Introduction to Synoptic Scale Dynamics

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1 General dynamics

The motion in the atmosphere and ocean is governed by a set of equations, known as the *Navier-Stokes* equations. These equations are used to produce our forecasts, for the weather and also for ocean currents. While there are details about these equations which are uncertain (for example, how we parametrize processes smaller than the grid size of the models), they are for the most part accepted as fact. Let's consider how these equations come about.

1.1 Derivatives

A fundamental aspect is how various fields (temperature, wind, density) change in time and space. Thus we must first specify how to take derivatives.

Consider a scalar, ψ, which varies in both time and space, i.e. $\psi = \psi(x, y, z, t)$. This could be the wind speed in the east-west direction, or the ocean density. By the chain rule, the total change in the ψ is:

$$d\psi = \frac{\partial}{\partial t}\psi\, dt + \frac{\partial}{\partial x}\psi\, dx + \frac{\partial}{\partial y}\psi\, dy + \frac{\partial}{\partial z}\psi\, dz \qquad (1)$$

so:

$$\frac{d\psi}{dt} = \frac{\partial}{\partial t}\psi + u\,\frac{\partial}{\partial x}\psi + v\,\frac{\partial}{\partial y}\psi + w\,\frac{\partial}{\partial z}\psi = \frac{\partial}{\partial t}\psi + \vec{u}\cdot\nabla\psi \qquad (2)$$

where (u, v, w) the components of the velocity in the (x, y, z) directions. On the left side, the derivative is a total derivative. That implies that ψ on the left side is only a function of time. This the case when ψ is observed *following the flow*. For instance, if you measure temperature in a balloon, moving with the winds, you only see changes in time. We call this the *Lagrangian* formulation. The derivatives on the right side though are partial derivatives. These are relevant for an observer at *a fixed location*. This person records temperature as a function of time, but her information also depends on her position. An observer at a different location will generally have a different records (depending on how far away she is). We call the right side the *Eulerian* formulation.

1.2 Continuity equation

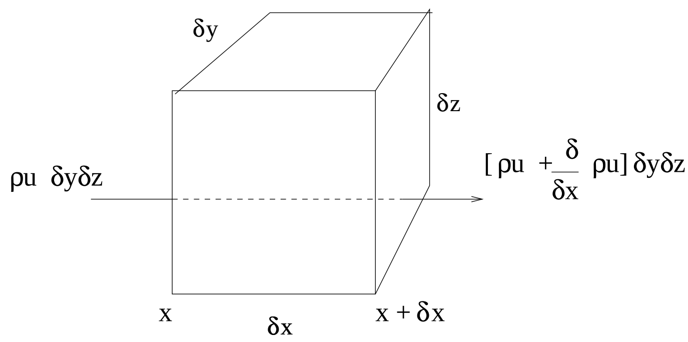

Figure 1: A infinitesimal element of fluid, with volume δV.

Consider a box fixed in space, with fluid (either wind or water) flowing through it. The flux of density through the left side is:

$$(\rho u)\,\delta y\,\delta z \tag{3}$$

Using a Taylor expansion, we can write the flux through the right side as:

$$[\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\,\delta y\,\delta z \tag{4}$$

If these density fluxes differ, then the box's mass will change. The net rate of change in mass is:

$$\frac{\partial}{\partial t}M = \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = (\rho u)\,\delta y\,\delta z - [\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\,\delta y\,\delta z$$
$$= -\frac{\partial}{\partial x}(\rho u)\delta x\,\delta y\,\delta z \tag{5}$$

The volume of the box is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u) \tag{6}$$

Taking into account all the other sides of the box we have:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = -\nabla\cdot(\rho\vec{u}) \tag{7}$$

We can rewrite the RHS as follows:

$$\nabla\cdot(\rho\vec{u}) = \rho\nabla\cdot\vec{u} + \vec{u}\cdot\nabla\rho \tag{8}$$

Thus the continuity equation can also be written:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = \frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho(\nabla\cdot\vec{u}) =$$
$$\frac{d\rho}{dt} + \rho(\nabla\cdot\vec{u}) = 0 \tag{9}$$

The first version of the equation is its Eulerian form. It states that the density at a location changes if there is a divergence in the flux into/out of the region. The last version is the Lagrangian form. This says that the density of a parcel of fluid advected by the flow will change if the flow is divergent, i.e. if:

$$\nabla\cdot\vec{u} \neq 0 \tag{10}$$

We can also obtain the continuity equation using a Lagrangian (moving) box. We assume the box contains a fixed amount of fluid, so that it conserves it mass, M. Then the *relative* change of mass is also conserved:

$$\frac{1}{M}\frac{d}{dt}M = 0 \tag{11}$$

The mass is the density times the volume of the box, so:

$$\frac{1}{\rho V}\frac{d}{dt}(\rho V) = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{V}\frac{dV}{dt} = 0 \tag{12}$$

Expanding the volume term by using the chain rule:

$$\frac{1}{V}\frac{dV}{dt} = \frac{1}{\delta x}\frac{\partial \delta x}{dt} + \frac{1}{\delta y}\frac{\partial \delta y}{dt} + \frac{1}{\delta z}\frac{\partial \delta z}{dt} =$$

$$\frac{1}{\delta x}\delta\frac{\partial x}{dt} + \frac{1}{\delta y}\delta\frac{\partial y}{dt} + \frac{1}{\delta z}\delta\frac{\partial z}{dt} \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{13}$$

as $\delta \rightarrow 0$. So:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{u} = 0 \tag{14}$$

which is the same as (9). Again, the density changes in proportion to the velocity divergence; the divergence determines whether the box shrinks or grows. If the box expands/shrinks, the density decreases/increases, to preserve the box's mass.

1.3 Equations of motion

The continuity equation pertains to mass. Now we consider the fluid velocities. We can derive expressions for these from Newton's second law:

$$\vec{a} = \vec{F}/m \tag{15}$$

The forces acting on a fluid parcel (a vanishingly small box) are:

- pressure gradients: $\frac{1}{\rho}\nabla p$
- gravity: $\vec{g}$
- friction: $\vec{\mathcal{F}}$

For a parcel with density ρ, we can write:

$$\frac{d}{dt}\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{\mathcal{F}} \tag{16}$$

This is the *momentum equation*, written in its Lagrangian form. Under the influence of the forcing terms, on the RHS, the air parcel will accelerate.

Actually, this is the momentum equation for a non-rotating earth. There are additional acceleration terms which come about due to rotation. As opposed to the *real* forces shown in (16), rotation introduces *apparent* forces. A stationary parcel on the earth will rotate with the planet. From the perspective of an observer in space, that parcel is traveling in circles, completing a circuit once a day. Since circular motion represents an acceleration (the velocity is changing direction), there is a corresponding force.

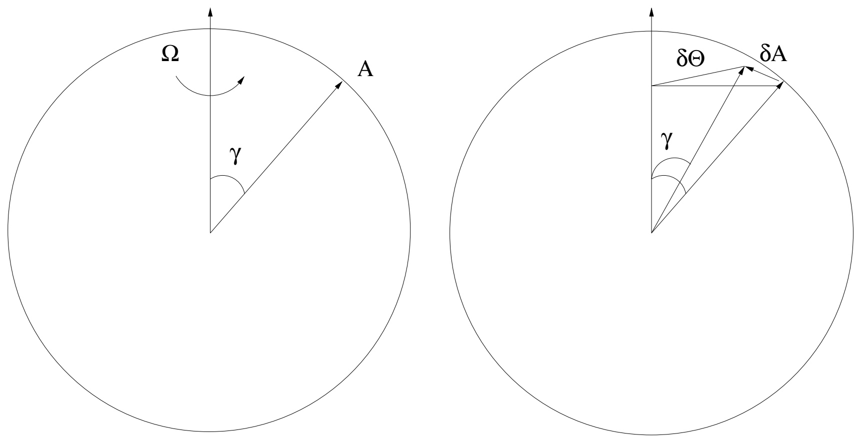

Figure 2: The effect of rotation on a vector, A, which is otherwise stationary. The vector rotates through an angle, $\delta\Theta$, in a time δt.

Consider such a stationary parcel, on a rotating sphere, with its position represented by a vector, $\vec{A}$ (Fig. 2). During the time, δt, the vector rotates through an angle:

$$\delta\Theta = \Omega\delta t \tag{17}$$

where Ω is the sphere's rotation rate. We will assume $\Omega = const.$, which is reasonable for the earth on weather time scales. The change in A is δA, the arc-length:

$$\delta\vec{A} = |\vec{A}|sin(\gamma)\delta\Theta = \Omega|\vec{A}|sin(\gamma)\delta t = (\vec{\Omega} \times \vec{A})\,\delta t \tag{18}$$

So we can write:

$$lim_{\delta\to 0}\,\frac{\delta\vec{A}}{\delta t} = \frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A} \tag{19}$$

If the vector is not stationary but moving in the rotating frame, one can show that:

$$(\frac{d\vec{A}}{dt})_F = (\frac{d\vec{A}}{dt})_R + \vec{\Omega} \times \vec{A} \tag{20}$$

The F here refers to the fixed frame and R to the rotating one. If $\vec{A} = \vec{r}$, the position vector, then:

$$(\frac{d\vec{r}}{dt})_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{21}$$

So the velocity in the fixed frame is just that in the rotating frame plus the velocity associated with the rotation.

Now consider that $\vec{A}$ is velocity in the fixed frame, $\vec{u}_F$. Then:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_F}{dt})_R + \vec{\Omega} \times \vec{u}_F \tag{22}$$

Substituting in the previous expression for u_F, we get:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d}{dt}[\vec{u}_R + \vec{\Omega} \times \vec{r}])_R + \vec{\Omega} \times [\vec{u}_R + \vec{\Omega} \times \vec{r}] \tag{23}$$

Collecting terms, we get:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} \tag{24}$$

We now have two additional terms: the *Coriolis* and *centrifugal* accelerations. Plugging these into the momentum equation, we obtain:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{25}$$

Consider the centrifugal acceleration. This is the negative of the centripetal acceleration and acts perpendicular to the axis of rotation (Fig. 3). The force projects onto both the radial and the N-S directions. This suggests that a parcel in the Northern Hemisphere would accelerate upward and southward. But these accelerations are balanced by gravity, which acts to pull the parcel toward the center *and* northward. The latter occurs because rotation changes the shape of the earth itself, making it ellipsoidal rather than spherical. The change in shape results in an exact cancellation of the N-S component of the centrifugal force.

The radial component on the other hand is overcome by gravity. If this weren't true, the atmosphere would fly off the earth. So the centrifugal force *modifies gravity*, reducing it over what it would be if the earth were stationary. Thus we can absorb the centrifugal force into gravity:

$$g' = g - \vec{\Omega} \times \vec{\Omega} \times \vec{r} \tag{26}$$

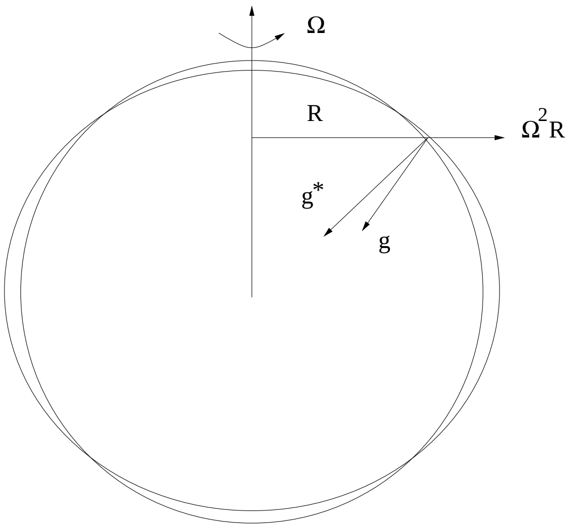

Figure 3: The centrifugal force and the deformed earth. Here is g is the gravitational vector for a spherical earth, and g^* is that for the actual earth. The latter is an *oblate spheroid*.

How big a modification is this? One can show that the centrifugal force at the equator is about $0.034m/sec^2$—or roughly 1/300th as large as g. The correction is so small in fact that we will ignore it (and drop the prime on g).

So the momentum equation can be written:

$$(\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{27}$$

There is only one rotational term to worry about, the Coriolis force.

There are three spatial directions and each has a corresponding momentum equation. In what follows, we will assume that we are in a localized region of the atmosphere, centered at a latitude, θ. Then we can define local coordinates (x, y, z) such that:

$$\delta x = acos(\theta)\delta\phi, \ \delta y = a\delta\theta, \ \delta z = \delta R$$

where ϕ is the longitude, a is the earth's radius and R is the radius. Thus x is the east-west coordinate, y the north-south coordinate and z the vertical coordinate. We define the corresponding velocities:

$$u \equiv \frac{dx}{dt},\ v \equiv \frac{dy}{dt},\ w \equiv \frac{dz}{dt}$$

The momentum equations will determine the accelerations in (x,y,z).

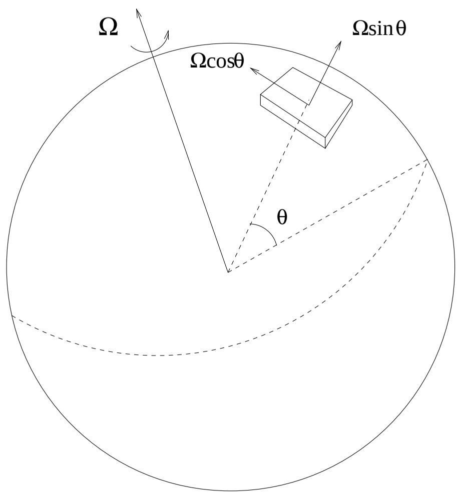

Figure 4: A region of the atmosphere at latitude θ. The earth's rotation vector projects onto the local latitudinal and radial coordinates.

The Coriolis term (which is a vector itself) projects onto both the y and z directions:

$$2\vec{\Omega} \times \vec{u} = (0, 2\Omega_y, 2\Omega_z) \times (u, v, w) =$$

$$2\Omega(w\,cos\theta - v\,sin\theta,\ u\,sin\theta, -u\,cos\theta) \tag{28}$$

Adding terms, we have:[1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + 2\Omega w\,cos\theta - 2\Omega v\,sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{29}$$

[1] If we had used spherical coordinates instead, we would have several additional *curvature* terms. However, these terms are negligible at the scales of interest and so are left out here.

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \qquad (30)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\Omega u\, cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \qquad (31)$$

where F_i is the frictional force acting in the i-direction.

The momentum equations are complex and *nonlinear*, involving products of velocities. As such, they are essentially unsolvable in this form. However, not all the terms are equally important. To see which ones dominate, we *scale* the equations. This means we estimate the sizes of the various terms in the equation by using reasonable values for the variables at the scales we're interested in.

1.4 Scaling the horizontal acceleration

For example, take the x-momentum equation, neglecting (for now) the friction term:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + 2\Omega w\, cos\theta - 2\Omega v\, sin\theta = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{D} \qquad 2\Omega W \qquad 2\Omega U \qquad \frac{\triangle p}{\rho L}$$

$$\frac{1}{2\Omega T} \qquad \frac{U}{2\Omega L} \qquad \frac{U}{2\Omega L} \qquad \frac{W}{2\Omega D} \qquad \frac{W}{U} \qquad 1 \qquad \frac{\triangle p}{2\Omega \rho U L}$$

In the third line, we have divided through by the scaling for the vertical component of the Coriolis acceleration, $2\Omega U$. By doing this, all the terms on the third line are *dimensionless* parameters, i.e. they have no units.

What we will do is to evaluate each and see how it compares to one (the size of the vertical Coriolis term). We have made an (educated) guess that this term is one of the largest. If any of the other terms is much less than one, we can neglect it. If however another term is much greater than one, our assumption that the Coriolis term was the largest was wrong and we will have to divide again, using the larger term.

To see how large the terms are, we plug in typical values. We will use values for the atmosphere, typical of weather disturbances (the result using typical oceanic values is the same):

$$U \approx 10\ m/sec, \quad 2\Omega = \frac{4\pi}{86400\,sec} = 1.45 \times 10^{-4} sec^{-1} \approx 10^{-4} sec^{-1},$$

$$L \approx 10^6\ m, \quad D \approx 10^4\ m, \quad T = L/U \approx 10^5\ sec \quad a \approx 6400\ km$$

$$\triangle_H P/\rho \approx 10^3\ m^2/sec^2, \quad W \approx 1\ cm/sec, \tag{32}$$

The horizontal scale, 1000 km, is known as the *synoptic scale* in the atmosphere. This is a typical horizontal scale for pressure systems. The time scale, proportional to the length scale divided by the velocity scale, is the *advective* time scale. This is what you'd expect, for example, if a front were advected by the winds past an observer. With an advective time scale, we have:

$$\frac{1}{2\Omega T} = \frac{U}{2\Omega L}$$

So the first term is the same size as the second and third terms. This ratio is a well-known quantity in meteorology and oceanography, and is known as the *Rossby number*. It has a value at synoptic scales of

$$\frac{U}{2\Omega L} = 0.1$$

Thus the first three terms are roughly 10 times smaller than the vertical Coriolis term.

However, the other terms are even smaller. The fourth term:

$$\frac{W}{2\Omega D} = 0.01$$

is about 10 times smaller than the Rossby number, and thus is 100 times smaller than the Coriolis term. And the fifth term, the other Coriolis term,

$$\frac{W}{U} = .001$$

is even smaller.

So is there anything to balance the vertical Coriolis term? The pressure gradient term scales as:

$$\frac{\triangle p}{2\Omega \rho U L} = 0.70 \approx 1$$

So the pressure gradient term is comparable in size to the vertical Coriolis term.

1.5 Geostrophic balance

The scaling suggests that the first order balance in the momentum equations is between the vertical component of the Coriolis acceleration and the pressure gradient:

$$-fv \approx -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{33}$$

$$+fu \approx -\frac{1}{\rho}\frac{\partial}{\partial y}p \qquad (34)$$

where

$$f \equiv 2\Omega sin\theta$$

This is the vertical component of the Coriolis parameter, the only one which is important at these scales. Relations (33, 34) represent the *geostrophic* balance. This is one of two fundamental balances at synoptic scales. The balance implies that if we know the pressure and density, we can deduce the velocities. So the winds or currents can be determined from maps of the pressure and density.

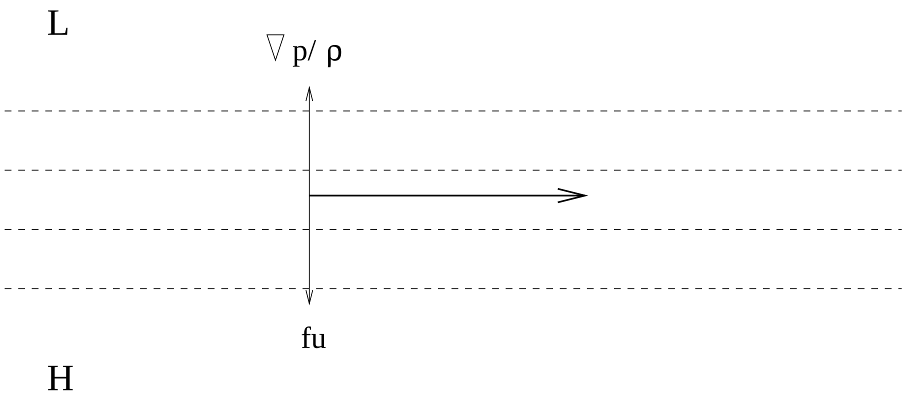

Figure 5: The geostrophic balance.

Consider the flow in Fig. (5). The pressure is high to the south and low to the north. Left alone, the pressure difference would force flow to the north. But the Coriolis force causes the flow to be *parallel* to the pressure contours. Because $\frac{\partial}{\partial y}p < 0$, we have from (34) that $u > 0$, so the flow is eastward. The Coriolis acceleration is to the right of the motion, and this exactly balances the pressure gradient force, which is to the left. Because the two forces balance, the motion is constant in time—there is no acceleration.

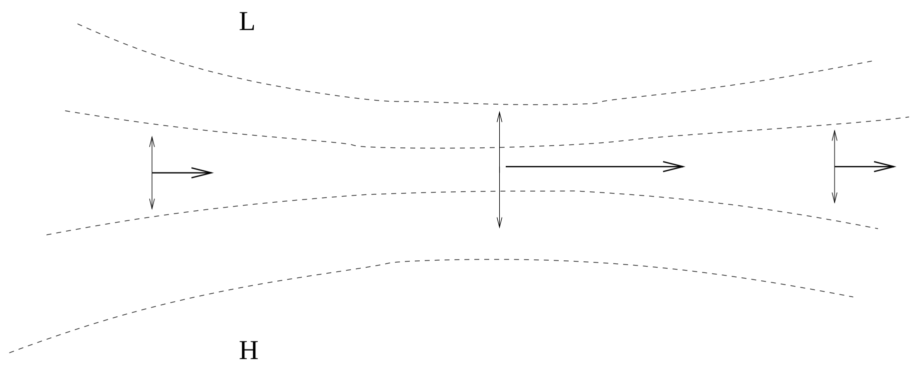

Figure 6: Geostrophic flow with non-constant pressure gradients.

If the pressure gradient changes in space, so will the geostrophic velocity. In Fig. (6), the flow accelerates into a region with more closely-packed pressure contours, then decelerates exiting the region.

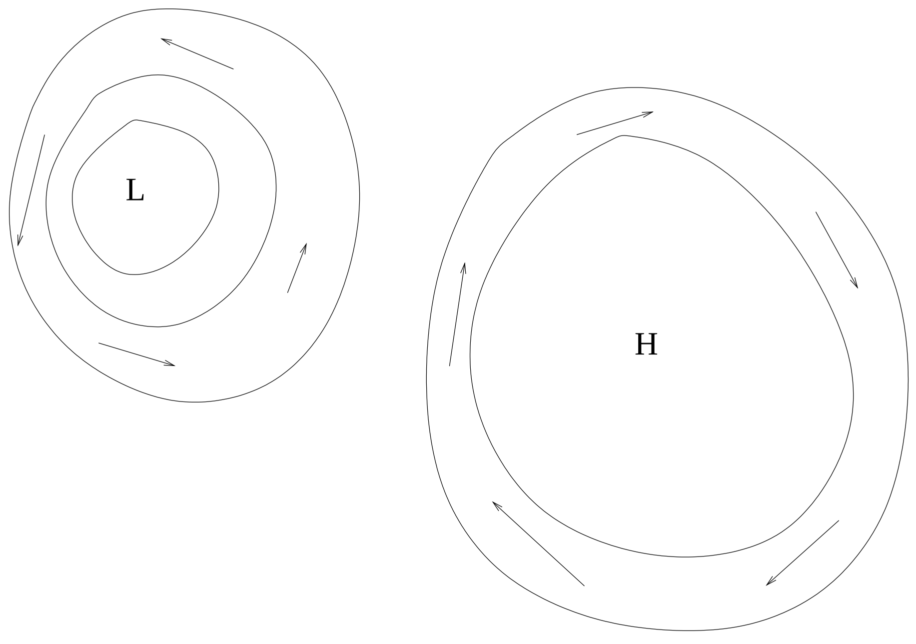

Figure 7: Geostrophic flow around pressure anomalies.

As a result of the geostrophic relations, we can take pressure maps and use them to estimate the winds, as in Fig. (7). From the previous arguments, the flow is counter-clockwise or *cyclonic* around a low pressure system. It is also clockwise or *anti-cyclonic* around a high pressure. So geostrophy is why the winds blow counter-clockwise around a hurricane.

Since $f = 2\Omega sin\theta$, the Coriolis force varies with latitude. It is strongest at high latitudes, and weaker at low latitudes. Note too that it is *negative* in the southern hemisphere! Thus the flow in Fig. (5) would instead be westward, with the Coriolis force acting to the left. In addition, the Coriolis force is identically *zero* at the equator. So the geostrophic balance *cannot hold* there.

1.6 The quasi-horizontal momentum equations

One problem with the geostrophic balances (33, 34) is that they cannot be used for prediction. Given the pressure field now, we can deduce the velocities now—but we can't predict what they will be in the future. This is because we have lost the time derivative terms in the momentum equations.

To do prediction, we must therefore include the next largest terms in the momentum equations, i.e. those which are of order Rossby number.

$$\frac{d_H u}{dt} - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{35}$$

The same reasoning yields:

$$\frac{d_H v}{dt} + fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p \tag{36}$$

where the Lagrangian derivative:

$$\frac{d_H}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

now does not include the vertical advection term. Thus the advection in these approximate equations is *quasi-horizontal*. This implies that synoptic scale motion is approximately *two-dimensional*, i.e. like motion confined to a tabletop. The vertical motion is generally much smaller.

1.7 Other momentum balances

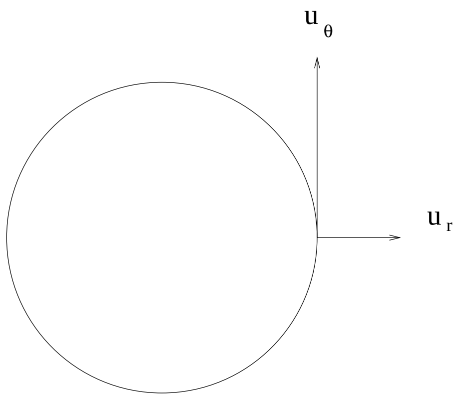

Figure 8: Circular flow.

The geostrophic balance occurs at synoptic scales, but other balances are possible at smaller scales. To see this, consider a perfectly circular flow (Fig. 8). The momentum equation in cylindrical coordinates (e.g. Batchelor, *Fluid Mechanics*) for the velocity in the radial direction is given by:

$$\frac{d}{dt}u_r - \frac{u_\theta^2}{r} - fu_\theta = -\frac{1}{\rho}\frac{\partial}{\partial r}p \tag{37}$$

The term u_θ^2/r is called the *cyclostrophic* term and is related to the centripetal acceleration. It is a curvature term like those found with spherical coordinates. If the flow is *steady* (not changing in time), then we have:

$$\frac{u_\theta^2}{r} + fu_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p \tag{38}$$

$$\frac{U^2}{R} \qquad 2\Omega U \qquad \frac{\triangle p}{\rho R}$$

$$\frac{U}{2\Omega R} \qquad 1 \qquad \frac{\triangle p}{2\rho\Omega U R}$$

We have scaled the equation as before. Note that the scale of the cyclostrophic term is determined by the Rossby number. Let's define that as ϵ.

1.7.1 Geostrophic flow

If $\epsilon \ll 1$, the cyclostrophic term is much smaller than the Coriolis term. Then, we must have:

$$\frac{\triangle p}{2\rho\Omega U R} \approx 1$$

and we have the geostrophic balance again:

$$f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r} p \tag{39}$$

Note that if the term on the RHS wasn't order one, the pressure gradient wouldn't be large enough to balance the Coriolis force and there would be no velocity.

1.7.2 Cyclostrophic flow

Now consider if $\epsilon \gg 1$. For example, a tornado at mid-latitudes has:

$$U \approx 30m/s, \quad f = 10^{-4} sec^{-1}, \quad R \approx 300m,$$

So $\epsilon = 1000$. Then the cyclostrophic term dominates over the Coriolis term. As we noted earlier, that means that we shouldn't have divided the scaling parameters by $2\Omega U$, but rather by U^2/R. Then we would have:

$$1 \qquad \frac{2\Omega R}{U} \qquad \frac{\triangle p}{\rho U^2}$$

Now the second term, which is just $1/\epsilon$, is very small (0.001 for the tornado) and we require:

$$\frac{\triangle p}{\rho U^2} \approx 1$$

In this case, we have the *cyclostrophic wind balance*:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial r}p \tag{40}$$

Notice that this is a *non-rotating* balance, because Ω doesn't enter. The pressure gradient is balanced by the centrifugal acceleration.

We can solve for the velocity after multiplying by r and then taking the square root:

$$u_\theta = \pm\sqrt{\frac{r}{\rho}\frac{\partial}{\partial r}p} \tag{41}$$

There are two interesting points about this. One is that only low pressure systems are permitted, because we require $\frac{\partial}{\partial r}p > 0$ in order to get a real solution. Second, *either sign* of wind is allowed. So our tornado can have either cyclonic or anti-cyclonic winds.

However, we know that tornadoes are low pressure systems with cyclonic flow. The reason the flow is cyclonic has to do with how the tornado spins up (i.e. how it forms). Indeed, the winds in the tornado aren't 30 m/sec all the time, but are much weaker while they are spinning up. The spin-up, it turns out, favors cyclonic winds.

1.7.3 Inertial flow

There is a third possibility, that there is no radial pressure gradient at all. This is called *inertial flow*. Then:

$$\frac{u_\theta^2}{r} + f u_\theta = 0 \quad \rightarrow \quad u_\theta = -fr \tag{42}$$

This corresponds to circular motion in "solid body rotation" (with the velocity increasing linearly from the center, as it would with a solid). The velocity is negative, implying the rotation is clockwise (anti-cyclonic) in the Northern Hemisphere. The time for a parcel to complete a full circle is:

$$\frac{2\pi r}{u_\theta} = \frac{2\pi}{f} = \frac{0.5\ day}{|sin\theta|}\ , \tag{43}$$

This is known as the "inertial period". "Inertial oscillations" are frequently seen at the ocean surface, and are excited by the winds and other forcing. They are much rarer in the atmosphere.

1.7.4 Gradient wind

The last possibility is that $\epsilon = 1$, in which case all three terms in (38) are important. This is the *gradient wind balance*. We can then solve for u_θ using the quadratic formula:

$$u_\theta = -\frac{1}{2}fr \pm \frac{1}{2}(f^2r^2 + \frac{4r}{\rho}\frac{\partial}{\partial r}p)^{1/2} = -\frac{1}{2}fr \pm \frac{1}{2}(f^2r^2 + 4\,r\,f\,u_g)^{1/2}\ , \tag{44}$$

after substituting in the definition of the geostrophic velocity. Note that if the pressure gradient vanishes, we recover the inertial velocity.

The gradient wind estimate clearly differs from the geostrophic estimate. The difference is typically about 10-20 % at mid-latitudes. To see this, we rewrite (38) thus:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p = f u_g \tag{45}$$

Then:

$$\frac{u_g}{u_\theta} = 1 + \frac{u_\theta}{fr} \tag{46}$$

The last term scales as the Rossby number. So if $\epsilon = 0.1$, the gradient wind estimate differs from the geostrophic value by 10 %. At low latitudes, where ϵ can be 1-10, the gradient wind estimate is more accurate.

1.8 Hydrostatic balance

Now we scale the vertical momentum equation. For this, we need an estimate of the vertical variation in pressure:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Neglecting the friction term, F_z, we have:

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \frac{u^2+v^2}{a} - 2\Omega u cos\theta = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g \tag{47}$$

$$\frac{WU}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad \frac{U^2}{a} \qquad 2\Omega U \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

$$\frac{UW}{gL} \qquad \frac{UW}{gL} \qquad \frac{UW}{gL} \qquad \frac{W^2}{gD} \qquad \frac{U^2}{ga} \qquad \frac{2\Omega U}{g} \qquad \frac{\triangle_V P}{g\rho D} \qquad 1$$

$$10^{-8} \qquad 10^{-8} \qquad 10^{-8} \qquad 10^{-11} \qquad 2\times 10^{-6} \qquad 10^{-4} \qquad 1 \qquad 1$$

Notice that we used the advective time scale, L/U, for the time scale T and we have divided through by g, which we assume will be large. The vertical pressure gradient and gravity terms are much larger than any of the others.

However this is somewhat misleading because we obtain the same balance if there is *no motion at all*! In particular, if $u = v = w = 0$, the vertical momentum equation is:

$$\frac{\partial}{\partial z}p = -\rho g \tag{48}$$

This is called the “hydrostatic balance” or literally the “non-moving fluid balance”. In addition, the two horizontal momentum equations reduce to:

$$\frac{1}{\rho}\frac{\partial}{\partial x}p = \frac{1}{\rho}\frac{\partial}{\partial y}p = 0 \tag{49}$$

So our non-moving atmosphere has $p = p(z)$, and no winds.

We aren’t particularly interested in this component of the flow, since we’re interested in the moving part. The latter comes from the dynamic (moving) portion of the pressure field. So we separate the pressure and density into static and dynamic components:

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t)$$

$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t) \tag{50}$$

Generally the dynamic components are much smaller than the static components, so that:

$$|p'| \ll |p_0| \tag{51}$$

Then we can write:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(p_0 + p') - g \approx -\frac{1}{\rho_0}\,(1 - \frac{\rho'}{\rho_0})\,\frac{\partial}{\partial z}(p_0 + p') - g$$

$$\approx -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + (\frac{\rho'}{\rho_0})\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g \tag{52}$$

Note we neglect terms proportional to the product of the dynamical variables, like $p'\rho'$.

Now the question is: how do we scale the dynamical pressure terms? Measurements suggest the vertical variation of p' is comparable to the horizontal variation, so:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2 \ .$$

The perturbation density, ρ', is roughly $1/100$ as large as the static density, so:

$$\frac{\rho'}{\rho_0}g \approx 10^{-1} m/sec^2 \ .$$

To scale these, we again divide by g. So both terms are of order 10^{-2}. So while they are smaller than the static terms, they are still *two orders of magnitude larger* than the next largest term in (47). So the approximate vertical momentum equation is still the hydrostatic balance, but for the perturbation pressure and density:

$$\frac{\partial}{\partial z}p' = -\rho' g \tag{53}$$

A model which uses this equation instead of the full vertical momentum equation is called a "hydrostatic model"; a model which uses the full vertical momentum is a "non-hydrostatic model". Notice that in the hydrostatic model, we have no information about $\frac{\partial}{\partial t}w$ and so have lost the ability to predict changes in the vertical velocity. Instead, w is usually diagnosed from the other variables.

1.9 The equations of state

In addition to the momentum and continuity equations, we require equations relating the density, temperature and, for the ocean, the salinity. In the atmosphere, the density and temperature are linked via the *Ideal Gas Law*:

$$p = \rho RT \tag{54}$$

where $R = 287\, Jkg^{-1}K^{-1}$ is the gas constant for dry air. The law is thus applicable to a dry gas, i.e. one without moisture, but a similar equation applies in the presence of moisture if one replaces the temperature with the so-called "virtual temperature".

In the ocean, both salinity and temperature affect the density. The dependence is expressed in an equation of state:

$$\rho = \rho(T,S) = \rho_c(1 - \alpha_T(T - T_{ref}) + \alpha_S(S - S_{ref})) + h.o.t. \tag{55}$$

where ρ_c is a constant, T_{ref} and S_{ref} are reference values for temperature and salinity and where $h.o.t.$ means "higher order terms". Increasing the temperature or decreasing the salinity reduces the density (makes lighter water). An important point is that the density is dominated by the first term, ρ_c, which is constant. We exploit this in the next section.

1.10 The Boussinesq Approximation

The fact that the oceanic density is dominated by ρ_c allows us to make the *Boussinesq* approximation. In this, we take the density to be constant, except in the "buoyancy term" on the RHS of the hydrostatic relation in (53).

Making this approximation, the geostrophic relations become:

$$-fv = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p \tag{56}$$

$$fu = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \tag{57}$$

where ρ_c is a constant.

In addition, with a constant density the continuity equation reduces to simply:

$$\frac{d\rho_c}{dt} + \rho_c(\nabla \cdot \vec{u}) = 0 \quad \rightarrow \quad \nabla \cdot \vec{u} = 0 \tag{58}$$

So oceanic flow is approximately *incompressible*.

These two alterations—having a constant density in the horizontal momentum equations and no density at all in the continuity equation—greatly simplify calculations.

Making the Boussinesq approximation apparently removes density variations. But what we do is to retain the variable density in the hydrostatic relation, i.e.:

$$\frac{\partial}{\partial z} = -\rho(x, y, z, t)g \tag{59}$$

This *buoyancy* term allows density variations to come into play in the flows.

1.11 Pressure Coordinates

We cannot responsibly apply the Boussinesq approximation to the atmosphere, except possibly in the planetary boundary layer (this is often done, for example, when considering the surface Ekman layer). But it is possible to achieve the same simplifications if we change the vertical coordinate to pressure instead of height.

We do this by exploiting the hydrostatic balance. Consider a pressure surface in two dimensions, (x, z). Applying the chain rule, we have:

$$\triangle p = \frac{\partial p}{\partial x} \triangle x + \frac{\partial p}{\partial z} \triangle z = 0 \tag{60}$$

on the surface. Substituting the hydrostatic relation, we get:

$$\frac{\partial p}{\partial x} \triangle x - \rho g \triangle z = 0 \tag{61}$$

so that:

$$\frac{\partial p}{\partial x}|_z = \rho g \frac{\triangle z}{\triangle x}|_p \equiv \rho \frac{\partial \Phi}{\partial x}|_p \tag{62}$$

where the subscripts indicate derivatives taken in vertical (z) and pressure (p) coordinates and where Φ is the *geopotential*:

$$\Phi \equiv \int_0^z g\, dz \tag{63}$$

Making this alteration removes the density from momentum equation because:

$$-\frac{1}{\rho} \nabla p|_z \quad \rightarrow \quad -\nabla \Phi|_p$$

So the geostrophic balance in pressure coordinates is simply:

$$f v = \frac{\partial}{\partial x} \Phi, \qquad f u = -\frac{\partial}{\partial y} \Phi \tag{64}$$

If we know the geopotential on a pressure surface, we can diagnose the velocities—without knowing the density.

In addition, the coordinate change simplifies the continuity equation. Consider a Lagrangian box with a volume:

$$\delta V = \delta x\,\delta y\,\delta z = -\delta x\,\delta y\,\frac{\delta p}{\rho g} \tag{65}$$

after substituting from the hydrostatic balance. The mass of the box is:

$$\rho\,\delta V = -\frac{1}{g}\delta x\,\delta y\,\delta p$$

Conservation of mass implies:

$$\frac{1}{\delta M}\,\frac{d}{dt}\,\delta M = \frac{g}{\delta x\delta y\delta p}\,\frac{d}{dt}\,(\frac{\delta x\delta y\delta p}{g}) = 0 \tag{66}$$

Rearranging:

$$\frac{1}{\delta x}\,\delta(\frac{dx}{dt}) + \frac{1}{\delta y}\,\delta(\frac{dy}{dt}) + \frac{1}{\delta p}\,\delta(\frac{dp}{dt}) = 0 \tag{67}$$

If we let $\delta \to 0$, we get:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{68}$$

where ω is the velocity perpendicular to the pressure surface (like w is perpendicular to a z-surface). Thus the change to pressure coordinates results in incompressible flow, as under the Boussinesq approximation.

Again it seems like the density has fallen out of the problem. But we retain it in the hydrostatic equation, which takes a slightly different form. Now we have that:

$$dp = -\rho g dz = -\rho d\Phi \tag{69}$$

So:

$$\frac{d\Phi}{dp} = -\frac{1}{\rho} = -\frac{RT}{p} \tag{70}$$

after invoking the Ideal Gas Law (54). Thus density (or equivalently temperature) variations are still important.

1.12 Thermal wind

If we combine the geostrophic and hydrostatic relations, we get the thermal wind relations. These tell us about the velocity shear. Take, for instance, the p-derivative of the geostrophic balance for v:

$$\frac{\partial v}{\partial p} = \frac{1}{f}\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{pf}\frac{\partial T}{\partial x} \tag{71}$$

after using (70). Note that the p passes through the x-derivative because it is constant on an isobaric surface. Likewise:

$$\frac{\partial u}{\partial p} = \frac{R}{pf}\frac{\partial T}{\partial y} \tag{72}$$

after using the hydrostatic relation (70). Thus the vertical shear is proportional to the lateral gradients in the temperature.

The thermal wind relations for the ocean derive from taking z-derivatives of the Boussinesq geostrophic relations (56-57), and then invoking the hydrostatic relation. The result is:

$$\frac{\partial v}{\partial z} = -\frac{1}{\rho_c f}\frac{\partial \rho}{\partial x} \tag{73}$$

$$\frac{\partial u}{\partial z} = \frac{1}{\rho_c f}\frac{\partial \rho}{\partial y} \tag{74}$$

Thus the shear in the ocean depends on lateral gradients in *density*, which can result from changes in either temperature or salinity.

The thermal wind is thus parallel to the temperature contours, with the warm wind on the right. In the ocean, the thermal wind is parallel to the

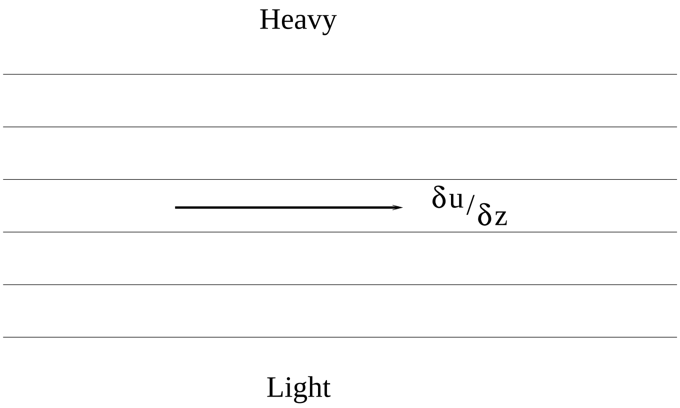

Figure 9: The thermal wind shear associated with a density gradient in the y-direction.

density contours, with light water on the right. This is illustrated in Fig. (9). There is a density gradient in y, meaning the shear is purely in the x-direction. The density increases to the north, so the gradient is positive. So $\partial u/\partial z$ is positive. Thus the zonal velocity is increasing going up, i.e. with the warm air to the right.

The analogous situation for the atmosphere is if the warm air is on the right. Then the temperature gradient in y is *negative*, so that $\partial u/\partial p < 0$. So u decreases as the pressure increases. However, the pressure increases going *downward*. So we would infer that $\partial u/\partial z$ is positive, just like in the ocean case.

If the temperature is only a function of pressure, $T = T(p)$, then T is constant on pressure surfaces in the atmosphere. It follows that there is no vertical shear— the geostrophic winds are constant with height. Likewise, if density is constant on z-surfaces, there is no vertical shear in the currents. A flow with zero vertical shear is called a *barotropic* flow. We will study barotropic flows in section (2).

There is also the possibility that the magnitude of the velocity changes with height but not the direction, as shown in Fig. (10). This is an *equi-*

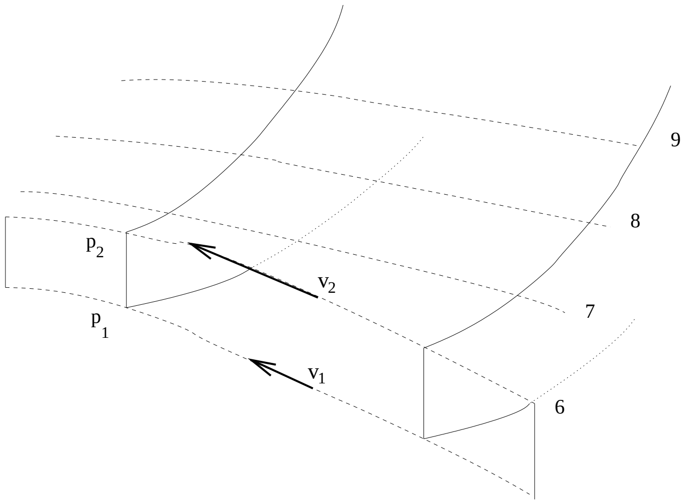

Figure 10: Vertical shear in the equivalent barotropic case. Note that the geostrophic velocities and the temperature contours are parallel.
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valent barotropic flow. In this case, the shear vectors are parallel to the velocities themselves and are also parallel to the temperature contours.

Using the thermal wind relation, we can estimate the strength of the jet stream, under the equivalent barotropic assumption. The zonally-averaged temperature decreases with latitude on the earth (the poles are colder than the equator). This means that $\partial T/\partial y < 0$, so that u should increase with height (Fig. 11).

To get an estimate of the wind speed aloft, we integrate the thermal wind balance in pressure. First we rewrite the zonal balance slightly:

$$p\frac{\partial u}{\partial p} = \frac{\partial u}{\partial ln(p)} = \frac{R}{f}\frac{\partial T}{\partial y} \tag{75}$$

Thus we have:

$$\int_{ps}^{p} du = u(p) - u(p_s) = \frac{R}{f}\int_{ps}^{p}\frac{\partial T}{\partial y}\, d\, ln(p) \tag{76}$$

T is a function of pressure, but let's replace it with the mean temperature

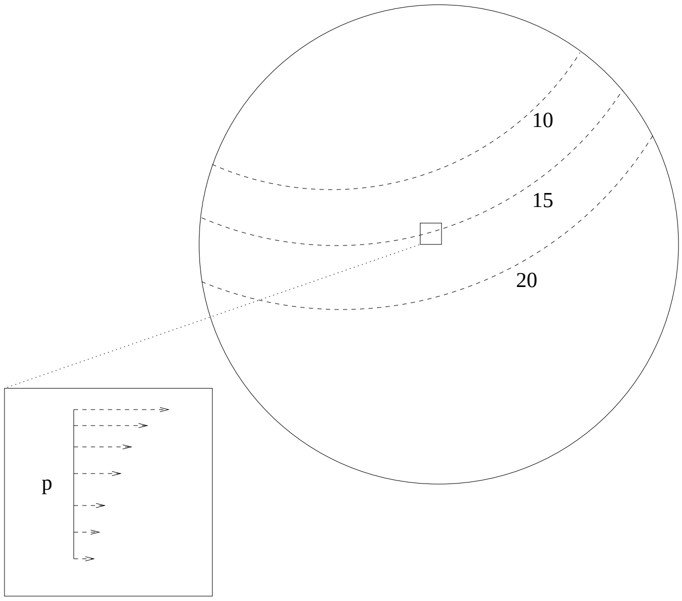

Figure 11: The jet stream on a zonally-average earth.

integrated over the height; call that $\overline{T}$. Then we get:

$$u(p) - u(p_s) = \frac{R}{f}\frac{\partial \overline{T}}{\partial y} ln(\frac{p}{p_s}) \quad (77)$$

At 30N, the zonally-averaged temperature gradient is roughly $0.75\ Kdeg^{-1}$. Assuming the wind is zero at the earth's surface, we can estimate the mean zonal wind at the level of the jet stream ($250\ hPa$):

$$u_g(250) = \frac{287}{2\Omega sin(30)}\, ln(\frac{250}{1000})\,(-\frac{0.75}{1.11 \times 10^5\, m}) = 36.8\ m/sec \quad (78)$$

This is comparable to the speed of the jet stream at this height.

The equivalent barotropic assumption is also used in simplified models of the atmosphere and ocean. A notable example is an equivalent barotropic model of the Antarctic Circumpolar Current, the large current which flows around Antarctica. However, in most cases the atmosphere and ocean are more complicated, with both the velocities' speed *and* direction changing with height. Thus we say that the atmosphere and ocean are *baroclinic*). So the temperature and geopotential contours are not parallel and the geostrophic wind can advect temperature.

In oceanography, relations (73) and (74) are routinely used to estimate ocean currents from density measurement made from ships. Ships collect *hydrographic* measurements of temperature and salinity, and these are then used to determine $\rho(x, y, z, t)$, from the equation of state (55). Then the thermal wind relations are integrated upward from chosen level to determine (u, v) above the level, for example:

$$u(x,y,z) - u(x,y,z_0) = \int_{z0}^{z} \frac{1}{\rho_c f} \frac{\partial \rho(x,y,z)}{\partial y} \, dz \qquad (79)$$

If (u, v, z_0) is set to zero at the lower level, it is known as a "level of no motion". Such thermal-wind derived estimates were used to map the global currents in the World Ocean Circulation Experiment (WOCE) during the 1990s.

1.13 The vorticity equation

We can obtain a very useful equation from the momentum equations if we *cross-multiply* them. Specifically, we take $\frac{\partial}{\partial x}$ of the y-momentum equation (36) and subtract $\frac{\partial}{\partial y}$ of the x-momentum equation (35). Doing this eliminates the pressure terms on the RHS, leaving (after some algebra):

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta + (f+\zeta)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) + v\frac{\partial}{\partial y}f = 0 \qquad (80)$$

where

$$\zeta = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u$$

is the curl of the horizontal velocities. This is the *vorticity*. Actually, it is the vertical component of the full vorticity:

$$\vec{\zeta} = \nabla \times \vec{u}$$

which has components:

$$\zeta_x = \frac{\partial}{\partial y}w - \frac{\partial}{\partial z}v = \frac{\partial}{\partial y}w$$

$$\zeta_y = \frac{\partial}{\partial z}u - \frac{\partial}{\partial x}w = -\frac{\partial}{\partial x}w$$

$$\zeta_z \equiv \zeta = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \quad (81)$$

Because the synoptic scale motion is quasi-two dimensional, the most important component is the vertical one, which is perpendicular to the plane of motion. Hereafter we refer to this as the *relative vorticity* and we drop the z subscript.

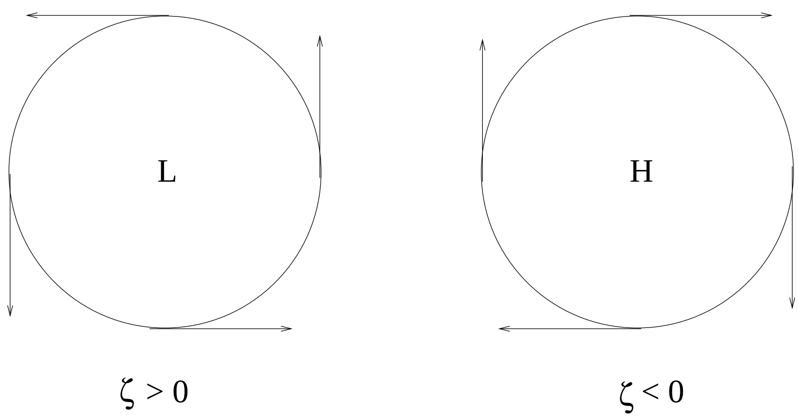

Figure 12: The relative vorticities of a low pressure (left) and high pressure (right) system.

What is vorticity? It is essentially a measure of shear or spin. Consider a low pressure system (Fig. 12). This has northward flow on the eastern side (from geostrophy) and southward flow on the western side. That implies that v is *increasing* with x. Likewise, it has westward flow to the north and eastward flow to the south, so u is decreasing with y. Thus

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial y}{\partial y} > 0$$

for the low pressure. Similarly, ζ is negative for a high pressure.

Note the vorticity equation involves a term which is:

$$v\frac{\partial}{\partial y}f$$

Actually, $f = 2\Omega sin(\theta)$, is a function of latitude. But latitude is related to y, by the relation:

$$y = a\theta$$

where a again is the earth's radius. So f is also a function of y. Later on we'll make approximations and write f explicitly in terms of y.

We can write the vorticity equation is a more compact form:

$$\frac{d_H}{dt}(\zeta + f) = -(f + \zeta)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) \tag{82}$$

We can do this because f is only a function of y, so that:

$$\frac{d_H}{dt}f = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})f = v\frac{\partial}{\partial y}f$$

Equation (82) suggests that the relative vorticity and f are basically on equal footing. So we often refer to f as the *planetary* vorticity. If we look down on the North Pole, we see the earth rotating cyclonically, like a low pressure system. So the planet appears to have a positive vorticity. This makes sense because $f = +2\Omega$ at the North Pole. Likewise, if we look instead at the South Pole, we would see the earth rotating anti-cyclonically. So we would say the planetary vorticity was negative there (where $f = -2\Omega$).

By cross-multiplying, we have eliminate the pressure on the right hand side of the momentum equations in the oceanic case. Similarly, we eliminate the geopotential on RHS in the atmospheric case. Thus equation (82) applies to *both* the atmosphere and ocean. The dynamics of the two systems are indeed very similar at synoptic scales; we can basically study both simultaneously. Another advantage with using vorticity is that it is a *scalar*, unlike the velocity which is a vector. This helps when visualizing complex flows.

Equation (82) states that the sum of the relative and planetary vorticities, which we call the *absolute vorticity*, is not conserved following a fluid parcel. Rather it changes in response to horizontal divergence. Consider a hypothetical case where the divergence is constant:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D = const. \tag{83}$$

First we let $D > 0$, which corresponds to a divergent flow (for example, below a downdraft at the surface; Fig. 13). Then the vorticity equation (82) is:

$$\frac{d}{dt}\zeta_a = -\zeta_a D \tag{84}$$

This implies:

$$\zeta_a = \zeta_a(0)\, e^{-Dt} \tag{85}$$

This implies that the absolute vorticity decays to zero, or that:

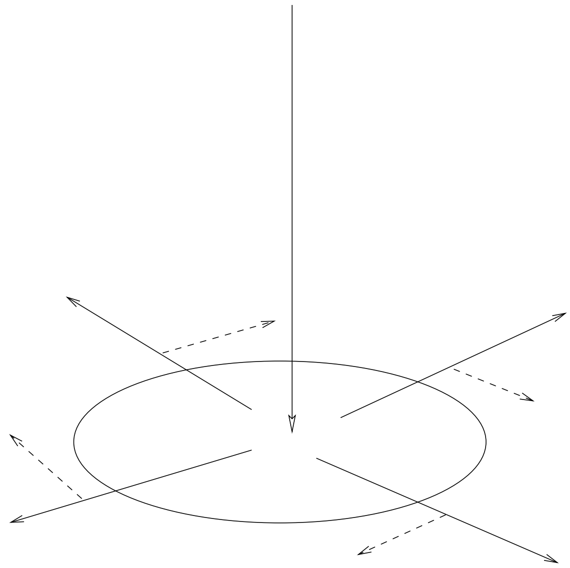

Figure 13: Divergent flow at the surface below a downdraft, producing anticyclonic circulation.

$$lim_{t\to\infty}\zeta \to -f \tag{86}$$

This is true regardless of the initial vorticity of the air parcel. Cyclonic and anticyclonic anomalies both become anticyclones with a vorticity approaching $-f$.

Physically, the outward flow associated with the divergence is diverted to the right by the Coriolis force (Fig. 13). This produces anticyclonic (clockwise) circulation.

Now consider convergent flow, with $D < 0$ (Fig. 14). In this case we have:

$$\zeta_a = \zeta_a(0)\, e^{Dt} \tag{87}$$

So the vorticity increases without bound. But it would appear that we could get either intense cyclones or anti-cyclones, depending on whether $\zeta(0) < 0$ or $\zeta(0) > 0$.

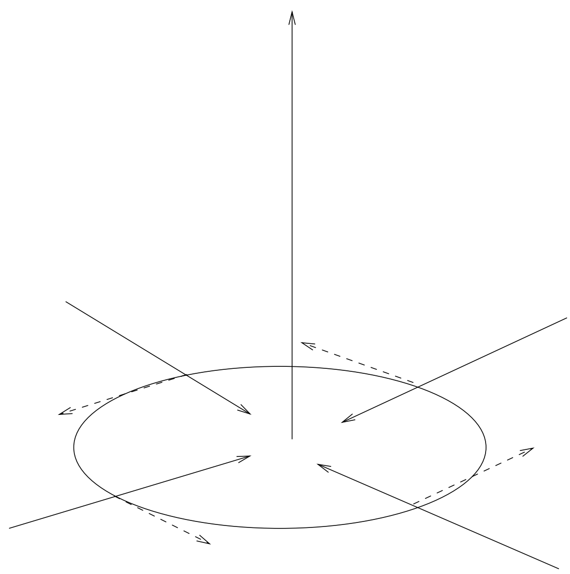

Figure 14: Convergent flow at the surface feeding an updraft.

To see which is more likely, we can scale the absolute vorticity:

$$\zeta + f$$

$$\frac{U}{L} \quad f$$

$$\frac{U}{fL} \quad 1$$

The relative vorticity scales as U/L, because it involves the velocity shear. Thus if the Rossby number is small, then:

$$\zeta + f \approx f > 0 \tag{88}$$

in the Northern Hemisphere. So the air outside the convergent region should have positive vorticity, due to the planetary rotation. Convergent flow thus favors intense *cyclones* rather than anticyclones. The inward flow in a convergence is steered to the right, generating cyclonic flow (Fig. 14). This is why intense storms (like hurricanes) are usually cyclonic.

2 Barotropic flows

As noted before, one can assume that the density in the ocean is approximately constant. This is not a reasonable assumption for the atmosphere, except perhaps in the planetary boundary layer (in the lowest kilometer over the surface). Nevertheless, the dynamics found in barotropic flows—flows without vertical shear—are largely the same as those with shear. Since the barotropic system is much simpler, it is useful to study.

2.1 Shallow Water Equations

It is convenient to focus on the equations in z-coordinates. As seen before, the momentum equations are well-approximated by their quasi-horizontal versions:

$$\frac{d_H u}{dt} - fv = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p \tag{89}$$

$$\frac{d_H v}{dt} + fu = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \tag{90}$$

These are just the Boussinesq equations. However, we will now also take the density constant in the hydrostatic relation:

$$\frac{\partial p}{\partial z} = -\rho_c g \tag{91}$$

In addition, we have the continuity equation for a constant density fluid, which is just the incompressible condition:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0 \tag{92}$$

Consider a layer of fluid, as shown in Fig. (15). The lower boundary is at $z = -H(x, y)$, where H is a function which represents bottom topography (i.e. mountains). The upper layer is at $z = \eta(x, y, t)$. This is a *free surface*, i.e. it can move, like the surface of the ocean or the tropopause in the atmosphere. If the upper surface were flat, then it would be at $z = 0$. If the bottom were flat, it would be at $z = -D$. So D is the *undisturbed* depth of the fluid.

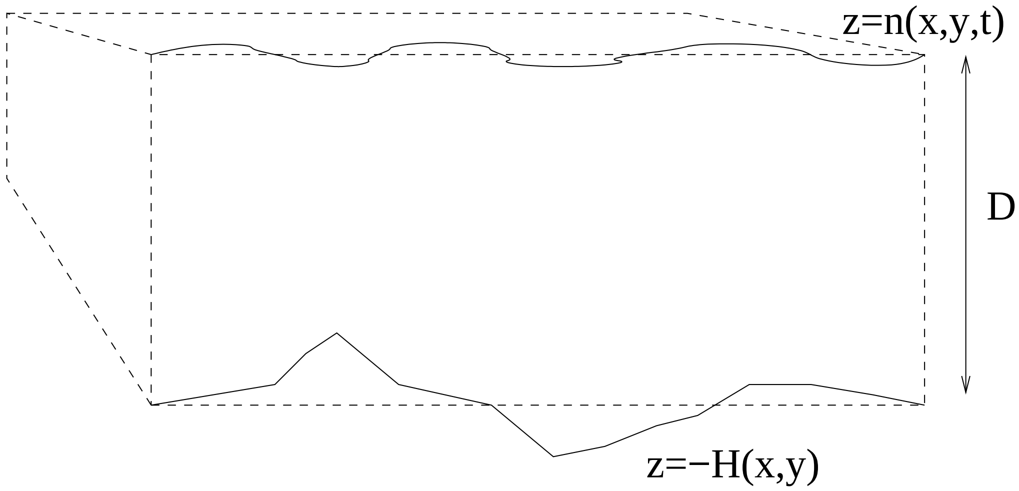

Figure 15: The fluid layer.

As we noted before, there is no vertical shear in a barotropic fluid. You can see this as follows. If we take the z-derivative of the momentum equations and substitute in from the hydrostatic relation, we get:

$$\frac{d_H}{dt}\frac{\partial u}{\partial z} - f\frac{\partial v}{\partial z} = -\frac{1}{\rho_c}\frac{\partial}{\partial x}\frac{\partial p}{\partial z} = \frac{g}{\rho_c}\frac{\partial}{\partial x}\rho_c = 0 \tag{93}$$

and similarly:

$$\frac{d_H}{dt}\frac{\partial v}{\partial z} + f\frac{\partial u}{\partial z} = 0 \tag{94}$$

These equations imply that the vertical shear can't change. If it is initially zero, it will stay zero. This is called the Taylor-Proudman theorem. Constant density flows have no vertical shear.

Now imagine we're at a depth, $-z$, in the fluid. We can integrate the hydrostatic relation up to the upper surface, thus:

$$\int_{-z}^{\eta} \frac{\partial p}{\partial z}\, dz = p(\eta) - p(-z) = -\rho_c g(\eta + z) \tag{95}$$

Here $p(\eta)$ is the pressure above the fluid layer, for example the atmospheric pressure above the ocean. We will ignore this. Taking the gradient of the integrated relation, we then get:

$$\nabla p(-z) = \rho_c g \nabla \eta \tag{96}$$

Using this in the momentum equations, we get:

$$\frac{d_H u}{dt} - fv = -g\frac{\partial}{\partial x}\eta \tag{97}$$

$$\frac{d_H v}{dt} + fu = -g\frac{\partial}{\partial y}\eta \tag{98}$$

So gradients in the upper surface height cause pressure gradients in the fluid interior, and these force the flow. These are the *shallow water momentum equations*. They can also be derived from the full momentum equations assuming a constant density and a small *aspect ratio*, the ratio between the depth of our fluid, D, and the typical horizontal length scale, L. The latter is usually 1000 times the former, so the aspect ratio for the ocean and atmosphere is very small.

The shallow water momentum equations have three unknowns, u, v and η—so the system is not closed. We need one additional equation. To get this, we turn to the incompressibility condition. Integrating that over the entire fluid depth, we get:

$$\int_{-H}^{\eta}(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v)\ dz + w(\eta) - w(-H) = 0 \tag{99}$$

Because the horizontal velocities have no vertical shear, we can move them through the integral, leaving:

$$(\eta + H)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) + w(\eta) - w(-H) = 0 \tag{100}$$

Now we need the vertical velocities at the upper and lower boundaries. We get these by noting that a fluid parcel on the boundary stays on the boundary. For a parcel on the upper surface:

$$z = \eta \tag{101}$$

We take the derivative of this, using the full Lagrangian derivative:

$$\frac{dz}{dt} = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})z = w(\eta) = \frac{d_H\eta}{dt} \tag{102}$$

Notice the last derivative is a horizontal one because $\eta = \eta(x, y, t)$. A similar relation applies at the lower boundary:

$$w(-H) = -\frac{d_H H}{dt} \tag{103}$$

The horizontal derivative occurs because $H = H(x, y)$. Of course the lower boundary isn't moving, but the term is non-zero because of the advective component, i.e.:

$$\frac{d_H H}{dt} = \vec{u}_H \cdot \nabla H \tag{104}$$

Putting these into the continuity equation, we get:

$$\frac{d_H}{dt}(\eta + H) + (\eta + H)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) = 0 \tag{105}$$

This is the Lagrangian form of the equation. In Eulerian terms, this is:

$$\frac{\partial}{\partial t}\eta + u\frac{\partial}{\partial x}(\eta + H) + v\frac{\partial}{\partial y}(\eta + H) + (\eta + H)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) =$$

$$\frac{\partial}{\partial t}\eta + \nabla \cdot (\vec{u}(\eta + H)) = 0 \tag{106}$$

These provide us with our third equation, involving u, v and η. Now we have a closed system.

2.2 Conservation of potential vorticity

If we cross-differentiate the shallow water equations (97-98), we obtain a vorticity equation, exactly like that in (82):

$$\frac{d_H}{dt}(\zeta + f) = -(f + \zeta)(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) \tag{107}$$

We can eliminate the horizontal divergence from this by using the Lagrangian form of the shallow water continuity equation (105). The result is:

$$\frac{d_H}{dt}(\zeta + f) = (\frac{\zeta + f}{\eta + H})\frac{d}{dt}(\eta + H) \tag{108}$$

or:

$$\frac{1}{H + \eta}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{(H + \eta)^2}\frac{d}{dt}(H + \eta) = 0$$

or equivalently:

$$\frac{d}{dt}(\frac{\zeta + f}{H + \eta}) = 0 \ . \tag{109}$$

Thus the absolute vorticity is conserved if we divide the sum by the total depth of the fluid. This is called the “potential vorticity” (PV), a fundamental quantity in shallow water dynamics. It is closely related to a more general quantity derived originally by Ertel (1942) and bearing his name (“the Ertel potential vorticity”). There are numerous examples of how PV conservation affects motion in rotating fluids.

2.3 Linear system

The three shallow water equations (97, 98, 106) are nonlinear because each has terms which involve products of the unknowns u, v and η. As such, they are difficult to solve analytically. However, solutions are possible if we linearize the equations. The linear solutions include phenomena like gravity and planetary waves, which are frequently observed in the atmosphere and ocean.

The central idea in linearizing the system is to assume the motion is weak. So, for example, we assume the height deviations are much smaller than the stationary (non-changing) water depth, i.e.

$$|\eta| \ll H(x, y) \; .$$

We assume too that the temporal changes in the velocity are greater than those due to advection, or

$$\frac{U}{T} \gg \frac{U^2}{L} \; ,$$

which implies that

$$U \ll \frac{L}{T} \; .$$

Making these approximations, the shallow water equations reduce to:

$$\frac{\partial}{\partial t}u - fv = -g\frac{\partial}{\partial x}\eta \tag{110}$$

$$\frac{\partial}{\partial t}v + fu = -g\frac{\partial}{\partial y}\eta \tag{111}$$

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0 \tag{112}$$

Note that H remains in the parentheses in (112) because we allow for spatial variations in the bottom depth, i.e. $H = H(x, y)$.

2.4 Constant f, flat bottom

We can simplify the system further if we assume that the rotation rate, f, is constant. This is known as the "f-plane approximation", and it applies if the area under consideration is small. To see this, we expand f in a Taylor series:

$$f = 2\Omega sin(\theta) \equiv 2\Omega[\, sin(\theta_0) + (\theta - \theta_0)cos(\theta_0) + O|(\theta - \theta_0)^2|] \,. \tag{113}$$

Here θ_0 is the central latitude of our plane of fluid. We see that the f-plane approximation applies when we can neglect all but the first term in the expansion, $2\Omega sin(\theta_0)$.

We will assume moreover that the bottom is flat. This isn't necessary, but simplifies the algebra a bit. With these two assumptions, we can reduce (110-112) to a single equation, as follows. If we cross-multiply the momentum equations, we get the linear version of the vorticity equation(82), with a constant f:

$$\frac{\partial}{\partial t}\zeta = -f\chi \tag{114}$$

where $\chi = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v$ is the horizontal divergence. We can get an equation for the divergence by taking $\frac{\partial}{\partial x}$ of (110) and adding $\frac{\partial}{\partial y}$ of (111). That yields:

$$\frac{\partial}{\partial t}\chi - f\zeta = -g\nabla^2\eta \tag{115}$$

We can then eliminate the vorticity by taking a time derivative of the divergence equation and substituting in from the vorticity equation:

$$\frac{\partial^2}{\partial t^2}\chi + f^2\chi = -g\frac{\partial}{\partial t}\nabla^2\eta \tag{116}$$

Then we eliminate χ using (112). The result is:

$$\frac{\partial}{\partial t}\,\{(\frac{\partial^2}{\partial t^2} + f^2)\,\eta - c_0^2\nabla^2\eta\} = 0\ . \tag{117}$$

Here $c_0 = \sqrt{gH}$ has the units of a velocity. We will see later this is related to the speed of gravity waves. This equation is linear and hence can be solved for η. Once we know the surface height, η, we can determine the velocities, u and v.

2.5 Gravity waves, no rotation

First let's examine the solutions where there is no rotation. Then equation (117) reduces to:

$$\frac{\partial}{\partial t}\{\frac{\partial^2}{\partial t^2}\eta - c_0^2\nabla^2\eta\} = 0\ . \tag{118}$$

This equation has three time derivatives and so admits three solutions. One is a *steady* solution in which η does not vary with time. Notice that if

$\eta = \eta(x, y)$, equation (118) is trivially satisfied. This is referred to as the "geostrophic mode", and we'll take this up later.

The other two solutions are time-varying and come from solving the portion of the equation in the braces. This is a second-order wave equation, and we can obtain a general solution if we Fourier transform the surface height:

$$\hat{\eta}(k, l, \omega) = \int \int \int \eta(x, y, t)\, e^{ikx+ily-i\omega t}\, dx\, dy\, dt$$

Here, k and l are *wavenumbers*. They are related to the wavelengths in the x and y directions:

$$\lambda_x = \frac{2\pi}{k}, \quad \lambda_y = \frac{2\pi}{l} \tag{119}$$

The constant ω is the *frequency*. This is related to the *period* of the wave, which is like a wavelength in time:

$$T = \frac{2\pi}{\omega} \tag{120}$$

Substituting the expansion into equation (118), we get:

$$-(\omega^2 - c_0^2 \kappa^2)\, \hat{\eta} = 0 \;, \tag{121}$$

where $\kappa \equiv (k^2 + l^2)^{1/2}$ is the modulus of the wavevector. So:

$$\omega = \pm c_0\, \kappa \,, \tag{122}$$

This expression thus relates the wave frequency to its wavenumber. It is referred to as the wave *dispersion relation*. We see that short wavelength (large wavenumber) waves have higher frequencies. So short waves seen from a beach will have shorter periods than long waves.

How quickly do the waves move? The *phase speed* of the wave, or the speed at which the crests move, is given by:

$$c = \frac{\omega}{\kappa} = \pm c_0 = \pm\sqrt{gH} \ . \tag{123}$$

So short and long waves propagate at the *same speed*. We say the waves are "non-dispersive" because an initial disturbance, which is generally composed of different wavelengths, will not separate into long and short waves. Any initial condition will produce waves moving with speed c_0.

As an example, consider the one-dimensional case, in which the motion is purely in the x-dimension:

$$\frac{\partial^2}{\partial t^2}\eta - c_0^2\frac{\partial^2}{\partial x^2}\eta = 0 \ . \tag{124}$$

All solutions to this equation have the form:

$$\eta = F_l(x + c_0 t) + F_r(x - c_0 t) \ .$$

Substituting this into (124) yields:

$$c_0^2(F_l'' + F_r'') - c_0^2(F_l'' + F_r'') = 0 \ ,$$

where the prime indicates differentiation with respect to the argument of the function. The function F_l represents a wave which propagates to the left, towards negative x, while F_r propagates to the right. One can see this by noting that the arguments of F_l and F_r remain constant with increasing time if x is decreasing in time and increasing, respectively.

Because there are two unknown functions, we require two sets of conditions to fully determine the solution. For instance, consider the case when

$$\eta(t = 0) = \mathcal{F}(x), \quad \frac{\partial}{\partial t}\eta(t = 0) = 0 \ .$$

This means the height has a certain shape at $t = 0$, and that the initial wave is not moving. Then we must have:

$$F_l = F_r = \frac{1}{2}\mathcal{F}(x) \ .$$

So the disturbance splits in two, with half propagating to the left and half to the right, both with speed c_0.

2.6 Gravity waves with rotation

Now consider what happens when $f \neq 0$. If we ignore the steady solution, the linearized shallow water equation (117) is:

$$(\frac{\partial^2}{\partial t^2} + f^2)\,\eta - c_0^2\,\nabla^2\eta = 0 \ . \tag{125}$$

Fourier transforming η, as before, we obtain the following dispersion relation:

$$\omega = \pm(f^2 + c_0^2\kappa^2)^{1/2} \, . \tag{126}$$

This is the dispersion relation for "Poincaré waves", which are gravity waves with rotation. For large wavenumbers (small waves), this is approximately:

$$\omega = \pm c_0\kappa \tag{127}$$

which is the same as the dispersion relation for non-rotating gravity waves. However, in the other limit, as $\kappa \rightarrow 0$, the relation is approximately:

$$\omega = \pm f \tag{128}$$

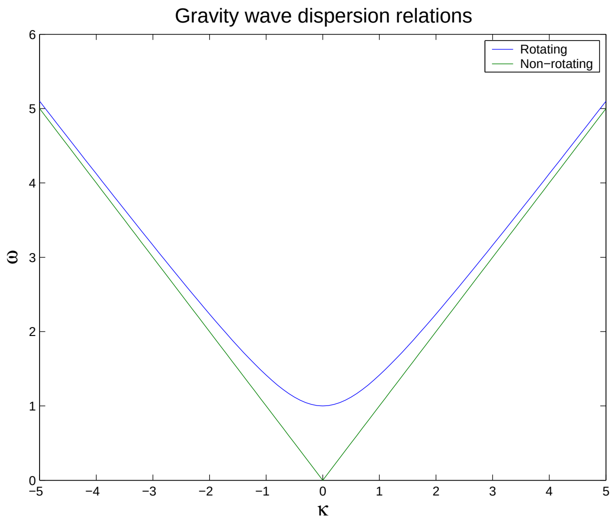

Figure 16: The gravity wave dispersion relations for the non-rotating and constant rotation cases.

So the frequency asymptotes to the inertial frequency (sec. 1.7.3) for large waves. We plot the non-rotating and rotating dispersion relations in Fig. (16).

The rotating and non-rotating frequencies are similar when:

$$\lambda = \frac{2\pi}{\kappa} \ll \frac{\sqrt{gH}}{f} \equiv L_D \ . \tag{129}$$

The scale, L_D, is the Rossby *deformation radius* (or sometimes the "external deformation radius"). At wavelengths small compared to the deformation radius, rotation is unimportant; the waves essentially do not "feel" the earth's rotation.

At scales larger than the deformation radius, the frequency asymptotes to f. In this case, gravity is unimportant because we would obtain the same result if we had ignored the pressure gradient terms in the momentum equations. Such waves are the inertial oscillations discussed in sec. (1.7.3); they have the same period equal to the Coriolis parameter, f.

We defined the phase speed as the ratio of the frequency to the wave number. Non-rotating gravity waves have the same phase speed regardless of their wavenumber and are thus non-dispersive. Rotation causes the gravity waves to disperse because the phase speed is now:

$$c = \frac{\omega}{\kappa} = \pm \frac{(f^2 + c_0^2 \kappa^2)^{1/2}}{\kappa} = c_0 \, (1 + (\frac{f}{c_0 \kappa})^2)^{1/2} \, .$$

In the short wave limit, the waves are approximately non-dispersive and traveling at speed, c_0. But the larger waves propagate faster. So an arbitrary initial disturbance will break into sinusoidal components, with the larger wavelengths moving away fastest.

As noted, rotation becomes important at scales larger than the deformation radius. But how big is the deformation radius? Using typical values and a latitude of 45 degrees, the deformation radius is:

$$\frac{\sqrt{gH}}{f} \approx \frac{\sqrt{10 * 4000}}{10^{-4}} = 2000 \; km \; ,$$

which is a rather large scale, corresponding to about 20 degrees of latitude. At such scales, our assumption of a constant Coriolis parameter is certainly not correct. So a more proper treatment of the large wave limit in such deep water would have to take the variation of f into account. Of course if the depth is less, the deformation radius will be smaller; this is the case for example in bays and shallow seas.

2.7 Geostrophic adjustment

We noted earlier that the linear shallow water equation (117) admits three solutions; two are propagating waves and one is independent of time. This third solution is trivial in the absence of rotation. But with rotation, the

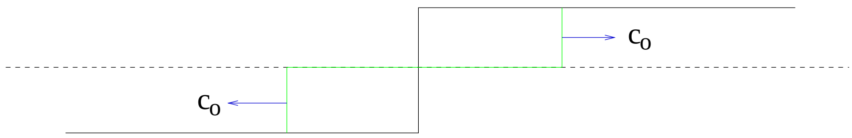

Figure 17: The evolution of a sea surface discontinuity in the absence of rotation.

solution, known as the "geostrophic mode", plays a central role in the atmosphere and ocean. To illustrate this mode, we consider an initial value problem for gravity waves, first without and then with rotation.

We consider again the one-dimensional problem, for simplicity. We assume the initial sea surface has a front (a discontinuity) at $x = 0$:

$$\eta(x, t = 0) = \frac{\eta_0}{2} sgn(x) ,$$

where the sgn function is 1 if the argument is greater than zero and -1 if not. This could represent, for example, the initial sea surface deviation generated by an earthquake. Without rotation, the solution follows from section (2.5):

$$\eta(x, t) = \frac{\eta_0}{4}(sgn(x - c_0 t) + sgn(x + c_0 t)) .$$

We see the discontinuity splits into two fronts, one propagating to the left and one to the right (Fig.17). The height in the wake of the two fronts is zero (since 1 + (-1) =0). There is no motion here.

The case with rotation is different because of the geostrophic mode. In this case, the relaxing front must conserve potential vorticity, and this will not allow a flat final state. If the PV is conserved, then from (109) we have:

$$\frac{\zeta + f}{H + \eta} = const. .$$

We assume there is no motion in the initial state. Then we have:

$$\frac{v_x + f}{H + \eta} = \frac{f}{H + (\eta_0/2) sgn(x)} \ . \tag{130}$$

where $v_x = \frac{\partial}{\partial x} v$. Notice the vorticity is just v_x because the front is one-dimensional. Cross-multiplying and re-arranging, we get:

$$[H + \frac{\eta_0}{2} sgn(x)] \, v_x - f\eta = -f \frac{\eta_0}{2} sgn(x) \ . \tag{131}$$

We will assume the initial interface displacement is much smaller than the total depth, i.e. $|\eta_0| \ll H$. Also, the final state is not time dependent. So the x-momentum equation is just the geostrophic balance:

$$fv = g\eta_x \ , \tag{132}$$

With these two changes, we get:

$$\eta_{xx} - \frac{1}{L_D^2}\eta = -\frac{\eta_0}{2L_D^2} sgn(x) \ . \tag{133}$$

where again, $L_D = \sqrt{gH}/f$ is the deformation radius.

Equation (133) is an ordinary differential equation. We solve it separately for $x > 0$ and $x < 0$. For $x > 0$, we have:

$$\eta_{xx} - \frac{1}{L_D^2}\eta = -\frac{\eta_0}{2L_D^2} \ . \tag{134}$$

The solution for this which decays as $x \to \infty$ is:

$$\eta = A \, exp(-x/L_D) + \frac{\eta_0}{2} \ . \tag{135}$$

The corresponding solution for $x < 0$ which decays as $x \to -\infty$ is:

$$\eta = B \, exp(x/L_D) - \frac{\eta_0}{2} \ . \tag{136}$$

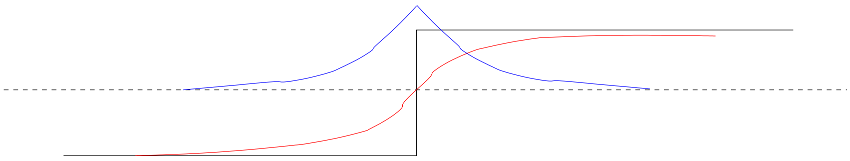

Figure 18: The sea surface height after geostrophic adjustment, beginning with a discontinuity, with rotation. The red curve shows the final interface shape and the blue curve the meridional velocity, v.

We thus have two unknowns, A and B. We find them by matching η and η_x at $x = 0$. That way the height and the velocity, v, will be continuous at $x = 0$. The result is:

$$\eta = \frac{\eta_0}{2}\,(1 - exp(-\frac{x}{L_D})) \qquad x \geq 0$$

$$= -\frac{\eta_0}{2}\,(1 - exp(\frac{x}{L_D})) \qquad x < 0 \tag{137}$$

The final state is plotted in Fig. (18). Recall that without rotation, the final state was flat, with no motion. With rotation, the initial front slumps, but does not vanish. Associated with this tilted height is a a meridional jet, intensified at $x = 0$:

$$v = \frac{g\eta_0}{2fL_D}\,exp(-\frac{|x|}{L_D})\ . \tag{138}$$

The flow is directed into the page. Thus the final state with rotation is one in motion, in which the tilted interface is supported by the Coriolis force acting on a meridional jet.

2.8 Kelvin waves

So far we have examined wave properties without worrying about boundaries. Boundaries can cause waves to reflect, changing their direction of propagation. But in the presence of rotation, boundaries can also support gravity waves which are *trapped* there; these are called Kelvin waves.

The simplest example of a Kelvin wave pertains to a infinitely long wall. Let's assume this is oriented parallel to the x-axis, and lies at $y = 0$. Because of the no-normal flow condition at the wall, we have $v = 0$ at $y = 0$. In fact, we can obtain solutions which have $v = 0$ everywhere. Because our linear wave equation (117) is expressed in terms of η, it is preferable to go back to the linearized shallow water equations (110-112) and set $v = 0$ there. This yields:

$$\frac{\partial}{\partial t}u = -g\frac{\partial}{\partial x}\eta, \tag{139}$$

$$fu = -g\frac{\partial}{\partial y}\eta \tag{140}$$

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}Hu = 0\ . \tag{141}$$

Notice that an equation for η can be derived with only the x-momentum and continuity equations:

$$\frac{\partial^2}{\partial t^2}\eta - c_0^2\frac{\partial^2}{\partial x^2}\eta = 0\ . \tag{142}$$

This is just the linearized shallow water equation (117) in one dimension *without* rotation. Indeed, rotation drops out of the x-momentum equation because $v = 0$. As a result, Kelvin waves will be non-dispersive, like non-rotating gravity waves. From section (2.5), we know the general solution to equation (142) involves two waves, one propagating towards negative x and one towards positive x:

$$\eta(x, y, t) = F_l(x + c_0t, y) + F_r(x - c_0t, y)\ .$$

We allow for structure in the y-direction and this remains to be determined.

It turns out that rotation enters in this way, in determining the structure of the wave away from the wall. From the x-momentum equation, (139), we can derive the velocity component parallel to the wall:

$$u = -\frac{g}{c_0}\left(F_l - F_r\right) .$$

Substituting this into the y-momentum equation, (140), we obtain:

$$\frac{\partial}{\partial y}F_l = \frac{f}{c_0}F_l, \quad \frac{\partial}{\partial y}F_r = -\frac{f}{c_0}F_r, \ .$$

The solutions are exponentials, with an e-folding scale of $\sqrt{gH}/f = L_D$, the deformation radius:

$$F_l \propto exp(y/L_D) \quad F_r \propto exp(-y/L_D) \ .$$

The exact solution depends on the where the wall is. We choose the solution which decays away from the wall (is trapped there) rather than one which grows indefinitely. If the wall covers the region $y > 0$, then the only solution decaying (in the negative y-direction) is F_l. If the wall covers the region $y < 0$, only F_r is decaying. Thus, in both cases, the Kelvin wave propagates at the gravity wave speed with the wall *to its right*.

Note though that the decay is also affected by the sign of f. If we are in the southern hemisphere, where $f < 0$, the Kelvin waves propagate with the wall to their left.

2.9 Rossby waves

In the previous cases, we took f to be constant. A result of this is that the "geostrophic mode" had no time dependence. It was a steady flow, as in the example in (2.7). But when we allow for variations of f, we pick up a low frequency oscillation.

So let's see what happens when f varies. The simplest way to do this is to extend the domain slightly, by retaining another term in the Taylor expansion of f:

$$f \approx 2\Omega sin(\theta_0) + 2\Omega cos(\theta_0)(\theta - \theta_0) \tag{143}$$

Using $y = a\theta$, we have:

$$f \approx f_0 + \beta(y - y_0) \tag{144}$$

where:

$$f_0 = 2\Omega sin(\theta_0)$$

and

$$\beta \equiv \frac{2\Omega}{a} cos(\theta_0)$$

Writing f like this is known as the β-plane approximation.

Now the shallow water continuity equation is given in (106). If the bottom is flat, H is constant. Moreover, let's assume that the upper layer is also flat and set $\eta = 0$. This is the *rigid lid approximation*; the effect is to filter out gravity waves. Then the continuity equation is simply:

$$(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) = 0 \tag{145}$$

With a rigid lid, the flow is *horizontally non-divergent*.

We can take advantage of this by writing the velocities in terms of a streamfunction. Specifically, we write:

$$u = -\frac{\partial}{\partial y}\psi, \quad v = \frac{\partial}{\partial x}\psi \tag{146}$$

Then:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = \frac{\partial}{\partial x}(-\frac{\partial}{\partial y}\psi) + \frac{\partial}{\partial y}(\frac{\partial}{\partial x}\psi) = 0 \tag{147}$$

With zero divergence, the vorticity equation (82) reduces to:

$$\frac{d_H}{dt}(\zeta + f) = \frac{d_H}{dt}(\zeta + \beta y) = 0 \tag{148}$$

using the β-plane approximation. Note we can ignore the f_0 term, a constant.

This equation is still nonlinear, involving the product of the velocities and the relative vorticity. So we will again linearize it. This time we will do it slightly different, by allowing for a constant, background zonal flow in addition to the weak perturbations. Specifically, we write:

$$u = U + u', \quad v = v' \tag{149}$$

where U is a constant. The mean flow, U, could be the Jet Stream in the atmosphere or the Gulf Stream in the mid-Atlantic. The relative vorticity is then:

$$\zeta = \frac{\partial}{\partial x}v' - \frac{\partial}{\partial y}u' \tag{150}$$

The U term vanishes because it's constant. Substituting these in and ignoring products of primed terms yields:

$$\frac{\partial}{\partial t}\zeta' + U\frac{\partial}{\partial x}\zeta' + v_g\frac{\partial}{\partial y}(\beta y) = 0 \tag{151}$$

We rewrite this in terms of our streamfunction, noting that $\zeta = \nabla^2\psi$. The result is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = 0 \tag{152}$$

This is the *Rossby wave equation*.

To solve it, we substitute in a Fourier wave solution:

$$\psi(x, y, t) \propto \hat{\psi}(k, l, \omega)e^{ikx+ily-i\omega t}$$

This yields:

$$(-i\omega + ikU)(-k^2 - l^2)\hat{\psi}e^{ikx+ily-i\omega t} + ik\beta\hat{\psi}e^{ikx+ily-i\omega t} = 0 \tag{153}$$

The wave part, as always, drops out leaving:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2} \tag{154}$$

This is the *Rossby wave dispersion relation*. Like the gravity wave dispersion relations, this connects the frequency with the wavenumbers. But this relation has several interesting features about this. Unlike with the gravity wave dispersion relation, where the frequency only depended on the magnitude of the wavevector, κ, the Rossby wave frequency is proportional to k, the zonal wavenumber. This means the frequency varies with the *direction* of wave propagation. Notice too that the waves cannot have $k = 0$, because then they would have zero frequency (they would not be wave-like).

The phase speed in the zonal direction is:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2} \equiv U - \frac{\beta}{\kappa^2} \tag{155}$$

where κ is the total wavenumber. Thus the phase speed depends on the wavenumber, so the waves are dispersive. The largest speeds occur when k and l are small, corresponding to long wavelengths. Thus large waves move faster than small waves.

Second, all waves propagate *westward* relative to the mean velocity, U. If $U = 0$, $c < 0$ for all (k, l). This is a distinctive feature of Rossby waves. Satellite observations of Rossby waves in the Pacific Ocean show that the waves, originating off of California and Mexico, sweep westward toward Asia.

Third, the wave speed depends on the orientation of the wave crests. The most rapid westward propagation occurs when the crests are oriented north-south, with $k \neq 0$ and $l = 0$. If the wave crests are oriented east-west, so that $k = 0$, then the wave frequency is zero and there is no wave motion at all.

The phase speed also has a meridional component, and this can be either towards the north or south:

$$c_y = \frac{\omega}{l} = \frac{Uk}{l} - \frac{\beta k}{l(k^2 + l^2)} \tag{156}$$

The sign of c_y thus depends on the signs of k and l. So Rossby waves can propagate northwest, southwest or west—but not east.

With a mean flow, the waves can be swept eastward, producing the appearance of eastward propagation. The short waves are more susceptible to eastward propagation. In particular, if

$$\kappa > \kappa_s \equiv (\frac{\beta}{U})^{1/2}$$

the wave moves eastward. Longer waves move westward, opposite to the mean flow. If $\kappa = \kappa_s$, the wave is *stationary* and the crests don't move at all—the wave is propagating west at exactly the same speed that the background flow is going east. Stationary waves can only occur if the mean flow is eastward, because the waves propagate westward.

Example: At what background wind velocity is an isotropic wave with a wavelength of 1000 km stationary? What about a wavelength of 5000 km? Assume we are at 45 degrees N.

An isotropic wave has the same scales in the x and y directions. So the wave has wavenumbers:

$$k = l = \frac{2\pi}{10^6} m^{-1} = 6.28 \times 10^{-6}\, m^{-1}$$

and:

$$\kappa^2 = k^2 + l^2 = 2k^2 = 7.90 \times 10^{-11}\, m^{-2}$$

At 45 N, we have:

$$\beta = \frac{1}{6.3 \times 10^6} \frac{4\pi}{86400} cos(45) = 1.63 \times 10^{-11}\, m^{-1} sec^{-1}$$

So:

$$\frac{\beta}{\kappa^2} = .21\, m/sec$$

So if $U = 0.21$ m/sec, the wave is stationary.

For $\lambda = 5000$ km, we find:

$$U_s = \frac{\beta \lambda^2}{2(4\pi^2)} = \frac{1.63 \times 10^{-11}(5 \times 10^6)^2}{8\pi^2} = 5.2 m/sec \tag{157}$$

What does a Rossby wave look like? Recall that ψ is proportional to the geopotential, or the pressure in the ocean. So a sinusoidal wave is a sequence of high and low pressure anomalies. An example is shown in Fig. (19). The wave is proportional to $cos(x)sin(y)$ and appears to be a grid of high and low pressure regions.

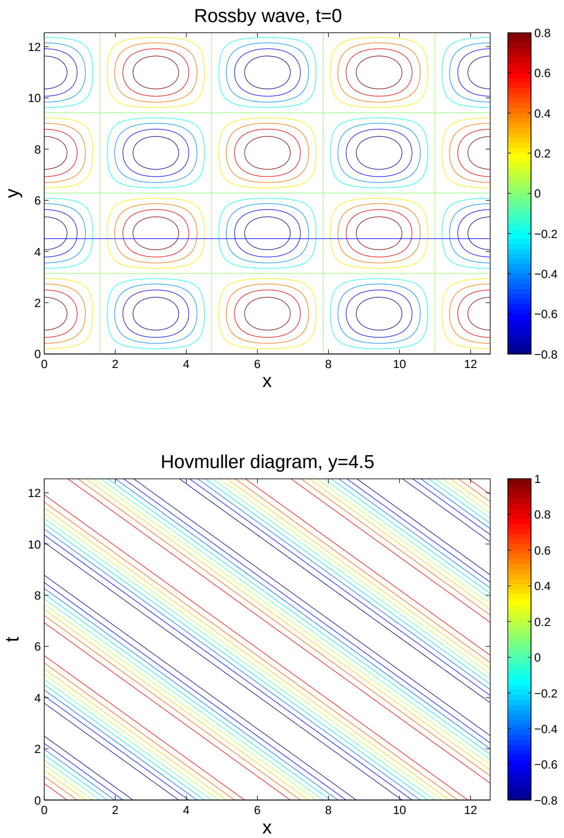

Figure 19: A Rossby wave, with $\psi = cos(x)sin(y)$. The red corresponds to high pressure regions and the blue to low. The lower panel shows a “Hovmuller” diagram of the phases at $y = 4.5$ as a function of time.

The whole wave in this case is propagating westward. Thus if we take a cut at a certain latitude, here $y = 4.5$, and plot $\psi(x, 4.5, t)$, we get the plot in the lower panel. This shows the crests and troughs moving westward at a constant speed (the phase speed). This is known as a “Hovmuller” diagram.

Three examples from the ocean are shown in Fig. (20). These are Hovmuller diagrams constructed from sea surface height in the Pacific, at

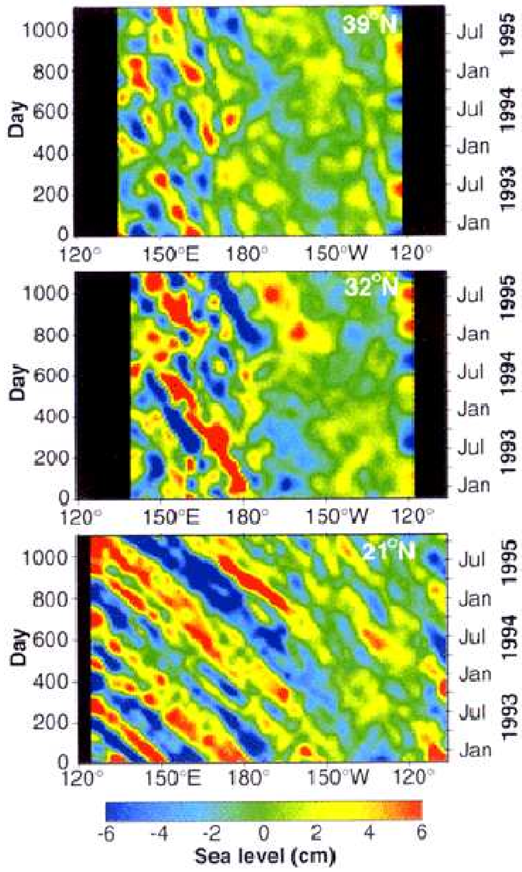

Figure 20: Three Hovmuller diagrams constructed from sea surface height in the North Pacific. From Chelton and Schlax (1996).

three different latitudes. We see westward phase propagation in all three cases. Interestingly, the phase speed (proportional to the tilt of the lines) differs in the three cases. To explain this, one needs to take stratification into account. In addition, the waves are more pronounced west of 150-180 W than in the east. The reason for this is still unknown.