IN3070/4070 – Logic – Autumn 2020

Lecture 3: LK: Soundness & Completeness

Martin Giese

3rd September 2020

Department of Informatics

University of Oslo

Today's Plan

- Semantics for Sequents
- Soundness
- Completeness

Outline

- Semantics for Sequents
- Soundness
- Completeness

Semantics for Sequents

Definition 1.1 (Valid sequent).

A sequent $\Gamma \implies \Delta$ is valid if all interpretations that satisfy all formulas in Γ satisfy at least one formula in Δ.

Example.

The following sequents are valid:

- $p \implies p$
- $p \to q, r \implies p \to q, s$
- $p, p \to q \implies q$
- $p \to q \implies \neg q \to \neg p$

Definition 1.2 (Countermodel/falsifiable sequent).

- *An interpretation $\mathcal{I}$ is a countermodel for the sequent $\Gamma \Longrightarrow \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$*
- *We say that a countermodel for a sequent falsifies the sequent.*
- *A sequent is falsifiable if it has a countermodel.*

Example.

The following sequents are falsifiable:

- $p \Longrightarrow q$ — *Countermodel:* $\mathcal{I}(p) = T, \mathcal{I}(q) = F$
- $p \lor q \Longrightarrow p \land q$ — *Countermodel: same, or* $\mathcal{I}(p) = F, \mathcal{I}(q) = T$
- $\Longrightarrow p$ — *Countermodel:* $\mathcal{I}(p) = F$
- $p \Longrightarrow$ — *Countermodel:* $\mathcal{I}(p) = T$
- $\Longrightarrow$ — *Countermodel: all interpretations!*

Summary

Valid

- $p, p \to q \Longrightarrow q$
- If $\mathcal{I} \models p$ and $\mathcal{I} \models p \to q$, then $\mathcal{I} \models q$.
- Validity is a semantic notion

Provable

$$\dfrac{p \Longrightarrow p, q \qquad q, q \Longrightarrow q}{p, p \to q \Longrightarrow q}$$

- Provability is a syntactic notion

Falsifiability

- $\neg p, p \to q \Longrightarrow \neg q$
- An interpretation $\mathcal{I}$ s.t. $\mathcal{I} \not\models p$ and $\mathcal{I} \models q$.

Not provable

$$\dfrac{\dfrac{q \Longrightarrow p, p}{\neg p \Longrightarrow p, \neg q} \qquad \dfrac{q, q \Longrightarrow p}{q, \neg p \Longrightarrow \neg q}}{\neg p, p \to q \Longrightarrow \neg q}$$

Outline

Soundness of LK

- We want all LK-provable sequents to be valid!
- If they are not, then LK would be incorrect or unsound ...

Definition 2.1 (Soundness).

The sequent calculus LK *is sound if every* LK*-provable sequent is valid.*

Theorem 2.1.

The sequent calculus LK *is sound.*

How to show the Soundness Theorem?

We show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Preservation of Falsifiability

Definition 2.2.

An LK-rule θ preserves falsifiability (upwards) if all interpretations that falsify the conclusion w of an instance $\frac{w_1 \cdots w_n}{w}$ of θ also falsify at least one of the premises w_i.

Lemma 2.1.

All LK-rules preserve falsifiability.

Proving Preservation of Falsifiability

- The proof has a separate case for each LK-rule.
- Consider for instance the $\to$-left-rule:

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta} \to\text{-left}$$

- We have to show that all instances of $\to$-left preserve falsifiability upwards.
- We let Γ, Δ, A and B in the rule stand for arbitrary (sets of) propositional formulae

Proof for $\neg$-right

Proof for $\neg$-right.

$$\frac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta} \neg\text{-right}$$

- Assume that $\mathcal{I}$ falsifies the conclusion.
- Then $\mathcal{I} \models \Gamma$, $\mathcal{I} \not\models \neg A$ and $\mathcal{I}$ falsifies all formulae in Δ.
- Per model semantics, we have $\mathcal{I} \models A$.
- Therefore, $\mathcal{I} \models \Gamma \cup \{A\}$ and $\mathcal{I}$ falsifies all formlae in Δ.
- Thus, $\mathcal{I}$ falsifies the premisse.

□

Proof for →-left

Proof for →-left.

$$\dfrac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta} \to\text{-left}$$

- Assume that $\mathcal{I}$ falsifies the conclusion.
- Then $\mathcal{I}$ satisfies $\Gamma \cup \{A \to B\}$ and falsifies all formlae in Δ.
- Since $\mathcal{I}$ satisfies $A \to B$, by definition of model semantics,
  (1) $\mathcal{I} \not\models A$, or
  (2) $\mathcal{I} \models B$.
- In case (1), $\mathcal{I}$ falsifies the left premisse.
- In case (2), $\mathcal{I}$ falsifies the right premisse.

□

Proving "for all"-statements

- Consider the statement "for all $x \in S$: $P(x)$".
- We can show this by showing $P(a)$ for each element $a \in S$.
- What if S is very large, or infinite?
- We can generalise from an arbitrary element:
  - Choose an arbitrary element $a \in S$.
  - Show that $P(a)$ holds.
  - Since a was arbitrarily chosen, the original statement must hold.

How to show the Soundness Theorem?

We show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Reminder: LK derivation

Definition 2.3 (LK Derivation).

1. *Let $\Gamma \implies \Delta$ be a sequent. Then*

$$\Gamma \implies \Delta$$

   is an LK-derivation of $\Gamma \implies \Delta$.

2. *Let $\dfrac{w_1 \quad \cdots \quad w_n}{\Gamma \implies \Delta}$ be an instance of an LK rule, and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ derivations of $w_1, \ldots, w_n$. Then*

$$\dfrac{\mathcal{D}_1 \quad \cdots \quad \mathcal{D}_n}{\Gamma \implies \Delta}$$

   is an LK-derivation of $\Gamma \implies \Delta$.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If an interpretation $\mathcal{I}$ falsifies the root sequent of an LK*-derivation δ, then $\mathcal{I}$ falsifies at least one of the leaf sequents of δ.*

Proof.

By structural induction on the LK-derivation δ.
Induction base: δ is a sequent $\Gamma \implies \Delta$:

$$\Gamma \implies \Delta$$

- Here, $\Gamma \implies \Delta$ is both root sequent and (only) leaf sequent.
- Assume $\mathcal{I}$ falsifies $\Gamma \implies \Delta$.
- Then $\mathcal{I}$ falsifies a leaf sequent in δ, namely $\Gamma \implies \Delta$.

□

Continued.

Induction step: δ is a derivation of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma_1 \implies \Delta_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \vdots \\ \Gamma_n \implies \Delta_n\end{array}}{\Gamma \implies \Delta}\, r$$

for some smaller derivations $\mathcal{D}_i$ with roots $\Gamma_i \implies \Delta_i$.

- Assume $\mathcal{I}$ falsifies $\Gamma \implies \Delta$.
- Rule r preserves falsifiability upwards.
- Therefore $\mathcal{I}$ falsifies $\Gamma_i \implies \Delta_i$ for some $i \in \{1, \ldots, n\}$.
- By induction, $\mathcal{I}$ falsifies one of the leaf sequents of $\mathcal{D}_i$.
- This is also a leaf sequent of δ

□

How to show the Soundness Theorem?

We show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

All axioms are valid

Lemma 2.3.

All axioms are valid.

Proof.

$$\Gamma, A \implies A, \Delta$$

- We will show that all interpretations that satisfy the antecedent also satisfy at least one formula of the succedent.
- Let $\mathcal{I}$ be an arbitrarily chosen interpretation that satisfies the antecedent.
- Then $\mathcal{I}$ satisfies the formula A in the succedent.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.
- Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \implies \Delta$.
- We know from the previous Lemma that $\mathcal{I}$ falsifies at least one leaf sequent of $\mathcal{P}$.
- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- So $\mathcal{P}$ cannot be an LK-proof.

$\square$

Analysis

- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- An axiom is never falsifiable
- Roots of LK-proofs are valid
- Most of this is independent of the actual rules.
- Central part is proving that every rule preserves falsifiability
- Shown individually for each rule
- Can add new rules, and just show "soundness" for those

Outline

- Semantics for Sequents
- Soundness
- Completeness

Completeness — Introduction

Definition 3.1 (Soundness).

The calculus LK *is sound if any* LK*-provable sequent is valid.*

Definition 3.2 (Completeness).

The calculus LK *is complete if every valid sequent is* LK*-provable.*

Validity	$\xleftarrow{\text{soundness}}$ / $\xrightarrow{\text{completeness}}$	**Provability**
(semantic)		(syntactic)
Universal statement:		Existential statement:
"for all interpretations..."		"there exists a proof..."

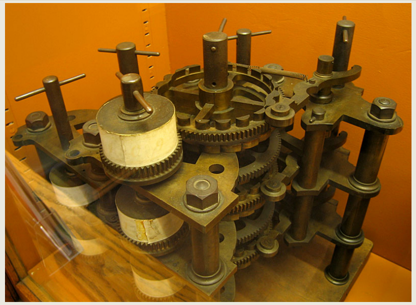
Completeness — Introduction

Soundnes: $\Gamma \implies \Delta$ provable $\Rightarrow$ $\Gamma \implies \Delta$ valid
Completeness: $\Gamma \implies \Delta$ valid $\Rightarrow$ $\Gamma \implies \Delta$ provable

- Soundness and Completeness are dual notions
- Soundness says that we cannot prove *more* than the valid sequents
- Completeness says that we can prove *all* valid sequents
- A sequent is valid if and only if it is not falsifiable
- We can therefore also express soundness and completeness as:

Soundness: $\Gamma \implies \Delta$ falsifiable $\Rightarrow$ $\Gamma \implies \Delta$ not provable
Completeness: $\Gamma \implies \Delta$ not provable $\Rightarrow$ $\Gamma \implies \Delta$ falsifiable

An LK-machine?



Soundness

All that is printed is valid.

Completeness

All that is valid will get printed.

- Something can be sound without being complete.
  - Then too little is shown.
  - Example with prime numbers: 2, 5, 7, 11, 17, 19, ...
- Something can be complete without being sound.
  - Then too much is shown
  - Example with prime numbers: 2, 3, 5, 7, 9, 11, 13, 15 ...
- We want both:
  - Not too much, not too little.
  - Example with prime numbers: 2, 3, 5, 7, 11, 13, 17, 19 ...

The Completeness Theorem

Theorem 3.1 (Completeness).

If $\Gamma \implies \Delta$ is valid, then it is provable in LK.

To show completeness of our calculus, we show the equivalent statement:

Lemma 3.1 (Model existence).

If $\Gamma \implies \Delta$ is not provable in LK, *then it is falsifiable.*

This means that there is an interpretation that makes all formulae in Γ true and all formulae in Δ false.

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a derivation $\mathcal{D}$ from $\Gamma \implies \Delta$ such that no further rule applications are possible. "A maximal derivation."
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom. We then have:
  - The leaf sequent of $\mathcal{B}$ contains only atomic formulae, and
  - the leaf sequent of $\mathcal{B}$ is not an axiom.
- We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

  $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on $\mathcal{B}$, and

  $\mathcal{B}^\perp$ be the set of formulae that occur in an succedent on $\mathcal{B}$, and

  $\mathcal{I}_\mathcal{B}$ be the interpretation that makes all atomic formulae in $\mathcal{B}^\top$ true and all other atomic formulae (in particular those in $\mathcal{B}^\perp$) false.

Example

$$\dfrac{\dfrac{\dfrac{p \Longrightarrow q, p \qquad q, p \Longrightarrow q}{p \to q, p \Longrightarrow q} \qquad \dfrac{r \Longrightarrow q, p \qquad q, r \Longrightarrow q}{p \to q, r \Longrightarrow q}}{p \to q, p \lor r \Longrightarrow q}}{p \to q \Longrightarrow (p \lor r) \to q}$$

We see that the branch $\mathcal{B}$ with leaf sequent $r \Longrightarrow q, p$ is not closed.

$\mathcal{B}^\top = \{r, p \to q, p \lor r\}$

$\mathcal{B}^\perp = \{q, p, (p \lor r) \to q\}$

$\mathcal{I}_\mathcal{B} =$ interpretation with $\mathcal{I}_\mathcal{B}(r) = T$ og $\mathcal{I}_\mathcal{B}(q) = \mathcal{I}_\mathcal{B}(p) = F$

To show: this interpretation falsifies the root sequent.

Proof of Completeness, cont.

- We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_\mathcal{B}$ makes all formulae in $\mathcal{B}^\top$ true, and all formulae in $\mathcal{B}^\perp$ false.
- We show for all propositional formulae A that

  If $A \in \mathcal{B}^\top$, then $\mathcal{I}_\mathcal{B} \models A$.

  If $A \in \mathcal{B}^\perp$, then $\mathcal{I}_\mathcal{B} \not\models A$.

Induction base: A is an atomic formula in $\mathcal{B}^\top / \mathcal{B}^\perp$.

- Our statment holds for $A \in \mathcal{B}^\top$ because that is how we defined $\mathcal{I}_\mathcal{B}$.
- For $A \in \mathcal{B}^\perp$, $A \notin \mathcal{B}^\top$ because atoms do not disappear from a branch and $\mathcal{B}$ contains no axiom. Therefore $\mathcal{I}_\mathcal{B} \not\models A$.

Induction step: From the assumption (IH) that the statement holds for A and B, we must show that it holds for $\neg A$, $(A \land B)$, $(A \lor B)$ og $(A \to B)$. These are four cases, of which we show three here.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

- $\neg A$ appears in an antecedent, it can't 'go away' unless $\neg$-left is applied
- Since the derivation is maximal, $\neg$-left *is* eventually applied
- A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\perp$.

- $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$-right is applied
- Since the derivation is maximal, $\neg$-right *is* eventually applied
- A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models \neg A$.

Case: Conjunction in antecedent/succedent

Assume that $(A \land B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \land B)$.

Assume that $(A \land B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\land$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \not\models B$
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models (A \land B)$

Case: Implication in antecedent/succedent

Assume that $(A \to B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, $\to$-left is eventually applied...
- ...introducing A in the succedent of one branch and B in the antecedent of the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \models B$
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \to B)$

Assume that $(A \to B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \not\models B$
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models (A \to B)$

Analysis

- If there is no proof for a sequent, there is a derivation...
  - Where all possible rules have been applied
  - At least one branch $\mathcal{B}$ has not been closed with an axiom
- We can use the atoms on $\mathcal{B}$ to construct an interpretation $\mathcal{I}_\mathcal{B}$
- $\mathcal{I}_\mathcal{B}$ makes atoms left true, and atoms right false
- $\mathcal{I}_\mathcal{B}$ also makes *all other* formulae left true and right false, because...
  - for every non-atomic formula, there is a rule that decomposes it
  - which must have been applied
  - and that guarantees that $\mathcal{I}_\mathcal{B}$ falsifies sequents, based on structural induction
- Structural induction on formulae, while soundness was by induction on derivations
- Not possible to prove completeness 'one rule at a time'

One-sided Sequent Calculus

- Only sequents with empty succedent: $\Gamma \Longrightarrow$
- To prove A, start with $\neg A \Longrightarrow$
- "Proof by contradiction" or "refutation"
- Negation rules combined with others:

$$\frac{\Gamma, \neg A, \neg B \Longrightarrow}{\Gamma, \neg(A \lor B) \Longrightarrow}\neg\lor \qquad \frac{\Gamma, \neg A \Longrightarrow \qquad \Gamma, \neg B \Longrightarrow}{\Gamma, \neg(A \land B) \Longrightarrow}\neg\land$$

- Double negation:

$$\frac{\Gamma, A \Longrightarrow}{\Gamma, \neg\neg A \Longrightarrow}\neg\neg$$

- Axiom:

$$\frac{}{\Gamma, A, \neg A \Longrightarrow}$$

- Can do the same with empty antecedents $\Longrightarrow \Delta$

Example with One-sided Sequents

- Instead of $p \to q \Longrightarrow (p \lor r) \to q$
- Start with $p \to q, \neg((p \lor r) \to q) \Longrightarrow$

$$\dfrac{\dfrac{\dfrac{}{\neg p, p, \neg q \Longrightarrow} \quad \dfrac{}{q, p, \neg q \Longrightarrow}}{p \to q, p, \neg q \Longrightarrow} \quad \dfrac{\dfrac{}{\neg p, r, \neg q \Longrightarrow} \quad \dfrac{}{q, r, \neg q \Longrightarrow}}{p \to q, r, \neg q \Longrightarrow}}{\dfrac{p \to q, p \lor r, \neg q \Longrightarrow}{p \to q, \neg((p \lor r) \to q) \Longrightarrow}}$$

- Soundness and completeness very similar to two-sided LK.

Semantic Tableaux (Ben-Ari 2.6)

- Others call these ‘block tableaux’
- Sequent arrow $\Longrightarrow$ not needed for one-sided calculus
- More handy to write top-down, like everybody else
- Mark ‘closed’ branches (with axioms) with $\times$

$$p \to q, \neg((p \vee r) \to q)$$

$$p \to q, p \vee r, \neg q$$

$$p \to q, p, \neg q \qquad\qquad p \to q, r, \neg q$$

$$\neg p, p, \neg q \quad q, p, \neg q \qquad \neg p, r, \neg q \quad q, r, \neg q$$

$$\times \qquad \times \qquad\qquad\qquad \times$$

Completeness

Short Hand Notation for Tableaux

- Only write the new formula in every node.
- Even more handy to write
- Close branch using literals A and $\neg A$ anywhere on a branch.
- Have to make sure that all rules were used on every branch!

$$p \to q, \neg((p \vee r) \to q)$$

$$p \vee r$$

$$\neg q$$

$$p \qquad\qquad r$$

$$\neg p \quad q \qquad \neg p \quad q$$

$$\times \quad \times \qquad\qquad \times$$

Completeness

Summary and Outlook

Until now:

- Propositional logic and model semantics
- LK Calculus
- Soundness
- Completeness

Next three weeks:

- First-order logic and model semantics
- LK Calculus for first-order logic
- Soundness
- Completeness

After that: resolution, DPLL, Prolog,. . .