# IN3070/4070 – Logic – Autumn 2020

## Lecture 2: Propositional Logic & Sequent Calculus

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# Today's Plan

- Motivation
- Syntax
- Semantics
- Logical Equivalence
- Satisfiability & Validity
- Summary
- Motivation
- Sequent Calculus
- Decision Procedure
- Summary

# Outline

# Propositional Logic

- simple "logical system" and basis for all others (first-order, description, modal, ...)
- logical systems formalize reasoning similar to programming languages that formalize computation

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- logical systems formalize reasoning similar to programming languages that formalize computation
- consequent separation of syntactical notions (formulae, proofs) and semantical notions (truth values, models)
- syntax defines what strings of symbols are “legal” formulae
- semantics assign meanings to legal formulae (through an interpretation of its symbols)

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# Syntax — Formulae

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**Definition 2.2 (Propositional Formulae).**

*The propositional formulae, denoted $A, B, C, F, G, H$, are inductively defined as follows:*

1. *Every atom $A \in \mathcal{P}$ is a formula.*
2. *If $A$ and $B$ are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.*

*Let $\mathcal{F}$ be the set of all (legal) formulae.*

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Examples:

$((p \to q) \leftrightarrow ((\neg p) \to (\neg q)))$ is a (legal) formula, identical to $(p \to q) \leftrightarrow (\neg p \to \neg q)$ and $p \to q \leftrightarrow \neg p \to \neg q$

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Alternative connectives: $\Rightarrow$ and $\supset$ (for $\rightarrow$), $\Leftrightarrow$ (for $\leftrightarrow$), & (for $\wedge$)

# Formula Trees

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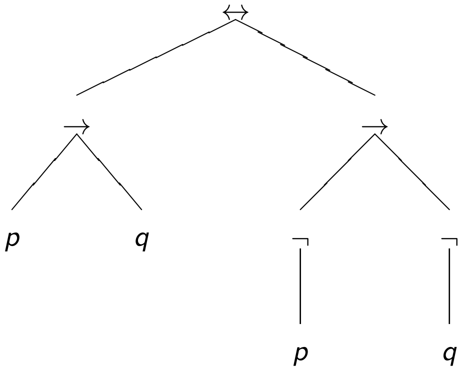

# Outline

# Semantics—Interpretation

Truth values are assigned to the atoms of a formula in order to evaluate the truth value of the formula.

**Definition 3.1 (Interpretation).**

*Let $\mathcal{P}$ be a set of atoms.*
*An interpretation is a total function $\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$ that assigns one of the truth values $T$ or $F$ to every atom in $\mathcal{P}$.*

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Simplfies some places, complicates things in others.

# Semantics—Truth Value

## Definition 3.2 (Truth Value).

*Let $\mathcal{I}$ be an interpretation. The truth value $v_{\mathcal{I}}(A)$ of $A$ under $\mathcal{I}$ is defined inductively as follows.*

- *If $A \in \mathcal{P}$ is an atom, then $v_{\mathcal{I}}(A) = \mathcal{I}(A)$*
- $v_{\mathcal{I}}(\neg A) = T$ *if* $v_{\mathcal{I}}(A) = F$ *and* $F$ *otherwise.*
- $v_{\mathcal{I}}(A \wedge B) = T$ *if* $v_{\mathcal{I}}(A) = T$ *and* $v_{\mathcal{I}}(B) = T$*, and* $F$ *otherwise.*
- $v_{\mathcal{I}}(A \vee B) = F$ *if* $v_{\mathcal{I}}(A) = F$ *and* $v_{\mathcal{I}}(B) = F$*, and* $T$ *otherwise.*
- $v_{\mathcal{I}}(A \rightarrow B) = F$ *if* $v_{\mathcal{I}}(A) = T$ *and* $v_{\mathcal{I}}(B) = F$*, and* $T$ *otherwise.*

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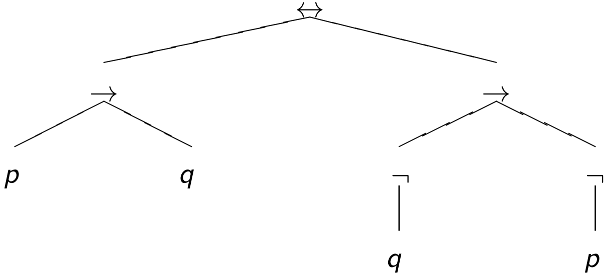
- *$v_{\mathcal{I}}(A \rightarrow B) = F$ if $v_{\mathcal{I}}(A) = T$ and $v_{\mathcal{I}}(B) = F$, and $T$ otherwise.*

Note: For the equivalence connective, it follows that

- $v_{\mathcal{I}}(A \leftrightarrow B) = T$ if $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$, and $F$ otherwise.

# Semantics — Truth Value

Example: Let $A = (p \to q) \leftrightarrow (\neg q \to \neg p)$
with $\mathcal{I}(p) = F$ and $\mathcal{I}(q) = T$.



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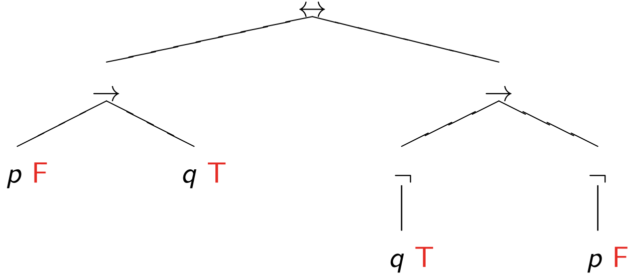

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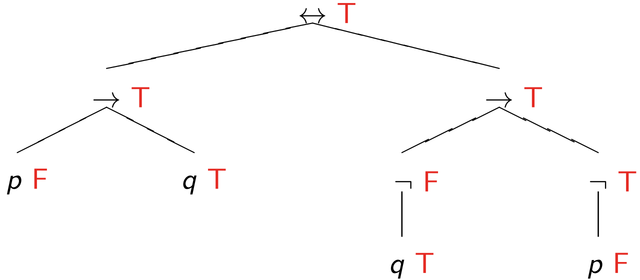

# Truth Tables

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**Definition 3.3 (Truth Table).**

*Let $A \in \mathcal{F}$ with $n$ atoms. A truth table has $n+1$ columns and $2^n$ rows. There is a column for each atom in $A$, plus a column for the formula $A$. The first $n$ columns specify all possible interpretations $\mathcal{I}$ that map atoms in $A$ to $\{T, F\}$. The last column shows $v_{\mathcal{I}}(A)$, the truth value of $A$ for each interpretation $\mathcal{I}$ .*

| $p_1$ | $p_2$ | $\dots$ | $p_n$ | $A$ |
|---|---|---|---|---|
| $T$ | $T$ | $\dots$ | $T$ | $v_{\mathcal{I}}(A)$ |
| $T$ | $T$ | $\dots$ | $F$ | $v_{\mathcal{I}}(A)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $F$ | $F$ | $\dots$ | $F$ | $v_{\mathcal{I}}(A)$ |

# Truth Tables

Example: $p \to q$

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Example: $p \rightarrow q$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| $T$ | $T$ | |
| $T$ | $F$ | |
| $F$ | $T$ | |
| $F$ | $F$ | |

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| $T$ | $T$ | $T$ |
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| $p$ | $q$ | $p \to q$ | $\neg p$ | $\neg q$ | $\neg q \to \neg p$ | $(p \to q) \leftrightarrow (\neg q \to \neg p)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | | | | | |
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| $p$ | $q$ | $p \to q$ | $\neg p$ | $\neg q$ | $\neg q \to \neg p$ | $(p \to q) \leftrightarrow (\neg q \to \neg p)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
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# Material Implication

The operator of $p \rightarrow q$ is called material implication.

- $p$ is the antecedent and $q$ is the consequent
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Example:

"Earth is farther from the sun than Venus" $\to$ "$1 + 1 = 3$"
is false since the antecedent is true and the consequent is false, but:

"Earth is farther from the sun than Mars" $\to$ "$1 + 1 = 3$"
is true(!) as the falsity of the antecedent by itself is sufficient to ensure the truth of the implication

# Outline

# Logical Equivalence

**Definition 4.1 (Logical Equivalence).**

*Let $A_1, A_2 \in \mathcal{F}$. If $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$, then $A_1$ is logically equivalent to $A_2$, denoted $A_1 \equiv A_2$.*

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- If $v_\mathcal{I}(A \vee B) = F$, then $v_\mathcal{I}(A) = F$ and $v_\mathcal{I}(B) = F$. Thus, $v_\mathcal{I}(B \vee A) = F$.

Since $\mathcal{I}$ was chosen arbitrarily, $v_\mathcal{I}(A \vee B) = v_\mathcal{I}(B \vee A)$ for all interpretations. □

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The following formulae are logical equivalent (more in [Ben-Ari, 2.3.3]):

| | |
|---|---|
| $A \vee true \equiv true$ | $A \wedge true \equiv A$ |
| $A \vee false \equiv A$ | $A \wedge false \equiv false$ |
| $A \rightarrow true \equiv true$ | $true \rightarrow A \equiv A$ |
| $A \rightarrow false \equiv \neg A$ | $false \rightarrow A \equiv true$ |
| $A \equiv A \wedge A$ | $A \equiv A \vee A$ |
| $A \vee B \equiv B \vee A$ | $A \wedge B \equiv B \wedge A$ |
| $A \vee (B \vee C) \equiv (A \vee B) \vee C$ | $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ |
| $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ | $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ |

Contrapositive: $A \rightarrow B \equiv \neg B \rightarrow \neg A$

# Outline

# Satisfiability and Validity

**Definition 5.1 (Satisfiable, Model, Valid, Unsatisfiable, Invalid).**

*Let* $A \in \mathcal{F}$.

- $A$ *is satisfiable iff* $v_{\mathcal{I}}(A) = T$ *for some interpretation* $\mathcal{I}$.
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- *A is invalid (or falsifiable), denoted $\not\models A$, iff it is not valid, that is, if $v_{\mathcal{I}}(A) = F$ for some interpretation $\mathcal{I}$.*
- *A set of formulae $U = \{A_1, \ldots\}$ is (simultaneously) satisfiable iff there exists an interpretation $\mathcal{I}$ such that $v_{\mathcal{I}}(A_i) = T$ for all $i$; otherwise U is unsatisfiable. The satisfying interpretation is a model of U.*

# Satisfiability and Validity

There is a close relation between these four semantical concepts.

**Theorem 5.1 (Satisfiable, Valid, Unsatisfiable, Invalid).**

*Let $A \in F$. $A$ is valid iff $\neg A$ is unsatisfiable. $A$ is satisfiable iff $\neg A$ is invalid.*

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*Let $\mathcal{I}$ be an arbitrary interpretation. $v_{\mathcal{I}}(A) = T$ if and only if $v_{\mathcal{I}}(\neg A) = F$ by definition of the truth value of negation. Since $\mathcal{I}$ was arbitrary, $v_{\mathcal{I}}(A) = T$ for all interpretations if and only if $v_{\mathcal{I}}(\neg A) = F$ for all interpretations, that is, iff $\neg A$ is unsatisfiable.*

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*If $A$ is satisfiable then for some interpretation $\mathcal{I}$, $v_{\mathcal{I}}(A) = T$. By definition of the truth value of negation, $v_{\mathcal{I}}(\neg A) = F$ so that $\neg A$ is invalid. Conversely, if $v_{\mathcal{I}}(\neg A) = F$ then $v_{\mathcal{I}}(A) = T$.* □

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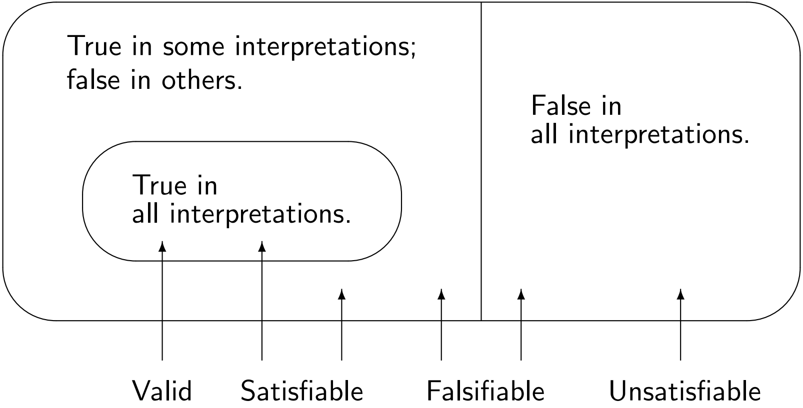

# Decidability

**Definition 5.2 (Decision Procedure).**

*Let $\mathcal{U} \subseteq \mathcal{F}$ be a set of (propositional) formulae. An algorithm is a decision procedure for $\mathcal{U}$ if given a formula $A \in \mathcal{F}$, it terminates and returns the answer "yes" if $A \in \mathcal{U}$ and the answer "no" if $A \notin \mathcal{U}$.*

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*For a given formula $A$ with $n$ atoms, use truth tables to evaluate truth values for $A$. If $v_{\mathcal{I}}(A) = T$ for all $2^n$ possible interpretations $\mathcal{I}$, then answer "yes"; otherwise answer "no".* □

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This method is not very efficient; more efficient procedures will be introduced later.

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**Definition 5.3 (Logical Consequence).**

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Example: Let $A = (p \vee r) \wedge (\neg q \vee \neg r)$. $A$ is a logical consequence of $\{p, \neg q\}$, denoted $\{p, \neg q\} \models A$, as $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$ such that $\mathcal{I}(p) = T$ and $\mathcal{I}(q) = F$. But $A$ is not valid, as $v_{\mathcal{I}}(A) = F$ for the interpretation $\mathcal{I}$ where $\mathcal{I}(p) = F$, $\mathcal{I}(q) = T$, $\mathcal{I}(r) = T$.

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*Proof.* *Left as an exercise.*

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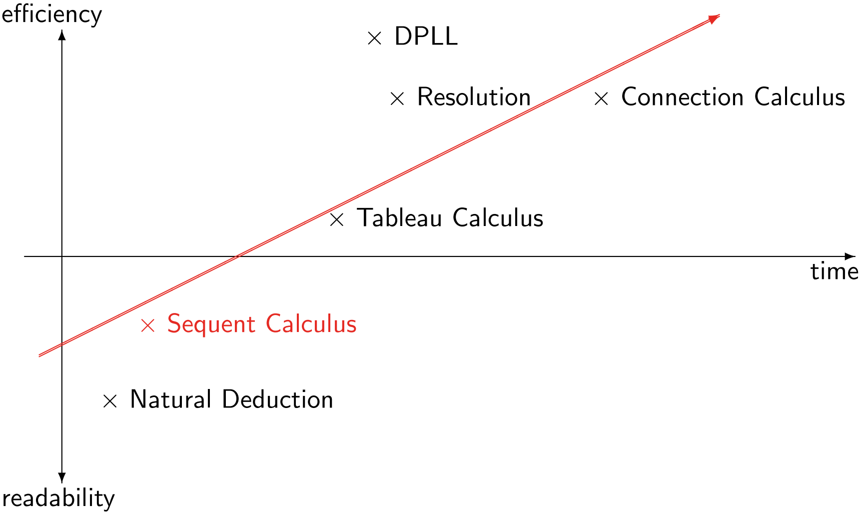

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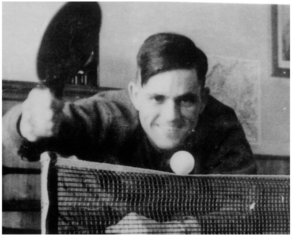

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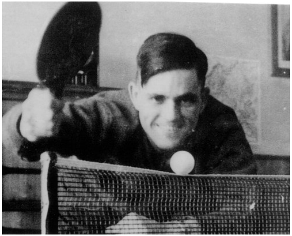

- Natural Deduction and Sequent calculus was developed by Gehard Gentzen in the 1930's
- Tools for investigating mathematical reasoning.

# Sequents

**Definition 8.1 (Sequent).**

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- The formula we try to show is at the root (bottom)
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- Sequent proofs are trees labeled with sequents.
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- So let's look at the rules in detail!

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# Calculus and Proof — General Definitions

**Definition 8.2 (Calculus/Deductive System).**

*A calculus consists of axioms and inference rules.*

*Axioms have the form* $\overline{\ w\ }$ *; rules have the form* $\dfrac{w_1 \cdots w_n}{w}$
*($w_1, \ldots, w_n$ are the premises, $w$ is the conclusion).*

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## Definition 8.3 (Proof, Derivation).

*Let $\mathcal{A}=\{A_1, \ldots\}$ be axioms and $\mathcal{R}=\{R_1, \ldots\}$ be rules of a calculus.*

1. *Let* $\overline{w}$ *be an instance of an axiom $A_i \in \mathcal{A}$. Then* $\overline{w}$ *is a proof of $w$.*
2. *Let* $\dfrac{w_1 \cdots w_n}{w}$ *be an instance of a rule $R_i \in \mathcal{R}$ and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ proofs of $w_1, \ldots, w_n$. Then* $\dfrac{\mathcal{D}_1 \cdots \mathcal{D}_n}{w}$ *is a proof of $w$.*

*A derivation is defined similarly, but leaves do not need to be axioms.*

# The Sequent Calculus LK

**Definition 8.4 (Proofs in LK).**

*A proof of a formula A in the LK calculus is a proof of the sequent $\Longrightarrow A$ using the rules and axiom of LK. A formula A is provable, written $\vdash A$, iff there is a proof for A.*

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**Theorem 8.1 (Soundness and Completeness of LK).**

*The calculus of natural deduction LK is sound and complete, i.e.*

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Proof.

*Next week!* □

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- Motivation
- Syntax
- Semantics
- Logical Equivalence
- Satisfiability & Validity
- Summary
- Motivation
- Sequent Calculus
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- **Summary**

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