GEF4610 – OBLIGATORY EXERCISE 1
All symbols are defined in the GEF4610 Lecture Notes

Problem 1
Conservation of potential vorticity

a) State the assumptions leading to the equations

$$\begin{aligned} u_t + uu_x + vu_y &= fv - g\eta_x - P_{Sx}/\rho_0, \\ v_t + uv_x + vv_y &= -fu - g\eta_y - P_{Sy}/\rho_0, \\ \eta_t + \left((H+\eta)u\right)_x + \left((H+\eta)v\right)_y &= 0, \end{aligned} \tag{1}$$

and derive the conservation equation for the potential vorticity:

$$\frac{DQ}{dt} \equiv \frac{D}{dt}\left[\frac{f+\zeta}{H+\eta}\right] = 0. \tag{2}$$

This means that Q is constant for a given column following the fluid motion. We will use these equations on an *f*-plane (*f* is constant), and assume steady motion ($\partial/\partial t = 0$).

b) Take that the motion is independent of the *y*-coordinate. Assume that $H = H_0, \zeta = 0, \eta = 0$ when $x \to -\infty$. Show from (2) that in this case

$$\frac{f+\zeta}{H+\eta} = \frac{f}{H_0}. \tag{3}$$

c) When $x \to -\infty$, the basic state is $u = u_0, v = 0$, $H = H_0, \eta = 0$, where u_0, H_0 are constants. As the fluid flows along the *x* axis, it encounters a symmetric sub-sea ridge $q(x)$ of width $2L$, centered at $x = 0$; see Figure 1. The ridge is infinitely long in the *y*-direction.

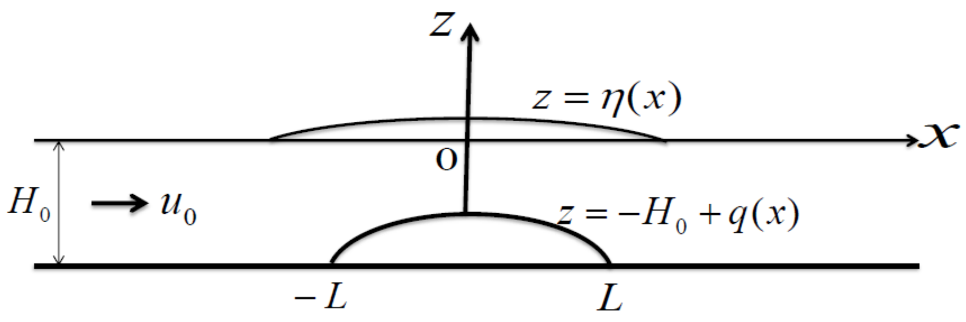

Figure 1. Model sketch.

In this problem the variables are independent of the *y*-coordinate. Show from (3) that

$$v_x = \frac{f}{H_0}(\eta - q)\,. \tag{4}$$

d) The presence of the ridge will alter the direction of the flow. Show from (4) that downstream of the ridge ($x > L$) we have

$$v(x) = \frac{f}{H_0}\int_{-\infty}^{x}\eta(x)dx - \frac{fA}{H_0}\,, \tag{5}$$

where $A = \int_{-L}^{L} q(x)dx$ is the cross-sectional area of the ridge.

e) Take that the height of the ridge is small. Then the change in velocity due to the ridge is small compared to u_0. Define the *x*-velocity component as $u = u_0 + \hat{u}$, where $|\hat{u}|/u_0 << 1$. Show from (1) that $P_{Sy} = -\rho_0 f u_0$ (what balance is referred to here?), and that

$$\hat{u} = \frac{u_0}{H_0}(q - \eta)\,. \tag{6}$$

f) If the free surface was prevented from moving vertically, i.e. $\eta \equiv 0$ everywhere (often referred to as the rigid lid approximation), calculate the velocity components after a fluid column has passed the ridge, that is when $x > L$.

g) Calculate the deflection angle between the flow direction before the ridge and after the ridge in the rigid lid case.

h) In the general case of $\eta \neq 0$, show that the condition for a flow only in the *x*-direction ($v = 0$) far downstream of the ridge is

$$\int_{-\infty}^{\infty}\eta(x)dx = \int_{-L}^{L} q(x)dx = A\,. \tag{7}$$

i) Take for the maximum values that $\eta(x=0) < q(x=0)$. What do you infer from (7) about the total fluid depth $H + \eta$ for $x < -L$ and $x > L$? (is it smaller or larger than H_0?)

j) Make a sketch in this case, and depict the same fluid column for (1): $x < -L$ (but near the ridge), (2): at the top of the ridge $x = 0$, and (3): just after the ridge for $x > L$. Indicate the sign of the relative vorticity at each position.

k) Use this information to sketch the trajectory (the path) of one single particle (column) as it moves from minus infinity across the ridge towards plus infinity.