IN3060/4060 – Semantic Technologies – Spring 2021

Lecture 5: Mathematical Foundations

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12th February 2021

Department of Informatics

University of Oslo

Mandatory exercises

- Remember: Hand-in Oblig 3 by Sunday .
- Oblig 4 published after next lecture.

Today's Plan

1. Repetition: SPARQL
2. Basic Set Algebra
3. Pairs and Relations
4. Propositional Logic

Outline

SPARQL

- SPARQL Protocol And RDF Query Language
- Standard language to query graph data represented by **RDF**
  - **SPARQL 1.0:** W3C Recommendation 15 January 2008
  - **SPARQL 1.1:** W3C Recommendation 21 March 2013

SPARQL – Example

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Answer:

?jd
<http://dbpedia.org/resource/Johnny_Depp>

Components of an SPARQL query

```
PREFIX foaf:  <http://xmlns.com/foaf/0.1/>
PREFIX dbo:   <http://dbpedia.org/ontology/>
SELECT DISTINCT ?collab
FROM <http://dbpedia_dataset>
WHERE {
    ?jd foaf:name "Johnny Depp"@en .
    ?pub dbo:starring ?jd .
    ?pub dbo:starring ?other .
    ?other foaf:name ?collab .
    FILTER (STR(?collab)!="Johnny Depp"@en)
}
ORDER BY ?collab
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Prologue: prefix definitions

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Results form specification: (1) variable list, (2) type of query (SELECT, ASK, CONSTRUCT, DESCRIBE), (3) remove duplicates (DISTINCT, REDUCED)

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Dataset specification

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Query pattern: graph pattern to be matched

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Solution modifiers: ORDER BY, LIMIT, OFFSET

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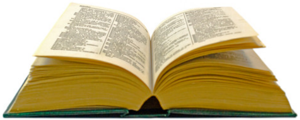

Image ©Colourbox.no

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- There are some problems with this, but it's good enough for us!

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- Sets with different elements are different:

$$\{1, 2\} \neq \{2, 3\}$$

Element of-relation

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- $x \notin \emptyset$, whatever x is!

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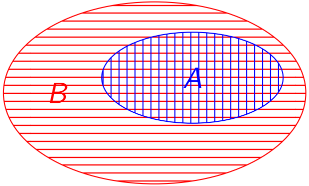

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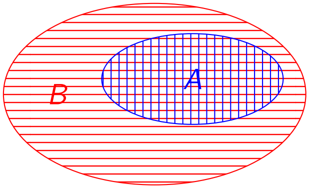

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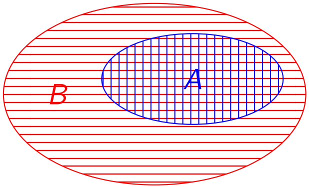

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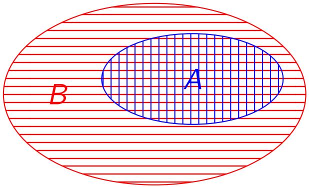

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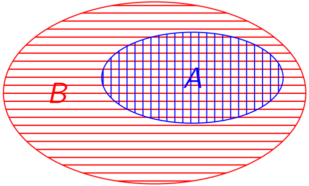

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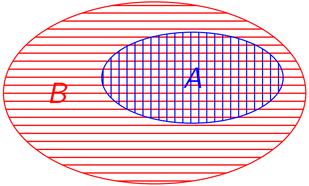

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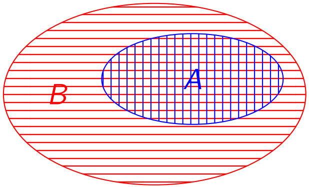

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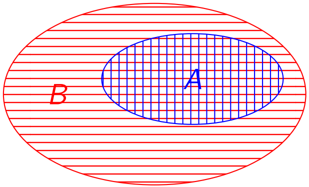

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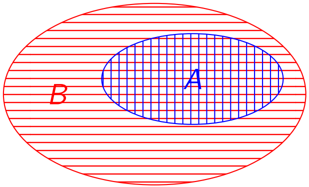

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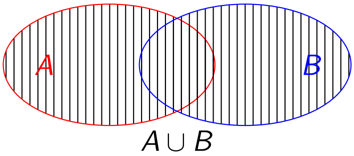
- The *union* of A and B contains



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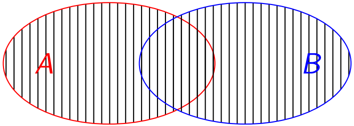

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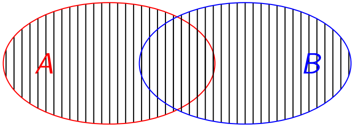

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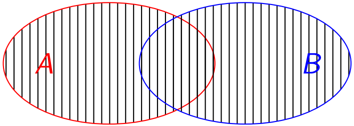

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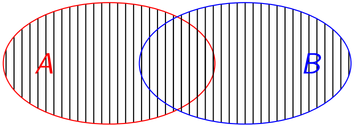

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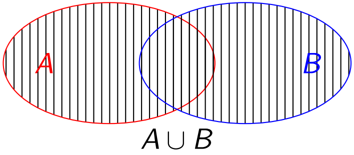

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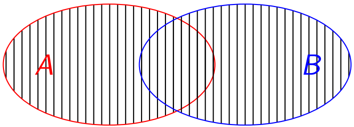

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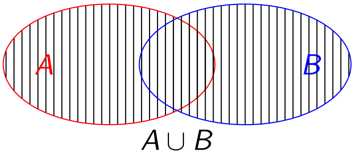

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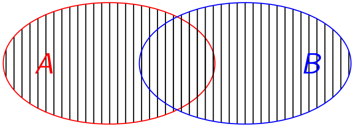

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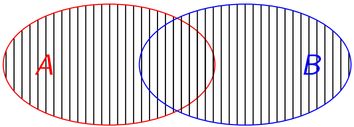

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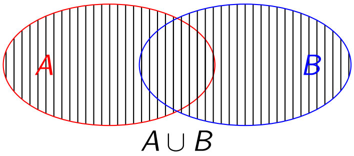

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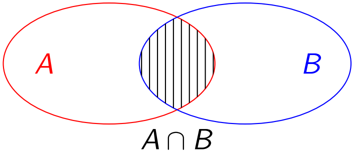

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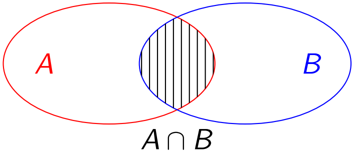

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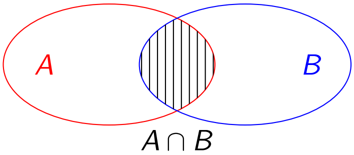

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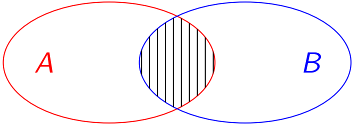

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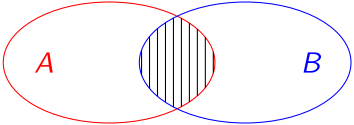

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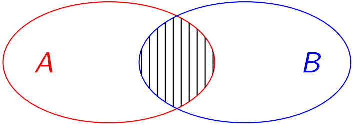

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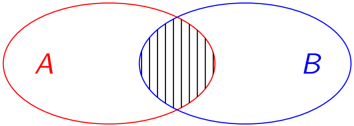

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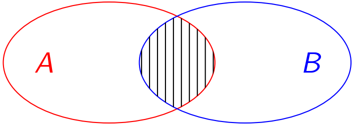

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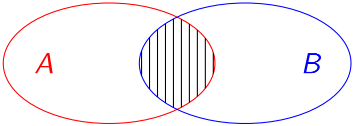

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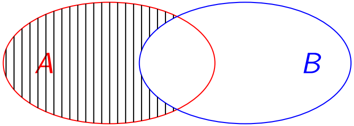

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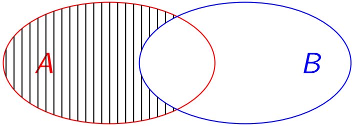

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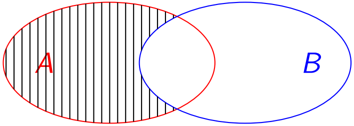

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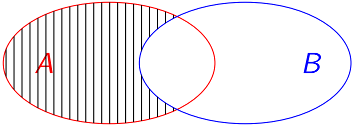

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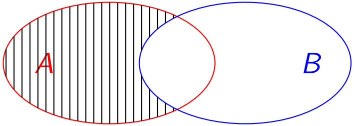

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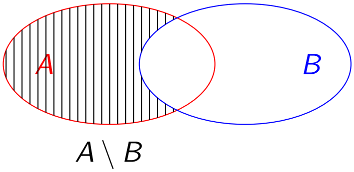

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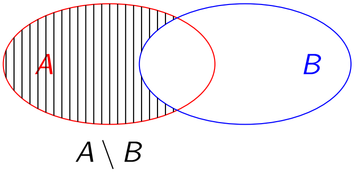

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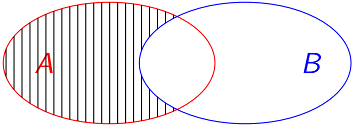

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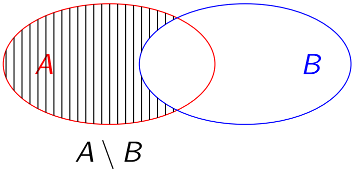

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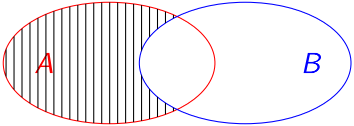

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Or:

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- Sets are good to group objects with some properties!
- How do we talk about relations between objects?

Pairs

- A pair is an *ordered* collection of two objects

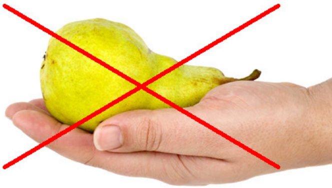

Image ©Colourbox.no

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  - And we can write:

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16 relations on A. Generally: $2^{(|A|^2)}$

Outline

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- For semantic technologies, description logic (DL) is most interesting

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    - proof theory: what is legal reasoning?
    - model semantics: declarative using set theory.
- For semantic technologies, description logic (DL) is most interesting
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Many Kinds of Logic

- In mathematical logic, many kinds of logic are considered
  - propositional logic (and, or, not)
  - description logic (a mother is a person who is female and has a child)
  - modal logic (Alice knows that Bob didn't know yesterday that...)
  - first-order logic (For all..., for some...)
- All of them formalizing different aspects of reasoning
- All of them defined mathematically
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  - talks about sets and relations
- Basic concepts can be explained using predicate logic

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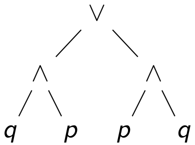
    - but every formula can be “parsed” uniquely.

$$((q \wedge p) \vee (p \wedge q))$$



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- Let's formalize this context, a.k.a. interpretation, a.k.a. model

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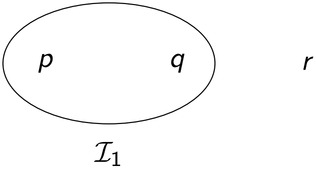

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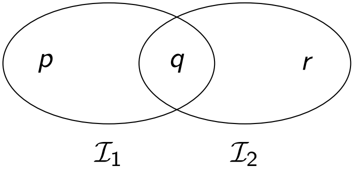

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- To say that p is true in $\mathcal{I}$, write

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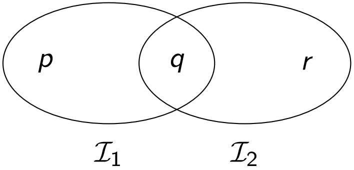

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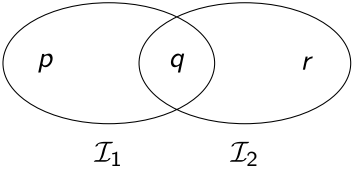

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- In other words, for all letters p:

$$\mathcal{I} \models p \qquad \text{if and only if} \qquad p \in \mathcal{I}$$

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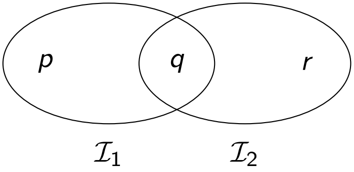

$\mathcal{I}_1 \models p \wedge q$ $\quad$ $\mathcal{I}_2 \not\models p \wedge q$

Validity of Compound Formulas, cont.

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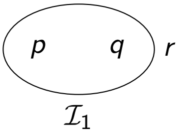
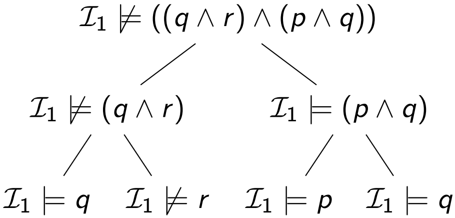
- For instance, if $\mathcal{I}_1 = \{p, q\}$:





Semantics for $\neg$, $\rightarrow$ and $\vee$

- The complete definition of $\models$ is as follows:
- For any interpretation $\mathcal{I}$, letter p, formulas A, B:
  - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$
  - $\mathcal{I} \models \neg A$ iff $\mathcal{I} \not\models A$
  - $\mathcal{I} \models (A \wedge B)$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
  - $\mathcal{I} \models (A \vee B)$ iff $\mathcal{I} \models A$ or $\mathcal{I} \models B$ (or both)
  - $\mathcal{I} \models (A \rightarrow B)$ iff $\mathcal{I} \models A$ implies $\mathcal{I} \models B$
- Semantics of $\neg$, $\wedge$, $\vee$, $\rightarrow$ often given as *truth table*:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$
f	f	t	f	f	t
f	t	t	f	t	t
t	f	f	f	t	f
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$$\mathcal{I}_2 \models (p \vee \neg p)$$

- $\mathcal{I}_2 \models p$
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- $(p \vee \neg p)$ is true in *all* interpretations!

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- Checking whether $\models A$ is the task of SAT-solving
- (co-)NP-complete in general (i.e. in practice exponential time)
- Small instances can be checked with a truth table:

$$\models (\neg p \vee (\neg q \vee (p \wedge q))) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \wedge q)$	$(\neg q \vee (p \wedge q))$	$(\neg p \vee (\neg q \vee (p \wedge q)))$
f	f	t	t	f	t	t
f	t	t	f	f	f	t
t	f	f	t	f	t	t
t	t	f	f	t	t	t

- Therefore: $(\neg p \vee (\neg q \vee (p \wedge q)))$ is a tautology!

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- So $(p \wedge \neg q) \models \neg(\neg p \vee q)$
- And $\neg(\neg p \vee q) \models (p \wedge \neg q)$

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- Any formula is equivalent to a formula containing only the connectives $\neg$ and $\wedge$.

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  - truth tables can be used for checking validity and etailment.