

IN3060/4060 – Semantic Technologies – Spring 2021

Lecture 5: Mathematical Foundations

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Mandatory exercises

- Remember: Hand-in Oblig 3 by Sunday .
- Oblig 4 published after next lecture.

Today's Plan

- 1 Repetition: SPARQL
- 2 Basic Set Algebra
- 3 Pairs and Relations
- 4 Propositional Logic

Outline

- 1 Repetition: SPARQL
- 2 Basic Set Algebra
- 3 Pairs and Relations
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SPARQL

- SPARQL Protocol And RDF Query Language
- Standard language to query graph data represented by **RDF**
 - **SPARQL 1.0**: W3C Recommendation 15 January 2008
 - **SPARQL 1.1**: W3C Recommendation 21 March 2013

SPARQL – Example

- DBpedia information about actors, movies, etc. <https://dbpedia.org/>
- Web interface for SPARQL writing: <http://dbpedia.org/sparql>

People called “Johnny Depp”

```
PREFIX foaf: <http://xmlns.com/foaf/0.1/>
SELECT ?jd WHERE {
  ?jd foaf:name "Johnny Depp"@en .
}
```

Answer:

?jd
< http://dbpedia.org/resource/Johnny_Depp >

Components of an SPARQL query

Prologue: prefix definitions **Results form specification:** (1) variable list, (2) type of query (SELECT, ASK, CONSTRUCT, DESCRIBE), (3) remove duplicates (DISTINCT, REDUCED)
Dataset specification **Query pattern:** graph pattern to be matched **Solution modifiers:** ORDER BY, LIMIT, OFFSET

```
PREFIX foaf: <http://xmlns.com/foaf/0.1/>
PREFIX dbo: <http://dbpedia.org/ontology/>
SELECT DISTINCT ?collab
FROM <http://dbpedia_dataset>
WHERE {
    ?jd foaf:name "Johnny Depp"@en .
    ?pub dbo:starring ?jd .
    ?pub dbo:starring ?other .
    ?other foaf:name ?collab .
    FILTER (STR(?collab)!="Johnny Depp"@en)
```

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- 1 Repetition: SPARQL
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Motivation

- The great thing about Semantic Technologies is. . .
- . . . Semantics!
- ~~“The study of meaning”~~
- RDF has a precisely defined semantics (=meaning)
- Mathematics is best at precise definitions
- RDF has a mathematically defined semantics



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Sets: Cantor's Definition

- From the inventor of Set Theory, Georg Cantor (1845–1918):

Unter einer „Menge“ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die „Elemente“ von M genannt werden) zu einem Ganzen.

- Translated:

A 'set' is any collection M of definite, distinguishable objects m of our intuition or intellect (called the 'elements' of M) to be conceived as a whole.

- There are some problems with this, but it's good enough for us!

Sets

- A set is a mathematical object like a number, a function, etc.
- Knowing a set is
 - knowing what is in it
 - knowing what is not
- Need to know whether elements are equal or not!
- There is no order between elements
- Nothing can be in a set several times
- Two sets A and B are equal if they contain the same elements
 - everything that is in A is also in B
 - everything that is in B is also in A

Elements, Set Equality

- Notation for finite sets:

$$\{ 'a', 1, \Delta \}$$

$$\{ \cdot \cdot \cdot \}$$

- Contains 'a', 1, and Δ , and nothing else.
- There is no order between elements

$$\{ 1, \Delta \} = \{ \Delta, 1 \}$$

- Nothing can be in a set several times

$$\{ 1, \Delta, \Delta \} = \{ 1, \Delta \}$$

- Sets with different elements are different:

$$\{ 1, 2 \} \neq \{ 2, 3 \}$$

Element of-relation

- We use \in to say that something is element of a set:

$$1 \in \{\text{'a'}, 1, \Delta\}$$

$$\text{'b'} \notin \{\text{'a'}, 1, \Delta\}$$

 \in

- $\{3, 7, 12\}$: a set of numbers
 - $3 \in \{3, 7, 12\}$, $0 \notin \{3, 7, 12\}$
- $\{\text{'a'}, \text{'b'}, \dots, \text{'z'}\}$: a set of letters
 - $\text{'y'} \in \{\text{'a'}, \text{'b'}, \dots, \text{'z'}\}$, $\text{'æ'} \notin \{\text{'a'}, \text{'b'}, \dots, \text{'z'}\}$,
- $\mathbb{N} = \{1, 2, 3, \dots\}$: the set of all natural numbers
 - $3060 \in \mathbb{N}$, $\pi \notin \mathbb{N}$.
- $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$: the set of all prime numbers
 - $257 \in \mathbb{P}$, $91 \notin \mathbb{P}$.
- The set P_{3060} of people in the zoom meeting right now
 - Jieying $\in P_{3060}$, Georg Cantor $\notin P_{3060}$.

Sets as Properties

- Sets are used a lot in mathematical notation
- Often, just as a short way of writing things
- More specifically, that something has a property
- E.g. “ n is a prime number.”
- In mathematics: $n \in \mathbb{P}$
- E.g. “Jieying is a human being.”
- In mathematics, $o \in H$, where
 - H is the set of all human beings
 - o is Jieying
- One *could* define $Prime(n)$, $Human(m)$, etc. but that is not usual
- Instead of writing “ x has property XYZ ” or “ $XYZ(x)$ ”,
 - let P be the set of all objects with property XYZ
 - write $x \in P$.

The Empty Set

- Sometimes, you need a set that has no elements.
- This is called the *empty set*
- Notation: \emptyset or $\{\}$
- $x \notin \emptyset$, whatever x is!

 \emptyset

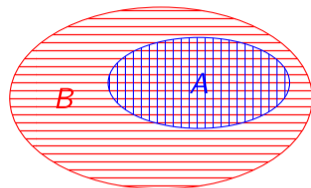
Subsets

- Let A and B be sets
- if every element of A is also in B
- then A is called a *subset* of B
- This is written

$$A \subseteq B$$

- Examples

- $\{1\} \subseteq \{1, 'a', \Delta\}$ vs. $1 \in \{1, 'a', \Delta\}$
- $\{1, 3\} \not\subseteq \{1, 2\}$
- $\mathbb{P} \subseteq \mathbb{N}$
- $\emptyset \subseteq A$ for any set A
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

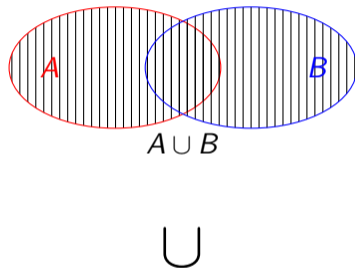


Set Union

- The *union* of A and B contains
 - all elements of A
 - all elements of B
 - also those in both A and B
 - and nothing more.
- It is written

$$A \cup B$$

- (A cup which you pour everything into)
- Examples
 - $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$
 - $\{1, 3, 5, 7, 9, \dots\} \cup \{2, 4, 6, 8, 10, \dots\} = \mathbb{N}$
 - $\emptyset \cup \{1, 2\} = \{1, 2\}$



Set Intersection

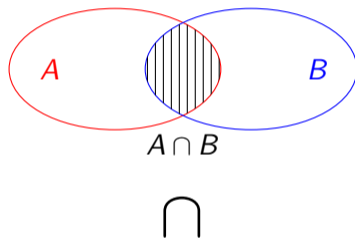
- The *intersection* of A and B contains
 - those elements of A
 - that are also in B
 - and nothing more.

- It is written

$$A \cap B$$

- Examples

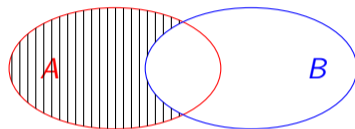
- $\{1, 2\} \cap \{2, 3\} = \{2\}$
- $\mathbb{P} \cap \{2, 4, 6, 8, 10, \dots\} = \{2\}$
- $\emptyset \cap \{1, 2\} = \emptyset$



Set Difference

- The *set difference* of A and B contains
 - those elements of A
 - that are *not* in B
 - and nothing more.
- It is written

$$A \setminus B$$



$$A \setminus B$$

- Examples

- $\{1, 2\} \setminus \{2, 3\} = \{1\}$
- $\mathbb{N} \setminus \mathbb{P} = \{1, 4, 6, 8, 9, 10, 12, \dots\}$
- $\emptyset \setminus \{1, 2\} = \emptyset$
- $\{1, 2\} \setminus \emptyset = \{1, 2\}$

Set Comprehensions

- Sometimes enumerating all elements is not good enough
- E.g. there are infinitely many, and “...” is too vague
- Special notation:

$$\{x \in A \mid x \text{ has some property}\}$$

- The set of those elements of A which have the property.
- Examples:

- $\{n \in \mathbb{N} \mid n = 2k \text{ for some } k \in \mathbb{N}\}$: the even numbers
- $\{n \in \mathbb{N} \mid n < 5\} = \{1, 2, 3, 4\}$
- $\{x \in A \mid x \notin B\} = A \setminus B$

$$\{\dots \mid \dots\}$$

Question

The *symmetric difference* $A \triangle B$ of two sets contains

- All elements that are in A or B ...
- ... but not in both.

Can you write $A \triangle B$ using \cap , \cup , \setminus ?

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$

Or:

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

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Motivation

- RDF is all about
 - Resources (objects)
 - Their properties (`rdf:type`)
 - Their relations amongst each other
- Sets are good to group objects with some properties!
- How do we talk about relations between objects?

Pairs

- A pair is an *ordered* collection of two objects
- Written

$$\langle x, y \rangle$$

$$\langle \cdot \cdot \cdot \rangle$$

- Equal if components are equal:

$$\langle a, b \rangle = \langle x, y \rangle \quad \text{if and only if} \quad a = x \quad \text{and} \quad b = y$$

- Order matters:

$$\langle 1, 'a' \rangle \neq \langle 'a', 1 \rangle$$

- An object can be twice in a pair:

$$\langle 1, 1 \rangle$$

- $\langle x, y \rangle$ is a pair, no matter if $x = y$ or not.

The Cross Product

- Let A and B be sets.
- Construct the set of all pairs $\langle a, b \rangle$ with $a \in A$ and $b \in B$.
- This is called the *cross product* of A and B , written

$$A \times B$$



- Example:
 - $A = \{1, 2, 3\}$, $B = \{\text{'a'}, \text{'b'}\}$.
 - $A \times B = \{ \langle 1, \text{'a'} \rangle, \langle 2, \text{'a'} \rangle, \langle 3, \text{'a'} \rangle, \langle 1, \text{'b'} \rangle, \langle 2, \text{'b'} \rangle, \langle 3, \text{'b'} \rangle \}$
- Why bother?
- Instead of “ $\langle a, b \rangle$ is a pair of a natural number and a person in this zoom meeting”...
- ... $\langle a, b \rangle \in \mathbb{N} \times P_{3060}$
- But most of all, there are subsets of cross products. . .

Relations

- A relation R between two sets A and B is...
- ... a set of pairs $\langle a, b \rangle \in A \times B$

$$R \subseteq A \times B$$

- We often write aRb to say that $\langle a, b \rangle \in R$
- Example:

- Let $L = \{\text{'a'}, \text{'b'}, \dots, \text{'z'}\}$
- Let \triangleright relate each number between 1 and 26 to the corresponding letter in the alphabet:

$$1 \triangleright \text{'a'} \quad 2 \triangleright \text{'b'} \quad \dots \quad 26 \triangleright \text{'z'}$$

- Then $\triangleright \subseteq \mathbb{N} \times L$:

$$\triangleright = \{\langle 1, \text{'a'} \rangle, \langle 2, \text{'b'} \rangle, \dots, \langle 26, \text{'z'} \rangle\}$$

- And we can write:

$$\langle 1, \text{'a'} \rangle \in \triangleright \quad \langle 2, \text{'b'} \rangle \in \triangleright \quad \dots \quad \langle 26, \text{'z'} \rangle \in \triangleright$$

More Relations

- A relation R on some set A is a relation between A and A :

$$R \subseteq A \times A = A^2$$

- Example: $<$
 - Consider the $<$ order on natural numbers:

$$1 < 2 \quad 1 < 3 \quad 1 < 4 \quad \dots \quad 2 < 3 \quad 2 < 4 \quad \dots$$

- $< \subseteq \mathbb{N} \times \mathbb{N}$:

$$\begin{aligned}
 < = \{ & \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 1, 4 \rangle, \dots \\
 & \langle 2, 3 \rangle, \langle 2, 4 \rangle, \dots \\
 & \langle 3, 4 \rangle, \dots \\
 & \dots \}
 \end{aligned}$$

- $< = \{ \langle x, y \rangle \in \mathbb{N}^2 \mid x \text{ is less than } y \}$

Family Relations

- Consider the set $S = \{\text{Homer, Marge, Bart, Lisa, Maggie}\}$.
- Define a relation P on S such that

$$x P y \quad \text{iff} \quad x \text{ is parent of } y$$

- For instance:

$$\text{Homer } P \text{ Bart} \quad \text{Marge } P \text{ Maggie}$$

- As a set of pairs:

$$P = \left\{ \langle \text{Homer, Bart} \rangle, \langle \text{Homer, Lisa} \rangle, \langle \text{Homer, Maggie} \rangle, \langle \text{Marge, Bart} \rangle, \langle \text{Marge, Lisa} \rangle, \langle \text{Marge, Maggie} \rangle \right\} \subseteq S^2$$

- For instance:

$$\langle \text{Homer, Bart} \rangle \in P \quad \langle \text{Marge, Maggie} \rangle \in P$$



Set operations on relations

- Since relations are just sets of pairs, we can use set operations and relations on them.
- We say that R is a subrelation P if $R \subseteq P$.
- E.g.: if F is the father-of-relation,

$$F = \{\langle \text{Homer}, \text{Bart} \rangle, \langle \text{Homer}, \text{Lisa} \rangle, \langle \text{Homer}, \text{Maggie} \rangle\}$$

then $F \subseteq P$.

- If M is the mother-of-relation,

$$M = \{\langle \text{Marge}, \text{Bart} \rangle, \langle \text{Marge}, \text{Lisa} \rangle, \langle \text{Marge}, \text{Maggie} \rangle\}$$

then $F \cup M = P$.

Special Kinds of Relations

- Certain properties of relations occur in many applications
- Therefore, they are given names
- $R \subseteq A^2$ is *reflexive*
 - $x R x$ for all $x \in A$.
 - E.g. “=”, “ \leq ” and “ \geq ” in mathematics, “has same color as”, etc.
- $R \subseteq A^2$ is *symmetric*
 - If $x R y$ then $y R x$.
 - E.g. “=” in mathematics, friendship in facebook, connected by rail, etc.
- $R \subseteq A^2$ is *transitive*
 - If $x R y$ and $y R z$, then $x R z$
 - E.g. “=”, “ \leq ”, “ $<$ ” in mathematics, “is ancestor of”, etc.



Question

Let $A = \{1, 2\}$, a set of two elements.

How many different relations on A are there?

$$A \times A = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$$

A relation on A is a subset of $A \times A$. So how many subsets are there?

$$\{\}, \quad \{\langle 1, 1 \rangle\}, \quad \{\langle 1, 2 \rangle\}, \quad \{\langle 1, 1 \rangle, \langle 1, 2 \rangle\}, \dots$$

16 relations on A . Generally: $2^{(|A|^2)}$

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Many Kinds of Logic

- In mathematical logic, many kinds of logic are considered
 - propositional logic (and, or, not)
 - description logic (a mother is a person who is female and has a child)
 - modal logic (Alice knows that Bob didn't know yesterday that. . .)
 - first-order logic (For all. . . , for some. . .)
- All of them formalizing different aspects of reasoning
- All of them defined mathematically
 - Syntax (\approx grammar. What is a formula?)
 - Semantics (What is the meaning?)
 - proof theory: what is legal reasoning?
 - model semantics: declarative using set theory.
- For semantic technologies, description logic (DL) is most interesting
 - talks about sets and relations
- Basic concepts can be explained using predicate logic

Propositional Logic: Formulas

- Formulas are defined “by induction” or “recursively”:

1 Any letter p, q, r, \dots is a formula

2 if A and B are formulas, then

- $(A \wedge B)$ is also a formula (read: “ A and B ”)
- $(A \vee B)$ is also a formula (read: “ A or B ”)
- $(A \rightarrow B)$ is also a formula (read “ A implies B ”)
- $\neg A$ is also a formula (read: “not A ”)

$$\wedge \vee \rightarrow \neg$$

- Nothing else is. Only what rules [1] and [2] say is a formula.
- Examples of formulae:

$$p \quad (p \wedge \neg q) \quad (q \wedge q) \quad (\neg p \rightarrow q) \quad ((p \vee \neg q) \wedge (\neg p \rightarrow q))$$

- Examples of non-formulas:

$$pqr \quad p\neg q \quad \wedge (p$$

Propositional Formulas, Using Sets

- The set of all formulas Φ is the least set such that

1 All letters $p, q, r, \dots \in \Phi$

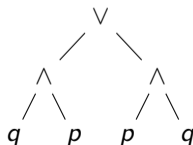
2 if $A, B \in \Phi$, then

- $(A \wedge B) \in \Phi$
- $(A \vee B) \in \Phi$
- $(A \rightarrow B) \in \Phi$
- $\neg A \in \Phi$

- Formulas are just a kind of strings until now:

- no meaning
- but every formula can be “parsed” uniquely.

$((q \wedge p) \vee (p \wedge q))$



Terminology

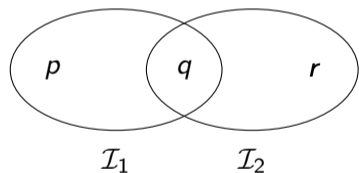
- $\neg, \wedge, \vee, \rightarrow$ are called *connectives*.
- A formula $(A \wedge B)$ is called a *conjunction*.
- A formula $(A \vee B)$ is called a *disjunction*.
- A formula $(A \rightarrow B)$ is called an *implication*.
- A formula $\neg A$ is called a *negation*.

Truth

- Logic is about things being true or false, right?
- Is $(p \wedge q)$ true?
- That depends on whether p and q are true!
- *If* p is true, and q is true, then $(p \wedge q)$ is true
- *Otherwise*, $(p \wedge q)$ is false.
- So truth of a formula depends on the truth of the letters
- We also say the “interpretation” of the letters
- In other words, in general, truth depends on the context
- Let’s formalize this context, a.k.a. interpretation, a.k.a. model

Interpretations

- Idea: put all letters that are “true” into a set!
- Define: An *interpretation* \mathcal{I} is a set of letters.
- Letter p is true in interpretation \mathcal{I} if $p \in \mathcal{I}$.
- E.g., in $\mathcal{I}_1 = \{p, q\}$, p is true, but r is false.
- But in $\mathcal{I}_2 = \{q, r\}$, p is false, but r is true.



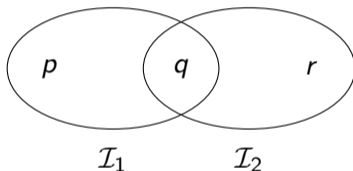
Semantic Validity \models

- To say that p is true in \mathcal{I} , write

$$\mathcal{I} \models p$$



- For instance



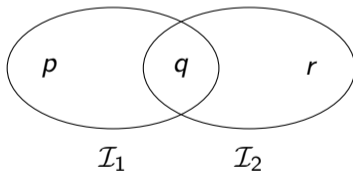
$$\mathcal{I}_1 \models p \quad \mathcal{I}_2 \not\models p$$

- In other words, for all letters p :

$$\mathcal{I} \models p \quad \text{if and only if} \quad p \in \mathcal{I}$$

Validity of Compound Formulas

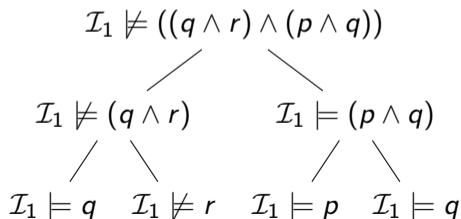
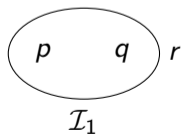
- So, is $(p \wedge q)$ true?
- That depends on whether p and q are true!
- And that depends on the interpretation.
- All right then, *given some* \mathcal{I} , is $(p \wedge q)$ true?
- Yes, if $\mathcal{I} \models p$ and $\mathcal{I} \models q$
- No, otherwise
- For instance



$$\mathcal{I}_1 \models p \wedge q \quad \mathcal{I}_2 \not\models p \wedge q$$

Validity of Compound Formulas, cont.

- That was easy, p and q are only letters...
- ...so, is $((q \wedge r) \wedge (p \wedge q))$ true in \mathcal{I} ?
- Idea: apply our rule recursively
- For any formulas A and B ,...
- ...and any interpretation \mathcal{I} ,...
- ... $\mathcal{I} \models A \wedge B$ if and only if $\mathcal{I} \models A$ and $\mathcal{I} \models B$
- For instance, if $\mathcal{I}_1 = \{p, q\}$:



Semantics for \neg , \rightarrow and \vee

- The complete definition of \models is as follows:
- For any interpretation \mathcal{I} , letter p , formulas A, B :
 - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$
 - $\mathcal{I} \models \neg A$ iff $\mathcal{I} \not\models A$
 - $\mathcal{I} \models (A \wedge B)$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
 - $\mathcal{I} \models (A \vee B)$ iff $\mathcal{I} \models A$ or $\mathcal{I} \models B$ (or both)
 - $\mathcal{I} \models (A \rightarrow B)$ iff $\mathcal{I} \models A$ implies $\mathcal{I} \models B$
- Semantics of \neg , \wedge , \vee , \rightarrow often given as *truth table*:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$
f	f	t	f	f	t
f	t	t	f	t	t
t	f	f	f	t	f
t	t	f	t	t	t

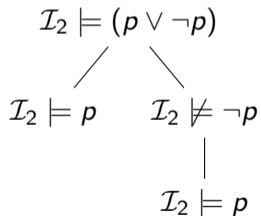
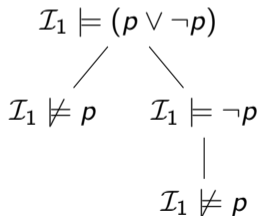
Some Formulas Are Truer Than Others

- Is $(p \vee \neg p)$ true?
- Only two interesting interpretations:

$$\mathcal{I}_1 = \emptyset$$

$$\mathcal{I}_2 = \{p\}$$

- Recursive Evaluation:



- $(p \vee \neg p)$ is true in *all* interpretations!

Tautologies

- A formula A that is true in *all* interpretations is called a *tautology*
- also *logically valid*
- also a *theorem* (of propositional logic)
- written:

$$\models A$$

- $(p \vee \neg p)$ is a tautology
- True whatever p means:
 - The sky is blue or the sky is not blue.
 - Lucas B. will win the race or Lucas B. will not win the race.
 - The slithy toves gyre or the slithy toves do not gyre.
- Possible to derive true statements mechanically. . .
- . . . without understanding their meaning!

Checking Tautologies

- Checking whether $\models A$ is the task of SAT-solving
- (co-)NP-complete in general (i.e. in practice exponential time)
- Small instances can be checked with a truth table:

$$\models (\neg p \vee (\neg q \vee (p \wedge q))) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \wedge q)$	$(\neg q \vee (p \wedge q))$	$(\neg p \vee (\neg q \vee (p \wedge q)))$
f	f	t	t	f	t	t
f	t	t	f	f	f	t
t	f	f	t	f	t	t
t	t	f	f	t	t	t

- Therefore: $(\neg p \vee (\neg q \vee (p \wedge q)))$ is a tautology!

Entailment

- Tautologies are true in all interpretations
- Some Formulas are true only under certain assumptions
- A entails B , written $A \models B$ if

$$\mathcal{I} \models B$$

for all interpretations \mathcal{I} with $\mathcal{I} \models A$

- Also: “ B is a logical consequence of A ”
- Whenever A holds, also B holds
- For instance:

$$p \wedge q \models p$$

- Independent of meaning of p and q :
 - If it rains and the sky is blue, then it rains
 - If L.B. wins the race and the world ends, then L.B. wins the race
 - If 'tis brillig and the slythy toves do gyre, then 'tis brillig

Checking Entailment

- SAT solvers can be used to check entailment:

$$A \models B \quad \text{if and only if} \quad \models (A \rightarrow B)$$

- We can check simple cases with a truth table:

$$(p \wedge \neg q) \models \neg(\neg p \vee q) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \wedge \neg q)$	$(\neg p \vee q)$	$\neg(\neg p \vee q)$
f	f	t	t	f	t	f
f	t	t	f	f	t	f
t	f	f	t	t	f	t
t	t	f	f	f	t	f

- So $(p \wedge \neg q) \models \neg(\neg p \vee q)$
- And $\neg(\neg p \vee q) \models (p \wedge \neg q)$

Equivalent formulas and redundant connectives

- In other words, $(p \wedge \neg q)$ and $\neg(\neg p \vee q)$ always have the same truth value, no matter the interpretation.
- We say that A and B are *equivalent* if A and B always have the same truth value.
- For this we often introduce another connective, \leftrightarrow .
- $\mathcal{I} \models (A \leftrightarrow B)$ iff $\mathcal{I} \models A$ if and only if $\mathcal{I} \models B$.
- To express that two formulas A, B are equivalent, we can write $\models (A \leftrightarrow B)$.
- We actually only need a subset of the connectives:
- E.g.:
 - $\models ((A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B))$.
 - $\models ((A \rightarrow B) \leftrightarrow (\neg A \vee B))$.
 - $\models ((A \leftrightarrow B) \leftrightarrow ((A \rightarrow B) \wedge (B \rightarrow A)))$.
- So we actually only need \neg and \wedge to express any formula!
- Any formula is equivalent to a formula containing only the connectives \neg and \wedge .

Recap

• Sets

- are collections of objects without order or multiplicity
- often used to gather objects which have some property
- can be combined using \cap, \cup, \setminus

• Relations

- are sets of pairs (subset of cross product $A \times B$)
- $x R y$ is the same as $\langle x, y \rangle \in R$
- can use set operations on relations, e.g. $F \subseteq P$.

• Predicate Logic

- has formulas built from letters, $\wedge, \vee, \rightarrow, \neg$ (*syntax*)
- which can be evaluated in an *interpretation* (*semantics*)
- interpretations are sets of letters
- recursive definition for semantics of $\wedge, \vee, \rightarrow, \neg$
- $\models A$ if $\mathcal{I} \models A$ for all \mathcal{I} (*tautology*)
- $A \models B$ if $\mathcal{I} \models B$ for all \mathcal{I} with $\mathcal{I} \models A$ (*entailment*)
- truth tables can be used for checking validity and entailment.