UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	INF1100 — Introduction to programming with scientific applications
Day of examination:	Tuesday, December 15, 2009
Examination hours:	14.30 – 17.30.

This examination set consists of 11 pages.

Appendices:	None.
Permitted aids:	None.

Make sure that your copy of the examination set is complete before you start solving the problems.

- Read through the complete exercise set before you start solving the individual exercises. If you miss information in an exercise, you can provide your own reasonable assumptions as long as you explain them in detail.
- Most of the exercises result in short code where there is little need for comments, unless you do something complicated or non-standard. In that case, comments should convey the idea behind the program constructions such that it becomes easy to evaluate the solution.
- Many exercises ask you to "write a function". A main program calling the function is then not required, unless it is explicitly stated. You may, in these types of exercises, also assume that necessary modules are already imported outside the function. On the other hand, if you are asked to write a complete program, explicit import of modules must be a part of the solution.
- The maximum possible score on the exam is 100 points. There are 10 exercises, and the number of points for each exercise is given in the heading.

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Exercise 1 (5 points)

Write a Python function `h(y)` for evaluating the mathematical function

$$\frac{2}{\sqrt{\pi}}e^{-y^2}\,.$$

Also write a main program where you call the Python function.
Solution:

```
from math import sqrt, pi, exp

def h(y):
    return 2/sqrt(pi)*exp(-y**2)

print h(1.0)
```

Exercise 2 (10 points)

Write a Python function for solving the following system of two difference equations:

$$\begin{aligned} v_i &= v_{i-1} + dw_{i-1}, \\ w_i &= w_{i-1} + d(A\sin(c(i-1)d) - p|w_{i-1}|w_{i-1} - q\sin(v_{i-1})), \end{aligned}$$

for $i = 1, \ldots, N$. The initial conditions read $v_0 = s$ and $w_0 = 0$. The parameters d, A, c, p, and q in the equations are prescribed constants. The Python function should return the sequences $v_0, v_1, \ldots, v_N$ and $w_0, w_1, \ldots, w_N$.
Solution:

```
def difference_equation(N, s, d, A, c, p, q):
    v = zeros(N+1)
    w = zeros(N+1)
    v[0] = s
    w[0] = 0
    for i in range(1, N+1):
        v[i] = v[i-1] + d*w[i-1]
        w[i] = w[i-1] + d*(A*sin(c*(i-1)*d) - \
                           p*abs(w[i-1])*w[i-1] - \
                           q*sin(v[i-1]))
    return v, w
```

(Continued on page 3.)

Exercise 3 (10 points)

The function in Exercise 2 stores all the values $v_0, v_1, \ldots, v_N$ and $w_0, w_1, \ldots, w_N$. If the aim is to compute just v_N and w_N, only four values of the sequences are strictly necessary to store during the calculations. Make a new version of the function where you minimize the storage. Return the final values v_N and w_N.
Solution:

```
def difference_equation_eff(N, s, d, A, c, p, q):
    v_prev = s
    w_prev = 0
    for i in range(1, N+1):
        v = v_prev + d*w_prev
        w = w_prev + d*(A*sin(c*(i-1)*d) - \
                        p*abs(w_prev)*w_prev - \
                        q*sin(v_prev))
        # update for next step:
        v_prev = v
        w_prev = w
    return v, w
```

Exercise 4 (10 points)

An integral

$$\int_a^b g(t)dt$$

can be approximated by the formula

$$\frac{b-a}{n+1}\sum_{i=0}^{n} g(t_i), \tag{1}$$

which arises from the Monte Carlo integration method. In this method, t_i are random variables uniformly distributed in the interval $[a, b]$. Write a Python function `MC(g, a, b, n=10000)` for computing an integral by the formula (1) (where the arguments `g`, `a`, `b`, and `n` correspond to the quantities $g(t)$, a, b, and n in the mathematical formula). Call the function to compute

$$\frac{2}{\sqrt{\pi}}\int_0^1 e^{-x^2}dx\,.$$

Solution:

```
import random
def MC(g, a, b, n=10000):
```

(Continued on page 4.)

```
    s = 0
    for i in range(0, n+1, 1):
        t = random.uniform(a, b)
        s += g(t)
    return (b-a)/float(n+1)*s

# The function to be integrated is already defined in
# exercise 1, can reuse that function h(y)

print 'Integral with MC method:', MC(h, 0, 1)
```

Exercise 5 (10 points)

Vectorize the `MC` function from the previous exercise. That is, make sure that there are no explicit Python loops in the code. Assume that the `g(t)` function can accept an array `t` as argument and (in that case) return an array. (Hint: use `numpy.random.uniform(a, b, n)` and `numpy.sum`.)
Solution:

```
def MC_vec(g, a, b, n=10000):
    from numpy import random, sum
    x = random.uniform(a, b, n+1)
    g_values = g(x)
    s = sum(g_values)
    return (b-a)/float(n+1)*s

# test with h(y) function from exercise 1:
print 'Integral with vectorized MC method:', MC_vec(h, 0, 1)
```

Exercise 6 (10 points)

Modify the function `MC` from Exercise 4 such that it also writes a file with information on how the approximation evolves as we increase the number of function evaluations. To be specific, define

$$I_k = \frac{b-a}{k+1}\sum_{i=0}^{k} g(x_i) \qquad (2)$$

as the approximation using $k+1$ function evaluations, and write to file the quantities $I(0)$, $I(1)$, ..., $I(n)$. (This can easily be done inside a loop in the `MC` function.) The resulting file, called `approx.dat`, looks as follows (only the first nine lines are shown here):

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```
k:        0,   approximation=0.509454
k:        1,   approximation=0.806124
k:        2,   approximation=0.905143
k:        3,   approximation=0.915171
k:        4,   approximation=0.837735
k:        5,   approximation=0.867419
k:        6,   approximation=0.834705
k:        7,   approximation=0.849747
k:        8,   approximation=0.822398
```

Here, `approximation` corresponds to the value of I_k.
Solution:

```
def MC_log(g, a, b, n=10000):
    output = open('approx.dat', 'w')
    s = 0
    for i in range(0, n+1, 1):
        x = random.uniform(a, b)
        s += g(x)
        k = i
        I = (b-a)/float(k+1)*s
        output.write('k:%8d,   approximation=%g\n' % (k, I))
    output.close()
    return I

value = MC_log(h, 0, 1, n=1000)  # h(x) given in previous exer.
```

Exercise 7 (10 points)

Consider the following class and an associated main program:

```
class Diffme:
    def __init__(self, g, dx=1E-7):
        self.g, self.dx = g, dx

    def __call__(self, x):
        g, dx = self.g, self.dx
        return (g(x+dx) - g(x-dx))/(2.*dx)

def h(t):
    return 3*t + 2

dhdt = Diffme(h)
print dhdt(1)
```

Explain the program flow. (You do not need to calculate a numerical value for `dhdt(1)`.)

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Solution:

```
First an instance of class Diffme is made, and the h function
is stored as attribute self.g.

The dhdt(1) call calls the __call__ method in class Diffme.
The attributes self.g and self.dx equal, from the initialization
of the instance, the h function and 1E-7 (the default value),
respectively. The returned expression is therefore

(h(1+self.dx) - h(1-self.dx))/(2*self.dx) with self.dx=1E-7
```

Exercise 8 (10 points)

A cylindrical tank of radius R is filled with water to a height h_0. By opening a valve of radius r at the bottom of the tank, water flows out, and the height of water at time t, denoted by $h(t)$, decreases with time. The function $h(t)$ is governed by the differential equation

$$\frac{dh}{dt} = -\left(\frac{R}{r}\right)^{-2}\left(1+\left(\frac{r}{R}\right)^{4}\right)^{-1/2}\sqrt{2gh}\,. \tag{3}$$

Write a program for computing and plotting $h(t)$, using the class `RungeKutta4` from the `ODESolver` hierarchy of methods for ordinary differential equations (see code below). Let $r = 1$ cm, $R = 30$ cm, $g = 9.81$ m/s^2, $h_0 = 0.5$ m in the program example. Use a time step of $\Delta t = 10$ s and simulate for six minutes.

A (slightly simplified) version of class `ODESolver` and two subclasses are listed here for reference:

```
class ODESolver:
    """
    Superclass for numerical methods solving ODEs

      du/dt = f(u, t)

    Attributes:
    t: array of time values
    u: array of solution values (at time points t)
    k: step number of the most recently computed solution
    f: callable object implementing f(u, t)
```

(Continued on page 7.)

```
    dt: time step (assumed constant)
    """
    def __init__(self, f, dt):
        self.f = lambda u, t: numpy.asarray(f(u, t), float)
        self.dt = dt

    def set_initial_condition(self, u0, t0=0):
        self.u = []    # u[k] is solution at time t[k]
        self.t = []    # time levels in the solution process

        self.u.append(numpy.asarray(u0, float))
        self.t.append(float(t0))
        self.k = 0  # time level counter

    def solve(self, T):
        """
        Advance solution from t = t0 to t = T, in steps of dt.
        """
        self.k = 0
        t = 0
        while t < T:
            unew = self.advance()

            self.u.append(unew)
            t = self.t[-1] + self.dt
            self.t.append(t)
            self.k += 1
        return numpy.array(self.u), numpy.array(self.t)


class ForwardEuler(ODESolver):
    def advance(self):
        u, dt, f, k, t = \
           self.u, self.dt, self.f, self.k, self.t[-1]

        unew = u[k] + dt*f(u[k], t)
        return unew

class RungeKutta4(ODESolver):
    def advance(self):
        u, dt, f, k, t = \
           self.u, self.dt, self.f, self.k, self.t[-1]
        dt2 = dt/2.0
        K1 = dt*f(u[k], t)
        K2 = dt*f(u[k] + 0.5*K1, t + dt2)
        K3 = dt*f(u[k] + 0.5*K2, t + dt2)
        K4 = dt*f(u[k] + K3, t + dt)
```

(Continued on page 8.)

```
        unew = u[k] + (1/6.0)*(K1 + 2*K2 + 2*K3 + K4)
        return unew
```

Solution:

```
def f(h, t):
    return -(R/r)**(-2)*\
        (1 + (r/R)**4)**(-0.5)*sqrt(2*g*h)


from ODESolver import RungeKutta4
from scitools.std import sqrt, plot
r = 0.01
R = 0.3
h0 = 0.5
g = 9.81
dt = 10
# 6 minutes = 360 seconds
T = 360
method = RungeKutta4(f, dt)
method.set_initial_condition(h0)
h, t = method.solve(T)
plot(t, h, title='height of water in a tank')

# Better implementation:
# class for the right-hand side

class Tank:
    def __init__(self, r, R, h0):
        self.r, self.R, self.h0 = \
                float(r), float(R), h0

    def __call__(self, h, t):
        r, R = self.r, self.R
        g = 9.81
        return -(r/R)**2*\
        (1 + (r/R)**4)**(-0.5)*sqrt(2*g*h)

tank = Tank(r=0.01, R=0.3, h0=1)
dt = 10
T = 360
method = ForwardEuler(tank, dt)
method.set_initial_condition(tank.h0)
h, t = method.solve(T)
```

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Exercise 9 (15 points)

The task in this exercise is to compute the solution $v(t)$ of the following second-order differential equation:

$$v'' + p|v'|v' + q\sin(v) = A\sin(ct), \quad v(0) = s,\ v'(0) = 0,$$

where $p \geq 0$, $q > 0$, $A \geq 0$, $c > 0$, and $s \in [0, \pi]$ are given constants. First we rewrite the equation as a system of two first-order equations

$$\begin{aligned}\frac{d}{dt}u^0 &= u^1,\\ \frac{d}{dt}u^1 &= A\sin(ct) - p|u^1|u^1 - q\sin(u^0)\,.\end{aligned}$$

The initial conditions for this system are $u^0(0) = s$ and $u^1(0) = 0$.

To solve the above first-order system, you shall apply a subclass, `ForwardEuler` or `RungeKutta4`, in the `ODESolver` hierarchy, listed in the previous exercise. These subclasses demand a right-hand side function `f(u, t)` defining the system of differential equations. Write a class for the relevant `f(u, t)` in this exercise. The class must have a `__call__` method and store p, q, A, c, and s as attributes. The equations are to be solved for $t \in [0, T]$. Show how to plot $v(t)$. You may set the following values of the parameters involved: $s = \pi/2$, $p = 0.1$, $q = 1$, $A = 1$, $c = 2$, time step $\Delta t = 2\pi/30$, and $T = 30\pi$. Figure 1 shows the corresponding solution $v(t)$ for these choices of parameters.

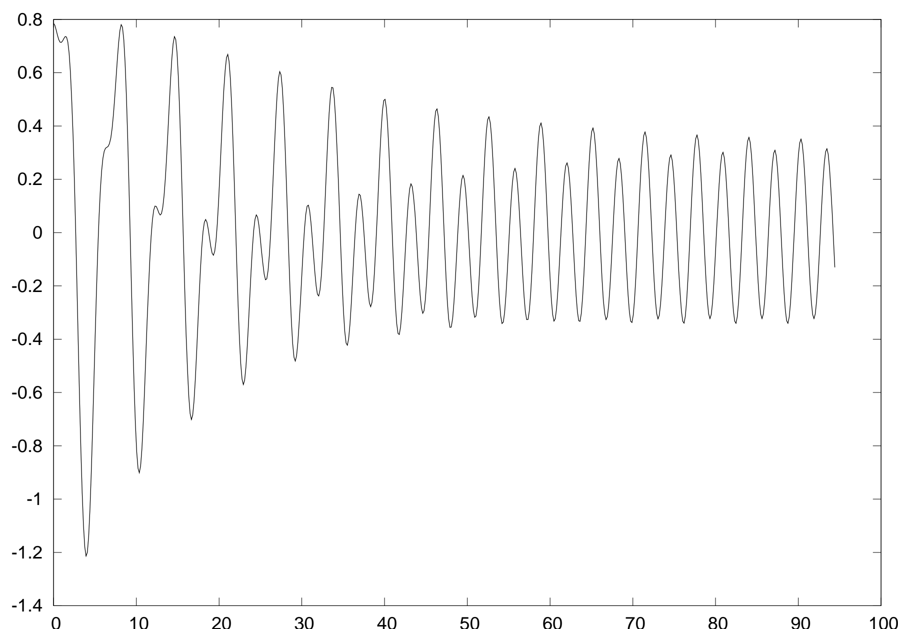

Figure 1: Plot of the solution of a 2nd-order differential equation.

Solution:

(Continued on page 10.)

```
class RHS:
    def __init__(self, p, q, A, c, s):
        self.p, self.q, self.A, self.c, self.s = \
                p, q, A, c, s

    def __call__(self, u, t):
        A, c, p, q = self.A, self.c, self.p, self.q
        return [u[1], A*sin(c*t) \
               -p*abs(u[1])*u[1] - q*sin(u[0])]

from ODESolver import RungeKutta4
from scitools.std import plot, sin, pi
f = RHS(p=0.1, q=1, A=1, c=2, s=pi/2)
dt = 2*pi/30
T = 30*pi
method = RungeKutta4(f, dt)
method.set_initial_condition([f.s, 0])
u, t = method.solve(T)
v = u[:,0]
plot(t, v, hardcopy='tmp.eps')
```

Exercise 10 (10 points)

One numerical method for solving an ordinary differential equation

$$u'(t) = f(u(t), t), \quad u(0) = U_0,$$

is the *midpoint* method:

$$u_{k+1} = u_{k-1} + 2\Delta t f(u_k, t_k), \tag{4}$$

where k is a time level, Δt the time step, and u_k is the approximation to u at time level k, i.e., when $t = t_k$. Equation (4) applies for $k = 1, 2, 3, \ldots$, while for $k = 0$ we use a simple Forward Euler approximation:

$$u_1 = u_0 + \Delta t f(u_0, t_0)\,. \tag{5}$$

The midpoint method defined by (4) and (5) is also valid for a system of ordinary differential equations when u and f are vectors.

Implement the midpoint method in a subclass of `ODESolver` (see Exercise 8 for relevant code).

Solution:

```
from ODESolver import ODESolver

class Midpoint(ODESolver):
```

(Continued on page 11.)

```
    def advance(self):
        u, dt, f, k, t = \
           self.u, self.dt, self.f, self.k, self.t[-1]

        if k >= 1:
            unew = u[k-1] + 2*dt*f(u[k], t)
        else:  # k == 0
            unew = u[k] + dt*f(u[k], t)
        return unew

# test the method on an easy problem: u'=-u
method = Midpoint(lambda u, t: -u, 0.01)
method.set_initial_condition(1)
u, t = method.solve(T=3)
from scitools.std import figure, plot
figure()
plot(t, u, title="Midpoint method for u'=-u")  # looks fine

#-------------------------------------------------------------
# longer integration in time triggers instabilities:
method.set_initial_condition(1)
u, t = method.solve(T=10)
figure()
plot(t, u, title="Midpoint method for u'=-u")

# try Midpoint on the oscillating system from the previous exercise:
method = Midpoint(f, dt)
method.set_initial_condition([f.s, 0])
u, t = method.solve(T)
v2 = u[:,0]
figure()
# the method exhibits numerical oscillations and becomes
# unstable:
plot(t[:-80], v2[:-80], 'r-',
     t, v, 'b-',
     legend=('Midpoint', 'RK4'),
     title='Midpoint vs RK4')
method = Midpoint(f, dt/100)   # shorter time step helps...
method.set_initial_condition([f.s, 0])
u, t = method.solve(T)
v2 = u[:,0]
figure()
# the method exhibits numerical oscillations and becomes
# unstable:
plot(t, v2, 'r-', title='Midpoint method')
```

END