IN3070/4070 – Logic – Autumn 2020

Lecture 2: Propositional Logic & Sequent Calculus

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Today's Plan

- Motivation
- Syntax
- Semantics
- Logical Equivalence
- Satisfiability & Validity
- Summary
- Motivation
- Sequent Calculus
- Decision Procedure
- Summary

Outline

Propositional Logic

- simple "logical system" and basis for all others (first-order, description, modal, ...)
- logical systems formalize reasoning similar to programming languages that formalize computation
- consequent separation of syntactical notions (formulae, proofs) and semantical notions (truth values, models)
- syntax defines what strings of symbols are "legal" formulae
- semantics assign meanings to legal formulae (through an interpretation of its symbols)

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Syntax — Formulae

Formulae are made up of atomic formulae and the logical connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication).

Definition 2.1 (Atomic Formulae).

Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be a countable set of symbols called atomic formulae (or atoms), denoted by lower case letters $p, q, r, \ldots$.

Definition 2.2 (Propositional Formulae).

The propositional formulae, denoted A, B, C, F, G, H, are inductively defined as follows:

1. *Every atom $A \in \mathcal{P}$ is a formula.*
2. *If A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.*

Let $\mathcal{F}$ be the set of all (legal) formulae.

Syntax — Formulae

Definition 2.3 (Equivalence Connective).

$A \leftrightarrow B := ((A \to B) \land (B \to A))$

In order make formulae easier to read, parentheses can be omitted:

- the order of precedence of the logical connectives is as follows (from high to low): $\neg$, $\land$, $\lor$, $\to$, $\leftrightarrow$
- connectives are assumed to be right-associative, i.e., $A \lor B \lor C$ means $(A \lor (B \lor C))$

Examples:

$((p \to q) \leftrightarrow ((\neg p) \to (\neg q)))$ is a (legal) formula, identical to

$(p \to q) \leftrightarrow (\neg p \to \neg q)$ and $p \to q \leftrightarrow \neg p \to \neg q$

$\#, f(a, P \to$ ☺ $!$ is *not* a formula

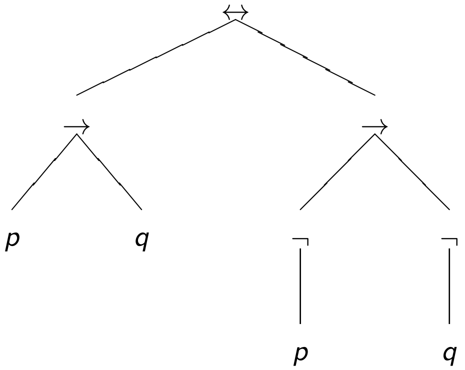
Alternative connectives: $\Rightarrow$ and $\supset$ (for $\to$), $\Leftrightarrow$ (for $\leftrightarrow$), & (for $\land$)

Formula Trees

Definition 2.4 (Formula Tree).

A formula can be presented as formula tree.

Example: $(p \to q) \leftrightarrow (\neg p \to \neg q)$



Outline

Semantics—Interpretation

Truth values are assigned to the atoms of a formula in order to evaluate the truth value of the formula.

Definition 3.1 (Interpretation).

Let $\mathcal{P}$ be a set of atoms.
An interpretation is a total function $\mathcal{I} : \mathcal{P} \to \{T, F\}$ that assigns one of the truth values T or F to every atom in $\mathcal{P}$.

Note: Ben-Ari defines $\mathcal{P}_A$, the atoms in A and $\mathcal{I}_A$ an "interpretation for A"

Simplfies some places, complicates things in others.

Semantics—Truth Value

Definition 3.2 (Truth Value).

Let $\mathcal{I}$ be an interpretation. The truth value $v_{\mathcal{I}}(A)$ of A under $\mathcal{I}$ is defined inductively as follows.

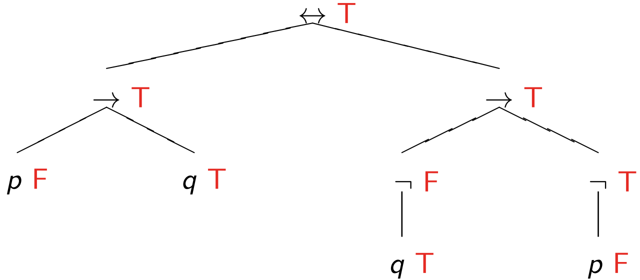
- *If $A \in \mathcal{P}$ is an atom, then $v_{\mathcal{I}}(A) = \mathcal{I}(A)$*
- *$v_{\mathcal{I}}(\neg A) = T$ if $v_{\mathcal{I}}(A) = F$ and F otherwise.*
- *$v_{\mathcal{I}}(A \wedge B) = T$ if $v_{\mathcal{I}}(A) = T$ and $v_{\mathcal{I}}(B) = T$, and F otherwise.*
- *$v_{\mathcal{I}}(A \vee B) = F$ if $v_{\mathcal{I}}(A) = F$ and $v_{\mathcal{I}}(B) = F$, and T otherwise.*
- *$v_{\mathcal{I}}(A \to B) = F$ if $v_{\mathcal{I}}(A) = T$ and $v_{\mathcal{I}}(B) = F$, and T otherwise.*

Note: For the equivalence connective, it follows that

- $v_{\mathcal{I}}(A \leftrightarrow B) = T$ if $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$, and F otherwise.

Semantics — Truth Value

Example: Let $A = (p \to q) \leftrightarrow (\neg q \to \neg p)$
with $\mathcal{I}(p) = F$ and $\mathcal{I}(q) = T$.



Truth Tables

A truth table is a format for displaying the semantics of a formula A by showing its truth value for every possible interpretation of A.

Definition 3.3 (Truth Table).

Let $A \in \mathcal{F}$ with n atoms. A truth table has $n+1$ columns and 2^n rows. There is a column for each atom in A, plus a column for the formula A. The first n columns specify all possible interpretations $\mathcal{I}$ that map atoms in A to $\{T, F\}$. The last column shows $v_{\mathcal{I}}(A)$, the truth value of A for each interpretation $\mathcal{I}$.

p_1	p_2	$\dots$	p_n	A
T	T	$\dots$	T	$v_{\mathcal{I}}(A)$
T	T	$\dots$	F	$v_{\mathcal{I}}(A)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	F	$\dots$	F	$v_{\mathcal{I}}(A)$

Truth Tables

Example: $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Material Implication

The operator of $p \to q$ is called material implication.

- p is the antecedent and q is the consequent
- it does not claim causation; i.e., it does not assert that the antecedent causes the consequent (or is even related to the consequent in any way)
- only states: if the antecedent is true, the consequent must be true
- it is false only if p is true and q is false

Example:

"Earth is farther from the sun than Venus" $\to$ "$1 + 1 = 3$"
is false since the antecedent is true and the consequent is false, but:

"Earth is farther from the sun than Mars" $\to$ "$1 + 1 = 3$"
is true(!) as the falsity of the antecedent by itself is sufficient to ensure the truth of the implication

Outline

Logical Equivalence

Definition 4.1 (Logical Equivalence).

Let $A_1, A_2 \in \mathcal{F}$. If $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$, then A_1 is logically equivalent to A_2, denoted $A_1 \equiv A_2$.

Example: $p \vee q \equiv q \vee p$ (proof by truth table)

Theorem 4.1 (Logical Equivalence "Commutativity").

Let $A, B \in \mathcal{F}$. Then $A \vee B \equiv B \vee A$.

Proof.

Let $\mathcal{I}$ be an arbitrary interpretation.

- If $v_{\mathcal{I}}(A \vee B) = T$, then $v_{\mathcal{I}}(A) = T$ or $v_{\mathcal{I}}(B) = T$. Thus, $v_{\mathcal{I}}(B \vee A) = T$.
- If $v_{\mathcal{I}}(A \vee B) = F$, then $v_{\mathcal{I}}(A) = F$ and $v_{\mathcal{I}}(B) = F$. Thus, $v_{\mathcal{I}}(B \vee A) = F$.

Since $\mathcal{I}$ was chosen arbitrarily, $v_{\mathcal{I}}(A \vee B) = v_{\mathcal{I}}(B \vee A)$ for all interpretations. □

Relationship between $\leftrightarrow$ and $\equiv$

- equivalence, $\leftrightarrow$, is a binary connective that appears in formulae
- logical equivalence, $\equiv$, is a property of pairs of formulae
- similar vocabulary, but $\leftrightarrow$ is part of the object language, whereas $\equiv$ is part of the metalanguage that we use to reason about the object language

Theorem 4.2 (Relation between $\equiv$ and $\leftrightarrow$).

$A \equiv B$ iff $v_{\mathcal{I}}(A \leftrightarrow B) = T$ for every interpretation $\mathcal{I}$.

Proof.

Suppose that $A \equiv B$ and let $\mathcal{I}$ be an arbitrary interpretation; then $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$ by definition of logical equivalence. From the Defn. of truth value, $v_{\mathcal{I}}(A \leftrightarrow B) = T$. Since $\mathcal{I}$ was arbitrary, $v_{\mathcal{I}}(A \leftrightarrow B) = T$ for all interpretations $\mathcal{I}$. The proof of the other direction is similar. □

Logically Equivalent Formulae

Extend syntax to include the two constant atoms *true* and *false*.

Definition 4.2 (Logical Constants).

Let true and false be two constant atoms with $\mathcal{I}(\mathit{true}) = T$ *and* $\mathcal{I}(\mathit{false}) = F$ *for any interpretation* $\mathcal{I}$ *(*$\top$ *and* $\bot$ *are also used).*

The following formulae are logical equivalent (more in [Ben-Ari, 2.3.3]):

$A \vee \mathit{true} \equiv \mathit{true}$	$A \wedge \mathit{true} \equiv A$
$A \vee \mathit{false} \equiv A$	$A \wedge \mathit{false} \equiv \mathit{false}$
$A \rightarrow \mathit{true} \equiv \mathit{true}$	$\mathit{true} \rightarrow A \equiv A$
$A \rightarrow \mathit{false} \equiv \neg A$	$\mathit{false} \rightarrow A \equiv \mathit{true}$
$A \equiv A \wedge A$	$A \equiv A \vee A$
$A \vee B \equiv B \vee A$	$A \wedge B \equiv B \wedge A$
$A \vee (B \vee C) \equiv (A \vee B) \vee C$	$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

Contrapositive: $A \rightarrow B \equiv \neg B \rightarrow \neg A$

Outline

Satisfiability and Validity

Definition 5.1 (Satisfiable, Model, Valid, Unsatisfiable, Invalid).

Let $A \in \mathcal{F}$.

- *A is satisfiable iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$. A satisfying interpretation $\mathcal{I}$ is a model for A.*
- *A is valid, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$. A valid propositional formula is also called a tautology.*
- *A is unsatisfiable iff it is not satisfiable, that is, if $v_{\mathcal{I}}(A) = F$ for all interpretations $\mathcal{I}$.*
- *A is invalid (or falsifiable), denoted $\not\models A$, iff it is not valid, that is, if $v_{\mathcal{I}}(A) = F$ for some interpretation $\mathcal{I}$.*
- *A set of formulae $U = \{A_1, ...\}$ is (simultaneously) satisfiable iff there exists an interpretation $\mathcal{I}$ such that $v_{\mathcal{I}}(A_i) = T$ for all i; otherwise U is unsatisfiable. The satisfying interpretation is a model of U.*

Satisfiability and Validity

There is a close relation between these four semantical concepts.

Theorem 5.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

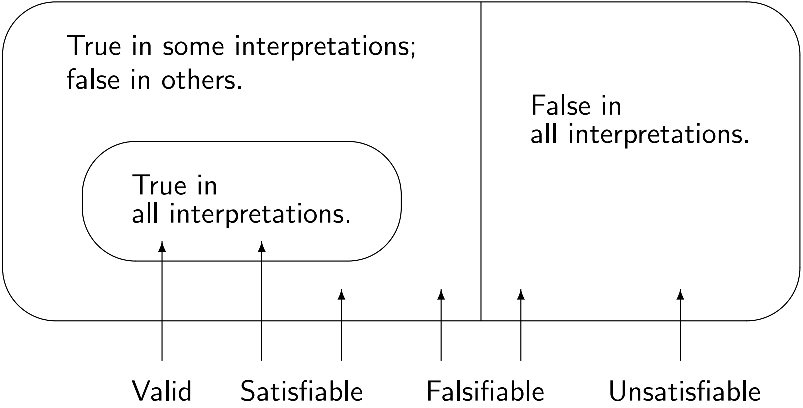
Let $A \in F$. A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is invalid.

Proof.

Let $\mathcal{I}$ be an arbitrary interpretation. $v_{\mathcal{I}}(A) = T$ if and only if $v_{\mathcal{I}}(\neg A) = F$ by definition of the truth value of negation. Since $\mathcal{I}$ was arbitrary, $v_{\mathcal{I}}(A) = T$ for all interpretations if and only if $v_{\mathcal{I}}(\neg A) = F$ for all interpretations, that is, iff $\neg A$ is unsatisfiable.

If A is satisfiable then for some interpretation $\mathcal{I}$, $v_{\mathcal{I}}(A) = T$. By definition of the truth value of negation, $v_{\mathcal{I}}(\neg A) = F$ so that $\neg A$ is invalid. Conversely, if $v_{\mathcal{I}}(\neg A) = F$ then $v_{\mathcal{I}}(A) = T$. □

Satisfiability and Validity



Decidability

Definition 5.2 (Decision Procedure).

Let $\mathcal{U} \subseteq \mathcal{F}$ be a set of (propositional) formulae. An algorithm is a decision procedure for $\mathcal{U}$ if given a formula $A \in \mathcal{F}$, it terminates and returns the answer "yes" if $A \in \mathcal{U}$ and the answer "no" if $A \notin \mathcal{U}$.

Theorem 5.2 (Truth Tables as Decision Procedure).

Truth tables are a decision procedure for $\{A \in \mathcal{F} \mid A \text{ is a tautology}\}$.

Proof.

For a given formula A with n atoms, use truth tables to evaluate truth values for A. If $v_{\mathcal{I}}(A) = T$ for all 2^n possible interpretations $\mathcal{I}$, then answer "yes"; otherwise answer "no". □

This method is not very efficient; more efficient procedures will be introduced later.

Logical Consequence

Definition 5.3 (Logical Consequence).

Let U be a set of formulas, A be a formula. A is a logical consequence of U, denoted $U \models A$, iff every model of U is a model of A.

Formula A need not be true in every possible interpretation, only in those interpretations which satisfy U, that is, only those which satisfy every formula in U. If U is empty, logical consequence is the same as validity.

Example: Let $A = (p \vee r) \wedge (\neg q \vee \neg r)$. A is a logical consequence of $\{p, \neg q\}$, denoted $\{p, \neg q\} \models A$, as $v_{\mathcal{I}}(A){=}T$ for all interpretations $\mathcal{I}$ such that $\mathcal{I}(p){=}T$ and $\mathcal{I}(q){=}F$. But A is not valid, as $v_{\mathcal{I}}(A){=}F$ for the interpretation $\mathcal{I}$ where $\mathcal{I}(p){=}F$, $\mathcal{I}(q){=}T$, $\mathcal{I}(r){=}T$.

Theorem 5.3 (Deduction Theorem).

Let $U = \{A_1, ..., A_n\}$. Then $U \models A$ iff $\models \bigwedge_i A_i \rightarrow A$.

Proof. *Left as an exercise.*

Outline

Summary

- syntax of propositional logic: atomic formulae, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- semantics of propositional logic: interpretation assigns truth value to atomic formulae and inductively to formulae in general
- truth tables can be used to evaluate the truth value of formulae
- material implication: not necessarily a causal relation between antecedent and consequent
- two formulae A and B are logically equivalent iff their truth value is identical for all interpretations
- four semantical concepts: satisfiable, valid, unsatisfiable, invalid
- these properties are decidable for propositional logic
- deduction theorem connects logical consequence and validity

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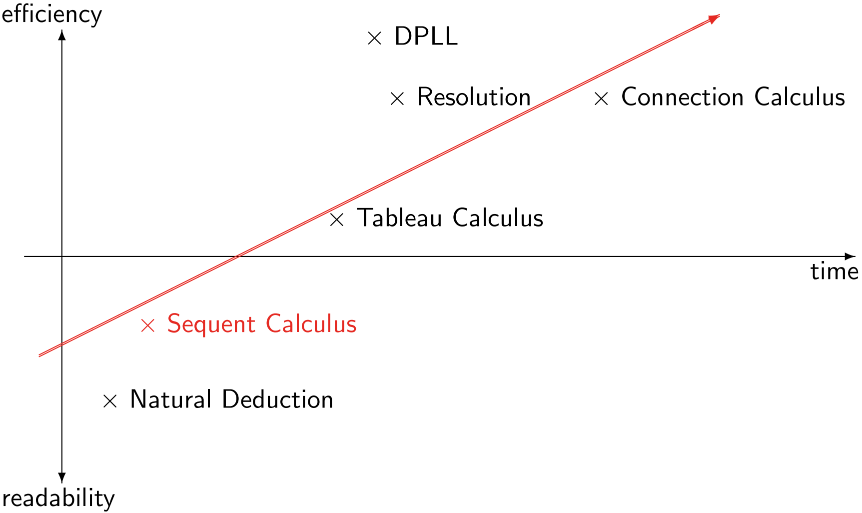
Proof Search Calculi



Outline

Natural Deduction: Rules for Implication and Negation

- rules for $\rightarrow$ (implication)

$$\dfrac{\begin{matrix}[A]^n \\ \vdots \\ B\end{matrix}}{A \rightarrow B} \rightarrow\text{-I}^n$$

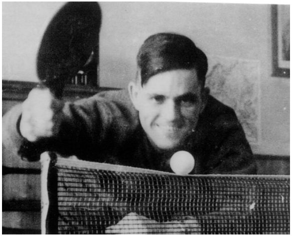
- rules for $\neg$ (negation)

$$\dfrac{\begin{matrix}[A]^n \\ \vdots \\ \mathit{false}\end{matrix}}{\neg A} \neg\text{-I}^n$$

Gentzen's Sequent Calculus

Goal: A derivation system similar to natural deduction but with "built-in" assumptions

"In order to prove the Hauptsatz, I had to introduce a suitable logical calculus. Hence, in this paper I will introduce another calculus of logical reasoning that has all desired properties." [G. Gentzen]



- Natural Deduction and Sequent calculus was developed by Gehard Gentzen in the 1930's
- Tools for investigating mathematical reasoning.

Sequents

Definition 8.1 (Sequent).

A sequent has the form $\Gamma \implies \Delta$ in which Γ and Δ are finite (possibly empty) multisets of formulae. The left side of the sequent is called the antecedent, the right side is called the succedent.

- $\Gamma \cup \{A\}$ or $\Delta \cup \{B\}$ are usually written as Γ, A and Δ, B, respectively
- intuitively, a sequent represents "provable from" in the sense that the formulae in Γ are assumptions for the set of formulae Δ to be proven
  - IF ALL of the formulae in Γ are true,
  - THEN SOME of the formulae in Δ are true

The Sequent Calculus LK

- Sequent proofs are trees labeled with sequents.
- Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \implies p, q}\text{ axiom} \qquad \dfrac{}{p, q \implies q}\text{ axiom}}{p, p \to q \implies q}\to\text{-left}}{p \wedge (p \to q) \implies q}\wedge\text{-left}}{\implies p \wedge (p \to q) \to q}\to\text{-right}$$

- The formula we try to show is at the root (bottom)
- Rules can cause branches to “grow”
- Some rules split a branch into two branches
- When we have a sequent like $A, \ldots \implies A, \ldots$ the branch is done

- So let’s look at the rules in detail!

LK — Rules for Conjunction and Disjunction

- rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \implies \Delta}{\Gamma, A \wedge B \implies \Delta}\ \wedge\text{-left} \qquad \frac{\Gamma \implies A, \Delta \qquad \Gamma \implies B, \Delta}{\Gamma \implies A \wedge B, \Delta}\ \wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \implies \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \vee B \implies \Delta}\ \vee\text{-left} \qquad \frac{\Gamma \implies A, B, \Delta}{\Gamma \implies A \vee B, \Delta}\ \vee\text{-right}$$

LK — Rules for Implication and Negation, Axiom

- rules for $\rightarrow$ (implication)

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \rightarrow B \implies \Delta} \rightarrow\text{-left} \qquad \frac{\Gamma, A \implies B, \Delta}{\Gamma \implies A \rightarrow B, \Delta} \rightarrow\text{-right}$$

- rules for $\neg$ (negation)

$$\frac{\Gamma \implies A, \Delta}{\Gamma, \neg A \implies \Delta} \neg\text{-left} \qquad \frac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta} \neg\text{-right}$$

- the axiom

$$\frac{}{\Gamma, A \implies A, \Delta} \text{axiom}$$

Examples of LK Proofs

Example: $(p \wedge q) \to p$

$$\dfrac{\dfrac{\dfrac{}{p, q \Longrightarrow p}\ \text{axiom}}{p \wedge q \Longrightarrow p}\ \wedge\text{-left}}{\Longrightarrow (p \wedge q) \to p}\ \to\text{-right}$$

Example: $p \wedge (p \to q) \to q$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \Longrightarrow p, q}\ \text{axiom} \qquad \dfrac{}{p, q \Longrightarrow q}\ \text{axiom}}{p, p \to q \Longrightarrow q}\ \to\text{-left}}{p \wedge (p \to q) \Longrightarrow q}\ \wedge\text{-left}}{\Longrightarrow p \wedge (p \to q) \to q}\ \to\text{-right}$$

Example: $(\neg p \vee q) \to (p \to q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \Longrightarrow p, q}\ \text{axiom}}{\neg p, p \Longrightarrow q}\ \neg\text{-left} \qquad \dfrac{}{q, p \Longrightarrow q}\ \text{axiom}}{\neg p \vee q, p \Longrightarrow q}\ \vee\text{-left}}{\neg p \vee q \Longrightarrow p \to q}\ \to\text{-right}}{\Longrightarrow (\neg p \vee q) \to (p \to q)}\ \to\text{-right}$$

Calculus and Proof — General Definitions

Definition 8.2 (Calculus/Deductive System).

A calculus consists of axioms and inference rules.

Axioms have the form $\overline{\;w\;}$ *; rules have the form* $\dfrac{w_1 \cdots w_n}{w}$

($w_1, \ldots, w_n$ are the premises, w is the conclusion).

An "instance" of a rule is the result of replacing all formula variables A, B, and set variables Γ, Δ by concrete formulae and sets of formulae

Definition 8.3 (Proof, Derivation).

Let $\mathcal{A}=\{A_1, \ldots\}$ be axioms and $\mathcal{R}=\{R_1, \ldots\}$ be rules of a calculus.

1. *Let $\overline{\;w\;}$ be an instance of an axiom $A_i \in \mathcal{A}$. Then $\overline{\;w\;}$ is a proof of w.*
2. *Let $\dfrac{w_1 \cdots w_n}{w}$ be an instance of a rule $R_i \in \mathcal{R}$ and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ proofs of $w_1, \ldots, w_n$. Then $\dfrac{\mathcal{D}_1 \cdots \mathcal{D}_n}{w}$ is a proof of w.*

A derivation is defined similarly, but leaves do not need to be axioms.

The Sequent Calculus LK

Definition 8.4 (Proofs in LK).

A proof of a formula A in the LK calculus is a proof of the sequent $\Longrightarrow A$ using the rules and axiom of LK. A formula A is provable, written $\vdash A$, iff there is a proof for A.

Theorem 8.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

- *if A is provable in LK, then A is valid (if $\vdash A$ then $\models A$)*
- *if A is valid, then A is provable in LK (if $\models A$ then $\vdash A$)*

Proof.

Next week! □

Outline

Sequent Calculus as Decision Procedure

The sequent calculus can be used as a decision procedure.

- Starting from the root $\Longrightarrow A$, apply the rules of the sequent calculus LK to every sequent until no more rules can be applied
  - induction: this will stop
  - magic: order does not matter. I.e. won't show this now
- now, the sequents in all leaves of the derivation contain only atomic formulae
- if all leaf sequents are axioms, then the formula is valid; otherwise, it is invalid (A is satisfiable iff $\neg A$ is invalid)

Example: $p \land (p \to q) \to r$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \Longrightarrow p, r}\text{axiom} \qquad \dfrac{}{p, q \Longrightarrow r}\ ?}{p, p \to q \Longrightarrow r}\to\text{-left}}{p \land (p \to q) \Longrightarrow r}\land\text{-left}}{\Longrightarrow p \land (p \to q) \to r}\to\text{-right}$$

Outline

Summary

- Gentzen's sequent calculus uses sequents $\Gamma \Longrightarrow \Delta$ to formalize logical reasoning; Γ are the assumptions in order to prove Δ
- it was originally invented as a tool to study natural deduction
- the sequent calculus consists of one axiom and two inference rules for each logical connective; it is sound and complete
- it can be used as a decision procedure for validity of propositional formulae in a straightforward way.
- Next week: Soundness and Completeness proofs