w'}\, dz = -\overline{u'w'}|_h + \overline{u'w'}|_0$$

We assume mixing vanishes at the top of the layer:

$$\overline{u'w'}|_h = 0$$

Convective boundary layer

Thus:

$$fh(\overline{v} - \overline{v}_g) = -\overline{u'w'}|_0$$

The RHS is proportional to the *stress* the bottom exerts on the atmosphere, i.e.:

$$-\overline{u'w'}|_0 = \frac{\tau_{xz}}{\rho_0}|_0$$

From surface measurements, we can parameterize this stress:

$$\frac{\tau_{xz}}{\rho_0}|_0 = -\overline{u'w'}|_0 = C_d \mathcal{V}\, u,$$

Convective boundary layer

Similarly, we have:

$$\frac{\tau_{yz}}{\rho_0}|_0 = -\overline{v'w'}|_0 = C_d \mathcal{V}\, v$$

Here C_d is the *drag coefficient*. It is determined by the data. Also:

$$\mathcal{V} \equiv (u^2 + v^2)^{1/2}$$

Notice the stress is proportional to the square of the velocity. We call this a *quadratic drag law*.

Convective boundary layer

Thus:

$$fh(\overline{v} - \overline{v}_g) = C_d \mathcal{V}\, \overline{u}$$

and:

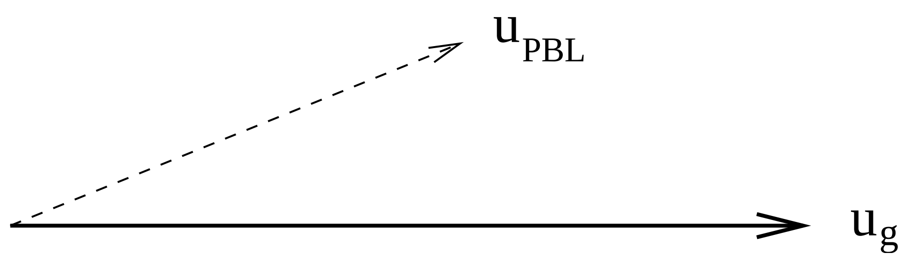
$$-fh(\overline{u} - \overline{u}_g) = C_d \mathcal{V}\, \overline{v}$$

Convective boundary layer

Say $v_g = 0$; then:

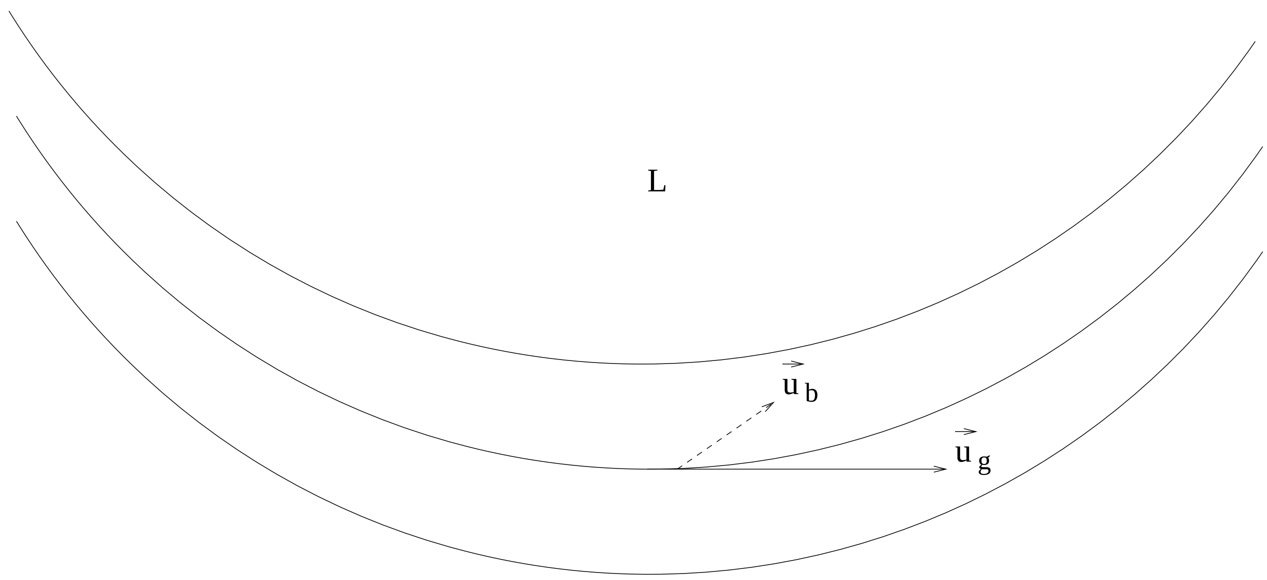
$$v = \frac{C_d}{fh} \mathcal{V} u,$$

$$u = u_g - \frac{C_d}{fh} \mathcal{V} v$$



Convective boundary layer

If $u > 0$, then $v > 0$



- Flow *down the pressure gradient*

Convective boundary layer

Solving the boundary layer equations is not so simple because $\mathcal{V} = \sqrt{u^2 + v^2}$

Coupled nonlinear equations

But we can use iterative methods

Make a first guess, then iteratively correct

Stable boundary layer

Now assume no large scale vertical mixing

Wind speed *and* direction can vary with height

General situation is very complicated

But we will examine a simple example

Stable boundary layer

Primary assumption is that the eddy mixing is proportional to the mean shear

$$\overline{u'w'} \equiv -\frac{\tau_{xz}}{\rho_0} = A_z \frac{\partial}{\partial z}\overline{u}$$

here A_z is the “eddy exchange coefficient” (with units of m^2/sec

- the stronger the shear, the stronger the mixing

Stable boundary layer

By the same argument:

$$\overline{v'w'} = -\frac{\tau_{yz}}{\rho_0} = -A_z\,\frac{\partial}{\partial z}\overline{v}$$

So we have:

$$-fv = -fv_g + \frac{\partial}{\partial z}[A_z(z)\frac{\partial}{\partial z}u]$$

$$fu = fu_g + \frac{\partial}{\partial z}[A_z(z)\frac{\partial}{\partial z}v]$$

These are linear equations and can be solved for (u, v)

Stable boundary layer

Simplest case is if $A_z(z)$ is constant

Studied by Swedish oceanographer V. W. Ekman (1905)

Solution illustrates the general behavior

We consider a boundary layer above a flat surface

Ekman layer

Boundary conditions: use the "no-slip condition":

$$u = 0, v = 0 \quad at \ z = 0$$

Far from the surface, the velocities approach their geostrophic values:

$$u \longrightarrow u_g, v \longrightarrow v_g \quad z \longrightarrow \infty$$

Assume the geostrophic flow is zonal and independent of height:

$$u_g = U, \qquad v_g = 0$$

Ekman layer

Boundary layer velocities vary only in the vertical:

$$u = u(z)\ , \quad v = v(z)\ , \quad w = w(z)$$

From continuity:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = \frac{\partial}{\partial z}w = 0\ .$$

With a flat bottom, this implies:

$$w = 0$$

Ekman layer

The system is linear, so can decompose the horizontal velocities:

$$u = U + \hat{u}, \quad v = 0 + \hat{v}$$

Then:

$$f\hat{v} = A_z \frac{\partial^2}{\partial z^2}\hat{u}$$

$$-f\hat{u} = A_z \frac{\partial^2}{\partial z^2}\hat{v}$$

Ekman layer

Boundary conditions:

$$\hat{u} = -U, \hat{v} = 0 \quad at \;\; z = 0$$

Introduce a new variable:

$$\chi \equiv \hat{u} + i\hat{v}$$

Then two equations reduce to:

$$\frac{\partial^2}{\partial z^2}\chi = i\frac{f}{A_z}\chi$$

Ekman layer

The solution is:

$$\chi = A\, exp(\frac{z}{\delta_E})\, exp(i\frac{z}{\delta_E}) + B\, exp(-\frac{z}{\delta_E})\, exp(-i\frac{z}{\delta_E})$$

where:

$$\delta_E = \sqrt{\frac{2A_z}{f}}$$

This is the “Ekman depth”

Corrections must decay going up, so:

$$A = 0$$

Ekman layer

Take the real part of the horizontal velocities:

$$u = Re\{\chi\} = Re\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

and

$$v = Im\{\chi\} = -Re\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

Ekman layer

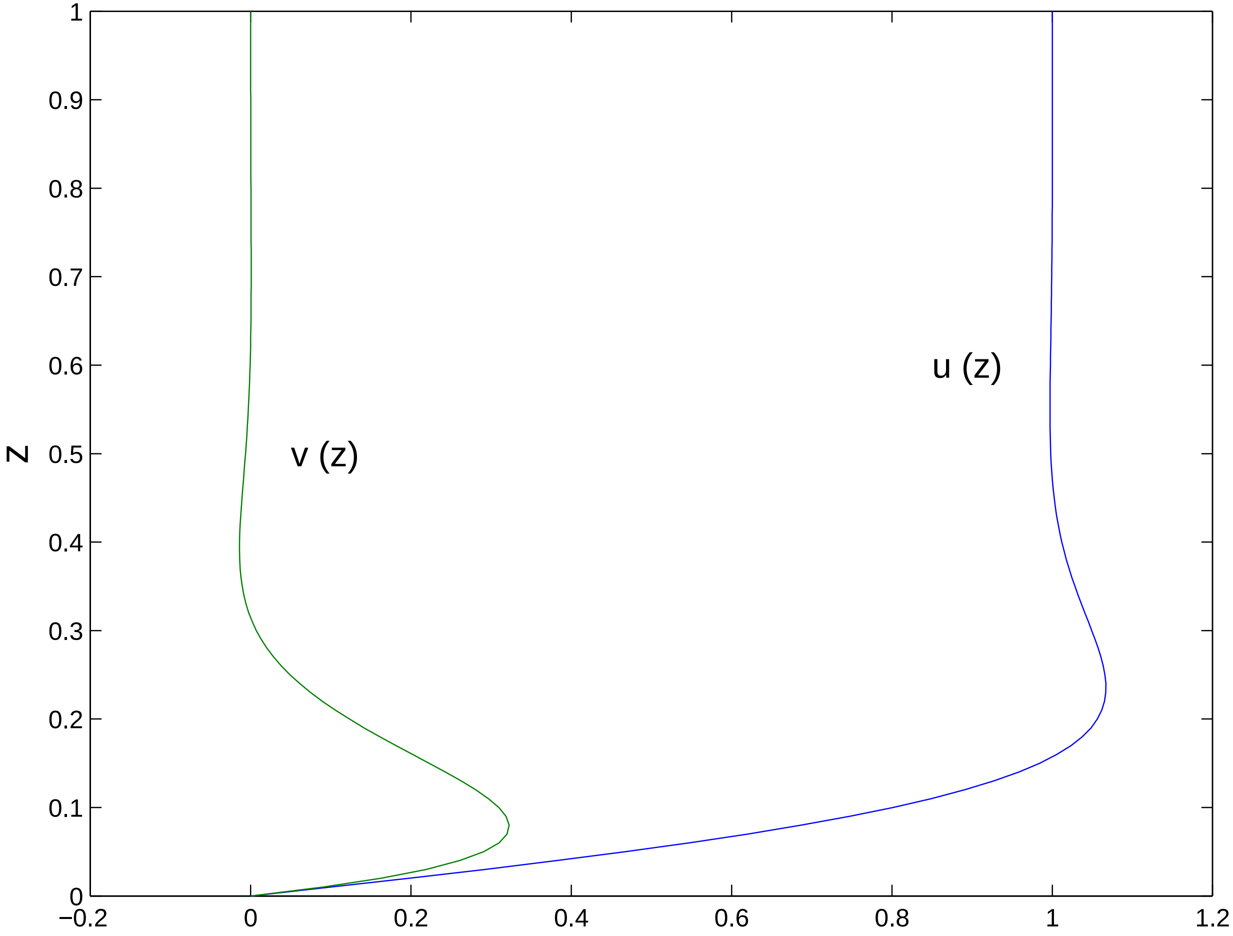
For zero flow at $z = 0$, require $Re\{B\} = -U$ and $Im\{B\} = 0$.

So:

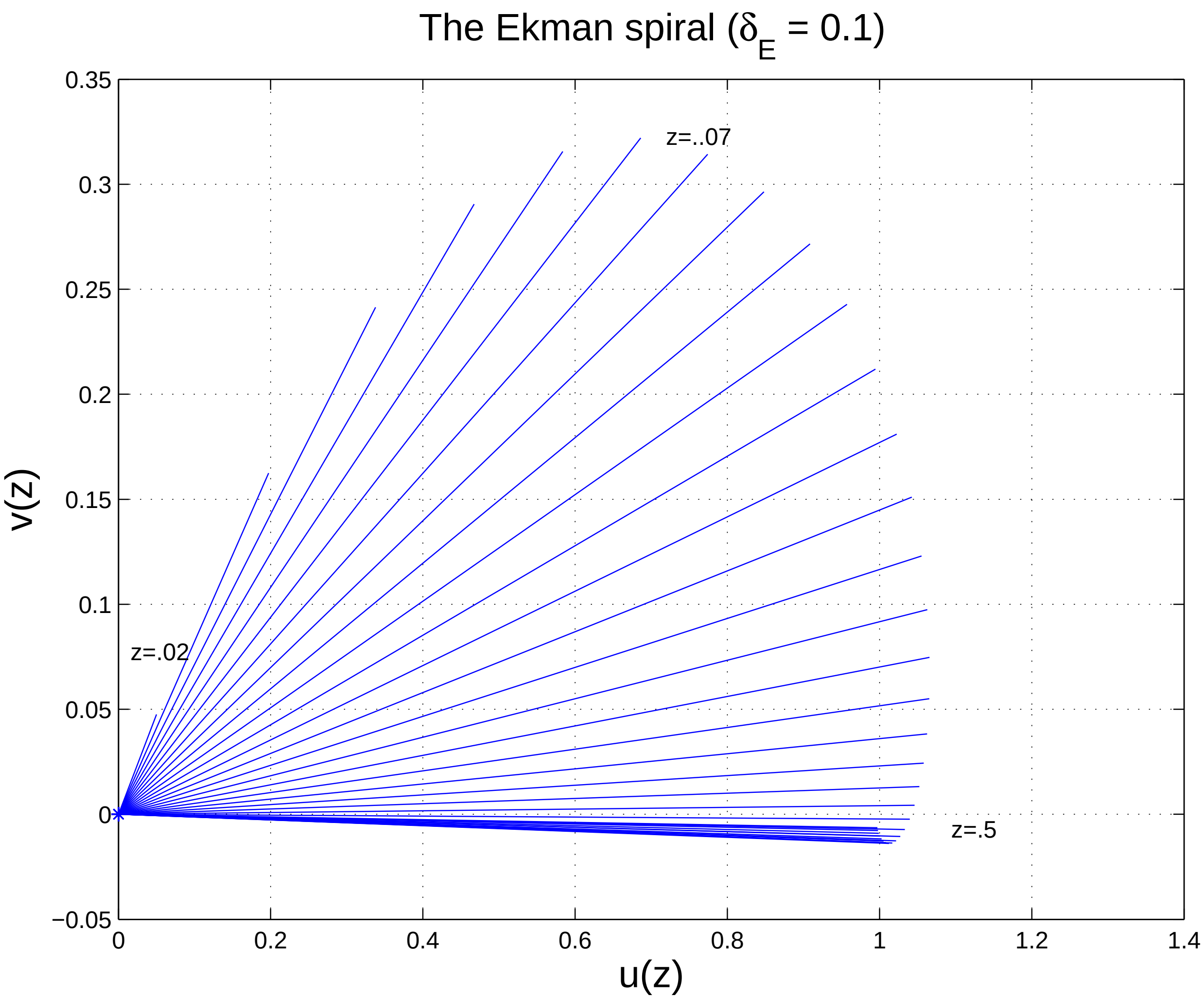
$$u = U + \hat{u} = U - U\,exp(-\frac{z}{\delta_E})\,cos(\frac{z}{\delta_E})$$

$$v = \hat{v} = U\,exp(-\frac{z}{\delta_E})\,sin(\frac{z}{\delta_E})$$

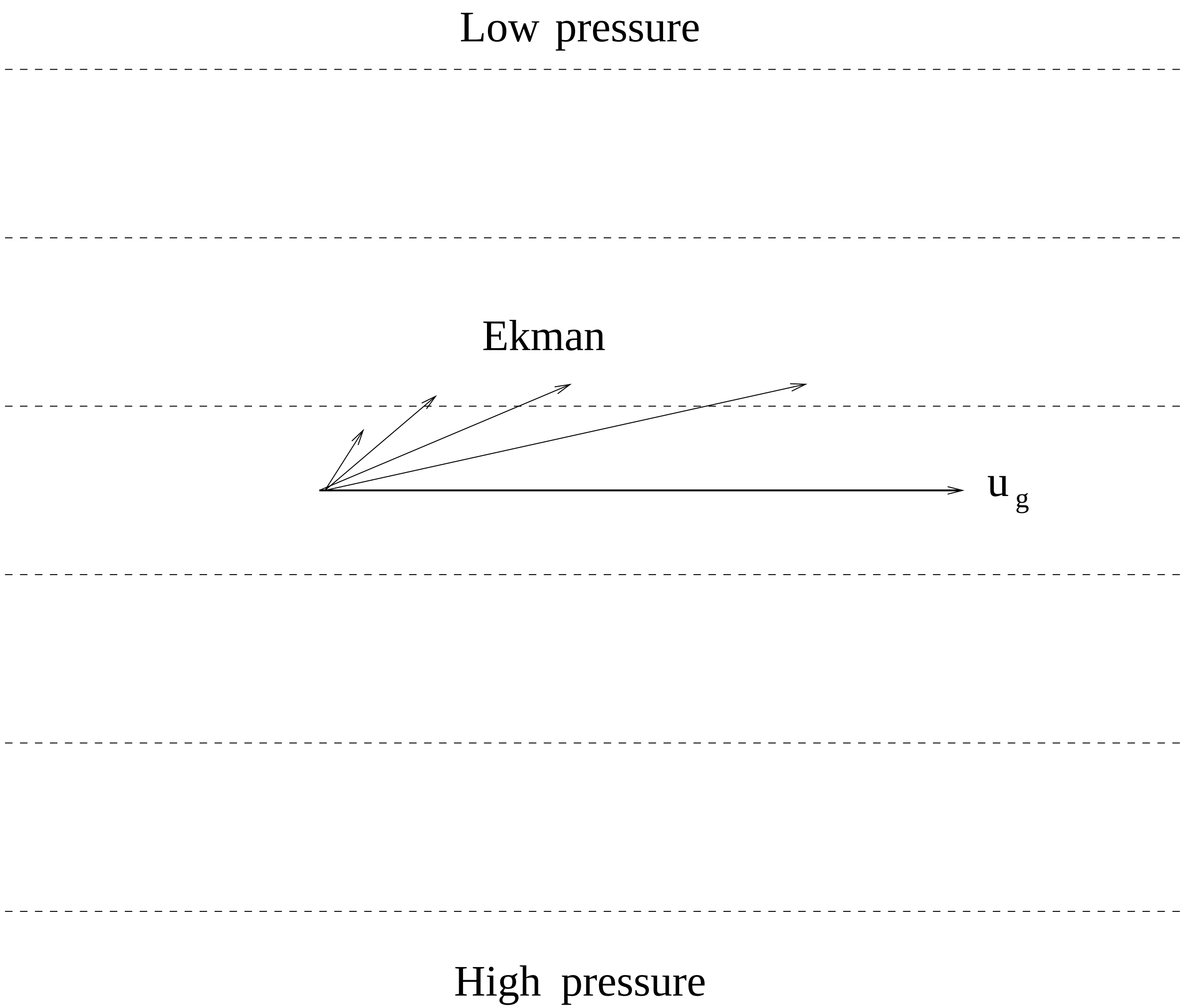
Ekman layer, $\delta_E = 0.1$



Ekman spiral, $\delta_E = 0.1$



Ekman velocities



Ekman spiral

The velocity veers to the *left*, toward low pressure

Observations suggest $u \longrightarrow u_g$ at $z = 1$ km.

If $f = 10^{-4}/sec$, then $A_z \approx 50\, m^2/sec$

As in the convective boundary layer, turbulence allows flow from high pressure to low pressure.

Spin-down

With flow down the pressure gradient, the boundary layer should *weaken* pressure systems

Consider how an Ekman layer causes a cyclone to decay in time

Or: what is the stress imposed by the Ekman layer on the overlying flow?

Spin-down

In the x-momentum equation, we have that:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p - \frac{\partial}{\partial z}\overline{u'w'}$$

or:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p + \frac{\partial}{\partial z}\frac{\tau_{xz}}{\rho_0}$$

or:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p + \frac{\partial}{\partial z}(A_z\frac{\partial}{\partial z}u)$$

Spin-down

Assume a barotropic flow. Then we can integrate the equation in the vertical:

$$h\frac{du}{dt} - fhv = -h\frac{1}{\rho_0}\frac{\partial}{\partial x}p + A_z\frac{\partial}{\partial z}u\Big|_0^h$$

Here h is the depth of the fluid (e.g. the tropopause). The stress vanishes at the top of the layer, so:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p - \frac{A_z}{h}\frac{\partial}{\partial z}u\Big|_0$$

Likewise:

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial}{\partial y}p - \frac{A_z}{h}\frac{\partial}{\partial z}v\Big|_0$$

Spin-down

Have the shear terms from the Ekman derivation

If $u_g = U$ and $v_g = 0$, we found:

$$u = U + \hat{u} = U - U\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$

$$v = \hat{v} = U\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$

So:

$$\frac{\partial}{\partial z} u|_0 = \frac{1}{\delta_e} U, \quad \frac{\partial}{\partial z} v|_0 = \frac{1}{\delta_e} U,$$

Spin-down

With $(u_g, v_g) = (0, V)$, you get:

$$(\frac{\partial}{\partial z}u, \frac{\partial}{\partial z}v)|_0 = \frac{1}{\delta_e}(-V, V)$$

So for a general flow (U, V), we have:

$$(\frac{\partial}{\partial z}u, \frac{\partial}{\partial z}v)|_0 = \frac{1}{\delta_e}(U - V, U + V)$$

Spin-down

We can put these into the momentum equations:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p - ru + rv$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial}{\partial y}p - ru - rv$$

where

$$r = \frac{A_z}{\delta_e h}$$

The Ekman layer acts like a *linear drag*

Spin-down

How does this affect the vorticity?

From Kelvin's theorem (pg 224):

$$\frac{d}{dt}\iint(\vec{\zeta}+2\vec{\Omega})\cdot\hat{k}\,dA=\iint(\nabla\times\vec{F})\,\cdot\hat{k}\,dA$$

for a barotropic fluid. For a small area, this is:

$$\frac{d}{dt}(\zeta+f)A=(\frac{\partial}{\partial x}F_y-\frac{\partial}{\partial y}F_x)\,A$$

Plugging in the Ekman terms and assuming $A=const.$:

$$\frac{d}{dt}(\zeta+f)=r\frac{\partial}{\partial x}(-u-v)-r\frac{\partial}{\partial y}(-u+v)$$

Spin-down

or:

$$\frac{d}{dt}(\zeta + f) = -r\zeta - r\chi$$

where χ is the divergence. For a geostrophic flow, $\chi = 0$. If we take $f = const.$, then:

$$\frac{d}{dt}\zeta = -r\zeta$$

So that:

$$\zeta(t) = \zeta(0)\, exp(-rt)$$

Spin-down

Thus the vorticity "spins down" with a time scale of $T = 1/r$

This is the Ekman spin-down time. How big is it?

$$1/r = \frac{\delta_e h}{A_z} \approx \frac{10^3(10^4)}{50} = 2 \times 10^5 \, sec$$

or about two days. This is *much* faster than with molecular damping, which gave a time of 3 x 10^9 years! (pg 227)

Spin-down

Ekman friction is much more potent than molecular

For baroclinic flows, we write:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p - ru|_0$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial}{\partial y}p - rv|_0$$

$\rightarrow$ Drag is determined by the bottom velocities

Note we drop the other two Ekman terms, as these contribute only to the divergence