GEF2500 - OBLIGATORY EXERCISE 2, SPRING 2014
All symbols are defined in the GEF2500 Lecture Notes

Problem 1

Consider interfacial waves in a two-layer system. The lower layer (label 2) has constant depth H_2, and constant density ρ_2, and the upper layer (label 1) is infinitely high and has constant density ρ_1 $(\rho_1 < \rho_2)$. The *z*-axis is vertical, and the *x*-axis is situated along the interface between the layers. The lower layer has an impermeable bottom at $z = -H_2$.

a) Explain why the kinematic boundary condition at the bottom becomes:

$$w_2 = 0, \quad z = -H_2. \tag{1}$$

b) Neglect the Coriolis force, and the viscous force. Show from the linearized equation of motion in the *x*, *z*-plane that

$$\begin{aligned} u_1 &= \frac{\partial \phi_1}{\partial x}, \quad w_1 = \frac{\partial \phi_1}{\partial z}, \\ u_2 &= \frac{\partial \phi_2}{\partial x}, \quad w_2 = \frac{\partial \phi_2}{\partial z}. \end{aligned} \tag{2}$$

c) Show that

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} &= 0, \\ \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} &= 0. \end{aligned} \tag{3}$$

d) We write the velocity potential in complex notation as

$$\phi_{1,2} = F_{1,2}(z)\exp(i(kx - \omega t)). \tag{4}$$

Show that

$$\begin{aligned} F_1 &= A_1 \exp(-kz), \\ F_2 &= A_2 \exp(kz) + B_2 \exp(-kz), \end{aligned} \tag{5}$$

where A_1, A_2, B_2 are constants (use that $\phi_1 \to 0,\ z \to \infty$).

e) At the interface we write for a complex wave component

$$\eta = a\exp(i(kx - \omega t)). \tag{6}$$

Apply that $w_1 = w_2 = \partial\eta / \partial t$ at $z = 0$, together with (1), and show that

$$\phi_1 = \frac{ia\omega}{k}\exp(-kz)\exp(i(kx-\omega t)),$$
$$\phi_2 = -\frac{ia\omega\cosh(k(z+H_2))}{k\sinh(kH_2)}\exp(i(kx-\omega t)). \quad (7)$$

f) Show that the dynamic pressure in this approximation can be written

$$p_{1,2} = -\rho_{1,2}\left(\frac{\partial\phi_{1,2}}{\partial t} + gz\right). \quad (8)$$

g) At the interface $z = \eta = a\exp(i(kx-\omega t))$, we have that $p_1 = p_2$ (the dynamic boundary condition). In this problem $k|\eta| << 1$ and $|\eta| << H_2$, which means that you can put $z = 0$ in $\phi_{1,2}$ when applying this condition. Show that the frequency in this problem is given by

$$\omega^2 = \frac{(\rho_2-\rho_1)gk}{\rho_2\coth(kH_2)+\rho_1}. \quad (9)$$

h) Find the frequency for the two extreme cases $kH_2 >> 1$ and $kH_2 << 1$. Hint in the latter case: expand $\exp(kH_2), \exp(-kH_2)$ in power series; see e.g. Rottmann.

i) Use the real part of (7) and determine the Stokes drift in the lower layer.