Logistic Map

A simple mathematical system; useful to learn some basic concepts:

- Bifurcation and bifurcation diagram
- Periodicity vs. chaos
- Period-doubling road to chaos
- Attractor
- Repellor
- Chaos and sensitive dependence

Recursive formula for time-development of the variable *x*:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n)$$

where $n = \frac{t - t_0}{\Delta t}$ is a discrete time – coordinate,

$x_0 \in [0,1]$ is a given initial condition,

and $r \in (0,4]$ is a parameter.

Fixed points (stationary solutions) and their ***stability***

$x_{n+1} = x_n$ for any n $\Leftrightarrow$ the solution of $x = f(x)$

Solution: $x = 0$ and $x = 1 - {}^{1}/_{r}$ for $r \in [1,4]$

Linear stability analysis of a fixed point: $\hat{x}$

Add a small perturbation: $x_n = \hat{x} + \delta x_n$ and neglect non-linear δx – terms:

$$\delta x_{n+1} = r(1 - 2\hat{x})\delta x_n = f'(\hat{x})\,\delta x_n = [\tfrac{df}{dx}]_{\hat{x}}\,\delta x_n$$

The fixed point's stability: if $\frac{|\delta x_{n+1}|}{|\delta x_n|} \begin{cases} < 0: & stable \\ = 0: & indifferent \\ > 0: & unstable \end{cases}$

The stability of $\hat{x}$ is determined by $f'(\hat{x})$.

$\hat{x} = 0$: $\delta x_{n+1} = r\, \delta x_n$. Stable: $r < 1$ (*attractor*); unstable: $r > 1$ (*repellor*)

$\hat{x} = 1 - \frac{1}{r}$: $\delta x_{n+1} = (2 - r\,)\delta x_n$. Stable for $r \in [1{,}3]$; unstable for > 3.

What happens for $r > 3$?

Periodicity. For å positive integer *N:*

The solution is periodic with period *N* if, for any *n*: $x_n = x_{\boldsymbol{n+N}} = f^{\boldsymbol{N}}(x_n)$

It can be shown that when *r* increases beyond 3, periodicity is obtained over an infinite range of intervals for *r*, where the intervals' width decreases:

N=2, N=4, N=8, N=16, …

until *N* becomes infinite at some critical value r_c .

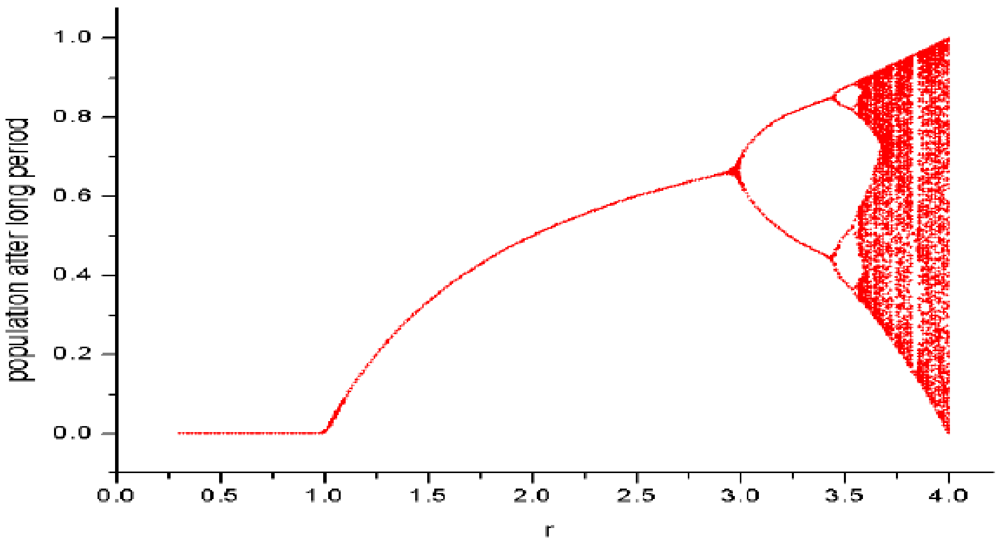
For $r > r_c$ the attractor contains infinitely many points (*strange attractor*) and there is no periodicity, but *sensitive dependence on* x_0*: Chaos*.

The strange attractor is *self-similar*.

In the interval $r \in [3, r_c]$ there is a *period-doubling road to chaos.*

For every value of *r* where *N* is doubled, there is a *bifurcation.*

The *bifurcation diagram* shows this:



Stability of time developments: Lyapunov exponent.

$\{x_n\}$ is a solution starting from x_0 : a non-linear trajectory (an orbit):

$$x_n = f^n(x_0)$$

Investigation of the sensitivity to initial state by adding a small perturbation: $x_0' = x_0 + \delta x_0$

Successive Taylor-development and removing non-linear δx_n – terms:

$$\delta x_n = \prod_{i=0}^{n-1} f'(x_i)\ \delta x_0$$

Define: $\Lambda_n = \prod_{i=0}^{n-1} f'(x_i) = {}^{\delta x_n}/_{\delta x_0}$; then

the Lyapunov exponent is: $\lambda = \lim_{n\to\infty}[ln\,|\Lambda_n|/n]$

$l = e^{\lambda}$ is the Lyapunov number.

$$\lambda = \lim_{n\to\infty}[ln\,|\prod_{i=0}^{n-1} f'(x_i)|/n] = lim_{n\to\infty}\frac{1}{n}[\sum_{i=0}^{n-1} ln\,|f'(x_i)|\,]$$

The Lyapunov exponent is the growth rate of small perturbations averaged over all states $\{x_n\}$***; i.e. the systems attractor.*** $\lambda < 0 =>$ ***chaos***