GEF2500

GEOPHYSICAL FLUID MECHANICS

Jan Erik H. Weber

Department of Geosciences,
Section for Meteorology and Oceanography,
University of Oslo.
E-mail: j.e.weber@geo.uio.no

Blindern, January 2013

Contents

1. FLUID MECHANICS

1.1 Particle motion relative to the rotating earth

To follow the motion of individual material particles, their positions must refer to some kind of reference or coordinate system. Let us consider the following reference systems:

(X, Y, Z) - Inertial reference (axes fixed in space).
Motion in (X, Y, Z) is called *absolute* motion.

(x, y, z) - Relative reference (axes fixed to the rotating earth).
Motion in (x, y, z) is called *relative* motion.

Fig. 1.1 shows a model sketch with the earth moving around the sun whilst rotating about its own axis with angular velocity $\vec{\Omega}$.

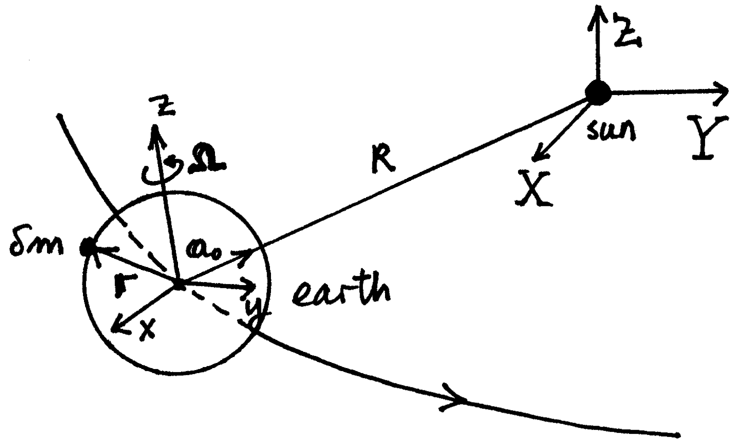

Fig. 1.1 Model sketch.

Consider the motion of a small particle with mass δm. We introduce the following notation:

$\vec{a}_{abs}$ - acceleration of δm measured in (X, Y, Z)
$\vec{a}_{rel}$ - acceleration of δm measured in (x, y, z).

Generally, if $\vec{\Omega}$ = const., we have (see the Appendix):

$$\vec{a}_{abs} = \vec{a}_{rel} + 2\vec{\Omega}\times\vec{v}_{rel} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + \vec{a}_0. \tag{1.1.1}$$

Here we have defined

$$\vec{v}_{rel} = \left(\frac{d\vec{r}}{dt}\right)_{(x,y,z)}, \quad \vec{a}_{rel} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{(x,y,z)}. \tag{1.1.2}$$

Newton's 2. law of motion can be stated as

$$\delta\vec{F} = \delta m\vec{a}_{abs}, \qquad (1.1.3)$$

where $\delta\vec{F}$ is the sum of the Newtonian forces acting on the particle. By inserting for the absolute acceleration from (1.1.1), we find that

$$\delta m\vec{a}_{rel} = -\delta m 2\vec{\Omega}\times\vec{v}_{rel} - \delta m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) - \delta m\vec{a}_0 + \delta\vec{F}. \qquad (1.1.4)$$

Here, by definition

$$\begin{aligned} -\delta m 2\vec{\Omega}\times\vec{v}_{rel} &\equiv \delta\vec{F}_{Cor} && \text{(the Coriolis force)} \\ -\delta m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) &\equiv \delta\vec{F}_{cen} && \text{(the centrifugal force)} \\ -\delta m\vec{a}_0 &\equiv \delta\vec{F}_i && \text{(the force of inertia).} \end{aligned}$$

The Newtonian forces can be subdivided in the following way:

$$\delta\vec{F} = \begin{cases} \delta\vec{F}_p \text{(the pressure–gradient force)} \\ \delta\vec{F}_f \text{ (the friction force)} \\ \delta\vec{K}_{ge} \text{(the gravitation force from earth)} \\ \delta\vec{K}_{gsm} \text{(the gravitation force from the sun and the moon)} \end{cases}$$

We define the apparent force of gravity:

$$\delta\vec{F}_g \equiv \delta\vec{K}_{ge} + \delta\vec{F}_{cen} = -\frac{G\delta m M}{r^2}\vec{i}_r - \delta m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}), \qquad (1.1.5)$$

where M is the mass of the earth, G is the universal gravity constant, and the unit vector $\vec{i}_r$ is defined in Fig.1.2.

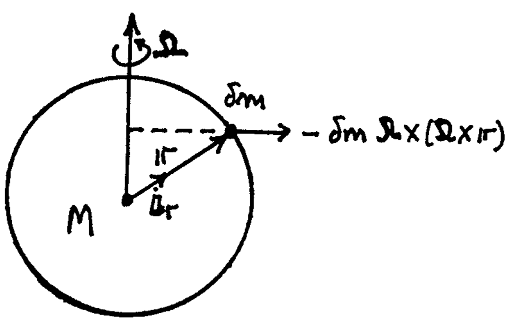

Fig. 1.2 Sketch of the centrifugal force.

The acceleration due to the (apparent) gravity force is denoted by $\vec{g}$, and defined by

$$\delta\vec{F}_g \equiv \delta m\vec{g}. \tag{1.1.6}$$

From (1.1.5) and (1.1.6) we obtain

$$\vec{g} = -\frac{GM}{r^2}\vec{i}_r - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \equiv -\nabla\Phi\,, \tag{1.1.7}$$

where Φ is called the *geopotential*. The tidal force is defined by

$$\delta\vec{F}_t \equiv \delta\vec{F}_i + \delta\vec{K}_{gsm}. \tag{1.1.8}$$

Here $\delta\vec{F}_i = -\delta m\vec{a}_0$ is the force of inertia (it is a centrifugal force due to the motion around the sun which is independent of the particle's position on the earth), and $\delta\vec{K}_{gsm}$ the force of gravity due to the sun and the moon (this force varies with the position of the particle on the earth). The tidal effect from the sun is sketched below.

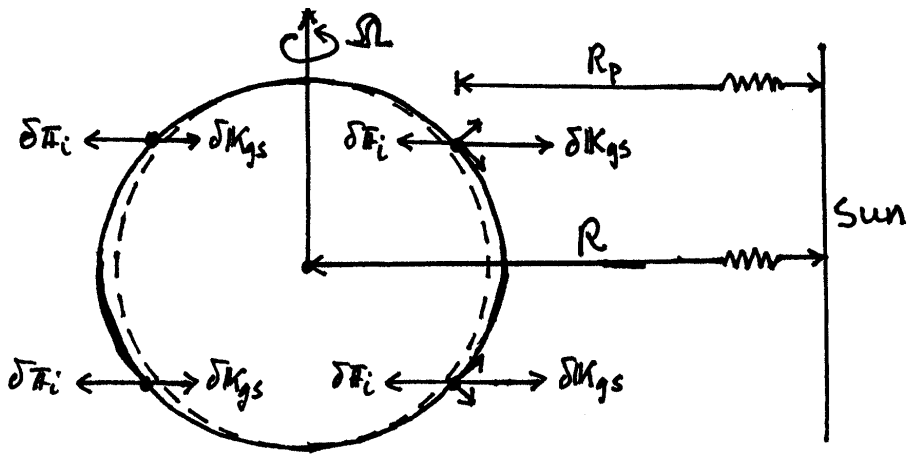

Fig. 1.3 Tidal effect from the sun.

Due to the large distance between the earth and the sun, the position vector $\vec{R}_p$ for a particle, measured from the common centre of mass (which in practice is that of the sun), is very nearly parallel to the vector $\vec{R}$ joining the two centres of mass. In Fig. 1.3 we have for the magnitudes of the forces

$$\left|\delta\vec{K}_{gs}\right| = \frac{GM_{sun}\delta m}{R_p^2} \tag{1.1.9}$$

and

$$\left|\delta\vec{F}_i\right| = \delta m\left|\vec{a}_0\right| = \frac{GM_{sun}\delta m}{R^2}. \tag{1.1.10}$$

The tangential component of the vector sum of these forces yields a tendency for the fluid (air, water) to assemble in areas closest to and furthest from the sun, as indicated in Fig. 1.3. This explains in principle the semi-diurnal *equilibrium* tide in the atmosphere and the ocean.

In summary we obtain for the motion of a particle relative to the rotating earth

$$\begin{aligned}\delta m\vec{a}_{rel} &= \delta\vec{F}_{Cor} + \delta\vec{F}_g + \delta\vec{F}_t + \delta\vec{F}_p + \delta\vec{F}_f \\ &= -\delta m 2\vec{\Omega}\times\vec{v}_{rel} + \delta m\vec{g} + \delta\vec{F}_t + \delta\vec{F}_p + \delta\vec{F}_f .\end{aligned} \tag{1.1.11}$$

When applied to a *fluid*, such as water or air, we must utilize the *theory of fluids* to determine $(\vec{v}_{rel}, \vec{a}_{rel})$ and the contact forces $\delta\vec{F}_p, \delta\vec{F}_f$. The tidal force $\delta\vec{F}_t$ is obtained from *astronomical* observations.

1.2 The concept of fluids and fluid particles

The matter relevant to the geophysical problems discussed here is composed of molecules. For solids like ice or rocks, the molecules are essentially bound in a lattice, and cannot move freely. In a gas, or a mixture of gases such as air, single molecules move independently in a random, chaotic way. In liquids like water, there are bonds between adjacent molecules, or groups of molecules. However, these bonds easily break and re-form, so we need not differentiate between gases and liquids in this context. They will both be referred to as *fluids*.

The (absolute) temperature of a fluid is a measure of the kinetic energy of the molecules; higher temperature means higher speed. When the absolute temperature is zero, the molecules are practically at rest, although they possess a finite amount of kinetic energy (zero point energy). The absolute temperature is measured in Kelvins (K). The conversion to deg. Celsius (^oC) is: 0K = –273.15^{o}C. Furthermore, since the molecules move in a chaotic, random way, any surface in contact with the fluid will be bombarded by molecules. Among other things, this bombardment gives rise to a force normal to the surface. This force per unit area is called *pressure*. The pressure becomes higher the more molecules there are per unit volume of the fluid. It should be mentioned that molecules do not actually have to be reflected at an impermeable, material surface to create pressure forces. Such forces also arise when molecules move across immaterial surfaces and the motion is accompanied by a rate of change of momentum across the surface.

Fortunately, we need not work on a molecular level to describe the macroscopic motion of a fluid. Instead, we introduce the concept of *fluid particles*. A fluid particle has a *given* mass. That means by definition that it does not exchange matter with the surroundings. The particles are taken to be so small that the fluid composed of them can be considered as a continuous medium in which the unknown quantities and their derivatives exist in every point. In practice, however, the particle dimensions will be large compared to the mean free path travelled by the molecules in the fluid. This means that each particle contains many molecules!

There are essentially two ways of studying fluid motion. The first is to try to follow the path of individual (labelled) fluid particles. This is called the *Lagrangian* method. The second is to determine the fluid motion at each geometrical point in fluid space without considering the whereabouts of individual particles. The latter is the *Eulerian* method. (They were actually both introduced by the Swiss mathematician Leonard Euler (1707-83).) Each method has its advantages (and disadvantages). In the analyses which follow later in this text, we shall use the Eulerian description of fluid motion. This approach usually turns out to be the simplest.

So why does a fluid move? Since a fluid can be considered as composed of individual particles, the fluid motion is quite simply the resultant motion of all these particles. Furthermore, since a fluid particle possesses inertia, it moves according to the laws of physics. The velocities in question are always much smaller than the speed of light, so the mechanics developed by the English mathematician and astronomer Sir Isaac Newton (1642-1727) can be applied to each fluid particle (in exactly the same way as to a single object in space such as a planet or satellite). According to Newton's 2. law of motion, it is the sum of forces on the particle that determines its acceleration, or the rate of change of velocity with respect to time as stated in (1.1.3).

One of the basic problems of geophysical fluid mechanics is to determine the forces in the fluid. This is by no means simple, since each fluid particle is surrounded by neighbouring particles that act to compress it as well as deform its shape. In terms of forces, the *compression* is associated with the effect of *pressure* and the change of *shape* with *friction* or *viscosity*.

1.3 Velocity and acceleration of fluid particles

As explained in the previous section, a fluid consists of infinitely many particles, each having a constant mass. Let an individual fluid particle δm move a distance $D\vec{r}$ in time dt; see the sketch below.

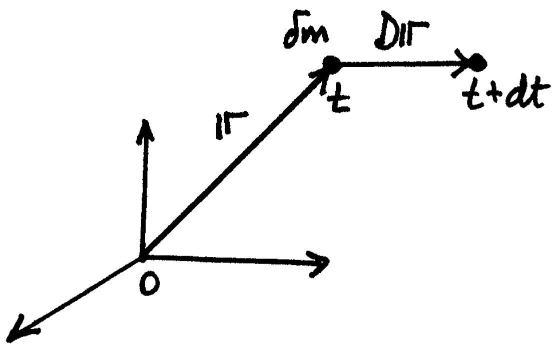

Fig. 1.4 Motion of an individual fluid particle.

The velocity, as $dt \to 0$, becomes:

$$\vec{v} = \left(\vec{v}_{rel}\right) = \frac{D\vec{r}}{dt}. \qquad (1.3.1)$$

In an Eulerian description we have

$$\vec{v} = \vec{v}(x,y,z,t), \qquad (1.3.2)$$

i.e. the velocity is a *field* variable. We choose a Cartesian reference, or coordinate system; see below.

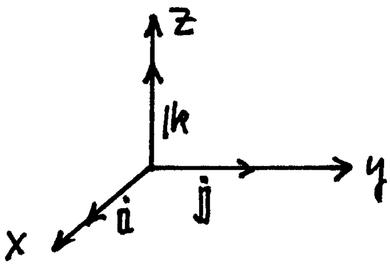

Fig. 1.5 Cartesian coordinate system.

Here $\vec{i}$, $\vec{j}$, $\vec{k}$ are constant unit vectors. In a Cartesian system we can write

$$\vec{v} = u(x,y,z,t)\vec{i} + v(x,y,z,t)\vec{j} + w(x,y,z,t)\vec{k} \,. \tag{1.3.3}$$

The rate of change of velocity for an individual fluid particle is written $D\vec{v}$. Hence, from a Taylor series development (Brook Taylor, 1685-1731):

$$D\vec{v} = \frac{\partial \vec{v}}{\partial t}dt + \frac{\partial \vec{v}}{\partial x}Dx + \frac{\partial \vec{v}}{\partial y}Dy + \frac{\partial \vec{v}}{\partial z}Dz \,. \tag{1.3.4}$$

The acceleration of a fluid particle then becomes, as $dt \to 0$:

$$\vec{a}\left(= \vec{a}_{rel}\right) = \frac{D\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\frac{Dx}{dt} + \frac{\partial \vec{v}}{\partial y}\frac{Dy}{dt} + \frac{\partial \vec{v}}{\partial z}\frac{Dz}{dt} \,. \tag{1.3.5}$$

But, from our definition (1.3.1) of the velocity of an individual fluid particle, we have

$$\vec{v} = \frac{D\vec{r}}{dt} = \frac{Dx}{dt}\vec{i} + \frac{Dy}{dt}\vec{j} + \frac{Dz}{dt}\vec{k} = u\vec{i} + v\vec{j} + w\vec{k}, \tag{1.3.6}$$

or

$$u = \frac{Dx}{dt}, \quad v = \frac{Dy}{dt}, \quad w = \frac{Dz}{dt}. \tag{1.3.7}$$

Hence we obtain from (1.3.5)

$$\vec{a} = \frac{D\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \,, \tag{1.3.8}$$

where the gradient operator ∇ is defined by

$$\nabla \equiv \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \,. \tag{1.3.9}$$

In summary we have

$$\vec{a} = \frac{D\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\,, \tag{1.3.10}$$

where $\partial \vec{v} / \partial t$ is termed the *local* acceleration and $\vec{v} \cdot \nabla \vec{v}$ the *convective* acceleration.

1.4 Conservation of mass for a fluid

Consider the geometrically fixed volume $\delta V = \delta x \delta y \delta z$ in Fig. 1.6.

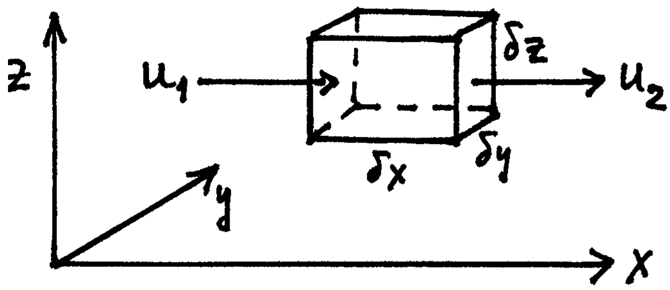

Fig. 1.6 Volume element fixed in space.

The mass δm inside the depicted volume can be written $\delta m = \rho \delta V$, where ρ is the mean *density* of the fluid within the volume. We obtain for the net flux of mass in the *x*-direction:

$$\delta Q^{(x)} = (\rho_1 u_1 - \rho_2 u_2)\delta y \delta z\,, \tag{1.4.1}$$

where ρ_1 and ρ_2 are the mean densities at the opposing surfaces. By a Taylor series development we readily obtain

$$\rho_2 u_2 = \rho_1 u_1 + \frac{\partial}{\partial x}(\rho u)\delta x + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\rho u)\delta x^2 + ... \tag{1.4.2}$$

Neglecting the higher order (small) terms, we have

$$\delta Q^{(x)} = -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z = -\frac{\partial}{\partial x}(\rho u)\delta V\,. \tag{1.4.3}$$

Similarly, in the *y*- and *z*-directions:

$$\delta Q^{(y)} = -\frac{\partial}{\partial y}(\rho v)\delta V$$
$$\delta Q^{(z)} = -\frac{\partial}{\partial z}(\rho w)\delta V. \qquad (1.4.4)$$

The net mass flux into the fixed volume δV can then be written

$$\delta Q = \delta Q^{(x)} + \delta Q^{(y)} + \delta Q^{(z)} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta V = -\nabla\cdot(\rho\vec{v})\delta V\ , \qquad (1.4.5)$$

where the gradient operator is defined by (1.3.9). The flux (1.4.5) is called the *advective* flux, since it is associated with the transport, or advection, of fluid particles.

At this stage we make the following, basic assumption: The increase of mass within δV can *only* be due to a net advective flux of mass through the boundaries. This means that

$$\frac{\partial}{\partial t}(\rho\delta V) = -\nabla\cdot(\rho\vec{v})\delta V, \qquad (1.4.6)$$

or

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{v}) = -\rho\nabla\cdot\vec{v} - \vec{v}\cdot\nabla\rho\ . \qquad (1.4.7)$$

In summary, by rearranging (1.4.7), conservation of *mass* leads to

$$\nabla\cdot\vec{v} = -\frac{1}{\rho}\frac{D\rho}{dt}, \qquad (1.4.8)$$

where $D\rho/dt \equiv \partial\rho/\partial t + \vec{v}\cdot\nabla\rho$ is the rate of change of density for an *individual* fluid particle. This equation is often called the *continuity equation*. In explicit form it can be written

$$\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} = 0\ . \qquad (1.4.9)$$

We shall return to this form of the mass conservation equation later on, and discuss the cases when it can be simplified.

1.5 Contact forces in fluids: the pressure-gradient force and the viscous force

We consider here the forces on a fluid particle due to the action or the presence of the surrounding particles.

a. The pressure-gradient force

The pressure (= p) is a *positive* quantity (it cannot be negative). The force due to the pressure on a surface element with *outward* normal vector $\vec{n}$ can be written

$$\delta\vec{K} = -p\delta A\vec{n} = -p\delta\vec{A} \tag{1.5.1}$$

i.e. the pressure force is directed *towards* the surface element; see the sketch in Fig. 1.7.

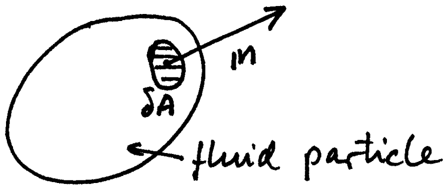

Fig. 1.7 Sketch of surface element.

Consider now the forces due to the pressure on the infinitesimal fluid element in Fig. 1.8:

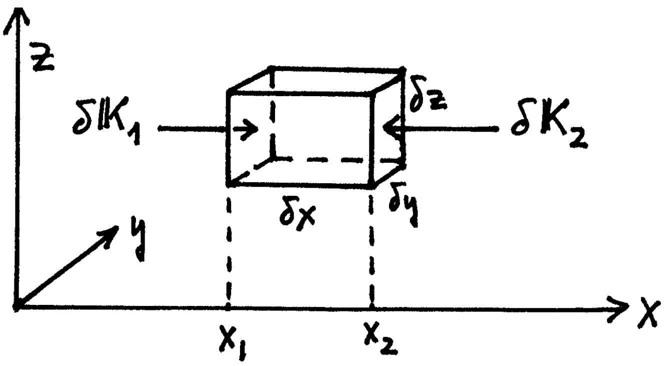

Fig. 1.8 Pressure force on an infinitesimal fluid element.

In the x-direction we have

$$\left.\begin{aligned} \delta\vec{K}_1 &= p_1\delta y\delta z\vec{i}\,, \\ \delta\vec{K}_2 &= -p_2\delta y\delta z\vec{i}\,. \end{aligned}\right\} \tag{1.5.2}$$

The *net* force in the x-direction due to the pressure can be written by using a Taylor series expansion:

$$\begin{aligned} \delta\vec{K}^{(x)} &= \delta\vec{K}_1 + \delta\vec{K}_2 = -(p_2 - p_1)\delta y\delta z\vec{i} \\ &= -\left(\frac{\partial p}{\partial x}\delta x\right)\delta y\delta z\vec{i} = -\frac{\partial p}{\partial x}\delta V\vec{i}\,, \end{aligned} \tag{1.5.3}$$

where we have neglected higher order terms. Similarly; in the *y*- and *z*-directions we find

$$\left.\begin{aligned}\delta\vec{K}^{(y)} &= -\frac{\partial p}{\partial y}\delta V\vec{j},\\ \delta\vec{K}^{(z)} &= -\frac{\partial p}{\partial z}\delta V\vec{k}.\end{aligned}\right\} \tag{1.5.4}$$

The net force on the fluid element due to the pressure can then be written

$$\delta\vec{F}_p = \delta\vec{K}^{(x)} + \delta\vec{K}^{(y)} + \delta\vec{K}^{(z)} = -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)\delta V = -\nabla p\delta V \tag{1.5.5}$$

Introducing $\delta m = \rho\delta V$, we find

$$\delta\vec{F}_p = -\frac{1}{\rho}\nabla p\delta m\,. \tag{1.5.6}$$

This is called the *pressure-gradient* force.

b. The viscous force

Friction acts on any surface in the fluid. The friction force per unit area is called the *viscous stress*. The viscous stress depends on the *orientation* of the surface. Stresses that act perpendicular to a surface are called *normal stresses,* and stresses that act along (parallel to) a surface are called *shear stresses;* see the sketch below.

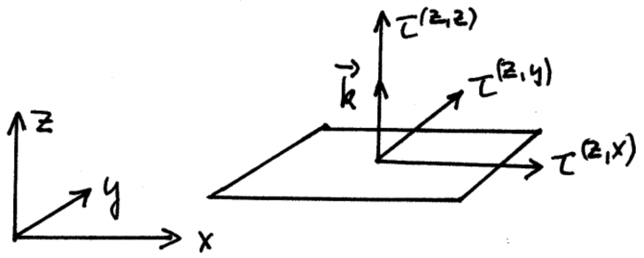

Fig. 1.9 Sketch of viscous stresses on a surface with unit normal $\vec{k}$.

It turns out that on the scales for which viscosity is important, the effect of compressibility can be neglected, i.e. in this case the continuity equation (1.4.8) reduces to $\nabla\cdot\vec{v} = 0$. On the surface in Fig. 1.9, we find for the stress components, when we assume that we have a Newtonian fluid:

$$\left.\begin{aligned}\tau^{(z,x)} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\\ \tau^{(z,y)} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\\ \tau^{(z,z)} &= 2\mu\frac{\partial w}{\partial z},\end{aligned}\right\} \tag{1.5.7}$$

where $\tau^{(z,x)}$ and $\tau^{(z,y)}$ are the *shear* stresses and $\tau^{(z,z)}$ is the *normal* stress. In (1.5.7) μ is the *molecular* viscosity coefficient, which is taken to be constant. Generally, we can write for the flows considered here:

$$\tau^{(m,n)} = \mu\left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n}\right), \quad m, n = 1, 2, 3. \tag{1.5.8}$$

By considering an infinitesimal element $\delta V = \delta x \delta y \delta z$, and applying a Taylor series expansion, as we did for the pressure forces in the beginning of this section, we find for the net viscous forces in the *x*-, *y*- and *z*-directions:

$$\left.\begin{aligned}\delta\vec{F}^{(x)} &= \mu\nabla^2 u\,\delta V\,\vec{i},\\ \delta\vec{F}^{(y)} &= \mu\nabla^2 v\,\delta V\,\vec{j},\\ \delta\vec{F}^{(z)} &= \mu\nabla^2 w\,\delta V\,\vec{k},\end{aligned}\right\} \tag{1.5.9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.5.10}$$

is called the *Laplacian* operator. The total viscous force on the fluid element then becomes

$$\begin{aligned}\delta\vec{F}_f &= \delta\vec{F}^{(x)} + \delta\vec{F}^{(y)} + \delta\vec{F}^{(z)}\\ &= \mu\nabla^2(u\vec{i} + v\vec{j} + w\vec{k})\delta V = \mu\nabla^2\vec{v}\,\delta V.\end{aligned} \tag{1.5.11}$$

Utilizing that $\delta m = \rho \delta V$, and introducing the *kinematic* viscosity coefficient $\nu \equiv \mu/\rho$, we find for the viscous force

$$\delta\vec{F}_f = \nu\nabla^2\vec{v}\,\delta m. \tag{1.5.12}$$

For water, the value of the molecular kinematic viscosity coefficient is about 0.012 cm^2 s^{-1}, while for air we have $\nu \approx 0.14$ cm^2 s^{-1}.

1.6 The Navier-Stokes equation

The tidal force $\delta\vec{F}_t$ can be omitted if we do not intend to study the generation of tides. Introducing $\vec{a}_{rel} = D\vec{v}/dt$, $\delta\vec{F}_p = -(1/\rho)\nabla p\delta m$, and $\delta\vec{F}_f = \nu\nabla^2\vec{v}\delta m$ into the general equation of motion (1.1.10), and dividing by δm, we obtain

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -2\vec{\Omega}\times\vec{v} + \vec{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v}\,. \tag{1.6.1}$$

For convenience we change the position of our coordinate system; see Fig.1.10.

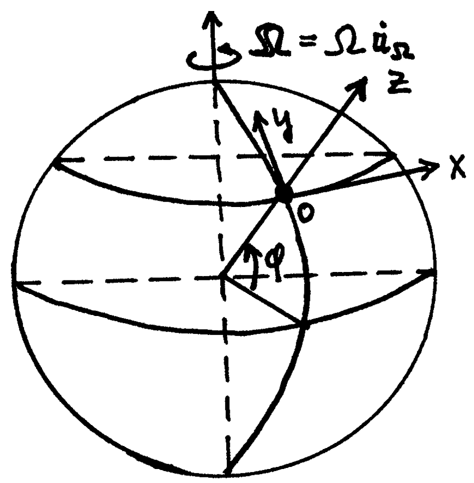

Fig. 1.10 Coordinate system with origin at the earth's surface.

We take the earth to be spherical in this approximation. It can be inferred from (1.1.7) that the centrifugal contribution to the acceleration of gravity is negligibly small (less than about 3 per mille) compared to the gravity part. Hence, to a good approximation we have

$$\vec{g} = -\nabla\Phi \approx -g\,\vec{k} \tag{1.6.2}$$

where g = 9.81 m s^{-2}. In this new coordinate system we can write

$$\Omega\,\vec{i}_\Omega = \Omega\cos\varphi\,\vec{j} + \Omega\sin\varphi\,\vec{k}\,, \tag{1.6.3}$$

where $\Omega \approx 2\pi/(24$ hrs) and φ is the latitude. Unless we are very close to the equator, an order of magnitude analysis shows that we can *neglect* the *y*-component of the rotation vector in the Coriolis force, i.e. we have approximately

$$\begin{aligned} -2\Omega\vec{i}_\Omega\times\vec{v} &\approx -2\Omega\sin\varphi\,\vec{k}\times\vec{v} \\ &= -f\,\vec{k}\times\vec{v}. \end{aligned} \tag{1.6.4}$$

Here $f \equiv 2\Omega\sin\varphi$ is the *Coriolis parameter*. With these simplifications, we can write (1.6.1):

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -f\vec{k} \times \vec{v} - \frac{1}{\rho}\nabla p - g\vec{k} + \nu \nabla^2 \vec{v} \,. \tag{1.6.5}$$

This is often called the *Navier-Stokes* equation.

In our Cartesian reference system defined in Fig. 1.10, the Coriolis parameter is only a function of the *y*-coordinate. We may write approximately that

$$f = f_0 + \left(\frac{df}{dy}\right)_0 y = f_0 + \beta y \,, \tag{1.6.6}$$

where

$$\begin{aligned} f_0 &= 2\Omega \sin \varphi_0, \\ \beta &= \frac{1}{R}\left(\frac{d}{d\varphi}\left(2\Omega \sin \varphi\right)\right)_{\varphi=\varphi_0} = \frac{2\Omega}{R}\cos \varphi_0. \end{aligned} \tag{1.6.7}$$

This is called the *beta-plane* approximation. As an alternative to solve our governing equations in spherical coordinates for large horizontal areas in the atmospheres or ocean, this approximation often proves very useful. If *f* is approximately constant in an (*x*, *y*)-area, we say that the motion occurs on an *f*-plane.

The Navier-Stokes equation (1.6.5) is a vector equation. It can be written in component form in the *x*, *y* and *z*-directions, respectively, as

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= fv - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu \nabla^2 w. \end{aligned} \tag{1.6.8}$$

Remark that the Coriolis force in our approximation only works in the horizontal directions, while the gravity acts in the vertical direction.

Together with the continuity equation (1.4.9), (1.6.8) yields 4 equations for the 5 unknowns u, v, w, p, ρ. Obviously, we need one more equation to close this system. This will be an *equation of state* that connects the pressure and the density. Every fluid has an equation of state. For a fluid like dry air it is simple (ideal gas), while for seawater it is very complicated. Unfortunately, the equation of state often involves temperature and salinity as well, so we need in fact two more equations; one for heat and one for salt, to close the system (7 unknowns, 7 equations). We shall not attempt to solve this full system here, but settle for simplified cases which shed light on important physical and dynamical processes in the ocean-atmosphere system.

Since we consider partial differential equations in time and space, a formal solution requires that we specify the variables at time equal zero, called the *initial conditions*. We must also apply conditions for the fluid motions at the boundaries of the fluid (the ground, the ocean bottom, the ocean surface, the coasts etc.) Such conditions are called *boundary conditions*. For example, at an impermeable boundary at rest, such as a vertical wall, the fluid cannot flow through it, and hence the velocity normal to the wall must be zero. This condition is independent of the viscous properties of the fluid, and is called a *kinematic* boundary condition. For fluid motion along a rigid boundary, the presence of viscosity requires that a fluid particle in contact with the boundary moves with the same speed as the boundary itself (no-slip condition). Hence, if the boundary is at rest, all the particles at the boundary must have zero velocity. Similarly, at the moving boundary between two fluids, like the air-water interface, the viscous stresses at the ocean surface must be equal to the viscous stresses in the air. The no-slip condition at a rigid boundary as well as the continuity of viscous stresses at the interface between two fluids, are termed *dynamic* boundary conditions. In the problems that follow, we shall show examples of the application of initial as well as boundary conditions in solving the equations for the fluid motion.

1.7 Simplifying the continuity equation

In the atmosphere and the ocean the surfaces of constant density are nearly horizontal. We can then write for the density

$$\rho = \rho_0(z) + \hat{\rho}(x, y, z, t), \tag{1.7.1}$$

where

$$\rho_0(z) >> |\hat{\rho}|. \tag{1.7.2}$$

The change of density following a fluid particle can then be approximated as

$$\frac{D\rho}{dt} = \frac{\partial\hat{\rho}}{\partial t} + u\frac{\partial\hat{\rho}}{\partial x} + v\frac{\partial\hat{\rho}}{\partial y} + w\left(\frac{d\rho_0}{dz} + \frac{\partial\hat{\rho}}{\partial z}\right) \approx w\frac{d\rho_0}{dz}, \tag{1.7.3}$$

which is the leading term in this equation. Inserting into (1.4.9), we obtain after neglecting the small terms that

$$\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial(\rho_0 w)}{\partial z} = 0, \tag{1.7.4}$$

or, since ρ_0 is only a function of *z*:

$$\nabla\cdot(\rho_0\vec{v}) = 0. \tag{1.7.5}$$

We realize that if

$$\rho_0\left|\frac{\partial w}{\partial z}\right| >> \left|w\frac{d\rho_0}{dz}\right|, \tag{1.7.6}$$

we obtain from (1.7.4) that the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.7.7}$$

In this case the velocity vector is *solenoidal*. If the vertical velocity typically varies over a vertical length scale *H*, we obtain from (1.7.6) that

$$H << \frac{\rho_0}{|d\rho_0 / dz|} \equiv H_s \, , \tag{1.7.8}$$

where H_s is the *scale height*. So, if the vertical velocity scale is much smaller than the scale height, the continuity equation reduces to (1.7.7), or $\nabla \cdot \vec{v} = 0$. In the ocean the scale height is about 40 times the ocean depth, which is the maximum length scale for the vertical motion. Hence (1.7.7) is always a good approximation for the ocean. However, for some atmospheric motions (1.7.8) is not fulfilled. In that case the form (1.7.5) of the continuity equation must be used. It must also be said that (1.7.7) presupposes that particle velocities and wave propagation velocities are much smaller than the speed of sound c_s, which in the atmosphere is about 340 m s^{-1}, and in the ocean about 1500 m s^{-1}.

2. STRUCTURE OF THE STATIC ATMOSPHERE AND OCEAN

2.1 Static stability

Before we proceed to discuss the fluid motions we find as solutions to our governing equations, we consider some properties of a fluid with no motion at all. When $\vec{v} = 0$ everywhere, the Navier-Stokes equation (1.6.5) reduces to

$$\nabla p = -\rho g \vec{k} \, . \tag{2.1.1}$$

In general, the set of points in space where the pressure at a given time has a certain value, e.g. $p = p_1$, constitutes an *isobaric surface*. From the horizontal components of (2.1.1), we realize immediately that we must have $p = p(z)$. The vertical component then yields $\rho = \rho(z)$. Accordingly, for a fluid at rest in the gravity field, the isobaric surfaces and the surfaces of constant density (the *isopycnal* sufaces) must be parallel and horizontal. Or more correctly, they must be parallel to the surfaces of constant gravity potential, which are called the *geopotential* surfaces; see (1.1.7).

In the discussion of the static stability of a fluid, one often uses the term fluid *parcel*, especially in meteorology, meaning a fluid sample of uniform temperature and composition. Besides, dry air and water vapour it can contain suspended or dissolved particles (*aerosols*) etc. In the present context there will be no need to distinguish between a fluid parcel and a fluid particle. They will both be treated as infinitesimal elements in a continuous field.

We shall displace a particle a small vertical distance δz from its original position z_0. We disregard any changes in velocity of the particle and the surrounding fluid, and just consider the net acceleration of the displaced particle at its new position $z_0 + \delta z$. The mass δm of the particle must be conserved, i.e. $D(\delta m) = 0$, and the gravity force acting on it is $\delta m g$. Furthermore, according to the principle discovered by Archimedes, the particle is subject to an upward force (an up-thrust) which equals the weight of the fluid it has displaced (the displaced fluid is the fluid which originally was situated at the position $z_0 + \delta z$). From our discussion of the pressure-gradient force in section 1.5 we understand that the force discovered by Archimedes is just the sum of external pressure forces over the entire surface of the particle. If the mass of the displaced fluid is $\delta m'$, the buoyant up-thrust is $\delta m' g$. Accordingly, if we neglect all other forces, which is acceptable, since we have assumed that the particle is at rest, it will be subject to a vertical acceleration $a^{(z)}$ given by Newton's 2. law of motion:

$$\delta m a^{(z)} = \delta m' g - \delta m g \, . \tag{2.1.2}$$

At the initial equilibrium position, where $z = z_0$, we have $\delta m' = \delta m$. According to (2.1.2), if we displace a particle <u>upward</u>, and $\delta m' < \delta m$ at the new position of the particle, then $a^{(z)} < 0$, and the particle has a tendency to move back towards its original position. The equilibrium state is then said to be *stable*. Similarly, it is stable if $\delta m' > \delta m$ for a <u>downward</u> displacement.

Since the volumes of the particle and that of the displaced fluid must always be the same everywhere, i.e. $\delta V < \delta V'$, it is actually the difference in densities that counts. Then, from (2.1.2) stability requires that $\rho'(z_0 + \delta z) < \rho(z_0 + \delta z)$ when $\delta z > 0$, and $\rho'(z_0 + \delta z) > \rho(z_0 + \delta z)$ when $\delta z < 0$. Utilizing a Taylor series expansion and the fact that $\rho'(z_0) = \rho(z_0)$, we obtain from the first two terms that the equilibrium is stable if

$$\left(\frac{d\rho}{dz}\right)_{env} < \left(\frac{d\rho}{dz}\right)_{par} , \tag{2.1.3}$$

where the subscript *env* now replaces the prime in denoting the environment.

If the particle acceleration becomes zero when the particle is displaced to a new position, the equilibrium is said to be *neutral*. From (2.1.2) this may be stated mathematically as

$$\left(\frac{d\rho}{dz}\right)_{env} = \left(\frac{d\rho}{dz}\right)_{par} . \tag{2.1.4}$$

Finally, the equilibrium is *unstable* if

$$\left(\frac{d\rho}{dz}\right)_{env} > \left(\frac{d\rho}{dz}\right)_{par} . \tag{2.1.5}$$

The density change with height of the surrounding fluid (the environment) can be obtained from measurements. It is the change of density for the particle on the right-hand side of these inequalities which poses problems. This rate of change depends on the thermodynamics of the process. We shall consider this problem in the next section.

The stability problem may conveniently be discussed in terms of oscillations. If we replace the small vertical displacement δz by a particle coordinate $\zeta(t)$, then $a^{(z)} = d^2\zeta / dt^2$. Hence, (2.1.2) can be written

$$\frac{d^2\zeta}{dt^2} = -\frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_{par} - \left(\frac{d\rho}{dz}\right)_{env}\right]\zeta \,. \tag{2.1.6}$$

We now define the *Brunt-Väisälä* frequency (or the *buoyancy* frequency) *N* by

$$N^2 = \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_{par} - \left(\frac{d\rho}{dz}\right)_{env}\right]. \tag{2.1.7}$$

Then (2.1.6) can be written

$$\frac{d^2\zeta}{dt^2} + N^2\zeta = 0 \,. \tag{2.1.8}$$

We see that when the equilibrium is stable, i.e. when (2.1.3) applies, then $N^2 > 0$. Hence *N* is real, and the solutions of (2.1.8) are periodic oscillations with frequency *N* and period $2\pi / N$. For an unstable equilibrium; see (2.1.5), $N^2 < 0$. In that case *N* becomes imaginary. One of the solutions of (2.1.8) for the small vertical displacement then increases exponentially in time, and we realize that very soon our assumption of a *small* displacement will be violated.

2.2 Thermodynamics

If a fluid particle receives an amount of heat Δq per unit mass (the mass is constant, remember), its volume α per unit mass ($\alpha = 1/\rho$) will change by $D\alpha$ and its internal energy *e* per unit mass will change by *De*. According to the first law of thermodynamics, these changes are related by

$$\Delta q - pD\alpha = De. \tag{2.2.1}$$

The parameter α is also termed the *specific* volume. We note from (2.2.1) that if heat is received ($\Delta q > 0$) and the work done by the pressure is positive (compressing the particle, $D\alpha < 0$), both effects will tend to increase the internal energy ($De > 0$). Since *e* is proportional to the absolute temperature, this means that the temperature of the particle will increase in this case. We use here Δ and not *D* for changes in the heat since *q* is not a proper function of the state variables. (The reader is referred to specific texts on thermodynamics for a discussion of this subject.)

At this stage we introduce yet another state variable; the *entropy s* per unit mass, or the *specific* entropy. According to the second law of thermodynamics, we must have

$$\Delta q = TDs \,. \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2) we obtain

$$TDs = De + pD\alpha \,. \tag{2.2.3}$$

In general, for any fluid (a gas or a liquid) we have an equation of state, expressing a relation between the state variables; i.e.

$$f\,(p,T,S,\alpha)=0, \tag{2.2.4}$$

where S denotes the composition (salinity for sea water, specific humidity for air). Solving this for α (or ρ) means that when p, T and S are specified, then ρ is *given* by (2.2.4). In our problem with displacement of fluid particles, we shall assume that the composition does not change, i.e. DS = 0. This means that our thermodynamic problem in fact only has two independent state variables, p and T, say. Accordingly, the change of the specific entropy may be written

$$Ds=\frac{\partial s}{\partial p}Dp+\frac{\partial s}{\partial T}DT\ . \tag{2.2.5}$$

We define the specific heat capacity c_p at constant pressure by

$$c_p=T\left(\frac{\partial s}{\partial T}\right)_p, \tag{2.2.6}$$

where the subscript p indicates that the pressure is kept constant. Inserting into (2.2.5), we obtain

$$T\,Ds=T\,\frac{\partial s}{\partial p}Dp+c_p DT\ . \tag{2.2.7}$$

At this stage it proves convenient to introduce the Gibbs function g_i per unit mass:

$$g_i=e+p\alpha-Ts\,. \tag{2.2.8}$$

For an individual fluid particle we have

$$Dg_i=\alpha Dp-sDT, \tag{2.2.9}$$

where we have utilized (2.2.3). Since also g_i is a function of the two independent state variables p and T, we may write

$$Dg_i=\frac{\partial g_i}{\partial p}Dp+\frac{\partial g_i}{\partial T}DT=\alpha Dp-sDT, \tag{2.2.10}$$

where the last expression has been substituted from (2.2.9). Accordingly, from (2.2.10):

$$\left.\begin{aligned} \alpha &= \frac{\partial g_i}{\partial p}, \\ s &= -\frac{\partial g_i}{\partial T}. \end{aligned}\right\} \tag{2.2.11}$$

By differentiating α partially with respect to T and s partially with respect to p, we realize immediately that

$$\frac{\partial s}{\partial p} = -\frac{\partial \alpha}{\partial T}. \tag{2.2.12}$$

We insert this result into (2.2.7) and obtain

$$TDs = c_p DT - \frac{\partial \alpha}{\partial T} TDp. \tag{2.2.13}$$

By introducing the *thermal expansion* coefficient γ_T, defined by

$$\gamma_T = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}, \tag{2.2.14}$$

we obtain from (2.2.13) that

$$TDs = c_p DT - \alpha \gamma_T TDp. \tag{2.2.15}$$

This particular form of the first law of thermodynamics is well suited for our purpose.

It should be mentioned that γ_T defined by (2.2.14) is positive for air. This is not always so for water, e. g. freshwater (S = 0) at surface level has a density maximum when $T \approx 4$°C. However, for sea water with salinity larger than 24.7 psu (psu = practical salinity unit ≈ gram salt per kilo of sea water), the density increases monotonically as the temperature decreases towards the freezing point. Then, in this case, $\gamma_T > 0$ for the entire range of ocean temperatures, −2°C < T < 30°C, say.

2.3 The adiabatic lapse rate

The thermodynamics we have used up to now is strictly speaking only valid for a fluid which is in equilibrium. However, we shall assume that the changes that occur in the ocean and the atmosphere are slow enough for this theory to apply. Furthermore, we shall assume that the time scales involved when we displace a particle vertically are so small that no heat (or salt or humidity) is exchanged between the particle and its surroundings. Such processes are said to be *adiabatic*. As seen from (2.2.2), when Δq = 0, the entropy must be constant. Therefore, such changes are also termed *isentropic*.

For an adiabatic process (Δq = 0 or Ds = 0) we obtain from (2.2.15)

$$DT = \frac{\gamma_T T}{\rho c_p} Dp. \qquad (2.3.1)$$

We realize that if we displace a particle vertically an infinitesimal distance *Dz*, its change in pressure *Dp* must be equal to the change *dp* of the surrounding fluid (if not, the particle would explode or implode!). Accordingly, we can write

$$Dp = dp = \frac{dp}{dz} dz = \frac{dp}{dz} Dz. \qquad (2.3.2)$$

For a fluid at rest we have from (2.1.1) that

$$\frac{dp}{dz} = -\rho g. \qquad (2.3.3)$$

Inserting (2.3.3) into (2.3.2) yields

$$Dp = -\rho g Dz. \qquad (2.3.4)$$

Finally, by substituting for *Dp* in (2.3.1), we obtain for the adiabatic rate of change of temperature with height, or the *adiabatic lapse rate*:

$$\left(\frac{DT}{Dz}\right)_{ad} = -\frac{g\gamma_T}{c_p} T \equiv -\Gamma. \qquad (2.3.5)$$

For fluids with a simple equation of state (which does not include sea water) it is easy to compute Γ. For dry air (ideal gas), we have

$$\rho = \frac{p}{RT}, \qquad (2.3.6)$$

where *R* is the gas constant for dry air ($R = 287.04\ \mathrm{J\,kg^{-1}\,K^{-1}}$)). Then, from (2.2.14), $\gamma_T = 1/T$, and hence

$$\Gamma = \frac{g}{c_p}. \qquad (2.3.7)$$

This is called the *dry* adiabatic lapse rate since we have not considered the effect of a possible condensation of water vapour inside the particle (or parcel). The discussion of the *moist* adiabatic lapse rate (accounting for the effect of condensation) will be left for specific courses in meteorology. It suffices here to say that the value of Γ for a dry atmosphere is about 1°C per 100 m, while the moist adiabatic lapse rate is generally somewhat smaller.

2.4 Explicit form of the Brunt-Väisälä frequency

Utilizing the concept of adiabatic processes for the fluid particle, we may discuss the static stability problem in a more explicit way. The Brunt-Väisälä frequency (2.1.7) may now be written

$$N^2 = \frac{g}{\rho}\left[\left(\frac{D\rho}{Dz}\right)_{ad} - \left(\frac{d\rho}{dz}\right)_{env}\right] \qquad . \tag{2.4.1}$$

Generally, we have

$$\left(\frac{D\rho}{Dz}\right)_{ad} = \frac{\partial\rho}{\partial p}\left(\frac{Dp}{Dz}\right)_{ad} + \frac{\partial\rho}{\partial T}\left(\frac{DT}{Dz}\right)_{ad} + \frac{\partial\rho}{\partial S}\left(\frac{DS}{Dz}\right)_{ad} \tag{2.4.2}$$

$$\left(\frac{d\rho}{dz}\right)_{env} = \frac{\partial\rho}{\partial p}\left(\frac{dp}{dz}\right)_{env} + \frac{\partial\rho}{\partial T}\left(\frac{dT}{dz}\right)_{env} + \frac{\partial\rho}{\partial S}\left(\frac{dS}{dz}\right)_{env} . \tag{2.4.3}$$

Utilizing (2.3.2), (2.3.5) and the fact that $(DS/Dz)_{ad} = 0$, we find by inserting (2.4.2) and (2.4.3) into (2.4.1)

$$N^2 = g\left[-\frac{1}{\rho}\frac{\partial\rho}{\partial T}\left(\left(\frac{dT}{dz}\right)_{env} + \Gamma\right) - \frac{1}{\rho}\frac{\partial\rho}{\partial S}\left(\frac{dS}{dz}\right)_{env}\right]. \tag{2.4.4}$$

Introducing the expansion coefficient γ_S for the composition

$$\gamma_S = \frac{1}{\rho}\frac{\partial\rho}{\partial S}, \tag{2.4.5}$$

and inserting for the thermal expansion coefficient γ_T from (2.2.14), we finally obtain from (2.4.4):

$$N^2 = g\gamma_T\left(\left(\frac{dT}{dz}\right)_{env} + \Gamma\right) - g\gamma_S\left(\frac{dS}{dz}\right)_{env} . \tag{2.4.6}$$

Here the adiabatic lapse rate is given by (2.3.5), i.e. $\Gamma = g\gamma_T T / c_p$.

In the atmosphere one can often neglect the effect of the specific humidity on the stability. Then the condition for stability ($N^2 > 0$) becomes

$$\left(\frac{dT}{dz}\right)_{env} > -\Gamma. \tag{2.4.7}$$

In the deep ocean we often find that $(dS/dz)_{env} \approx 0$ and $S \approx 35$ psu. Here the criterion for stability conforms to (2.4.7). In the upper ocean (upper 1000 meters, say), the contribution from the salinity to the stability criterion cannot be neglected. In fact, in coastal and polar waters the salinity contribution in (2.4.6) may dominate.

In the upper ocean we usually find that $(dT/dz)_{env} >> \Gamma \approx 0.04°\text{C km}^{-1}$. Hence the Brunt-Väisälä frequency (2.4.6) may be written

$$N^2 = -\frac{g}{\rho}\left(\frac{\partial\rho}{\partial T}\frac{dT}{dz} + \frac{\partial\rho}{\partial S}\frac{dS}{dz}\right), \tag{2.4.8}$$

where we have reinstated the density and the subscript for the environment is understood. In the ocean, where the density is close to 1000 kg m^{-3}, it is common to introduce the parameter σ_t for the density at constant pressure, i.e.

$$\sigma_t = \rho(T, S, p = p_0) - 1000, \tag{2.4.9}$$

where p_0 is the constant surface pressure. Hence, we realize that (2.4.8) can simply be written

$$N^2 = -\frac{g}{\rho}\frac{d\sigma_t}{dz}. \tag{2.4.10}$$

This formula is often used to determine the buoyancy frequency in the oceanic pycnocline. Typically, both in the lower atmosphere and in the upper ocean we find $N \approx 0.01\,\text{s}^{-1}$. This means a period $2\pi / N$ for vertical oscillations of about 10 minutes.

2.5 Potential temperature and potential density

The concept of adiabatic (or isentropic) processes naturally leads to the definition of *potential* temperature. This is defined as the temperature a particle (or parcel) of fluid would acquire if moved adiabatically to a reference pressure level p_r, usually taken as 1 bar. The potential temperature θ can be obtained from (2.3.1) by integration, i.e.

$$\int_T^\theta \frac{dT'}{T'} = \int_p^{p_r} \frac{\gamma_T}{\rho c_p} dp'. \tag{2.5.1}$$

Hence

$$\theta = T \exp\left(\int_p^{p_r} \frac{\gamma_T}{\rho c_p} dp'\right). \tag{2.5.2}$$

For the ocean, the integral on the right-hand side may be calculated using tabulated values for the integrand. For the atmosphere, an explicit expression can be obtained if ideal gas behaviour is assumed. Utilizing (2.3.6) and the fact that c_p is constant for an ideal gas, we readily obtain from (2.5.2) that

$$\frac{\theta}{T} = \left(\frac{p_r}{p}\right)^{\frac{R}{c_p}} . \tag{2.5.3}$$

In the calculation of the Brunt-Väisälä frequency (2.4.1), we now may use p and θ as independent state variables in place of p and T. By definition, $D\theta = 0$ for a particle that is displaced adiabatically. As before, we assume that there is no change in composition for a particle during this process ($DS = 0$). Hence, we have for the rate of change of density for a particle

$$\left(\frac{D\rho}{Dz}\right)_{ad} = \frac{\partial\rho}{\partial p}\left(\frac{Dp}{Dz}\right)_{ad} . \tag{2.5.4}$$

The corresponding change for the environment can be written

$$\left(\frac{d\rho}{dz}\right)_{env} = \frac{\partial\rho}{\partial p}\left(\frac{dp}{dz}\right)_{env} + \frac{\partial\rho}{\partial\theta}\left(\frac{d\theta}{dz}\right)_{env} + \frac{\partial\rho}{\partial S}\left(\frac{dS}{dz}\right)_{env} . \tag{2.5.5}$$

For the Brunt-Väisälä frequency defined by (2.4.1) we then obtain

$$N^2 = g\hat{\gamma}_T\left(\frac{d\theta}{dz}\right)_{env} - g\hat{\gamma}_S\left(\frac{dS}{dz}\right)_{env} . \tag{2.5.6}$$

Here we have defined

$$\hat{\gamma}_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\theta}\right)_{p,S} , \tag{2.5.7}$$

$$\hat{\gamma}_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{p,\theta} . \tag{2.5.8}$$

These expansion coefficients are slightly different from those defined by (2.2.14) and (2.4.5) since they involve the potential temperature θ and not T. The subscripts here are stated as reminders of what are kept constant during the differentiation.

In cases where the effect of composition (salt, specific humidity) can be neglected, the stability criterion (2.4.7) can be stated in terms of potential temperature as

$$\left(\frac{d\theta}{dz}\right)_{env} > 0 . \tag{2.5.9}$$

Typically, in the lower part of the atmosphere (the *troposphere*) the mean distribution of T and θ with height can be sketched as in Fig. 2.1.

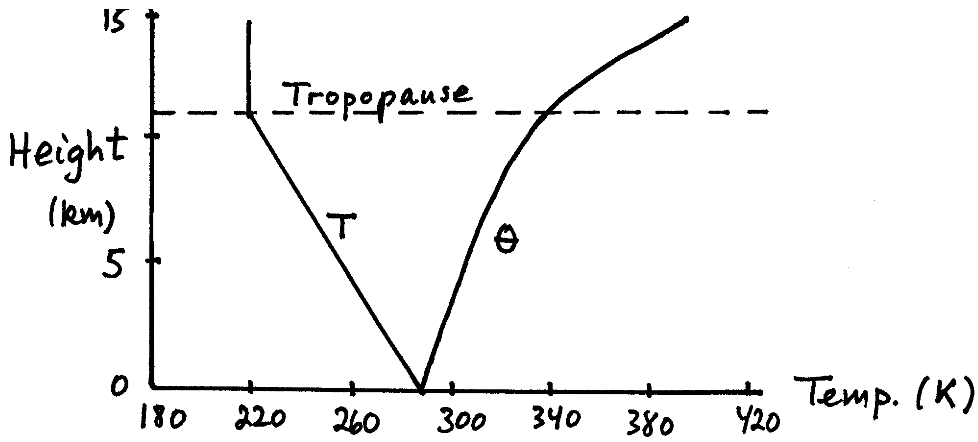

Fig. 2.1 Typical mean profiles of T and θ in the troposphere.

In deep ocean trenches typical distributions of S, T and θ may be found as displayed in Fig. 2.2.

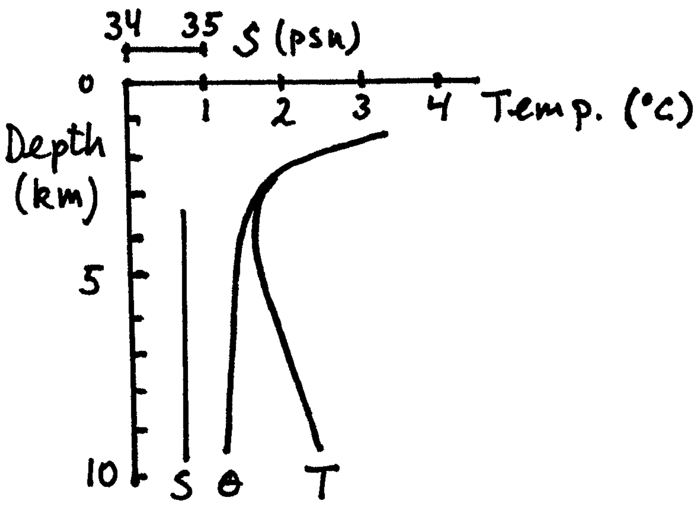

Fig. 2.2 Profiles of S, T and θ in the Mindanao Trench.

Despite the apparent instability due to the decrease in the *in situ* temperature with height in the entire troposphere and in the deepest part of the ocean, we note from the corresponding potential temperature distribution that the equilibrium actually is stable in both cases.

It often proves convenient to use *potential* density ρ_{pot} instead of the *in situ* density ρ. The potential density is defined as the density a particle of fixed composition would acquire if moved adiabatically to a reference pressure level p_r, usually taken as 1 bar (= 10^5 Pa). Since the particle then would have a potential temperature θ, the potential density is simply obtained by replacing T by θ in the general equation of state (2.2.4), solved for the density, and inserting $p = p_r$. We then find

$$\rho_{pot} = \rho(p_r, \theta, S). \tag{2.5.10}$$

Utilizing the ideal gas concept for air, we obtain from (2.3.6) that

$$\rho_{pot} = \frac{p_r}{R\theta}. \tag{2.5.11}$$

Hence the potential density is inversely proportional to the potential temperature in this case.

In the ocean the value of ρ_{pot}, like ρ, is close to 1000 kg m^{-3}. It is therefore convenient to define a parameter σ_θ by

$$\sigma_\theta = \rho_{pot} - 1000 . \tag{2.5.12}$$

For the deep ocean, where $dS/dz \approx 0$, the stability criterion (2.5.9) can be written

$$\left(\frac{d\theta}{dz}\right)_{env} = \left(\frac{d\theta}{d\rho_{pot}}\right)\left(\frac{d\rho_{pot}}{dz}\right)_{env} > 0 . \tag{2.5.13}$$

Utilizing the fact that $d\theta / d\rho_{pot} < 0$; see for example (2.5.11) for an ideal gas, (2.5.13) yields as a condition for stability:

$$\left(\frac{d\sigma_\theta}{dz}\right)_{env} < 0 \ . \tag{2.5.14}$$

3. OSCILLATORY MOTION

We noticed in Section 2 that by displacing a particle vertically in a stably stratified fluid, it tended to oscillate up and down with the buoyancy frequency *N*, e.g. (2.1.8). In this section we shall investigate oscillations in fluids more accurately. We shall not disturb single particles from equilibrium positions as in Section 2, but study oscillatory fluid motion on the basis of the equations of momentum and mass conservation that we have derived in Section 1. This is reasonable, since it is virtually impossible to move one single particle without disturbing neighbouring particles as well. The associated fluid motion must then satisfy our governing equations.

3.1 Purely horizontal motion: Inertial oscillations

We consider motion on an *f*-plane, i.e. we assume that the Coriolis parameter is constant. Furthermore, there are no pressure gradients in the horizontal direction, i.e. $\partial p/\partial x = \partial p/\partial y = 0$, and the effect of friction is so small that it can be neglected. We also take that the vertical velocity component *w* is zero everywhere, and that the horizontal velocity components do not vary in space. Then from (1.6.5):

$$\frac{du}{dt} = fv , \tag{3.1.1}$$

$$\frac{dv}{dt} = -fu . \tag{3.1.2}$$

Since u and v now only vary with time, we have used the symbol for ordinary differentiation in these equations. Multiplying (3.1.2) with the imaginary unit, $i = (-1)^{1/2}$, and introducing the complex velocity $W \equiv u + iv$, we obtain from (3.1.1) and (3.1.2) that

$$\frac{dW}{dt} + ifW = 0\,. \tag{3.1.3}$$

We see immediately that this equation has the solution

$$W = W_0 e^{-ift}\,, \tag{3.1.4}$$

where W_0 is an integration constant. For solving this problem we need to specify the initial conditions. Assuming that $u(t = 0) = 0$ and $v(t = 0) = v_0$, we find from (3.1.4) that $W_0 = iv_0$. Utilizing that $\exp(-ix) = \cos x - i \sin x$, ($x$ real), we can write (3.1.4) on real form as

$$\left.\begin{aligned} u &= v_0 \sin\, ft, \\ v &= v_0 \cos\, ft\,. \end{aligned}\right\} \tag{3.1.5}$$

From this result we note that the velocity vector $\vec{v} = u\vec{i} + v\vec{j}$ has constant magnitude, i.e.

$$|\vec{v}| = \left(v_0^2 \sin^2\, ft + v_0^2 \cos^2\, ft\right)^{1/2} = v_0\,. \tag{3.1.6}$$

Furthermore, from (3.1.5) we note that in the northern hemisphere, where $f > 0$, the velocity vector rotates clockwise with time and turns 360° after a period T_i, given by

$$T_i = \frac{2\pi}{f}. \tag{3.1.7}$$

This is called the *inertial* period, and the motion (3.1.5) having this period is referred to as *inertial oscillations*. If we introduce the *pendulum* day T_p, defined by

$$T_p = \frac{2\pi}{\Omega \sin\varphi}\,, \tag{3.1.8}$$

where $\Omega\sin\varphi$ is the local vertical component of the earth's angular velocity at a location with latitude φ, we find that

$$T_i = \frac{2\pi}{f} = \frac{2\pi}{2\Omega\sin\varphi} = \frac{1}{2}T_p\,, \tag{3.1.9}$$

i.e. the inertial period equals half a pendulum day. Accordingly, at the north pole, $T_i \approx 12$ hrs, which at the equator, $T_i \to \infty$.

Let (x_L, y_L) be coordinates for an individual fluid particle (Lagrangian coordinates) in this case. Then

$$u = \frac{dx_L}{dt}, \qquad v = \frac{dy_L}{dt}, \tag{3.1.10}$$

where u and v are given by (3.1.5). By integration, we obtain

$$\begin{aligned} x_L &= -\frac{v_0}{f}\cos ft + x_0 , \\ y_L &= \frac{v_0}{f}\sin ft + y_0 , \end{aligned} \tag{3.1.11}$$

where (x_0, y_0) are integration constants. By re-arranging (3.1.11), squaring and adding, we obtain

$$(x_L - x_0)^2 + (y_L - y_0)^2 = \frac{v_0^{\;2}}{f^2}. \tag{3.1.12}$$

Hence we see that each particle moves in a circle, with centre at (x_0, y_0) and with radius $r_i = v_0 / f$. The radius r_i is called the *inertial radius*. In the ocean we have typical velocities $v_0 = 0.1\,\mathrm{m\,s}^{-1}$. At mid latitude $f = 10^{-4}\,\mathrm{s}^{-1}$. We then find $r_i \sim 1\,\mathrm{km}$ in the ocean. The corresponding value of the inertial period, $T_i = 2\pi / f$, is here about 15 hrs. In the atmosphere we have typically $v_0 = 10\,\mathrm{m\,s}^{-1}$. Then we find that $r_i \sim 100\,\mathrm{km}$ in the atmosphere. Obviously, from (3.1.11), a particle moves in a clockwise sense (when $f > 0$); see Fig. 3.1.

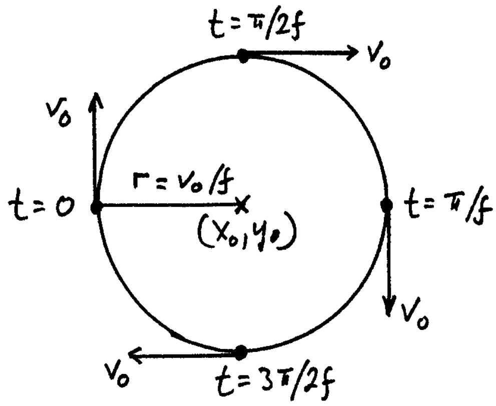

Fig. 3.1 Particle position and velocity vector at various times for inertial motion.

If the fluid, in addition to the inertial oscillations, also possesses a uniform velocity V in the y-direction, say, the coordinates of an individual particle can be written

$$\begin{aligned} x_L &= -\frac{v_0}{f}\cos ft , \\ y_L &= \frac{v_0}{f}\sin ft + Vt , \end{aligned} \tag{3.1.13}$$

where we have taken $x_0 = y_0 = 0$. In Fig. 3.2 we have sketched the particle trajectory in this case. (*a*): the translation velocity is smaller than the orbital speed, i.e. $V < v_0$, and (*b*): $V > v_0$.

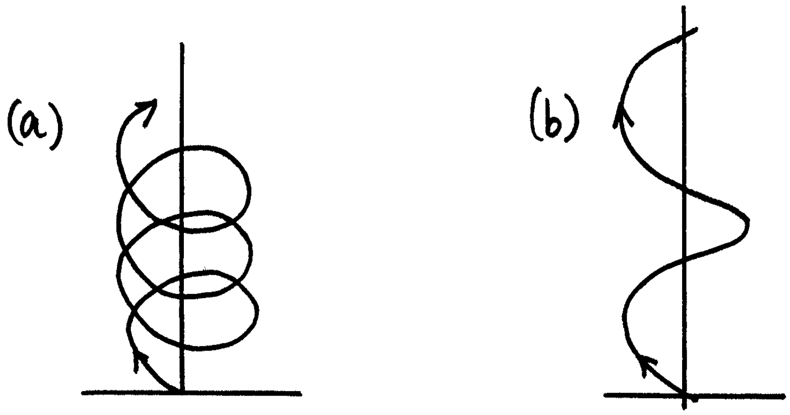

Fig. 3.2 Sketch of particle trajectories for translation plus inertial oscillation.

In Fig. 3.3 we have displayed the results of some observations in the Baltic Sea by Gustafson and Kullenberg (1933).

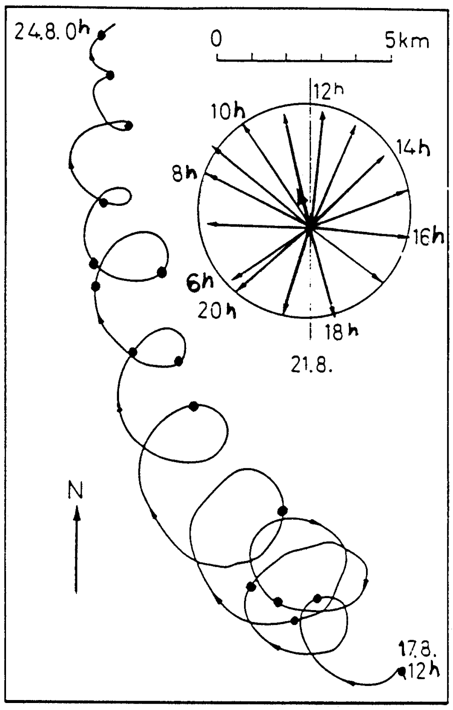

Fig. 3.3 Observed inertial currents in the Baltic Sea (Gustafson & Kullenberg 1933). The insert shows the vector diagram of the current during one inertial period (one half pendulum day) and the scale of the inertial circles.

We notice the similarity between the observations in Fig. 3.3 and the sketch in Fig. 3.2.a. The fact that the radii of the inertial circles in the observations decrease in time is due to the effect of friction, which is present in the real ocean and a real atmosphere, but which has been neglected in the theoretical development presented here.

3.2 Oscillations in the vertical plane: interfacial gravity waves

In the previous section we studied oscillatory motion that was entirely due to the earth's rotation. Since the motion was purely horizontal, the effect of gravity did not enter the problem. We now consider the other extreme. Here the oscillations occur in the vertical plane, and they are so rapid that the effect of the earth's rotation can be neglected. So in this analysis we put $f = 0$. Again we neglect the effect of friction. We consider a fluid that consists of two horizontal, homogeneous layers with constant densities ρ_1 and ρ_2, respectively, and we study the motion that takes place when we disturb the interface between these two layers. In a crude manner these two layers could be the atmosphere and the ocean. The interface would then be the ocean surface. Alternatively, this configuration could model the mixed surface layer in the ocean overlaying denser deep water, or a cold atmospheric inversion layer near the ground below warmer air. In this analysis we assume that the amplitudes of the oscillations are so small that we, in (1.6.5), can take

$$\left|\vec{v}\cdot\nabla\vec{v}\right| << \left|\frac{\partial\vec{v}}{\partial t}\right|. \tag{3.2.1}$$

By neglecting the convective acceleration term, which contains products of the velocity and the velocity gradients, and with constant density, the momentum equation becomes *linear* (contains no products of dependent variables). In each layer we then obtain from (1.6.5)

$$\frac{\partial\vec{v}}{\partial t} = -\frac{1}{\rho}\nabla p - g\vec{k}\,. \tag{3.2.2}$$

From the vorticity of this equation we readily obtain

$$\frac{\partial}{\partial t}\left(\nabla\times\vec{v}\right) = 0\,. \tag{3.2.3}$$

Hence if the vorticity was zero when the motion started, which we assume, it will remain zero for all times. A velocity field that has zero vorticity can be expressed in terms of a velocity potential ϕ:

$$\vec{v} = \nabla\phi\,. \tag{3.2.4}$$

From the continuity equation (1.4.8) with constant density we then obtain

$$\nabla^2\phi = 0\,. \tag{3.2.5}$$

This is called the Laplace equation. We insert (3.2.4) into (3.2.2), and use that $g\vec{k} = \nabla(gz)$. By integrating the resulting equation, we easily obtain

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - gz\,, \tag{3.2.6}$$

where we have put an arbitrary function of time equal to zero. This constitutes the linearized version of the Euler equation.

In fact, if the vorticity in this problem is zero, we can utilize the identity $\vec{v}\cdot\nabla\vec{v} = \nabla(\vec{v}^2/2) - \vec{v}\times(\nabla\times\vec{v}) = \nabla(\vec{v}^2/2)$. The proof that $\nabla\times\vec{v} = 0$ for all times, if it was initially zero, follows from Kelvin's circulation theorem. The proof will not be given here, but is found in standard textbooks in fluid mechanics. By integrating the Navier-Stokes equation with $f = 0$ in space, we now obtain

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - \frac{1}{2}(\nabla\phi)^2 - gz\,, \tag{3.2.7}$$

which is the complete version of the Euler equation. In our studies here we shall be content with using the linearized version (3.2.6).

3.3 Boundary conditions

We consider an oscillatory disturbance η in the form of a wave with wavelength λ, and amplitude *a* along the interface between the two layers; see Fig. 3.4.

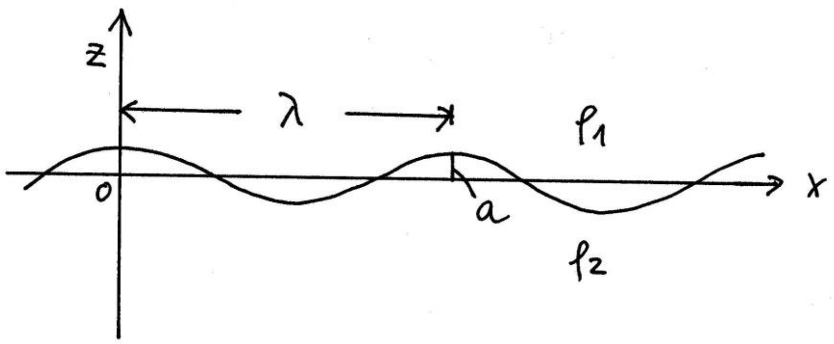

Fig. 3.4 Model sketch

Such waves are called interfacial waves. For simplicity we consider two-dimensional motion, and let the waves propagate along the *x*-axis. Mathematically we can write

$$\eta = a\cos(kx - \omega t)\,. \tag{3.3.1}$$

Here $k = 2\pi/\lambda$ is the wave number, and $\omega = 2\pi/T$ is the wave frequency, where T is the wave period. The motion in the upper layer cannot be independent of the motion in the lower layer. The variables are connected through the boundary conditions at the moving interface, given by $z = \eta$. For $k > 0, \omega > 0$, any part of the wave (3.3.1) (crests, troughs etc.) travels in the positive *x*-direction with speed $c = \omega/k$, which is called the phase speed. Such travelling waves are generally referred to as *progressive* waves.

Consider a small material element with base δA that has thickness h_1 and h_2 on both sides of a part of the interface; see Fig. 3.5.

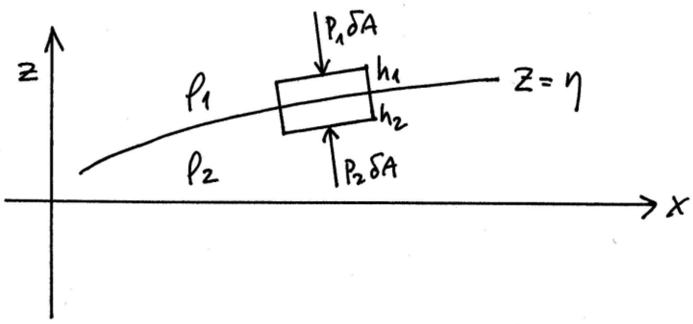

Fig. 3.5 Material element at the interface.

The mass δm of the element in Fig. 3.5 is $\delta m = (\rho_1 h_1 + \rho_2 h_2)\delta A$. The net force due to the pressure in the direction normal to the interface is $(p_2 - p_1)\delta A$. The acceleration in this direction can then be written

$$a_n = \frac{p_2 - p_1}{\rho_1 h_1 + \rho_2 h_2}. \tag{3.3.2}$$

If now $p_1 \neq p_2$ when $h_1 \to 0$ and $h_2 \to 0$, then $a_n \to \infty$. This would lead to infinitely large velocities in the fluid, which is physically impossible. Hence, when $h_1, h_2 \to 0$, we must have that $p_1 \to p_2$. Accordingly, at an infinitely thin material surface the pressure must be the same on both sides. Remember, the effect of friction will modify the normal stress; see (1.5.7), but that has been neglected here. Hence we must require that

$$p_1 = p_2, \quad \text{at} \quad z = \eta(x,t). \tag{3.3.3}$$

This is the dynamic boundary condition in this problem.

We assume in this problem that the interface between the two layers is a material surface, i.e. there is no exchange of mass across the interface. The material surface is given by $F \equiv z - \eta = 0$. At time t it has a given shape in space given by $F(t) = 0$. At a little later time $t + \Delta t$ all particles have move slightly, so now $F(t + \Delta t) = 0$. Since Δt is very small, we can write $F(t + \Delta t) = F(t) + (DF/dt)\Delta t$. Hence $DF/dt = D(z - \eta)/dt = 0$. But since $Dz/dt = w$ per definition, and $D\eta/dt = \partial\eta/\partial t + u\partial\eta/\partial x + v\partial\eta/\partial y$, we must have at both sides of the material interface:

$$\begin{aligned} w_1 &= \frac{\partial\eta}{\partial t} + u_1\frac{\partial\eta}{\partial x} + v_1\frac{\partial\eta}{\partial y}, \quad z = \eta, \\ w_2 &= \frac{\partial\eta}{\partial t} + u_2\frac{\partial\eta}{\partial x} + v_2\frac{\partial\eta}{\partial y}, \quad z = \eta. \end{aligned} \tag{3.3.4}$$

This constitutes the kinematic boundary conditions at the interface. For waves with small amplitudes we can neglect the nonlinear terms, and evaluate the velocities at the mean position $z = 0$. Hence we have approximately that

$$w_1 = w_2 = \frac{\partial \eta}{\partial t}, \quad \text{at} \quad z = 0. \tag{3.3.5}$$

In the present problem we assume that the wavelength is much smaller than the thickness of the two layers. At large distances from the interface, the effect of the interfacial wave will not be felt in the fluid. Mathematically this can be expressed as

$$\begin{aligned} \phi_1 &\to 0, \quad z \to \infty, \\ \phi_2 &\to 0, \quad z \to -\infty. \end{aligned} \tag{3.3.6}$$

These are the last two boundary conditions needed to close the system.

3.4 Solutions for a Fourier component

To solve this problem it is, as in the previous section, convenient to introduce complex variables as a helping tool, and then choose real parts to represent the physical solution. Accordingly, we write the interfacial displacement (3.3.1) as complex Fourier component

$$\eta = a e^{i(kx - \omega t)}. \tag{3.4.1}$$

We assume that the variables separate, and seek two-dimensional solutions of the Laplace equation for the velocity potential of the form

$$\phi = F(z) e^{i(kx - \omega t)}. \tag{3.4.2}$$

Then (3.2.5) reduces to

$$\frac{d^2 F}{dz^2} - k^2 F = 0, \tag{3.4.3}$$

which is an ordinary second order differential equation with constant coefficients. The solution is readily obtained as

$$F = A e^{kz} + B e^{-kz}. \tag{3.4.5}$$

In the upper and lower layer we then obtain

$$\begin{aligned} \phi_1 &= (A_1 e^{kz} + B_1 e^{-kz}) e^{i(kx - \omega t)}, \quad z \ge 0, \\ \phi_2 &= (A_2 e^{kz} + B_2 e^{-kz}) e^{i(kx - \omega t)}, \quad z \le 0. \end{aligned} \tag{3.4.6}$$

Utilizing (3.3.6), we realize immediately that to avoid infinitely large solutions, we must require

$$A_1 = 0, B_2 = 0. \tag{3.4.7}$$

Using that $w_{1,2} = \partial\phi_{1,2} / \partial z$ from (3.2.4), we find by inserting (3.4.6) and (3.4.7) into (3.3.5) that

$$-kB_1 = kA_2 = -i\omega a\,. \tag{3.4.8}$$

Hence

$$\begin{aligned}\phi_1 &= \frac{i\omega a}{k} e^{-kz + i(kx - \omega t)},\\ \phi_2 &= -\frac{i\omega a}{k} e^{kz + i(kx - \omega t)}.\end{aligned} \tag{3.4.9}$$

Finally, from (3.2.6), we find for the complex pressures in the upper and lower layer respectively:

$$\begin{aligned}\frac{p_1}{\rho_1} &= -\frac{\omega^2 a}{k} e^{-kz + i(kx - \omega t)} - gz,\\ \frac{p_2}{\rho_2} &= \frac{\omega^2 a}{k} e^{kz + i(kx - \omega t)} - gz.\end{aligned} \tag{3.4.10}$$

Utilizing the dynamic boundary condition (3.3.3), and assuming that $\exp(\pm k\eta) \approx 1$, we obtain that

$$\omega^2 = \left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gk\,. \tag{3.4.11}$$

This is called the *dispersion* relation. In our statically stable situation $(\rho_2 > \rho_1)$, it yields two real frequencies

$$\omega = \pm (g^* k)^{1/2}, \tag{3.4.12}$$

where $g^* = g(\rho_2 - \rho_1)/(\rho_1 + \rho_2)$ is termed the *reduced* gravity. We can write (3.3.1) as

$$\eta = a\cos(k(x - ct)), \tag{3.4.13}$$

where c is the phase speed given by

$$c = \frac{\omega}{k}\,. \tag{3.4.14}$$

The positive value of the frequency in (3.4.12) corresponds to positive phase speed, i.e. a wave component that propagates in the positive x-direction, and the negative one to a wave propagating in the negative x-direction. These waves can only exist in the presence of vertical density differences in the gravity field. They are referred to as *interfacial gravity* waves.

From (3.4.12) and (3.4.14) we realize that the phase speed of interfacial waves depends on the wavelength. Such waves are called *dispersive* waves. For the waves considered here the phase speed is proportional to the square root of the wavelength, which means that longer waves move faster than shorter waves.

Up to now we have considered one single wave component. If we have two wave components of the same amplitude, but with slightly different wave numbers and frequencies, they can be written in complex form as

$$\eta_+ = \frac{1}{2} a \exp i\{(k+\Delta k)x - (\omega + \Delta\omega)t\},$$
$$\eta_- = \frac{1}{2} a \exp i\{(k-\Delta k)x - (\omega - \Delta\omega)t\}, \tag{3.4.15}$$

where $|\Delta k / k| << 1, \quad |\Delta\omega / \omega| << 1$. Each of the two components above is a solution to our wave problem. Since we work with linear theory, also the sum $\eta_+ + \eta_-$ of the two components becomes a solution. This superposition can be written

$$\eta_+ + \eta_- = \frac{1}{2} a \exp i(kx - \omega t)\left[\exp i(\Delta kx - \Delta\omega t) + \exp(-i(\Delta kx - \Delta\omega t))\right]$$
$$= a \cos \Delta k\left(x - \frac{\Delta\omega}{\Delta k} t\right) \exp i(kx - \omega t). \tag{3.4.16}$$

We denote the real part of (3.4.16) by η, representing the physical solution. We then find

$$\eta = a \cos\left(\Delta k\left(x - \frac{\Delta\omega}{\Delta k} t\right)\right) \cos\left(k\left(x - \frac{\omega}{k} t\right)\right). \tag{3.4.17}$$

This shows that η is an amplitude-modulated wave train consisting of series of wave groups, as shown in Fig. 3.6, where we have plotted η / a as a function of kx for $\Delta k / k = 0.1$.

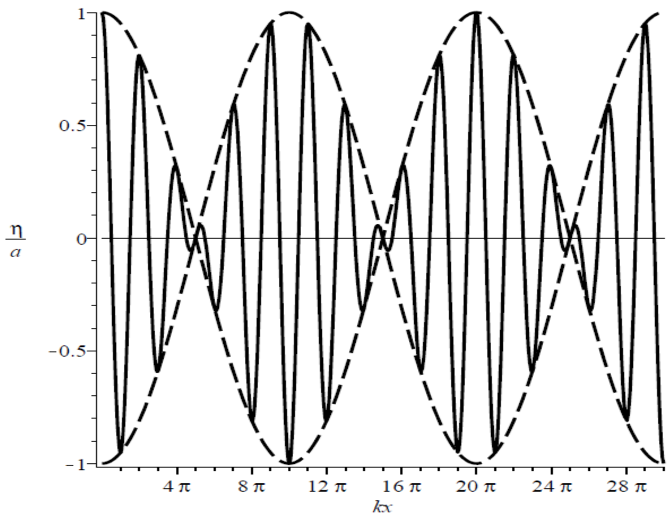

Fig. 3.6. Sketch of wave groups at a specific time (dashed envelope), moving with the group velocity. The solid line is the modulated wave, moving through the groups with the phase speed.

The individual waves in the group will propagate with the ordinary phase speed $c = \omega / k$, while the group itself will propagate with the group velocity $c_g = \Delta\omega / \Delta k$. In the limit when $\Delta k \to 0$, the group velocity becomes the derivative of the frequency with respect to the wave number, i.e.

$$c_g = \frac{d\omega}{dk} . \tag{3.4.18}$$

Since $\omega = kc$, and $k = 2\pi / \lambda$, we note that (3.4.18) can be written as

$$c_g = c - \lambda \frac{dc}{d\lambda} . \tag{3.4.19}$$

So, if the phase speed increases with increasing wavelength (normal dispersion), then $c_g < c$. If the phase speed is independent of the wavelength (non-dispersive waves), we have that $c_g = c$.

3.5 Surface waves in a layer of finite depth

When we consider waves on the interface between the atmosphere and the ocean, that is the sea surface, we refer to these waves as *surface* waves. In this case the density ρ_2 of the ocean is about one thousand times larger than the density ρ_1 of the atmosphere, and we have that $g^* \approx g$ in (3.4.12). Gravity waves of the form considered here is generated at the ocean surface by the action of the wind, and they are often called wind waves. When surface waves generated by the wind in the open sea approach the coast, they feel the presence of the ocean bottom. If the ocean bed is flat, which we assume here, and situated at $z = -H$, we must have at the ocean bottom

$$w = \frac{\partial \phi}{\partial z} = 0, \quad z = -H , \tag{3.5.1}$$

where we for simplicity have dropped the label 2 for the lower layer. This constitutes the kinematic boundary condition at the ocean bottom. Utilizing that at the surface

$$\frac{\partial \eta}{\partial t} = w = \frac{\partial \phi}{\partial z}, \quad z = 0 , \tag{3.5.2}$$

we find from (3.4.1) and (3.4.5) that

$$\phi = -\frac{i\omega a \cosh(k(z+H))}{k \sinh(kH)} \exp i(kx - \omega t) . \tag{3.5.3}$$

Hence, the real parts of the velocities in the ocean can be written

$$u = \frac{\omega a \cosh(k(z+H))}{\sinh(kH)}\cos(kx - \omega t),$$
$$w = \frac{\omega a \sinh(k(z+H))}{\sinh(kH)}\sin(kx - \omega t). \tag{3.5.4}$$

For the real part of the pressure we find from (3.2.6) that

$$\frac{p}{\rho} = \frac{\omega^2 a \cosh(k(z+H))}{k \sinh(kH)}\cos(kx - \omega t) - gz\,. \tag{3.5.5}$$

For surface waves in the ocean we can neglect the effect of the air above the water. This means that we can take $p = 0$ at the surface. Hence, from the dynamic boundary condition $p(\eta) = 0$, (3.5.5) yields

$$\eta = a\cos(kx - \omega t) = \frac{\omega^2 a \cosh(k(\eta + H))}{gk \sinh(kH)}\cos(kx - \omega t)\,. \tag{3.5.6}$$

Utilizing that $\eta << H$, we obtain for the frequency

$$\omega^2 = gk \tanh(kH)\,. \tag{3.5.7}$$

For waves propagating in the positive *x*-direction, we find for the phase speed of deep-water waves ($kH >> 1$) that

$$c = \frac{\omega}{k} = \left(\frac{g\lambda}{2\pi}\right)^{1/2}. \tag{3.5.8}$$

For waves in shallow water ($kH << 1$), we find

$$c = (gH)^{1/2}\,. \tag{3.5.9}$$

We note that shallow-water waves are non-dispersive. It is a simple exercise to show from (3.4.18) and (3.5.7) that the general relation between the group velocity and the phase velocity for surface waves becomes

$$\frac{c_g}{c} = \frac{1}{2}\left(1 + \frac{2kH}{\sinh(2kH)}\right). \tag{3.5.10}$$

3.6 Standing waves

In section 3.4 we added two wave components travelling in the same direction with slightly different frequencies and wave numbers to obtain wave groups. The principle of superposition for linear waves must also be true for waves travelling in opposite directions.

For complex progressive wave components of equal amplitude *a*, the surface shape for this case can be written

$$\eta = ae^{i(kx-\omega t)} + ae^{i(kx+\omega t)} . \tag{3.6.1}$$

Take that $\omega > 0$. Then the first component travels in the positive *x*-direction, while the second component travels in the negative *x*-direction. We can also write the elevation (3.6.1) as

$$\eta = 2ae^{ikx} \cos \omega t . \tag{3.6.2}$$

The velocity potential ϕ for the superposition of waves is given by the Laplace equation: $\partial^2\phi / \partial x^2 + \partial^2\phi / \partial z^2 = 0$. This suggests that we can write

$$\phi = (Ae^{kz} + Be^{-kz})e^{ikx} \sin \omega t . \tag{3.6.3}$$

The $\sin \omega t$ in this expression comes from the fact that ϕ must satisfy the kinematic boundary condition (3.5.2) at the surface. At the ocean bottom $z = -H$, (3.5.1) leads to

$$B = Ae^{-2kH} . \tag{3.6.4}$$

From (3.5.2) and (3.6.2) we obtain

$$A - B = -\frac{2a\omega}{k} . \tag{3.6.5}$$

Using (3.6.4) and (3.6.5), the velocity potential for this problem can be written

$$\phi = -\frac{2a\omega \cosh(k(z+H))}{k \sinh(kH)} e^{ikx} \sin \omega t . \tag{3.6.6}$$

The dynamic boundary condition is $p(z = \eta) = 0$. Hence, from (3.2.6) for linear waves: $(\partial\phi / \partial t)_{z=0} + g\eta = 0$, which yields the same dispersion relation as before

$$\omega^2 = gk \tanh(kH) . \tag{3.6.7}$$

However, the motion in the fluid layer is now very different from that of a progressive wave. This is most easily seen by introducing the stream function for this problem. Since here

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 , \tag{3.6.8}$$

we can define a stream function ψ such that

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x} . \tag{3.6.9}$$

We note that by this definition, (3.6.8) is identically satisfied. In vector notation we can write for the velocity in the fluid

$$\vec{v} = -\frac{\partial \psi}{\partial z}\vec{i} + \frac{\partial \psi}{\partial x}\vec{k} = -\vec{j} \times \nabla \psi \,. \tag{3.6.10}$$

Since $\nabla \psi$ is always perpendicular to the lines of constant ψ, we note from (3.6.10) that for two-dimensional motion, the velocity vector is tangential to lines of $\psi = \text{constant}$ at every instant of time. Accordingly, such lines are *stream lines*, and that is why ψ is called the stream function. Since now $\partial\psi / \partial x = w = \partial\phi / \partial z$, we easily obtain from (3.6.6), taking the real part:

$$\psi = -\frac{2a\omega \sinh(k(z+H))}{k \sinh(kH)} \sin kx \sin \omega t \,. \tag{3.6.11}$$

By plotting lines of constant ψ, we can visualize the flow field in this case. We note that $\psi = 0$ for $z = -H$, so the ocean bottom is a stream line. Also, for $x = n\pi / k, n = 0,1,2,3..$, we have that $\psi = 0$, so the layer is divided into cells of width π / k, or alternatively $\lambda / 2$ (half a wavelength). This wave system is called *standing waves*. In the figure below we depict two (of infinitely many) cells

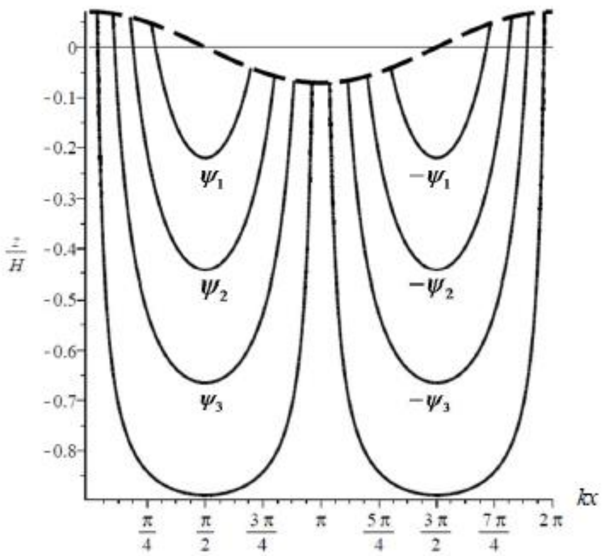

Fig. 3.7. Sketch of stream lines at a particular instant of time (solid lines). The dashed line on top represents the non-dimensional surface elevation at the same instant.

From the real part of (3.6.2), we note that the surface elevation is zero for all times when $\cos kx = 0$, i.e. $x = (2n+1)\pi/(2k), n = 0,1,2..$. In Fig. 3.7 we observe two such points for $kx = \pi / 2$, and $kx = 3\pi / 2$. Points in space where the surface elevation in standing waves always is zero are called *nodal points*, or *nodes*.

Often this kind of wave system originates in closed basins with lateral walls. If we have impermeable vertical boundaries at $x = 0$, and $x = L$, say, we must require that

$u = -\partial\psi/\partial z = 0$ at these walls. From (3.6.11) we realize that this is fulfilled at $x = 0$, while the condition at the other wall requires that $kL = n\pi$. Accordingly, this is not possible for an arbitrary wave number. We note that *k* must belong to a discrete set

$$k = k_n = n\pi/L, \quad n = 1,2,3... \tag{3.6.12}$$

The corresponding possible frequencies (often called eigen-frequencies) become

$$\omega_n = \left(gk_n \tanh(k_n H)\right)^{1/2}. \tag{3.6.13}$$

Such standing waves are sometimes observed in large lakes. They are often called *seiches*. We note that for the lowest mode $n = 1$, the surfaces oscillates with frequency $\omega_1 = ((g\pi/L)\tanh(\pi H/L))^{1/2}$ about the mid-point of the lake (which is a node), with maximum amplitudes at the two ends. For most lakes we have $H << L$. Then the lowest mode reduces to $\omega_1 = \pi(gH)^{1/2}/L$ (the shallow water approximation).

Seiches can also occur in harbors with one open end. In this case we have standing waves with a node at the end $(w = \partial\psi/\partial x = 0, x = L)$, and hence $k_n = (2n+1)\pi/L$, $n = 0,1,2,...$ If the frequency of the tidal motion in the ocean outside the harbor coincides with one of the admissible eigen-frequencies, resonance can occur, leading to large vertical motion within the harbor.

3.7 Energy considerations

A local wind event in the open deep ocean generates wind waves with many different wavelengths. Since such waves are dispersive, as seen from (3.5.8), the longest waves will travel fastest. For example, for a wavelength of 300 m, we find that the phase speed is nearly 22 m/s. These waves may propagate faster than the low pressure system that generated then, and hence escape from the storm region. Such waves are called *swell*, and may propagate for hundreds of kilometres through the ocean till they finally reach the coast, gradually transforming to shallow-water waves. Finally, they break in the surf zone on the beach, and loose their mechanical energy. In this way we understand that waves are carriers of energy. They get their energy from the wind, propagate the energy over large distances, and loose it by doing work on the beaches in the form of beach erosion processes etc. If there is any rest mechanical energy, it is transferred to heat in the breaking process.

The total mechanical energy *E* per unit area in surface waves is the sum of the mean kinetic energy E_k and the mean potential energy E_p. Per definition

$$E_k = \frac{1}{T}\int_0^T \left(\tfrac{1}{2}\rho \int_{-H}^{\eta} (u^2 + w^2)dz \right) dt \approx \frac{1}{T}\int_0^T \left(\tfrac{1}{2}\rho \int_{-H}^{0} (u^2 + w^2)dz \right) dt, \tag{3.7.1}$$

where $T = 2\pi/\omega$ is the wave period. For periodic wave motion we assume that the potential energy is zero at the mean surface level. Hence

$$E_p = \frac{1}{T}\int_0^T \left(\rho g \int_0^\eta z dz \right) dt \,. \tag{3.7.2}$$

Inserting from (3.3.1) and (3.5.4), we obtain after some algebra that

$$E_k = E_p = \frac{1}{4}\rho g a^2 \,. \tag{3.7.3}$$

Hence, the mechanical energy is equally partitioned between kinetic and potential energy. The total energy per unit area, often referred to as the energy density, becomes

$$E = E_k + E_p = \frac{1}{2}\rho g a^2 \,. \tag{3.7.4}$$

The mean horizontal energy flux F_E is the work per unit time done by the dynamic (fluctuating) pressure in displacing particles horizontally. By definition

$$F_E = \frac{1}{T}\int_0^T \left(\int_{-H}^\eta pudz \right) dt \approx \frac{1}{T}\int_0^T \left(\int_{-H}^0 pudz \right) dt \,. \tag{3.7.5}$$

Applying the horizontal velocity in (3.5.4) and the dynamic pressure in (3.5.5) (leaving out the static part $\rho g z$), we find

$$F_E = \frac{\rho \omega^3 a^2}{8k^2 \sinh^2 kH}\left(\sinh(2kH) + 2kH \right). \tag{3.7.6}$$

Utilizing the phase speed definition (3.14.4), the dispersion relation (3.5.7), and the group velocity given by (3.5.10), we can write the mean energy flux (3.7.6) as

$$F_E = c_g E \,. \tag{3.7.7}$$

In our earlier treatment of the group velocity it was defined from a purely kinematic point of view. We understand from (3.7.7) that the group velocity has a much deeper significance: It is the velocity that the mean energy in the wave motion travels with. Accordingly, to receive a signal that propagates over a distance L in the form of a wave, we must wait a time $t = L/c_g$, before the receiver picks up the signal.

3.8 Particle motion in progressive surface waves

In progressive waves it is the wave form which moves with the phase speed. The individual fluid particles move with a much smaller velocity. For surface waves, where $z \le 0$, we obtain from (3.5.4) in the deep-water limit ($kH >> 1$):

$$u = \omega a e^{kz} \cos(kx - \omega t),$$
$$w = \omega a e^{kz} \sin(kx - \omega t). \tag{3.8.1}$$

We note that the maximum velocity in the fluid occur at the surface $z = 0$. Here

$$\left(u^2 + w^2\right)^{1/2} = \omega a = kac. \tag{3.8.2}$$

The quantity *ka* is referred to as the wave *steepness*. For the small-amplitude waves considered here, we always have for the steepness that

$$ka = 2\pi a / \lambda << 1. \tag{3.8.3}$$

We thus see from (3.8.2) that the magnitude of the fluid velocity is much smaller than the phase speed of the wave.

We may define Lagrangian particle coordinates (x_L, z_L) in the vertical plane in the same way as we did for inertial oscillations in the horizontal plane. Utilizing that

$$\frac{\partial x_L}{\partial t} = u, \quad \frac{\partial z_L}{\partial t} = w, \tag{3.8.4}$$

we find from (3.8.1) that

$$x_L = -a e^{kz_0} \sin(kx_0 - \omega t) + x_0,$$
$$z_L = a e^{kz_0} \cos(kx_0 - \omega t) + z_0, \tag{3.8.5}$$

where (x_0, z_0) is the mean position of the particle $(z_0 \le 0)$. We then obtain

$$(x_L - x_0)^2 + (z_L - z_0)^2 = a^2 e^{2kz_0}. \tag{3.8.6}$$

Hence individual fluid particles move in closed circles with radius $r = a\exp(kz_0)$. At the surface $(z_0 = 0)$, the radius has its maximum value *a*. It is easy to show from (3.5.4) that for finite depth, the individual particles move in closed ellipses, where the long axis is horizontal and the short axis is vertical.

3.9 The Stokes drift

The result above that the particles in deep water progressive waves move in closed circles is correct in the present linear approach (remember we have linearized our equations). In reality, if we do our calculations without linearization, we find that that the individual fluid particles have a slow net drift in the wave propagation direction. This is because the velocity of the fluid particle is a little larger when it is closest to the surface, than when it is farthest away from it. Hence, it moves a little more forward than it moves backward. The resulting motion will be a forward spiral; see the sketch below. The net particle velocity described her is called the *Stokes drift*.

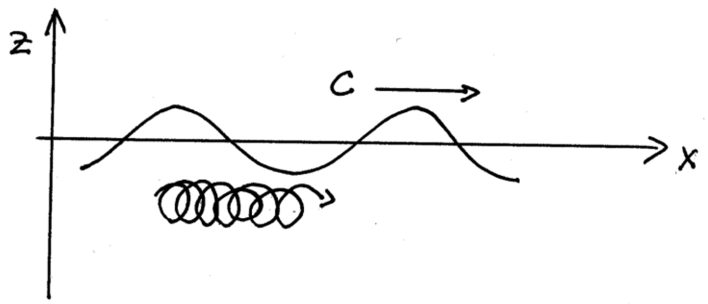

Fig. 3.8 Sketch of nonlinear motion of a fluid particle due to waves.

The Stokes drift can be obtained by considering the *Lagrangian* velocity, which is the velocity following an individual fluid particle. We denote it by $\vec{v}_L$. Then $\vec{v}_L(\vec{r}_0, t)$ is the velocity of a fluid particle whose position at time $t = t_0$ is $\vec{r}_0 = (x_0, y_0, z_0)$. At a later time *t*, the particle has moved to a new position

$$\vec{r}_L = \vec{r}_0 + D\vec{r}\,. \tag{3.9.1}$$

where

$$D\vec{r} = \int_{t_0}^{t} \vec{v}_L(\vec{r}_0, t')dt'\,. \tag{3.9.2}$$

In our former Eulerian specification the fluid velocity at time *t* is $\vec{v}(\vec{r}_L, t)$, e.g. (1.3.2). Hence

$$\vec{v}_L(\vec{r}_0, t) = \vec{v}(\vec{r}_L, t)\,. \tag{3.9.3}$$

By inserting for $\vec{r}_L$ from (3.9.1), we obtain

$$\vec{v}_L(\vec{r}_0, t) = \vec{v}\left(\vec{r}_0 + D\vec{r}, t\right). \tag{3.9.4}$$

We assume that the distance $D\vec{r} = \vec{r}_L - \vec{r}_0$ travelled by the particle in the time interval $t - t_0$ is small. Hence, from the two first terms of a Taylor series expansion we obtain

$$\vec{v}_L(\vec{r}_0, t) = \vec{v}(\vec{r}_0, t) + \frac{\partial \vec{v}}{\partial x_0}Dx + \frac{\partial \vec{v}}{\partial y_0}Dy + \frac{\partial \vec{v}}{\partial z_0}Dz = \vec{v}(\vec{r}_0, t) + D\vec{r}\cdot\nabla_L\vec{v}\,, \tag{3.9.5}$$

where $\nabla_L \equiv \vec{i}\,\partial/\partial x_0 + \vec{j}\,\partial/\partial y_0 + \vec{k}\,\partial/\partial z_0$. If we use (3.9.2), we can write (3.9.5) as

$$\vec{v}_L(\vec{r}_0, t) = \vec{v}(\vec{r}_0, t) + \left(\int_{t_0}^{t} \vec{v}_L(\vec{r}_0, t')dt'\right)\cdot\nabla_L\vec{v}(\vec{r}_0, t)\,. \tag{3.9.6}$$

The last part of the velocity on the right-hand side of (3.9.6) is called the Stokes velocity $\vec{v}_S$, while the first term is the traditional Eulerian velocity at a fixed position. Hence, in general

$$\vec{v}_L = \vec{v} + \vec{v}_S \,. \tag{3.9.7}$$

For surface waves with small wave steepness the difference between $\vec{v}_L$ and $\vec{v}$ is small, so to second order in wave steepness we can substitute the Lagrangian velocity by the Eulerian velocity in the integral of (3.9.6), i.e.

$$\vec{v}_S = \left(\int_{t_0}^{t} \vec{v}(\vec{r}_0, t')dt' \right) \cdot \nabla_L \vec{v}(\vec{r}_0, t) \,. \tag{3.9.8}$$

For waves with period *T*, the averaged Stokes velocity (denoted by an over-bar) becomes

$$\bar{\vec{v}}_S = \frac{1}{T} \int_0^T \vec{v}_S dt \,. \tag{3.9.9}$$

The averaged Stokes velocity (3.9.9) is the aforementioned Stokes drift, and constitutes a mean current induced by the waves.

Let us return to the two-dimensional Eulerian wave field for deep water waves (3.8.1). Then, for calculating the Stokes drift, we have

$$\begin{aligned} u &= u(x_0, z_0, t) = \omega a e^{kz_0} \cos(kx_0 - \omega t), \\ w &= w(x_0, z_0, t) = \omega a e^{kz_0} \sin(kx_0 - \omega t). \end{aligned} \tag{3.9.10}$$

In this problem t_0 is arbitrary, so we take $t_0 = 0$. When we average the Stokes velocity in time, we only get non-zero contributions from $\cos^2(kx_0 - \omega t), \sin^2(kx_0 - \omega t)$ in (3.9.9). It is then easily seen that the Stokes drift components become

$$\begin{aligned} \bar{u}_S &= \frac{1}{T} \int_0^T \left((\int_0^t u dt') \frac{\partial u}{\partial x_0} + (\int_0^t w dt') \frac{\partial u}{\partial z_0} \right) dt = \omega k a^2 e^{2kz_0}, \\ \bar{w}_S &= \frac{1}{T} \int_0^T \left((\int_0^t u dt') \frac{\partial w}{\partial x_0} + (\int_0^t w dt') \frac{\partial w}{\partial z_0} \right) dt = 0. \end{aligned} \tag{3.9.11}$$

We note that the vertical component of the Stokes drift is zero, while the horizontal component decays exponentially with depth. Furthermore, $\bar{u}_S$ has a maximum at the surface, where $z_0 = 0$. For strong wind sea, with wavelength $\lambda = 100\,\mathrm{m}$, and amplitude $a = 2\,\mathrm{m}$, we find from (3.9.11) that $\bar{u}_S = 20\,\mathrm{cm\,s^{-1}}$ at the surface. This wave-driven current is comparable in magnitude to the wind- and tidally-driven currents we usually find in the world's oceans. The variation with depth of the Stokes drift in this example is plotted in Fig. 3.9.

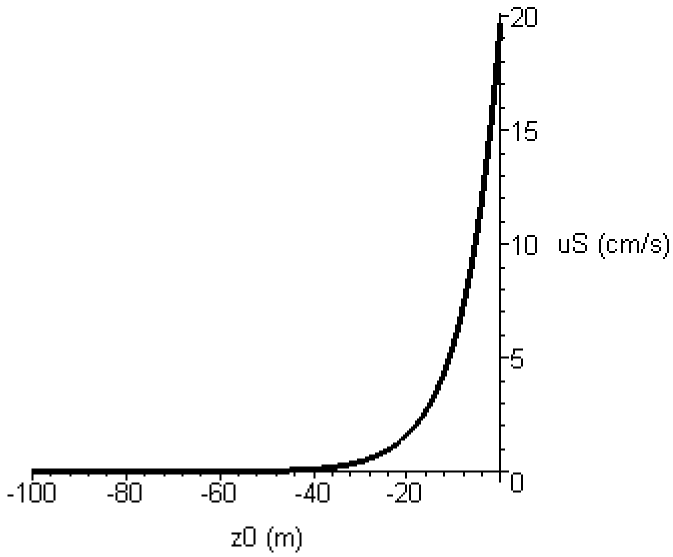

Fig. 3.9 Stokes drift from (3.9.11) as function of depth for deep water waves with wavelength 100 m and amplitude 2 m.

Obviously, the Stokes drift for the standing wave in section 3.6 is identically zero.

4. LARGE-SCALE MOTION AWAY FROM BOUNDARIES

4.1 The hydrostatic approximation

In an attempt to simplify our governing dynamical equations, we look at a two-dimensional model; see the sketch below.

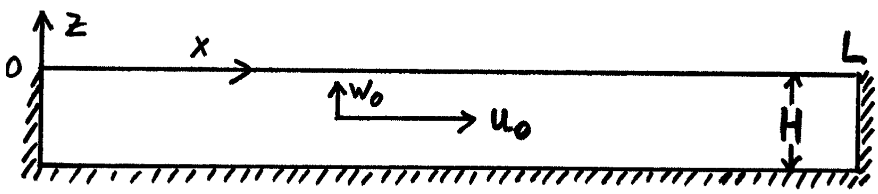

Fig. 4.1 A two-dimensional fluid model.

Typical oceanic length scales in the model are $L \sim 10^3$ km and $H \sim 1$ km, i.e.

$$\frac{H}{L} \sim 10^{-3}. \tag{4.1.1}$$

A typical horizontal velocity scale for the ocean is given by

$$u \sim u_0 \sim 0.1\,\mathrm{m\,s}^{-1}, \tag{4.1.2}$$

and a typical time scale

$$T \sim 10\ \mathrm{days} \sim 10^6\,\mathrm{s}. \tag{4.1.3}$$

For this scale analysis we may apply the two-dimensional continuity equation in the form (1.7.7). Hence

$$\left|\frac{\partial w}{\partial z}\right| = \left|\frac{\partial u}{\partial x}\right| . \tag{4.1.4}$$

Typically, we have

$$\left|\frac{\partial u}{\partial x}\right| \sim \frac{u_0}{L}, \quad \left|\frac{\partial w}{\partial z}\right| \sim \frac{w_0}{H} . \tag{4.1.5}$$

Then, from (4.1.4)

$$w_0 \sim \frac{H}{L} u_0 . \tag{4.1.6}$$

Hence, with these oceanic scales the typical vertical velocity is *very* much smaller than the typical horizontal velocity. This is also found to be the case in the atmosphere.

Away from the ground in the atmosphere and the surface/bottom in the ocean, the viscous forces are negligible in the momentum equation. The vertical component of the Navier-Stokes equation then becomes in the two-dimensional case

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g . \tag{4.1.7}$$

Then, by inserting typical values into (4.1.7):

$$
\begin{array}{ccccccccc}
\dfrac{\partial w}{\partial t} & + & u\dfrac{\partial w}{\partial x} & + & w\dfrac{\partial w}{\partial z} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & - & g \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dfrac{Hu_0}{LT} & & \dfrac{u_0{}^2 H}{L^2} & & \dfrac{u_0{}^2 H}{L^2} & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
10^{-10}\,\dfrac{\mathrm{m}}{\mathrm{s}^2} & & 10^{-11}\,\dfrac{\mathrm{m}}{\mathrm{s}^2} & & 10^{-11}\,\dfrac{\mathrm{m}}{\mathrm{s}^2} & & ? & & 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}
\end{array}
$$

Obviously, balance here requires to a high degree of accuracy that

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\ . \tag{4.1.8}$$

This simplification of the vertical component of the momentum equation is called the *hydrostatic* approximation. It expresses the balance between the vertical component of the pressure-gradient force and the gravity force (per unit mass), i.e. the *hydrostatic* balance. Equivalently; we may write

$$\frac{\partial p}{\partial z} = -\rho g, \tag{4.1.9}$$

which is called the *hydrostatic* equation. This equation is similar to (2.3.3) for a fluid at rest, except for the partial derivative in (4.1.9). Now the pressure and the density may be functions of the horizontal coordinates and time, besides z.

Although the inserted values for the parameters in this example are typical for the ocean, it turns out that the hydrostatic approximation also works very well for large scale motion in the atmosphere. As indicated here, this means that the typical horizontal length scale of the problem must be much larger than the fluid depth, which, admittedly, is more difficult to define in the atmosphere than in the ocean.

4.2 Isobaric coordinates

In cases where the hydrostatic approximation (4.1.9) applies, we may use the pressure p as an independent coordinate instead of z. This is widely used in meteorology. Then for any variable F in the fluid we can write $F = F(x, y, p, t)$. The variables (x, y, p) are known as *isobaric* coordinates. In meteorology one also sometimes uses the potential temperature θ instead of z. The variables (x, y, θ) are then known as *isentropic* coordinates. We here discuss some of the advantages by using isobaric coordinates, which is most commonly done.

Introducing the geopotential Φ defined by (1.6.2), the hydrostatic equation (4.1.9) now becomes, for fixed values of *x* and *y*:

$$\frac{\partial p}{\partial z} = -\rho \frac{\partial \Phi}{\partial z} \ . \tag{4.2.1}$$

By multiplying (4.2.1) by dz, we obtain $dp = -\rho d\Phi$ for fixed *x* and *y*. Hence, in isobaric coordinates:

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} \tag{4.2.2}$$

The fluid velocity (u, v, ϖ) in isobaric coordinates is defined as the rate of change of the isobaric coordinates of a fluid particle, i.e.,

$$u = \frac{Dx}{dt}, \quad v = \frac{Dy}{dt}, \quad \varpi = \frac{Dp}{dt}, \tag{4.2.3}$$

where now $Dq/dt \equiv \partial q/\partial t + u\partial q/\partial x + v\partial q/\partial y + \varpi \partial q/\partial p$ is the material derivative of any dependent variable *q*.

In general, we write for the total differential of the pressure in Cartesian coordinates

$$dp = \left(\frac{\partial p}{\partial x}\right)_{y,z} dx + \left(\frac{\partial p}{\partial y}\right)_{x,z} dy + \left(\frac{\partial p}{\partial z}\right)_{x,y} dz \ , \tag{4.2.4}$$

where the subscripts indicate what is kept fixed during the partial differentiation. Obviously, along an isobaric surface we must have $dp = 0$. Denoting the coordinates of the isobaric surface by (x_p, y_p, z_p), we have from (4.2.4) that

$$0 = \left(\frac{\partial p}{\partial x}\right)_{y,z} dx_p + \left(\frac{\partial p}{\partial y}\right)_{x,z} dy_p + \left(\frac{\partial p}{\partial z}\right)_{x,y} dz_p \ . \tag{4.2.5}$$

Geometrically, the isobaric surface $p = p_0$ may be written as $z = z(x, y, p_0)$. Hence, along this surface:

$$dz_p = \left(\frac{\partial z}{\partial x}\right)_{y,p} dx_p + \left(\frac{\partial z}{\partial y}\right)_{x,p} dy_p \ . \tag{4.2.6}$$

From (4.2.5) we then obtain

$$0 = \left[\left(\frac{\partial p}{\partial x}\right)_{y,z} + \left(\frac{\partial p}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_{y,p}\right] dx_p + \left[\left(\frac{\partial p}{\partial y}\right)_{x,z} + \left(\frac{\partial p}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial y}\right)_{x,p}\right] dy_p \ . \tag{4.2.7}$$

Here dx_p and dy_p are independent variables, so the expressions inside each bracket in (4.2.7) must be zero. Utilizing the hydrostatic equation, this leads to

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = \rho\left(\frac{\partial (gz)}{\partial x}\right)_{y,p}, \quad \left(\frac{\partial p}{\partial y}\right)_{x,z} = \rho\left(\frac{\partial (gz)}{\partial y}\right)_{x,p}. \tag{4.2.8}$$

Finally, by introducing the geopotential into (4.2.8), we obtain for the horizontal pressure-gradient force per unit mass:

$$\begin{aligned} -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{y,z} &= -\left(\frac{\partial \Phi}{\partial x}\right)_{y,p}, \\ -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_{x,z} &= -\left(\frac{\partial \Phi}{\partial y}\right)_{x,p}. \end{aligned} \tag{4.2.9}$$

For an inviscid fluid the horizontal components of the momentum equation (1.6.6) in isopbaric coordinates reduce to

$$\begin{aligned} \frac{Du}{dt} - fv &= -\left(\frac{\partial \Phi}{\partial x}\right)_{y,p}, \\ \frac{Dv}{dt} + fu &= -\left(\frac{\partial \Phi}{\partial y}\right)_{x,p}. \end{aligned} \tag{4.2.10}$$

In later use, we simplify, and keep only the subscript p to remind ourselves that the pressure is constant during that specific operation. The form (4.2.10) involves no reference to the density, and is therefore much more amenable to use. The removal of the density from the expression for the pressure gradient is a major advantage of isobaric coordinates and provides one of the motivations for their application.

The mass of a small material element δm can be written in isobaric coordinates as

$$\delta m = \rho \delta x \delta y \delta z = -\delta x \delta y \delta p / g \,. \tag{4.2.11}$$

Since the mass must be conserved, we have that

$$\frac{D(\delta m)}{dt} = 0 \,. \tag{4.2.12}$$

By dividing (4.2.12) by $\delta x \delta y \delta p$, and applying (4.2.3) as well as the fact that g is constant, we find that

$$\frac{1}{\delta x}\left(\delta \frac{Dx}{dt}\right) + \frac{1}{\delta y}\left(\delta \frac{Dy}{dt}\right) + \frac{1}{\delta p}\left(\delta \frac{Dp}{dt}\right) = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \varpi}{\delta p} = 0 \,. \tag{4.2.13}$$

Here we have utilized that $D(\delta q)/dt = \delta Dq/dt$. In the limit $\delta x \delta y \delta p \to 0$, (4.2.13) becomes

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p+\frac{\partial \varpi}{\partial p}=0\,. \tag{4.2.14}$$

This form of the continuity equation does not contain any reference to the density field, and does not involve time derivatives. It is valid for compressible fluids, although it has the same form as the incompressible version (1.7.7) in Eulerian coordinates. The form (4.2.14) provides an additional motivation for the use of isobaric coordinates. However, it must be kept in mind that (4.2.14) is only valid when we can make the hydrostatic approximation.

4.3 Geostrophic flows

As mentioned before, away from the ground in the atmosphere, and from the surface/bottom/coast in the ocean, the effect of the viscosity is negligible. From (1.6.8) the horizontal components of the momentum equation can then be written

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=fv-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=-fu-\frac{1}{\rho}\frac{\partial p}{\partial y}.\end{aligned} \tag{4.3.1}$$

In such regions the time, length and velocity scales of section 4.1 are typical for large scale, slowly varying oceanic motion. By inserting these scales into (4.2.1), we find that the local and convective acceleration terms typically are of the order $10^{-7}\,\mathrm{m\,s^{-2}}$ and $10^{-8}\,\mathrm{m\,s^{-2}}$, respectively. At mid-latitudes we have $f \sim 10^{-4}\,\mathrm{s^{-1}}$. Hence, with a horizontal velocity scale of $10^{-1}\,\mathrm{m\,s^{-1}}$, we find that the Coriolis terms in (4.3.1) are typically of the order $10^{-5}\,\mathrm{m\,s^{-2}}$. Obviously, since this is about one thousand times larger than the acceleration terms, the only possible balance under these circumstances can occur between the Coriolis force and the horizontal components of the pressure-gradient force. This is called *geostrophic* balance, and can be stated mathematically as

$$\begin{aligned}0&=fv-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ 0&=-fu-\frac{1}{\rho}\frac{\partial p}{\partial y}.\end{aligned} \tag{4.3.2}$$

Here the velocity $\vec{v}_g=u\vec{i}+v\vec{j}$ is the geostrophic wind or current. It can be expressed explicitly from (4.3.2) as

$$\vec{v}_g=\frac{1}{f\rho}\vec{k}\times\nabla p\,. \tag{4.3.3}$$

Thus, the knowledge of the pressure distribution at any time determines the geostrophic current. It is important to note that the geostrophic balance (4.3.2) is only valid away from the equator (where $f=0$).

The common visual representation of the pressure field in a horizontal plane is a map showing a set of *isobars*. An isobar is a curve containing all points in which the pressure attains some chosen value. The isobars are drawn for pressure values at equidistant intervals, i.e. $p_1,\ p_1+\Delta p,\ p_1+2\Delta p$, etc. We note from (4.3.3) that the geostrophic wind blows along the isobars. If the low pressure minimum was concentrated in a point, surrounded with air of higher pressure, the horizontal component of the pressure gradient will be directed outwards from the centre. Accordingly, from (4.3.3), the geostrophic wind in the northern hemisphere would blow around the low pressure centre in an anti-clockwise direction. Similarly, the wind would blow around a high pressure centre in a clockwise manner. Such circulations are called *cyclonic* and *anti-cyclonic*, respectively; see the sketches in Fig. 4.2.

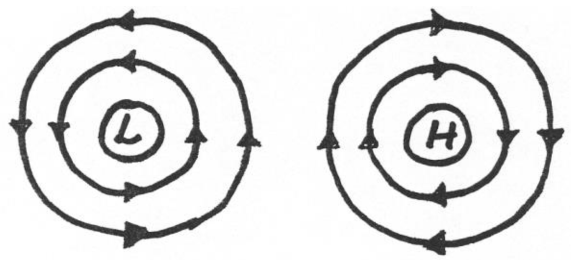

Fig. 4.2 Cyclonic circulation around a low pressure centre (L), and anti-cyclonic circulation around a high pressure centre (H) in the northern hemisphere. The solid lines are isobars.

An alternative way to depict the pressure field is by letting horizontal planes at equidistant heights $H_1,\ H_1+\Delta H,\ H_1+2\Delta H$ etc., intersect an isobaric surface. This yields a map showing a set of *contour lines* for this constant pressure surface. We note from (4.3.3) that the geostrophic flow is directed along the contour lines. The more closely spaced the contour lines become, the stronger is the geostrophic wind. In summary, the isobars yield the spatial pressure distribution at a particular height, and hence the geostrophic velocity at that height, while the contour lines yield the geostrophic flow along an isobaric surface at different heights. Obviously, the single isobar $p = p_1$ drawn on $z = H_1$ coincides with the contour line at $z = H_1$ for the isobaric surface $p = p_1$.

The ratio between the convective acceleration and the Coriolis force is called the Rossby number *Ro*, i.e.

$$Ro \equiv \left| \frac{\vec{v}\cdot\nabla\vec{v}}{f\vec{k}\times\vec{v}} \right| . \qquad (4.3.4)$$

We realize that geostrophic motion implies small Rossby numbers. In the order of magnitude analysis above we find that $Ro \sim 10^{-3}$, which is typical for large scale motion in the atmosphere and the ocean.

For such flows in the atmosphere one often uses isobaric coordinates. From (4.2.10) the geostrophic current (4.3.3) can then be written

$$\vec{v}_g = \frac{1}{f}\vec{k}\times\left(\nabla\Phi\right)_p . \qquad (4.3.5)$$

The advantage with this approach is that the density does not appear explicitly on the right-hand side of the expression for the geostrophic velocity. Furthermore, if f is constant, we note from (4.3.5) that the horizontal divergence of the geostrophic wind at constant pressure is zero:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p=0\,. \qquad (4.3.6)$$

Then, from (4.2.14), ϖ is independent of p.

4.4 Cyclostrophic flow

In some intense circulation systems, like tornadoes in the atmosphere, the Coriolis force is too small to balance the pressure gradient force. For quasi-steady motion in such cases the balance of forces on a fluid particle is between the centrifugal force (the negative convective acceleration) and the pressure gradient force. In a typical tornado the tangential velocity V may be 30 $\mathrm{m\,s^{-1}}$ at a distance $D=300\,\mathrm{m}$ from the centre of the vortex. For circular motion the centrifugal force per unit mass becomes

$$F_{cent}=\left|-\vec{v}\cdot\nabla\vec{v}\right|=\frac{V^2}{D}=3\,\mathrm{m s^{-2}}\,. \qquad (4.4.1)$$

At mid-latitudes with $f=10^{-4}\mathrm{s}^{-1}$, the Coriolis force per unit mass is

$$F_{Cor}=\left|fV\right|=3\times10^{-3}\mathrm{ms^{-2}}\,, \qquad (4.4.2)$$

which is much less than the centrifugal force in this case. Motions in which there is a balance between the centrifugal force and the pressure gradient force are referred to as *cyclostrophic* flows. We note from (4.3.4) that the Rossby number in this case becomes

$$Ro\sim F_{cent}/F_{Cor}\sim10^3\,. \qquad (4.4.3)$$

Hence, cyclostrophic flows are characterized by large Rossby numbers. Such flows must always have a pressure minimum in the centre, yielding a pressure gradient force that is directed inward. The circulation can in principle be cyclonic or anti-cyclonic. In both cases the centrifugal force becomes directed outwards, which balances the pressure gradient force.

4.5 Barotropic motion

When the isobaric and isopycnal surfaces in the fluid are parallel, the mass field is said to be *barotropic*. In this case the pressure can be written as a function of the density only, i.e. $p=p(\rho)$. We may define the work dW done by the pressure-gradient force per unit mass in displacing a particle a small distance $d\vec{r}$ by

$$dW = -\frac{1}{\rho}\nabla p \cdot d\vec{r} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) = -\frac{1}{\rho}dp\ . \qquad (4.5.1)$$

Let $d\vec{r}$ be aligned along a closed curve γ in space. Generally the total work done by the pressure in displacing the particle around the closed curve will be non-zero, i.e.

$$\oint_{\gamma} dW \neq 0\ . \qquad (4.5.2)$$

However, when the mass field is barotropic, we can write

$$\frac{1}{\rho}dp = \frac{1}{\rho}\frac{dp}{d\rho}d\rho\ . \qquad (4.5.3)$$

When we know $p = p(\rho)$, we can always find a function $G(\rho)$ such that

$$G(\rho) = \frac{1}{\rho}\frac{dp}{d\rho}, \qquad (4.5.4)$$

If H is the anti-derivative of *G*, i.e. $G = dH / d\rho$, we can write

$$\frac{1}{\rho}dp = G(\rho)d\rho = dH\ . \qquad (4.5.5)$$

Accordingly, the line integral in (4.5.2) along the closed curve γ now becomes

$$\oint_{\gamma} dW = -\oint_{\gamma} dH(\rho) = H(\rho_s) - H(\rho_e)\ . \qquad (4.5.6)$$

Here the value of the density ρ_s at the starting point of the integration is the same as the value ρ_e at the end point (closed curve). Furthermore, from physical considerations H, which is related to the work, must be a single valued function. Hence, in the barotropic case

$$\oint_{\gamma} dW \equiv -\oint_{\gamma} \frac{1}{\rho}\nabla p \cdot d\vec{r} = 0\ . \qquad (4.5.7)$$

This means that we can write

$$\frac{1}{\rho}\nabla p = \nabla H\ . \qquad (4.5.8)$$

For synoptic scale motion we can apply the hydrostatic balance equation (4.1.9). Combined with (4.5.8):

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g = \frac{\partial H}{\partial z} \tag{4.5.9}$$

Hence

$$H = -gz + F(x, y) \ . \tag{4.5.10}$$

The geostrophic balance equations (4.3.2) reduce in this case to

$$\begin{aligned} u &= -\frac{1}{f}\frac{\partial F}{\partial y} , \\ v &= \frac{1}{f}\frac{\partial F}{\partial x} . \end{aligned} \tag{4.5.11}$$

In our analysis the Coriolis parameter is constant, or varies with linearly with y in the beta-plan approximation, see (1.6.7). Since $F = F(x, y)$, we realize that the right-hand sides of (4.5.11) do not vary with the vertical z-coordinate. Accordingly, for a barotropic mass field we find for the geostrophic flow components that

$$\begin{aligned} u &= u(x, y) , \\ v &= v(x, y) . \end{aligned} \tag{4.5.12}$$

Such flows are called *barotropic* flows.

4.6 Baroclinic motion: density currents and the thermal wind

When the isopycnal and the isobaric surfaces intersect (mathematically when $\nabla\rho \times \nabla p \neq 0$) the mass field is said to be *baroclinic*. The resulting geostrophic motion is called *baroclinic* motion. In this case the horizontal velocity components vary with height. By differentiating both equations in (4.3.2) with respect to z:

$$\begin{aligned} \frac{\partial u}{\partial z} &= -\frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) , \\ \frac{\partial v}{\partial z} &= \frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) . \end{aligned} \tag{4.6.1}$$

In the ocean one often approximate the density on the right-hand side by a constant reference value ρ_r (the Boussinesq approximation; see section 5.1). By substituting the pressure from the hydrostatic equation (4.1.9), we obtain right away that the vertical variation of the baroclinic current is proportional to the horizontal gradients of the density field, i.e.

$$\frac{\partial u}{\partial z} = -\frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho_r}\frac{\partial p}{\partial y}\right) = -\frac{1}{f\rho_r}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = \frac{g}{f\rho_r}\left(\frac{\partial \rho}{\partial y}\right),$$
$$\frac{\partial v}{\partial z} = \frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho_r}\frac{\partial p}{\partial x}\right) = \frac{1}{f\rho_r}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = -\frac{g}{f\rho_r}\left(\frac{\partial \rho}{\partial x}\right). \tag{4.6.2}$$

Such currents are called *density currents*.

In the atmosphere, applying the ideal gas law, the hydrostatic equation (4.2.1) in isobaric coordinates can be written

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p}. \tag{4.6.3}$$

Differentiating the geostrophic balance (4.3.5) with respect to p, and applying (4.6.3), we find

$$p\frac{\partial u}{\partial p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p,$$
$$p\frac{\partial v}{\partial p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p, \tag{4.6.4}$$

or equivalently

$$\frac{\partial u}{\partial(\ln p)} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p,$$
$$\frac{\partial v}{\partial(\ln p)} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p. \tag{4.6.5}$$

These equations are often referred to as the *thermal wind* equations.

It is important to realize that it is only the *vertical shear* of the density current or the thermal wind that can be obtained from the geostrophically balanced equations in a fluid when the isobaric and isopycnal surfaces intersect. Accordingly, we can only determine uniquely velocity difference between two levels in the fluid. The velocity itself is not unique, because we can always add an arbitrary barotropic flow (4.5.12) to our solution.

We demonstrate this for an example that works equally well for the atmosphere as for the ocean. The horizontal geostrophic balance (4.3.2) and the vertical hydrostatic balance can be written in vector form as

$$0 = -f\vec{k}\times\vec{v} - \alpha\nabla \mathrm{p} - \nabla\Phi, \tag{4.6.6}$$

where $\alpha = 1/\rho$ and $\Phi = gz$ is the geopotential. We now integrate (4.6.6) along a closed path in the vertical plane; see Fig. 4.3.

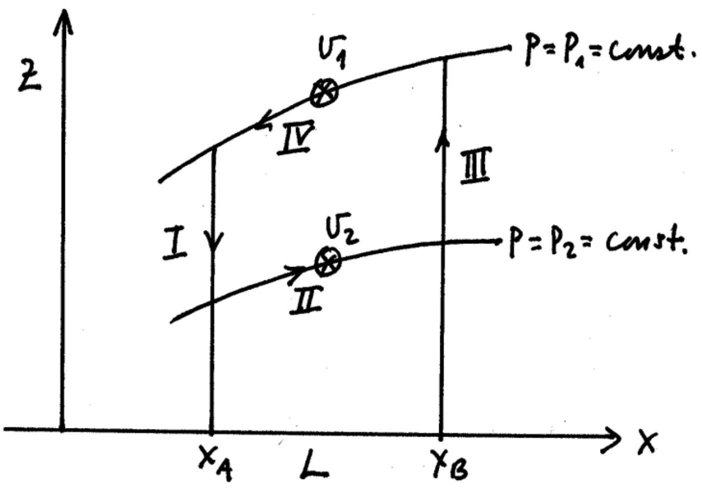

Fig. 4.3 Sketch of isobaric surfaces above horizontal positions A and B. Arrows indicate direction of integration along closed path.

The closed path here consists of the verticals *I* and *III* and the isobaric surfaces *II* and *IV* in Fig. 4.3. We integrate (4.6.6) along the closed path *I+II+III+IV.* Introducing a small vector displacement $d\vec{r} = dx\vec{i} + dz\vec{k}$ along the integration path, and utilizing that $\nabla p \cdot d\vec{r} = dp$ and $\nabla\Phi \cdot d\vec{r} = d\Phi$, we obtain

$$f\oint(\vec{k}\times\vec{v})\cdot d\vec{r} = -\oint \alpha dp - \oint d\Phi\,. \tag{4.6.7}$$

The last integral on the right-hand side is obviously zero, since Φ is single-valued. Furthermore, it is easy to show that

$$d\vec{r}\cdot(\vec{k}\times\vec{v}) = \vec{v}\cdot(d\vec{r}\times\vec{k})\,. \tag{4.6.8}$$

Hence, from (4.6.7)

$$f\oint \vec{v}\cdot(d\vec{r}\times\vec{k}) = -\oint \alpha dp. \tag{4.6.9}$$

Equivalently:

$$f\left[\int_I \vec{v}\cdot(d\vec{r}\times\vec{k}) + \int_{II} \vec{v}\cdot(d\vec{r}\times\vec{k}) + \int_{III} \vec{v}\cdot(d\vec{r}\times\vec{k}) + \int_{IV} \vec{v}\cdot(d\vec{r}\times\vec{k})\right]$$
$$= -\int_I \alpha dp - \int_{II} \alpha dp - \int_{III} \alpha dp - \int_{IV} \alpha dp\,. \tag{4.6.10}$$

Along *I* and *III* we have that $d\vec{r} \parallel \vec{k}$, while along *II* and *IV* we find $d\vec{r}\times\vec{k} = -dx\vec{j}$. Furthermore, along *II* and *IV* we have that *p* is constant, so here $dp = 0$. Equation (4.6.10) then reduces to

$$- f\left[\int_{II} v_2 dx + \int_{IV} v_1 dx\right] = -\int_{I} \alpha dp - \int_{III} \alpha dp\,. \qquad (4.6.11)$$

Introducing the x-coordinates x_A and x_B , we obtain

$$f\left[\int_{x_A}^{x_B} v_1 dx - \int_{x_A}^{x_B} v_2 dx\right] = \int_{p_1}^{p_2} \alpha_B dp - \int_{p_1}^{p_2} \alpha_A dp\,, \qquad (4.6.12)$$

where subscripts A and B refer to the horizontal positions. We can write

$$\int_{x_A}^{x_B} v_1 dx = V_1 L \quad , \quad \int_{x_A}^{x_B} v_2 dx = V_2 L\,, \qquad (4.6.13)$$

where V_1, V_2 are mean velocities in the y-direction along the considered isobaric surfaces.

In the atmosphere we can apply the equation of state, i.e. $\alpha = RT/p$, in the right-hand side of (4.6.12). Furthermore, it is usual to define $V \equiv V_1 - V_2$ as the *thermal wind*. We then obtain that

$$V = \frac{R}{f L}\left[\int_{p_1}^{p_2} T_B d\ln p - \int_{p_1}^{p_2} T_A d\ln p\right]. \qquad (4.6.14)$$

We note from (4.6.14) that the thermal wind, which is the difference in geostrophic velocity between two isobaric surfaces, is given by the horizontal difference of the integrated temperature between these two surfaces.

For oceanic applications it is usual to introduce the *specific volume anomaly* Δ defined by

$$\alpha = \alpha_{s,t,p} = \alpha_{35,0,p} + \Delta\,, \qquad (4.6.15)$$

where $\alpha_{35,0,p}$ is a reference value for seawater with a salinity of 35 psu and a temperature of $0^{\circ}\mathrm{C}$. Since $\alpha_{35,0,p}$ only varies with the pressure, the contributions from this term cancel in (4.6.12). Inserting from (4.6.13) and (4.6.15), we then finally obtain

$$V_1 - V_2 = \frac{1}{f L}\left[\int_{p_1}^{p_2} \Delta_B dp - \int_{p_1}^{p_2} \Delta_A dp\right]. \qquad (4.6.16)$$

This equation is often called *Helland-Hansen's* formula. It has been developed for calculating geostrophic currents from observational data. To use it, we need observed values of α at neighbouring ocean stations A and B as function of depth.

It is the same problem her as in the atmosphere; we only find uniquely the difference in velocity between two isobaric surfaces. If we know the geostrophic velocity somewhere, for example V_2 from direct velocity measurements, (4.6.16) yields the velocity everywhere in the water column. Often V_2 has been put equal to zero and the corresponding depth is referred to as the *level of zero motion*. This is a dubious approach, however, since it may very well happen that this level does not exist in the observational area.

The use of satellites to measure the sea surface slope appears to be a more reliable way of obtaining reference geostrophic velocities in the ocean. The forces (in the geostrophic approximation) on a fluid particle on an isobaric surface are depicted in Fig. 4.4.

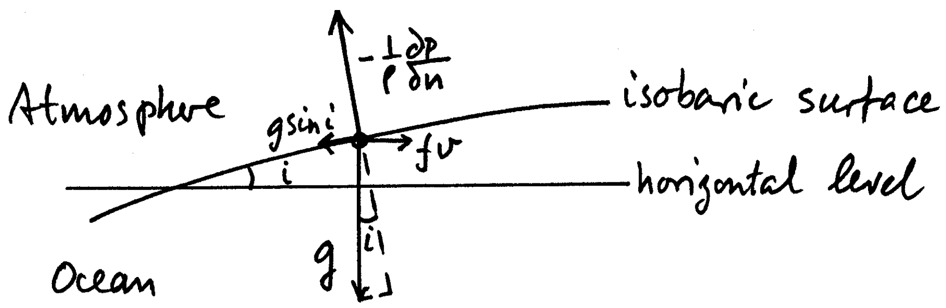

Fig. 4.4 Sketch of flow along an isobaric surface.

In the sketch above, the isobaric surface is sloping an angle i with respect to the horizontal. Balance of forces on a particle in the direction along the isobaric surface yields

$$fv\cos i = g\sin i\,. \tag{4.6.17}$$

Here v is the geostrophic velocity (into the paper). Hence

$$v = \frac{g\tan i}{f}\,. \tag{4.6.18}$$

This could have been obtained directly from (4.3.5) in isobaric coordinates, e.g.,

$$v = \frac{1}{f}\left(\frac{\partial\Phi}{\partial x}\right)_p = \frac{g}{f}\left(\frac{\partial z}{\partial x}\right)_p = \frac{g}{f}\tan i\,. \tag{4.6.19}$$

For example, for $f \sim 10^{-4}\ \text{s}^{-1}$, $g = 10\ \text{m s}^{-2}$ and $v \sim 1\ \text{m s}^{-1}$ (strong surface currents at mid-latitudes), we find

$$\tan i = \frac{1\,\text{m}}{100\,\text{km}}\,. \tag{4.6.20}$$

This kind of surface slopes (typical for the Gulf Stream) can be obtained from satellite altimeter data. Then, if $V_1 = g\tan i / f$ from (4.6.18) at the surface where the atmospheric pressure P_a is constant, Helland-Hansen's formula (4.6.16) can be written

$$V_g = \frac{g\tan i}{f} - \frac{1}{fL}\left[\int_{P_a}^{p}\Delta_B dp - \int_{P_a}^{p}\Delta_A dp\right]. \qquad (4.6.21)$$

Here V_g is the mean geostrophic velocity at the level where the ocean pressure is *p*.

4.7 Vertical velocity

For synoptic-scale motions in the atmosphere the vertical component of the velocity is of the order of a few centimetres per second. This is too small to be measured *in situ*. For large scale motion in the ocean the vertical velocity is $10^{-2} - 10^{-3}$ times the atmospheric values. Hence, the vertical velocity in the atmosphere and the ocean must be inferred from the fields that can be measured directly. One apparent disadvantage of using isobaric coordinates in the atmosphere is that $\varpi(p)$ is not directly equal to the vertical velocity $w(z)$. In Eulerian coordinates we can write

$$\varpi = \frac{Dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}. \qquad (4.7.1)$$

For synoptic-scale motion the horizontal velocity is basically geostrophic. We can therefore write

$$u = u_g + u_a, \quad v = v_g + v_a, \qquad (4.7.2)$$

where the *ageostropic* wind (u_a, v_a) is small. We use the hydrostatic approximation, and the fact that $\vec{v}_g \cdot \nabla p = 0$ from (4.3.2). By inserting into (4.7.2) into (4.7.1) we obtain

$$\varpi = \frac{\partial p}{\partial t} + u_a\frac{\partial p}{\partial x} + v_a\frac{\partial p}{\partial y} - g\rho w. \qquad (4.7.3)$$

For synoptic-scale motion we typically have

$$\Delta p \sim 10\ \text{hPa}, T \sim 1\,\text{d}, \text{L} \sim 1000\ \text{km},\ u_a \sim v_a \sim 1\,\text{ms}^{-1}, w \sim 10^{-2}\ \text{ms}^{-1}. \qquad (4.7.4)$$

An order of magnitude estimate then yields

$$\left|\frac{\partial p}{\partial t}\right| \sim \frac{\Delta p}{T} \sim 10\ \text{hPa d}^{-1},$$
$$\left|u_a\frac{\partial p}{\partial x} + v_a\frac{\partial p}{\partial y}\right| \sim (1\,\text{ms}^{-1}) \times (1\,\text{Pa km}^{-1}) \sim 1\text{hPa d}^{-1}, \qquad (4.7.5)$$
$$|g\rho w| \sim 100\ \text{hPa d}^{-1}.$$

From this assessment we note that is a good approximation to let

$$\varpi \approx -g\rho w\,. \tag{4.7.6}$$

5. BOUNDARY-LAYER FLOWS

5.1 The Boussinesq approximation

Winds in the atmosphere which is influenced by the ground, or currents in the ocean near the ocean surface or ocean bottom, are turbulent. This will be discussed in more detail in the next section. The influence of the boundaries is the reason why such flows are called boundary-layer flows. In the atmosphere the density varies across the lowest 1000 m by about 10 per cent, and the fluctuating component of the density deviates only a few per cent from the basic horizontal state, e.g. (1.7.1), (1.7.2). In the ocean the density variations in the upper layer, near the bottom and near the coast are even smaller, and so is also the deviation from the basic state. In these areas we may do some important simplifications in the dynamical equations. First, we take that the density in the Navier-Stokes equation is constant except in connection with the buoyancy term. This means that (1.6.5) is approximated by

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -f\vec{k}\times\vec{v} - \frac{1}{\rho_r}\nabla p - \frac{\rho}{\rho_r}g\vec{k} + \nu\nabla^2\vec{v} \tag{5.1.1}$$

where ρ_r is a constant reference value for the density. This approximation is called the *Boussinesq approximation.* In this approximation we can also assume that the velocity vector is solenoidal, i.e. we use the continuity equation in the form

$$\nabla\cdot\vec{v} = 0\,. \tag{5.1.2}$$

5.2 Turbulence and Reynolds averaging

At small fluid velocities, the motion of individual fluid particles is regular. The particles often move in parallel sheets. Such motion is called *laminar* motion.

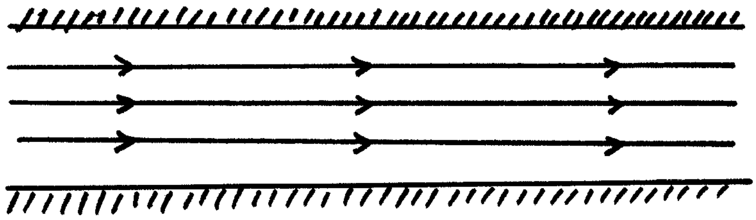

Fig. 5.1 Particle trajectories in laminar motion.

Let typical scales for velocity and length be U and d, respectively. We define a dimensionless parameter Re by

$$Re = \frac{Ud}{\nu}. \tag{5.2.1}$$

In 1880 Osborn Reynolds published his experimental results, showing that the particle motion becomes *irregular* when Re increases. For sufficiently large values of Re, the particle motion turns out to be *random* and *chaotic*. Such flows are called *turbulent* flows. The parameter Re has become known as the *Reynolds* number.

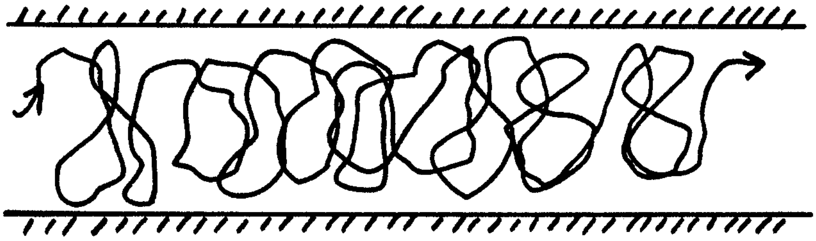

Fig. 5.2 Sketch of trajectory of an individual particle in turbulent flow.

Turbulence occurs for flow in pipes and channels when $Re \geq O(10^3)$, where O means the "order of".

Let us consider two extreme cases in geodynamics; the motion of an alpine glacier, and the atmospheric wind near the ground. In both cases we take the layer thickness d to be typically 100 m, but the viscosities and the velocities are quite different. For ice, which is very viscous, we take that $\nu_i \sim 10^{11}\ \mathrm{m^2 s^{-1}}$, and assume a typical velocity $U_i \sim 0.1\,\mathrm{m\,day^{-1}}$. For air near the ground typical values are $\nu_a \sim 10^{-5}\mathrm{m^2 s^{-1}}$, and $U_a \sim 10\ \mathrm{m\,s^{-1}}$. We then obtain from (5.2.1) for a glacier that $Re \sim 10^{-15}$, while for the wind $Re \sim 10^8$. Obviously, the streaming of the ice is laminar, and the air flow is turbulent.

The presence of turbulence is commonly seen in measurements of environmental flows. From time series (made at a fixed location) we often find *two* distinct periods; one short, t_1, and one much longer, t_2; see below for the velocity component u:

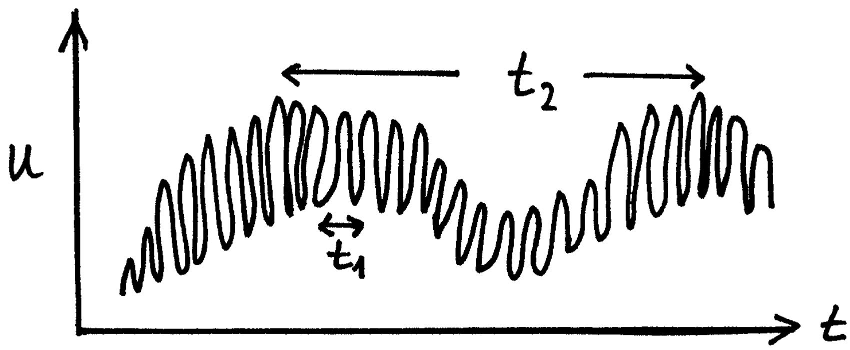

Fig. 5.3 Sketch of time series.

Here t_1 represents the period of the *turbulent* motion. We define a *mean* velocity component by

$$\overline{u} = \frac{1}{t_1}\int_{t-t_1/2}^{t+t_1/2} u(\xi)d\xi \ . \tag{5.2.2}$$

The mean quantity varies with time, but in a *slow* fashion. The typical time scale (period) is t_2, where $t_2 >> t_1$.

We write any of the dependent variables, Q, in the following way:

$$Q = \overline{Q} + Q'. \tag{5.2.3}$$

Here Q' represents the turbulent fluctuation, which typically varies over the period t_1. Hence

$$\overline{Q'} = 0. \tag{5.2.4}$$

Accordingly:

$$\overline{\overline{Q}} = \overline{Q}. \tag{5.2.5}$$

For the dependent variables in (5.1.1) we then may write

$$\left.\begin{aligned} \vec{v} &= \overline{\vec{v}} + \vec{v}', \\ p &= \overline{p} + p', \\ \rho &= \overline{\rho} + \rho'. \end{aligned}\right\} \tag{5.2.6}$$

Inserting (5.2.6) into (5.1.1), we find

$$\begin{aligned} \frac{\partial}{\partial t}(\overline{\vec{v}} + \vec{v}') + (\overline{\vec{v}} + \vec{v}')\cdot\nabla(\overline{\vec{v}} + \vec{v}') &= -f\vec{k}\times(\overline{\vec{v}} + \vec{v}') - \frac{1}{\rho_r}\nabla(\overline{p} + p') \\ &\quad - \left(\frac{\overline{\rho} + \rho'}{\rho_r}\right) g\vec{k} + \nu\nabla^2(\overline{\vec{v}} + \vec{v}'). \end{aligned} \tag{5.2.7}$$

By averaging each term in this equation, i.e. integrate it from $t - t_1/2$ to $t + t_1/2$, we obtain

$$\frac{\partial}{\partial t}\overline{\vec{v}} + \overline{\vec{v}}\cdot\nabla\overline{\vec{v}} + \overline{\vec{v}'\cdot\nabla\vec{v}'} = -f\vec{k}\times\overline{\vec{v}} - \frac{1}{\rho_r}\nabla\overline{p} - \frac{\overline{\rho}}{\rho_r}g\vec{k} + \nu\nabla^2\overline{\vec{v}}\ . \tag{5.2.8}$$

We have here utilized the fact that

$$\left.\begin{aligned} \overline{\overline{A}\,\overline{B}} &= \overline{A}\,\overline{B}, \\ \overline{\overline{A}B'} &= \overline{A}\,\overline{B'} = 0, \\ \overline{A'\overline{B}} &= \overline{A'}\,\overline{B} = 0. \end{aligned}\right\} \tag{5.2.9}$$

From the Boussinesq form (5.1.2) of the continuity equation, we find

$$\nabla \cdot (\overline{\vec{v}} + \vec{v}') = 0 . \tag{5.2.10}$$

By averaging we obtain

$$\left.\begin{aligned} \nabla \cdot \overline{\vec{v}} &= 0, \\ \nabla \cdot \vec{v}' &= 0. \end{aligned}\right\} \tag{5.2.11}$$

Then, for the advection term

$$\overline{\vec{v}' \cdot \nabla \vec{v}'} = \nabla \cdot (\overline{\vec{v}'\vec{v}'}) - \overline{\vec{v}' \nabla \cdot \vec{v}'} = \nabla \cdot (\overline{\vec{v}'\vec{v}'}) . \tag{5.2.12}$$

Accordingly, (5.2.8) can be written

$$\frac{\partial \overline{\vec{v}}}{\partial t} + \overline{\vec{v}} \cdot \nabla \overline{\vec{v}} = - f \vec{k} \times \overline{\vec{v}} - \frac{1}{\rho_r} \nabla \overline{p} - \frac{\overline{\rho}}{\rho_r} g \vec{k} + \nu \nabla^2 \overline{\vec{v}} - \nabla \cdot (\overline{\vec{v}'\vec{v}'}) . \tag{5.2.13}$$

We define the *turbulent* Reynolds stress tensor $\Re$ per unit mass:

$$\Re \equiv -\overline{\vec{v}'\vec{v}'} . \tag{5.2.14}$$

We note that the *divergence* of this tensor acts as a *force* per unit mass on the mean motion (it tends to accelerate a particle with mean velocity $\overline{\vec{v}}$). The problem now is how to express $\Re$ in terms of $\overline{\vec{v}}$.

We shall here use a very simple approach to this problem. Assuming that the functional dependence of the turbulent Reynolds stresses resemble those of the viscous stresses (1.5.8) for a Newtonian fluid, we take for the nine components of $\Re$:

$$\left.\begin{aligned} -\overline{u'u'} &= 2A^{(x)} \frac{\partial \overline{u}}{\partial x}, \\ -\overline{v'u'} &= A^{(y)} \frac{\partial \overline{u}}{\partial y} + A^{(x)} \frac{\partial \overline{v}}{\partial x} = -\overline{u'v'}, \\ -\overline{w'u'} &= A^{(z)} \frac{\partial \overline{u}}{\partial z} + A^{(x)} \frac{\partial \overline{w}}{\partial x} = -\overline{u'w'}, \\ -\overline{w'v'} &= A^{(z)} \frac{\partial \overline{v}}{\partial z} + A^{(y)} \frac{\partial \overline{w}}{\partial y} = -\overline{v'w'}, \\ -\overline{v'v'} &= 2A^{(y)} \frac{\partial \overline{v}}{\partial y}, \\ -\overline{w'w'} &= 2A^{(z)} \frac{\partial \overline{w}}{\partial z}. \end{aligned}\right\} \tag{5.2.15}$$

Here $A^{(x)}$, $A^{(y)}$, $A^{(z)}$ are called the *turbulent* eddy viscosity coefficients in the *x*-, *y*-, and *z*-directions respectively (or for short; *eddy viscosities*). The eddy viscosities $A^{(x)}$, $A^{(y)}$ and $A^{(z)}$ are generally different, but they are all much larger than the *molecular* viscosity ν. Usually we have

$$A^{(x)} \sim A^{(y)} > A^{(z)} >> \nu \quad . \tag{5.2.16}$$

5.3 Equations for the mean motion

a. Variable eddy viscosity and the logarithmic wind profile

We consider the steady mean wind above a horizontal surface situated at $z = 0$. In a thin, turbulent layer near the ground we may neglect the horizontal pressure gradient force, the Coriolis force, and the effect of molecular viscosity. The thickness of this surface layer depends on the vertical stability, but it is usually less than 10 per cent of the total planetary boundary-thickness. We take that the mean wind is parallel to the *x*-axis, and that the turbulent stresses only vary with the *z*-coordinate. Then the *x*-component of (5.2.13) in the surface layer reduces to

$$\frac{d}{dz}\left(-\overline{w'u'}\right) = 0 \, . \tag{5.3.1}$$

Hence, $-\overline{w'u'}$ is constant in this region. The corresponding constant turbulent shear stress (the turbulent Reynolds stress) is $\tau = -\rho_r \overline{w'u'}$. We can express the turbulent shear stress at the ground in terms of the *friction* velocity u_* as

$$\tau_0 = \left(-\rho_r \overline{w'u'}\right)_0 \equiv \rho_r u_*^2 \, . \tag{5.3.2}$$

Alternatively, for a wind in the *x*-direction, the stress at the ground can be written empirically as:

$$\tau_0 = \rho_r c_D \overline{u}_{10}^2 , \tag{5.3.3}$$

where $\overline{u}_{10}$ is the mean wind at 10 m height, and the drag coefficient c_D is of order 10^{-3}. For a characteristic wind velocity of $10\,\mathrm{m\,s^{-1}}$, we find from (5.3.2) and (5.3.3) that typically $u_* \sim 0.3\,\mathrm{m\,s^{-1}}$ in the atmosphere.

From the definitions (5.2.15), and the fact that the turbulent shear stress is constant, we then obtain

$$A^{(z)} \frac{d\overline{u}}{dz} = -\overline{w'u'} = u_*^2 \, . \tag{5.3.4}$$

In this expression it is not reasonable to assume that the eddy viscosity is constant. Very close to the ground the turbulent eddies are very small, and then increase in size as we move upward. The eddy viscosity will also increase with increasing friction velocity. We can then assume that in this region

$$A^{(z)} = \kappa u_* z\,, \tag{5.3.5}$$

where $\kappa \approx 0.4$ is von Kármán’s constant. A more solid basis for this assumption follows from Prandtl’s *mixing length* theory, which we leave for more advanced courses. From (5.3.4) and (5.3.5) we obtain

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\kappa z}\,. \tag{5.3.6}$$

We realize that the turbulent region cannot extend right to the ground, so at some very small height z_0 the mean wind vanishes. Integrating (5.3.6) we then obtain

$$\bar{u} = \frac{u_*}{\kappa}\ln\left(\frac{z}{z_0}\right). \tag{5.3.7}$$

We thus see that in the region of constant turbulent shear stress the wind profile is logarithmic. The length scale z_0 is often called the *roughness length*. It is related to the friction velocity by

$$z_0 = b u_*^2 / g\,, \tag{5.3.8}$$

where b is a dimensionless constant (Charnock’s formula). The value of z_0 depends condition at the ground. For grassy fields, typical values are in the range 1 – 4 cm.

b. Constant eddy viscosity

In many cases we can assume that the eddy viscosities are constant, especially when we are away from rigid surfaces. Then it is easy to compute the turbulent Reynolds stress terms in (5.2.13). For the *x*-component we find by applying (5.2.15):

$$\begin{aligned} -\nabla\cdot(\overline{\vec{v}'u'}) &= -\frac{\partial}{\partial x}(\overline{u'u'}) - \frac{\partial}{\partial y}(\overline{v'u'}) - \frac{\partial}{\partial z}(\overline{w'u'}) \\ &= A^{(x)}\frac{\partial^2\bar{u}}{\partial x^2} + A^{(y)}\frac{\partial^2\bar{u}}{\partial y^2} + A^{(z)}\frac{\partial^2\bar{u}}{\partial z^2} + A^{(x)}\frac{\partial}{\partial x}\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right) \end{aligned} \tag{5.3.9}$$

Utilizing (5.2.11), we finally obtain

$$-\nabla\cdot(\overline{\vec{v}'u'}) = A^{(x)}\frac{\partial^2\bar{u}}{\partial x^2} + A^{(y)}\frac{\partial^2\bar{u}}{\partial y^2} + A^{(z)}\frac{\partial^2\bar{u}}{\partial z^2}\,. \tag{5.3.10}$$

In the same way we obtain for the *y*- and *z*-components:

$$-\nabla\cdot(\overline{\vec{v}'v'}) = A^{(x)}\frac{\partial^2\overline{v}}{\partial x^2} + A^{(y)}\frac{\partial^2\overline{v}}{\partial y^2} + A^{(z)}\frac{\partial^2\overline{v}}{\partial z^2}\,, \tag{5.3.11}$$

$$-\nabla\cdot(\overline{\vec{v}'w'}) = A^{(x)}\frac{\partial^2\overline{w}}{\partial x^2} + A^{(y)}\frac{\partial^2\overline{w}}{\partial y^2} + A^{(z)}\frac{\partial^2\overline{w}}{\partial z^2}\,. \tag{5.3.12}$$

We define an operator ∇_A^2 by

$$\nabla_A^2 \equiv A^{(x)}\frac{\partial^2}{\partial x^2} + A^{(y)}\frac{\partial^2}{\partial y^2} + A^{(z)}\frac{\partial^2}{\partial z^2}\,. \tag{5.3.13}$$

Then we can write

$$-\nabla\cdot(\overline{\vec{v}'\vec{v}'}) = \nabla_A^2\overline{\vec{v}}\,. \tag{5.3.14}$$

The equation (5.2.13) for the mean motion now becomes:

$$\frac{\partial\overline{\vec{v}}}{\partial t} + \overline{\vec{v}}\cdot\nabla\overline{\vec{v}} = -f\vec{k}\times\overline{\vec{v}} - \frac{1}{\rho_r}\nabla\overline{p} - \frac{\overline{\rho}}{\rho_r}g\vec{k} + \nabla_A^2\overline{\vec{v}} + \nu\nabla^2\overline{\vec{v}}\,. \tag{5.3.15}$$

Since the molecular viscosity ν is much smaller than $A^{(x)}$, $A^{(y)}$, $A^{(z)}$, we can usually neglect the molecular viscosity term in (5.3.15). For the rest of the analysis we shall only consider the *mean* quantities. With that in mind, we can for simplicity delete the over-bars. Our governing momentum equation for the Reynolds averaged motion with the Boussinesq approximation then becomes

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -f\vec{k}\times\vec{v} - \frac{1}{\rho_r}\nabla p - \frac{\rho}{\rho_r}g\vec{k} + \nabla_A^2\vec{v}\,. \tag{5.3.16}$$

5.4 The oceanic Ekman current driven by wind stress

When we have a wind blowing along the sea surface, it will induce motion in the ocean with a vertical length scale that usually is much smaller than the ocean depth. Since the wind-stress can be transferred to the water only through the effect of viscosity, the turbulent friction terms in (5.2.13) now become important. (The additional effect of wind-generated surface waves complicates the problem and is left out here). If there are no horizontal pressure gradients, the friction force can only be balanced by the Coriolis force for non-accelerating flows. In this case we simplify, and assume that the eddy viscosities are constant. Hence, from (5.3.16), balance of forces requires

$$\left|\nabla_A^2\vec{v}_H\right| \sim \left|f\vec{k}\times\vec{v}_H\right|. \tag{5.4.1}$$

Obviously we here must be away from the equator, where $f = 0$. Let typically $|u| \sim |v|$ and take the current only to vary with the *z*-coordinate. In this direction the typical length scale is D_E. From the definition (5.3.5), we then obtain from (5.4.1) that

$$D_E \sim \left(\frac{A^{(z)}}{|f|} \right)^{1/2} . \tag{5.4.2}$$

This length scale, characterizing the depth limit of the direct wind influence in the ocean, is called the *Ekman* depth. A more precise definition is obtained from the subsequent mathematical treatment.

In this problem we assume that the horizontal axes are situated at the ocean surface, so here $-\infty < z \leq 0$. For steady horizontal motion with velocity components that only vary with *z*, and with no horizontal pressure gradients, (5.3.16) reduces to

$$\frac{d^2u}{dz^2} + \frac{fv}{A^{(z)}} = 0 , \tag{5.4.3}$$

$$\frac{d^2v}{dz^2} - \frac{fu}{A^{(z)}} = 0 . \tag{5.4.4}$$

Since f and $A^{(z)}$ here are taken to be constants, this system of coupled, second-order, ordinary differential equations can be solved by inserting for v, say, in (5.4.3), thus yielding one fourth-order equation for u. However, to simplify the algebra, we introduce the complex velocity $W = u + iv$, as we did for the inertial oscillations in section 3.1. Then, by multiplying (5.4.4) by the imaginary unit *i*, and adding the two equations, we obtain

$$\frac{d^2W}{dz^2} - \frac{ifW}{A^{(z)}} = 0 . \tag{5.4.5}$$

Hence, we need only solve one second-order differential equation for the complex velocity, instead of the more labour taking fourth-order equations for the velocity components. To solve (5.4.5) we need <u>two</u> boundary conditions. At the surface we let for simplicity the wind blow along the *y*-axis, exerting a constant wind-stress τ_0 on the water in this direction. The wind-stress in the *x*-direction is zero. The dynamic boundary conditions require that these stresses must equal the shear stresses in the water at the surface, i.e.

$$\rho_r A^{(z)} \frac{du}{dz} = 0 \quad , \quad z = 0 , \tag{5.4.6}$$

$$\rho_r A^{(z)} \frac{dv}{dz} = \tau_0 \quad , \quad z = 0 . \tag{5.4.7}$$

Here the turbulent shear stresses in the water on the left-hand side have been obtained from (5.2.15), assuming no *x*- or *y*-dependence of the variables. By multiplying (5.3.7) by *i* and adding the two expressions, the dynamic boundary condition at the surface reduces to

$$\frac{dW}{dz} = \frac{i\,\tau_0}{\rho_r A^{(z)}} \quad , \quad z = 0. \tag{5.4.8}$$

To obtain a second condition, we assume that the Ekman depth is much smaller than the ocean depth. Hence, the wind-induced current will be vanishingly small at large ocean depths. Mathematically, this means that $u \to 0$ and $v \to 0$ when $z \to -\infty$, or

$$W \to 0\,, \quad z \to -\infty\,. \tag{5.4.9}$$

The solution of (5.4.5), subject to the boundary conditions (5.4.8) and (5.4.9) is readily obtained. Define

$$a^2 = \frac{if}{A^{(z)}}\,, \tag{5.4.10}$$

where we assume that $f > 0$. Then, from (5.4.5)

$$W = C_1 \exp(az) + C_2 \exp(-az)\ . \tag{5.4.11}$$

Here C_1 and C_2 are integration constants. Since we can write $i = \exp(i\pi/2)$, we obtain from (5.4.10) that

$$a = \left(\frac{f}{A^{(z)}}\right)^{1/2} e^{i\pi/4} = \left(\frac{f}{2A^{(z)}}\right)^{1/2} (1+i)\,. \tag{5.4.12}$$

We note that the real part of a is positive. Hence, W in (5.4.11) will grow beyond limits when $z \to -\infty$, unless $C_2 = 0$. So, to have a finite solution everywhere, we must require that $C_2 = 0$. Furthermore, by differentiating (5.4.11), we find

$$\frac{dW}{dz} = a\,C_1, \quad z = 0. \tag{5.4.13}$$

By substitution from (5.4.8), we obtain

$$C_1 = \frac{i\,\tau_0}{\rho_r\ A^{(z)} a}\,. \tag{5.4.14}$$

To make the results easy to discuss, we define the Ekman depth (5.4.2) more precisely as

$$D_E = \pi \left(\frac{2A^{(z)}}{f}\right)^{1/2}\,. \tag{5.4.15}$$

Then the complex velocity (5.4.11) can be written

$$W = V_0 \exp[\pi z / D_E + i(\pi z / D_E + \pi / 4)] , \qquad (5.4.16)$$

where

$$V_0 = \frac{\sqrt{2}\, \pi\, \tau_0}{\rho_r\, f\, D_E} . \qquad (5.4.17)$$

Since by definition $W = u + iv$, we find from (5.4.16) by equating real and imaginary parts that

$$\left. \begin{aligned} u &= V_0 \exp(\pi z / D_E) \cos(\pi z / D_E + \pi / 4) , \\ v &= V_0 \exp(\pi z / D_E) \sin(\pi z / D_E + \pi / 4) \ . \end{aligned} \right\} \qquad (5.4.18)$$

The current given by (5.4.18) is called the *Ekman* current, after the Swedish oceanographer Valfried W. Ekman who first published this result in 1905.

The solution (5.4.18) has some interesting properties. We note by squaring and adding the two components that

$$(u^2 + v^2)^{1/2} = V_0 \exp(\pi z / D_E) , \qquad (5.4.19)$$

i.e. the magnitude of the current vector decreases exponentially with depth. At the surface $(z = 0)$, we have $u = v = V_0 / 2^{1/2}$. Since the wind here is blowing along the *y*-axis, this means that the Ekman current at the surface is deflected 45° to the right of the wind direction (in the northern hemisphere). Furthermore, the two velocity components behave differently when we move downward from the surface. Since the cosine-term increases with increasing depth $(z < 0)$ while the sine-term decreases, the velocity vector spirals to the right as we move down into the ocean (with exponentially decreasing magnitude). This behaviour is depicted in Fig. 5.4.

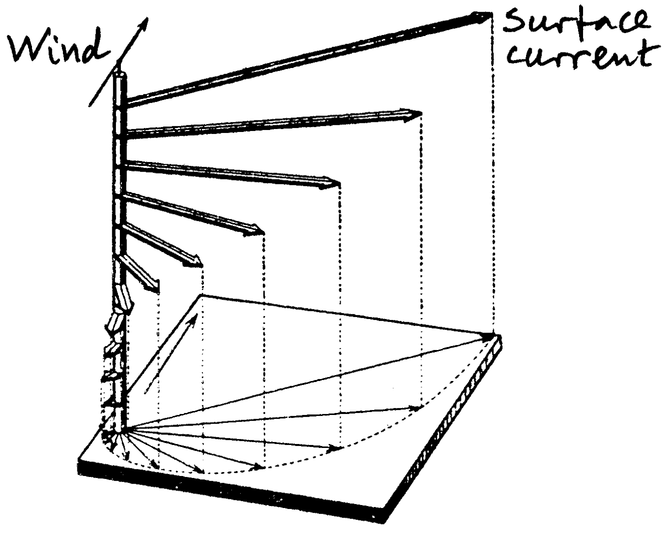

Fig. 5.4 Sketch of the Ekman current vector at various depths.

By projecting the current vectors on a horizontal plane, the arrows define a curve called a *hodograph* (the dotted curve in Fig. 5.4). This curve is referred to as the *Ekman spiral*.

At the Ekman depth $z = -D_E$, we find that $u = v = -(V_0 / 2^{1/2})\exp(-\pi)$. This means that the current here is oppositely directed to surface current, while its magnitude is reduced by a factor $\exp(-\pi) \approx 1/23$. Accordingly, for all practical applications, the Ekman current is confined to a layer between the surface and $z = -D_E$. This layer is known as the *Ekman* layer.

It was actually Fridtjof Nansen who suggested the problem discussed here to Ekman. From his observations with the *Fram* in the Polar Sea 1893-1896, Nansen found that the ice drifted somewhat to the right of the wind direction. This Nansen attributed to the effect of the Coriolis force. Physically, he argued that a balance of forces on an ice floe between the wind stress, the bottom drag and the Coriolis force must lead to a drift direction that was to the right of the wind-stress direction. Similarly, he argued that a balance of forces on any thin fluid layer below the surface with a certain drift direction would induce a drift velocity of the layer beneath that was further deflected to the right, which in essence explains the Ekman spiral.

Since the Ekman current effectively vanishes below the Ekman layer, the ocean need not be "infinitely" deep for the solution (5.4.18) to be valid. It suffices that $D_E << H$, where H is the ocean depth. With a maximum estimate of $A^{(z)} \sim 10^{-1}\,\mathrm{m^2\,s^{-1}}$ and $f \sim 10^{-4}\,\mathrm{s^{-1}}$, we find that $D_E \sim 140\,\mathrm{m}$, which is much less than the average ocean depth of about 4000 m.

In shallow seas, where $H < D_E$, the direct wind influence will be felt through the entire water column. An Ekman spiral will also develop in this case, but it will be modified by the presence of the bottom. In particular, the surface current will be deflected less than 45° to the right of the wind direction.

The presence of the sea bottom will also influence the currents in the deep ocean. This is entirely analogous to how the ground modifies the wind in the atmosphere. We leave this problem to section 5.8, where we discuss the planetary boundary layer.

5.5 The Ekman transport in the surface layer

Sometimes oceanographers are more interested in the total transport of seawater through a certain vertical section in the ocean than in the detailed current variation. For this purpose we define horizontal mass transport (or mass flux) components $(q^{(x)}, q^{(y)})$, by

$$q^{(x)} = \int_{-H}^{0} \rho u dz \quad , \quad q^{(y)} = \int_{-H}^{0} \rho v dz . \tag{5.5.1}$$

Here $q^{(x)}$ is the mass transport along the *x*-axis per unit length in the *y*-direction, while $q^{(y)}$ is the mass transport along the *y*-axis per unit length in the *x*-direction, and *H* is the ocean depth. Obviously, the total mass flux $Q^{(x)}$ through a section parallel to the *y*-axis of width *L* is given by

$$Q^{(x)} = \int_0^L q^{(x)} dy = \int_0^L \int_{-H(y)}^{0} \rho u \, dz \, dy, \tag{5.5.2}$$

see the sketch in Fig. 5.5.

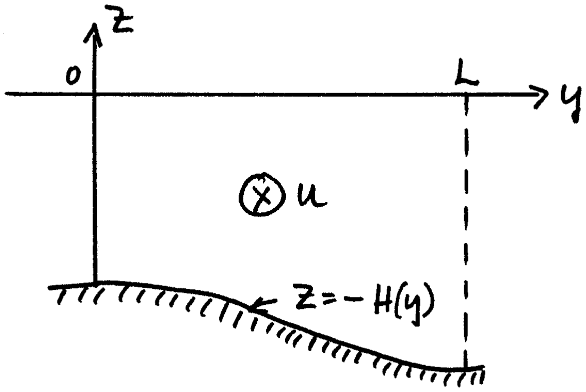

Fig. 5.5 Flow through an oceanographic section.

Since the density in the ocean actually varies very little, it can, as far as the mass transport is concerned, be set equal to a constant reference value ρ_r (= 1000 kg m^{-3}). We can then relate the mass and volume fluxes U, V per unit length by

$$q^{(x)} = \int_{-H}^{0} \rho u dz \approx \rho_r \int_{-H}^{0} u dz = \rho_r U,$$
$$q^{(y)} = \int_{-H}^{0} \rho v dz \approx \rho_r \int_{-H}^{0} v dz = \rho_r V. \quad (5.5.3)$$

Again, the total volume flux U_{tot} through the section in Fig. 5.5 is given by

$$U_{tot} = \int_0^L U dy = \int_0^L \int_{-H(y)}^{0} u dz\, dy. \quad (5.5.4)$$

We note that the total volume flux has dimension (length)^3 (time)$^{-1}$. Since the volume fluxes in the ocean usually are very large, we introduce the unit Sverdrup (Sv) by

$$1\,\text{Sv} = 10^6\ \text{m}^3\text{s}^{-1}. \quad (5.5.5)$$

Typically, for the Gulf Stream outside Florida, we have $U_{tot} \sim 30\,\text{Sv}$, while the Vest-Spitzbergen Current and the East-Greenland Current both have estimated transports $U_{tot} \sim 7\,\text{Sv}$.

The transport associated with the Ekman current in deep water $(D_E << H)$ is particularly interesting. This transport can be obtained directly from the governing equations (5.4.3) and (5.4.4). By integrating in the vertical, we find

$$f \int_{-D_E}^{0} u dz = A^{(z)} \left(\frac{dv}{dz}\right)_{z=0} - A^{(z)} \left(\frac{dv}{dz}\right)_{z=-D_E}, \quad (5.5.6)$$

$$f\int_{-D_E}^{0} v\,dz = -A^{(z)}\left(\frac{du}{dz}\right)_{z=0} + A^{(z)}\left(\frac{du}{dz}\right)_{z=-D_E} . \tag{5.5.7}$$

Since the current and the viscous stress practically vanish below the Ekman layer, the last terms on the right-hand side are zero. Furthermore, if the wind-stress components are $\tau_0^{(x)}$ and $\tau_0^{(y)}$ respectively, the dynamic boundary conditions require that

$$\left.\begin{aligned} \tau_0^{(x)} &= \rho_r A^{(z)}\frac{du}{dz} \quad , \quad z=0 \\ \tau_0^{(y)} &= \rho_r A^{(z)}\frac{dv}{dz} \quad , \quad z=0 \end{aligned}\right\} \tag{5.5.8}$$

i.e. the shear stresses on both sides of the (infinitely thin) material surface are equal. Here the turbulent shear stresses in the water (the right-hand side) have been obtained from (5.2.15). From (5.5.6) and (5.5.7) the Ekman transport components $(q_E^{(x)}, q_E^{(y)})$ then become

$$\begin{aligned} q_E^{(x)} &= \rho_r \int_{-D_E}^{0} u\,dz = \frac{\tau_0^{(y)}}{f}, \\ q_E^{(y)} &= \rho_r \int_{-D_E}^{0} v\,dz = -\frac{\tau_0^{(x)}}{f} , \end{aligned} \tag{5.5.9}$$

where we have utilized (5.5.8). We note that the Ekman transport is directed 90° to the right of the wind-stress vector when $f > 0$. This is easily seen from (5.5.9) using the vector notation $\vec{q}_E = q_E^{(x)}\vec{i} + q_E^{(y)}\vec{j}$ and $\vec{\tau}_0 = \tau_0^{(x)}\vec{i} + \tau_0^{(y)}\vec{j}$. Then (5.5.9) can be written

$$\vec{q}_E = -\frac{1}{f}\vec{k}\times\vec{\tau}_0 , \tag{5.5.10}$$

which proves the point. In Fig. 5.6 we have depicted this situation.

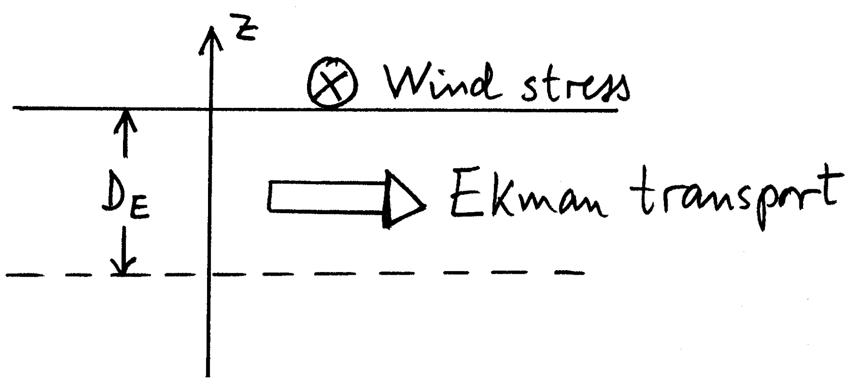

Fig. 5.6 Ekman transport in the northern hemisphere. The wind is directed into the paper.

In fact, the expression for the Ekman transport (5.5.10) can be derived without introducing the concept of eddy viscosity. If we go back to our Reynolds averaged equations for the turbulent mean motion (5.2.13), the balance between the Coriolis force and the turbulent Reynolds stresses becomes

$$\begin{aligned} -f\rho_r v &= -\frac{\partial(\rho_r \overline{w'u'})}{\partial z}, \\ f\rho_r u &= -\frac{\partial(\rho_r \overline{w'v'})}{\partial z}, \end{aligned} \qquad (5.5.11)$$

where (u, v) are turbulent mean velocities. We integrate (5.5.11) in the vertical across the Ekman layer, and apply the boundary conditions (5.5.8) in Reynolds stress form:

$$\left.\begin{aligned} \tau_0^{(x)} &= -\rho_r \overline{w'u'} \quad , \quad z = 0 \\ \tau_0^{(y)} &= -\rho_r \overline{w'v'} \quad , \quad z = 0 \end{aligned}\right\}, \qquad (5.5.12)$$

in addition to the no-stress bottom condition:

$$\overline{w'u'} = \overline{w'v'} = 0, \quad z = -D_E \,. \qquad (5.5.13)$$

It now follows readily from (5.5.11)-(5.5.13) that $f\,\vec{q}_E = -\vec{k} \times \vec{\tau}_0$. This makes the transport expression (5.5.10) more general than the Ekman current system (5.4.18), which was derived using a constant eddy viscosity $A^{(z)}$.

5.6 Storm surge at a straight coast

In discussing the Ekman flow, we have up to now taken the ocean to be of unlimited extent in the lateral directions. Therefore, near the coasts, the Ekman current system derived in section 5.4 is not valid. However, far away from the shore-line our solution (5.5.10) for the Ekman transport can be applied. By combining this with the fact that the on- or offshore volume transport must vanish at the coast, it is easy to realize physically what must happen in this case. When a constant wind blows along the coast, with the shore-line to the right, our computed Ekman transport (5.5.10), valid far away from the coast, must be directed onshore. In this analysis we assume that the ocean depth H is comparable to the Ekman depth (5.4.15). Neglecting any return flow, the total onshore transport will gradually approach zero as we get closer to the coast, resulting in a change of the surface elevation. This change in surface elevation due to wind is called a *storm surge*.

We consider a simplified situation where a constant wind is blowing along a straight coast in an ocean of constant depth; see Fig. 5.7.

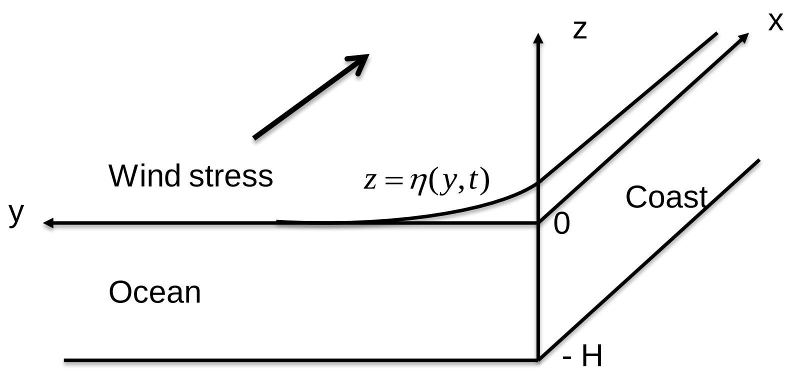

Fig. 5.7 The storm surge at the coast for the northern hemisphere.

We take that the density is constant in this problem. Then the continuity equation reduces to (1.7.7). By integrating this equation in the vertical from the horizontal bottom to the surface $z = \eta$, we obtain

$$\int_{-H}^{\eta} \frac{\partial u}{\partial x} dz + \int_{-H}^{\eta} \frac{\partial v}{\partial y} dz + w(z = \eta) - w(z = -H) = 0 \, . \tag{5.6.1}$$

Obviously, the last term in (5.6.1) is zero since the bottom is horizontal and at rest. By applying the general rule for differentiating an integral where the upper limit is not a constant, (5.6.1) can be written

$$\frac{\partial}{\partial x} \int_{-H}^{\eta} u dz + \frac{\partial}{\partial y} \int_{-H}^{\eta} v dz - u(\eta) \frac{\partial \eta}{\partial x} - v(\eta) \frac{\partial \eta}{\partial y} + w(z = \eta) = 0 \, . \tag{5.6.2}$$

From the boundary conditions (3.3.4), we have that

$$\frac{\partial \eta}{\partial t} = -u(\eta) \frac{\partial \eta}{\partial x} - v(\eta) \frac{\partial \eta}{\partial y} + w(z = \eta) \, . \tag{5.6.3}$$

Applying (5.6.3), and defining the volume fluxes

$$U = \int_{-H}^{\eta} u dz, \quad V = \int_{-H}^{\eta} v dz \, , \tag{5.6.4}$$

we can write (5.6.2) as

$$\frac{\partial \eta}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} \, . \tag{5.6.5}$$

We assume that the horizontal scale of this problem is so much larger than the vertical scale that we can apply the hydrostatic approximation (4.1.9). Taking that the air pressure is constant and equal to p_0 along the sea surface $z = \eta$, we have

$$p = -\rho g(z-\eta) + p_0 . \tag{5.6.6}$$

By applying (5.6.6), and assuming that the velocity is so small that we can neglect the nonlinear acceleration terms, the horizontal components of (5.3.16) reduce to

$$\begin{aligned} \frac{\partial u}{\partial t} &= fv - g\frac{\partial \eta}{\partial x} + A^{(z)}\frac{\partial^2 u}{\partial z^2}, \\ \frac{\partial v}{\partial t} &= -fu - g\frac{\partial \eta}{\partial y} + A^{(z)}\frac{\partial^2 v}{\partial z^2}. \end{aligned} \tag{5.6.7}$$

Here we have assumed that the variation of the frictional stresses is much larger in the vertical direction than in the horizontal direction. As depicted in Fig. 5.7, we assume that a constant wind stress $\tau_0^{(x)} = \tau_0$ is acting in the *x*-direction (along the coast). The boundary conditions at the surface then become:

$$A^{(z)}\frac{\partial u}{\partial z} = \frac{\tau_0}{\rho}, \quad A^{(z)}\frac{\partial v}{\partial z} = 0, \quad z = \eta . \tag{5.6.8}$$

At the bottom we take that the frictional stresses are zero, i.e.

$$A^{(z)}\frac{\partial u}{\partial z} = 0, \quad A^{(z)}\frac{\partial v}{\partial z} = 0, \quad z = -H . \tag{5.6.9}$$

As indicated in Fig. 5.7, we assume that the variables do not depend on the *x*-coordinate, i.e. $\partial / \partial x = 0$. By integrating (5.6.7) from the bottom to the surface, utilizing (5.6.8), (5.6.9), and assuming that $H >> \eta$, we finally obtain

$$\begin{aligned} \frac{\partial U}{\partial t} &= fV + \frac{\tau_0}{\rho}, \\ \frac{\partial V}{\partial t} &= -fU - gH\frac{\partial \eta}{\partial y}. \end{aligned} \tag{5.6.10}$$

We have here used that $\int_{-H}^{\eta}(\partial u/\partial t)dz \approx \partial U/\partial t$ and $\int_{-H}^{\eta}(\partial v/\partial t)dz \approx \partial V/\partial t$ for linearized motion. The integrated continuity equation now reduces to

$$\frac{\partial \eta}{\partial t} = -\frac{\partial V}{\partial y} . \tag{5.6.11}$$

The set of equations (5.6.10) and (5.6.11) has a solution where the flux *V* in the cross-shore direction is independent of time (while *U* and η are not). By taking $\partial V/\partial t = 0$, we can eliminate *U* and η to obtain

$$\frac{\partial^2 V}{\partial y^2} - \frac{f^2}{gH}V = \frac{f\tau_0}{\rho g H} . \tag{5.6.12}$$

We have a vertical coast at $y = 0$. Here $V = 0$. Furthermore, far out in the open ocean (mathematically, when $y \to \infty$) we must require that V is finite. Applying these conditions, the solution of (5.6.12) becomes

$$V = -\frac{\tau_0}{\rho f}\left(1 - e^{-y/a_0}\right), \qquad (5.6.13)$$

where

$$a_0 = (gH)^{1/2} / f\,. \qquad (5.6.14)$$

We note that the volume flux in the cross-shore direction changes over a typical length scale a_0, given by (5.6.14). This scale is referred to as the barotropic *Rossby radius* of deformation. Far away from the coast (when $y \to \infty$), we have that $V = -\tau_0 /(\rho f)$, which is just the open ocean Ekman transport perpendicular to the wind stress. This is so because we have assumed no return flow, and neglected the frictional stress at the ocean bottom.

Assuming that $\eta = 0$ when $t = 0$, we readily find from (5.6.11) and (5.6.13):

$$\eta(y,t) = \frac{\tau_0 t}{\rho (gH)^{1/2}} e^{-y/a_0}\,. \qquad (5.6.15)$$

In this problem we have geostrophic balance in the direction normal to the coast. From (5.6.10), with $\partial V / \partial t = 0$, we find

$$U = -\frac{gH}{f}\frac{\partial \eta}{\partial y} = \frac{\tau_0 t}{\rho} e^{-y/a_0}\,. \qquad (5.6.16)$$

We thus get an alongshore velocity $u = U / H$ in the form of a jet which is limited laterally by the Rossby radius.

We note that the surface elevation (the surge) and the alongshore velocity increase linearly in time. Obviously, this cannot represent the solution for very long times. But for the relatively short duration of a passing storm, (5.6.15) is shown to yield realistic values for the surge at the shore (Gill, 1982, p. 397). To obtain a solution for constant wind that is valid when $t \to \infty$, we must include bottom friction in the problem. This will be discussed in the next section.

5.7 Downwelling/upwelling

In the previous section we assumed that the ocean depth was comparable to the thickness of the surface Ekman layer D_E, and derived a transient solution for the sea level rise. For a deep ocean, where $H >> D_E$, we can imagine a steady solution of this problem ($\partial / \partial t = 0$ for all variables). Then, from (5.6.11), $\partial V / \partial y = 0$. Since $V(y = 0) = 0$, V must be zero everywhere. But we do have a non-zero Ekman transport towards the shore in the surface layer. Hence, there must be a compensating and equally large transport in the interior ocean in the offshore direction. The momentum balance in the y-direction must now be between the Coriolis force caused by the alongshore jet, the pressure-gradient force from the sloping surface, and the bottom stress. Obviously, there must be a downward vertical motion near the coast to maintain this circulation. We shall not go into a detailed discussion of this problem, but refer to Fig. 5.8 which qualitatively depicts the situation. The process, in which sinking water near the coast feeds the offshore interior transport, is called *downwelling*.

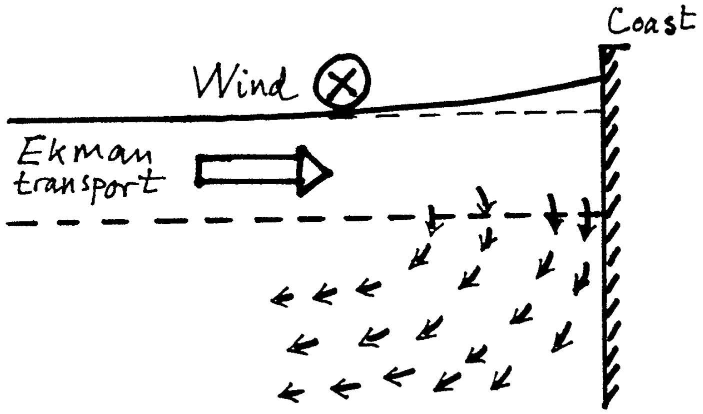

Fig. 5.8 Downwelling at a coast. The wind direction is into the paper. (The along-shore jet is not depicted).

In downwelling areas water which is saturated with oxygen and carbon dioxide is transported from the surface to deeper layers where it spreads horizontally. This process is very important for marine life in the sub-surface zone as well as for a possible storage of climate gases that dissolve in sea water at the ocean surface.

When a constant wind blows with the shore-line to the left, the Ekman transport (5.5.10) must be off-shore. The balance of forces is as before, but now the flow in the interior must be directed towards the coast. Near the coast, the in-flowing deeper water must rise to maintain the circulation. This process is called *upwelling*; see Fig. 5.9.

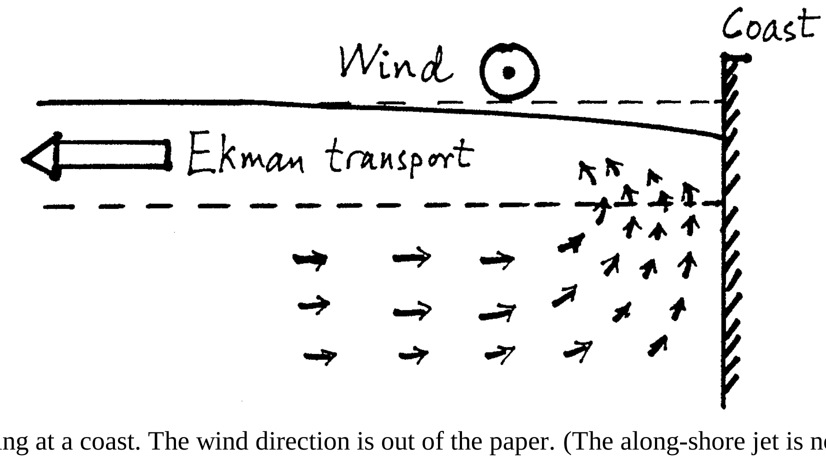

Fig. 5.9 Upwelling at a coast. The wind direction is out of the paper. (The along-shore jet is not depicted).

The upwelling process brings colder, nutrient-rich water to the euphotic zone, which is essential for the production of phytoplankton. This means that upwelling areas are rich in biologic activity. Some of the world's largest catches of fish are made here, e.g. off the coast of Peru and Chile.

In our discussion up to now, we have assumed that the density is constant. But in the real ocean we have less dense water above denser water. The density stratification determines the width of the upwelling zone, which typically is of the order 100 km, and the upwelling speed may be 5 to 10 m/day. Also the depth, from which the upwelled water actually comes, depends very much on the stratification of the water column below the well-mixed Ekman layer. In many cases it may not be more than a few hundred meters.

If the wind along the coast stops blowing, the Ekman transport, and hence the upwelling/downwelling process, will cease. However, the sloping surface near the coast will not return to its initial horizontal position. It may stay inclined, and the balance between the associated pressure-gradient force and the Coriolis force will yield geostrophic currents along the coast, as discussed in sections 4.3, and 5.6. Of course, in the real ocean the effect of friction will reduce the velocity of this coastal current over time, and the surface slope will gradually collapse. Therefore, in reality, we need re-occurring similar wind events or statistically permanent wind fields to sustain such coastal currents. This is for example thought to be the case in the Antarctic. Here prevailing easterly winds due to the permanent high pressure field over the cold continent supports an ocean current close to the Antarctic coast. Since this current is related to on-shore Ekman transport (remember, $f < 0$ here), the current direction is counter-clockwise, i.e. it flows in the wind direction. It is therefore called the *East wind* drift.

5.8 The planetary Ekman layer

As explained in Section 5.3, we actually have a thin layer near the ground where the mean turbulent shear stress is constant and the mean wind profile is logarithmic. Above this shear-layer, we find the atmospheric Ekman layer. In this section we simplify, and let the Ekman layer extend down to the ground. We place the horizontal axes along the ground, so in this case $0 \le z < \infty$. Above the Ekman layer, where the influence of friction can be neglected, we have a geostrophic barotropic wind (u_g, v_g). The flow in the Ekman layer then becomes

$$\begin{aligned} u &= u_g + u_E, \\ v &= v_g + v_E, \end{aligned} \tag{5.8.1}$$

where (u_E, v_E) is the part of the wind in the Ekman layer that varies with height. The equation for the complex Ekman flow $W = u_E + iv_E$ becomes exactly the same as in the ocean:

$$\frac{d^2W}{dz^2} - \frac{ifW}{A^{(z)}} = 0 \ . \tag{5.8.2}$$

We assume for simplicity that the geostrophic wind is directed along the *x*-axis, i.e. $v_g = 0.$ Since we have neglected the effect of the constant-shear layer, we can assume that the velocity vanishes at the ground, i.e.

$$\left.\begin{aligned} u_E &= -u_g, \quad z = 0 \\ v_E &= 0, \quad z = 0 \end{aligned}\right\} \Rightarrow W = -u_g, \quad z = 0. \tag{5.8.3}$$

In more realistic conditions we may instead specify the turbulent stress at the ground, and relate it to the geostrophic wind. However, to make this problem as simple as possible, we assume that the velocity vanishes, which is an ideal case. Above the Ekman layer, we have the geostrophic wind. Mathematically, this means that

$$u_E, v_E \to 0, \quad z \to \infty \Rightarrow W \to 0, \quad z \to \infty \quad . \tag{5.8.4}$$

Now, with $a^2 = if / A^{(z)}$ as before, and $z \geq 0$, the solution of (5.8.2) satisfying (5.8.4) becomes

$$W = C \exp(-az) . \tag{5.8.5}$$

From (5.8.3) we obtain that $C = -u_g$. Hence

$$u_E + iv_E = -u_g \exp(-az) . \tag{5.8.6}$$

Utilizing the definition (5.4.15) of the Ekman depth, we find that

$$a = \frac{\pi(1+i)}{D_E} . \tag{5.8.7}$$

We can then write for the mean wind in the planetary boundary layer

$$\begin{aligned} u &= u_g - u_g \exp\left(-\frac{\pi z}{D_E}\right)\cos\left(\frac{\pi z}{D_E}\right), \\ v &= u_g \exp\left(-\frac{\pi z}{D_E}\right)\sin\left(\frac{\pi z}{D_E}\right). \end{aligned} \tag{5.8.8}$$

This is exactly the same solution as we obtain for the Ekman current near the bottom of the ocean, when the ocean depth is much larger than the Ekman depth. We realize from (5.8.8) that if we look along the direction of the geostrophic wind, the flow in the Ekman layer will be directed somewhat to the left of this direction in the northern hemisphere $(f > 0)$. Hence we have a left-ward spiralling flow as we move downward towards the ground.

5.9 The transport in the planetary Ekman layer

The motion in the planetary Ekman layer to the left of the geostrophic wind is easily understood in terms of the horizontal balance of forces on a fluid particle. Since the effect of friction reduces the speed of the particle, the Coriolis force will no longer be able to balance pressure gradient force alone. The new balance on the particle will then be between the pressure gradient force, the Coriolis force and the friction force. Since the pressure gradient force is perpendicular to the isobars, and the friction acts against the velocity, the new velocity must be turned somewhat to the left of the isobars in order to generate the Coriolis force that is necessary for the balance of forces on the particle; see the sketch below.

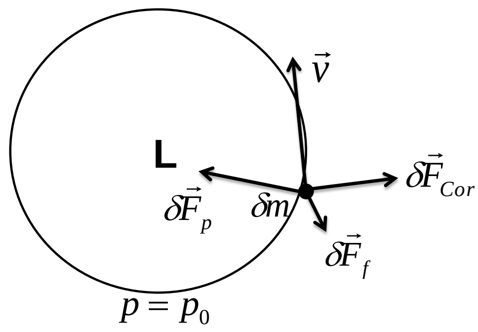

Fig. 5.10 Balance between the pressure-gradient force, the Coriolis force, and the friction force on a small particle in the planetary Ekman layer.

Quantitatively we find the mass transport in the planetary Ekman layer by integrating (5.8.8) vertically over the layer:

$$q^{(x)} = \rho_r \int_0^{D_E} u dz = \left(1 - \frac{1}{2\pi}\right)\rho_r D_E u_g ,$$
$$q^{(y)} = \rho_r \int_0^{D_E} v dz = \frac{1}{2\pi}\rho_r D_E u_g , \tag{5.9.1}$$

where we have used that $\exp(-\pi) << 1$. We define the angle α_E between the geostrophic current direction and the direction of the mass transport in the Ekman layer by

$$\alpha_E = \arctan(q^{(y)} / q^{(x)}) . \tag{5.9.2}$$

By inserting from (5.9.1), we find that $\alpha_E = \arctan(1/(2\pi - 1)) = 10.7^o$. Accordingly, when we look in the direction of the geostrophic current, we notice that the mass transport in the Ekman layer is directed about 11 degrees to the *left* of this direction in the northern hemisphere.

This has important consequences for large scale cyclonic or anti-cyclonic motion. In cyclonic motion we thus get a net mass transport near the ground towards the centre of the cyclone. This motion must in turn result in an upward and outward motion in the cyclone above the Ekman layer, making the cyclone wider. Since we can neglect the effect friction in the region above the Ekman layer, the absolute velocity circulation along horizontal material curves in cyclonic motion must be conserved. This follows from Kelvin's circulation theorem in hydrodynamics. The upward and outward motion induced by the mass convergence in the Ekman layer thus makes these material curves above the Ekman layer wider, and hence longer. Therefore, the horizontal velocity in the cyclone must decrease in order to conserve the circulation. This means that the intensity of the cyclone will decrease, i.e. the cyclone will become damped. If left alone, it will finally cease to exist. For an anti-cyclone, with divergence in the Ekman layer, the material curves will shrink in the region above the bottom layer. Hence the negative relative velocity will increase till it finally becomes zero. In both cases the final state will thus be that of the atmosphere following the solid rotating earth, with no motion relative to the ground. The phenomenon discussed her is called Ekman *suction*, or Ekman *pumping*, and will be dealt with in detail in more advanced courses in geophysical fluid dynamics.

APPENDIX

Relation between time derivatives of vectors in fixed and rotating frames

The derivations here refer to the reference systems defined in Section 1.1. To simplify the sketches, we consider two-dimensional motion, i.e. motion in the plane. The derived results, however, are generally valid in three dimensions. In Fig. 1A we have depicted the displacement of a small particle with mass δm. Whether this is a solid particle, or a fluid particle, is irrelevant at this stage. The angular velocity $\vec{\Omega}$ of frame (x, y) is constant in time. It is here perpendicular to the plane, and directed out of the paper.

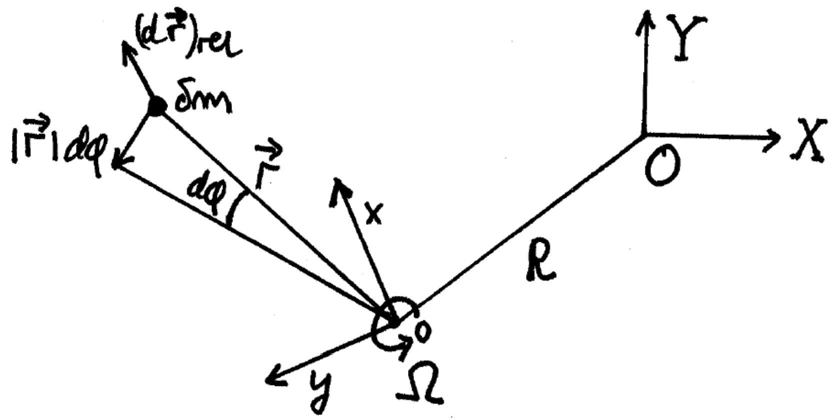

Fig. 1A Frames and particle displacements.

First, let the particle δm be at rest in the relative frame (x, y). As seen from (X, Y), the particle gets a displacement $|\vec{r}|d\varphi$ in time dt due to the rotation of (x, y). From Fig. 1A we note that this displacement is perpendicular to the position vector $\vec{r}$ in (x, y). The associated velocity, as seen from (X, Y), becomes $|\vec{r}|d\varphi/dt = |\vec{r}|\Omega$. In vector notation, this can be written $\vec{\Omega}\times\vec{r}$.

Second, assume that the particle does move in the frame (x, y). We denote the displacement of δm in (x, y) by $(d\vec{r})_{rel}$. Accordingly, the absolute velocity of δm, observed from (X, Y), can be written

$$\vec{v}_{abs} \equiv \left(\frac{d\vec{r}}{dt}\right)_{abs} = \left(\frac{d\vec{r}}{dt}\right)_{rel} + \vec{\Omega}\times\vec{r} \,. \tag{A.1}$$

This relation is not only valid for the position vector. We have in general, for any vector $\vec{A}(t)$, that

$$\left(\frac{d\vec{A}}{dt}\right)_{abs} = \left(\frac{d\vec{A}}{dt}\right)_{rel} + \vec{\Omega}\times\vec{A}. \tag{A.2}$$

From (A.1) and (A.2) we then obtain for the absolute acceleration

$$\begin{aligned}
\vec{a}_{abs} &\equiv \left(\frac{d\vec{v}_{abs}}{dt}\right)_{abs} \\
&= \left(\frac{d\vec{v}_{abs}}{dt}\right)_{rel} + \vec{\Omega}\times\vec{v}_{abs} \\
&= \left(\frac{d}{dt}\left(\vec{v}_{rel} + \vec{\Omega}\times\vec{r}\right)\right)_{rel} + \vec{\Omega}\times\left(\vec{v}_{rel} + \vec{\Omega}\times\vec{r}\right) \\
&= \underbrace{\left(\frac{d\vec{v}_{rel}}{dt}\right)_{rel}}_{\vec{a}_{rel}} + \underbrace{\vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rel}}_{\vec{\Omega}\times\vec{v}_{rel}} + \vec{\Omega}\times\vec{v}_{rel} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})
\end{aligned} \tag{A.3}$$

Accordingly,

$$\vec{a}_{abs} = \vec{a}_{rel} + 2\vec{\Omega}\times\vec{v}_{rel} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\,. \tag{A.4}$$

If the frame (x, y), in addition to rotation, also moves with a translational velocity $\vec{v}_0(t)$ relative to the fixed frame (X, Y), the absolute acceleration of δm gets an additional term, which is

$$\vec{a}_0 = \left(\frac{d\vec{v}_0}{dt}\right)_{abs}. \tag{A.5}$$

Hence, the relation between accelerations measured in fixed (absolute) frames and relative frames may be written

$$\vec{a}_{abs} = \vec{a}_{rel} + 2\vec{\Omega}\times\vec{v}_{rel} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + \vec{a}_0\,. \tag{A.6}$$

As mentioned before, we have here assumed that the angular velocity is constant in time.

BOOKS TO READ

Bachelor, G. K.: *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, 1967, 615 pp.

Gill, A. E.: *Atmosphere – Ocean Dynamics*, Academic Press, 1982, 662 pp.

Holton, J. R.: *An introduction to Dynamic Meteorology*, 4th Ed., Elsevier, 2004, 535 pp.

Krauss, W.: *Methods and Results of Theoretical Oceanography. Volume I: Dynamics of the Homogeneous and Quasi homogenous Ocean*, Gebrüder Borntraeger, 1973, 302 pp.

Pedlosky, J.: *Geophysical Fluid Dynamics*, 2nd Ed., Springer-Verlag, 1987, 710 pp.