IN3070/4070 – Logic – Autumn 2020

Lecture 3: LK: Soundness & Completeness

Martin Giese

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Department of Informatics

University of Oslo

Today's Plan

- Semantics for Sequents
- Soundness
- Completeness

Outline

- Semantics for Sequents
- Soundness
- Completeness

Semantics for Sequents

Definition 1.1 (Valid sequent).

A sequent $\Gamma \implies \Delta$ *is* *valid* *if all interpretations that satisfy all formulas in* Γ *satisfy at least one formula in* Δ.

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- $p, p \to q \implies q$
- $p \to q \implies \neg q \to \neg p$

Definition 1.2 (Countermodel/falsifiable sequent).

- *An interpretation* $\mathcal{I}$ *is a* countermodel *for the sequent* $\Gamma \implies \Delta$ *if* $v_{\mathcal{I}}(A) = T$ *for all formulae* $A \in \Gamma$ *and* $v_{\mathcal{I}}(B) = F$ *for all formulae* $B \in \Delta$

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- $p \implies$ — *Countermodel:* $\mathcal{I}(p) = T$
- $\implies$ — *Countermodel: all interpretations!*

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Not provable

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Definition 2.1 (Soundness).

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Theorem 2.1.

The sequent calculus LK *is sound.*

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Preservation of Falsifiability

Definition 2.2.

An LK*-rule* θ *preserves falsifiability (upwards) if all interpretations that falsify the conclusion* w *of an instance* $\frac{w_1 \cdots w_n}{w}$ *of* θ *also falsify at least one of the premises* w_i*.*

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Lemma 2.1.

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- We let Γ, Δ, A and B in the rule stand for arbitrary (sets of) propositional formulae

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- Thus, $\mathcal{I}$ falsifies the premisse.

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- Since $\mathcal{I}$ satisfies $A \rightarrow B$

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- What if S is very large, or infinite?
- We can generalise from an arbitrary element:
  - Choose an arbitrary element $a \in S$.
  - Show that $P(a)$ holds.
  - Since a was arbitrarily chosen, the original statement must hold.

How to show the Soundness Theorem?

We show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Reminder: LK derivation

Definition 2.3 (LK Derivation).

1. *Let* $\Gamma \implies \Delta$ *be a sequent. Then*

$$\Gamma \implies \Delta$$

is an LK-derivation of $\Gamma \implies \Delta$.

2. *Let* $\dfrac{w_1 \quad \cdots \quad w_n}{\Gamma \implies \Delta}$ *be an instance of an* LK *rule, and* $\mathcal{D}_1, \ldots, \mathcal{D}_n$ *derivations of* $w_1, \ldots, w_n$. *Then*

$$\frac{\mathcal{D}_1 \quad \cdots \quad \mathcal{D}_n}{\Gamma \implies \Delta}$$

is an LK-derivation of $\Gamma \implies \Delta$.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If an interpretation $\mathcal{I}$ falsifies the root sequent of an LK*-derivation δ, then $\mathcal{I}$ falsifies at least one of the leaf sequents of δ.*

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- Then $\mathcal{I}$ falsifies a leaf sequent in δ, namely $\Gamma \implies \Delta$.

□

Continued.

Induction step: δ is a derivation of the form

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for some smaller derivations $\mathcal{D}_i$ with roots $\Gamma_i \implies \Delta_i$.

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- This is also a leaf sequent of δ

□

How to show the Soundness Theorem?

We show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

All axioms are valid

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- Then $\mathcal{I}$ satisfies the formula A in the succedent.

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Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.

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- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- So $\mathcal{P}$ cannot be an LK-proof.

□

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Outline

- Semantics for Sequents
- Soundness
- Completeness

Completeness — Introduction

Definition 3.1 (Soundness).

The calculus LK *is* *sound* *if any* LK*-provable sequent is valid.*

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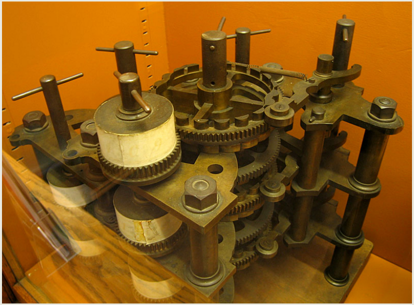
Soundness: $\Gamma \implies \Delta$ falsifiable $\Rightarrow$ $\Gamma \implies \Delta$ not provable
Completeness: $\Gamma \implies \Delta$ not provable $\Rightarrow$ $\Gamma \implies \Delta$ falsifiable

An LK-machine?



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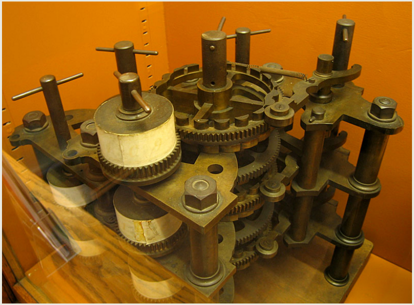

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All that is printed is valid.

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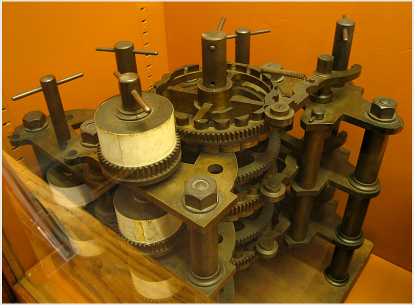

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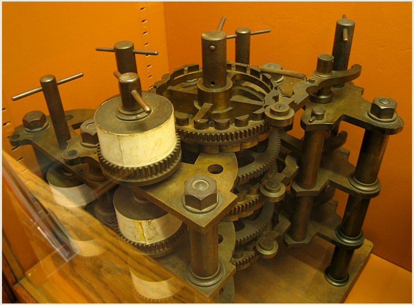

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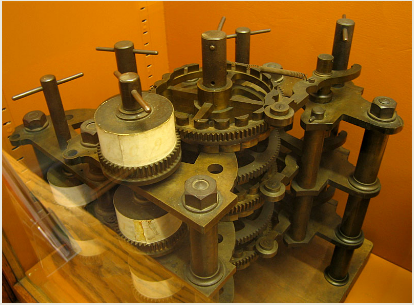

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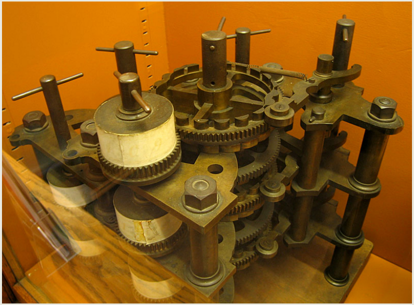

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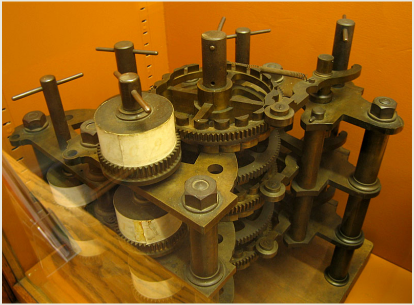

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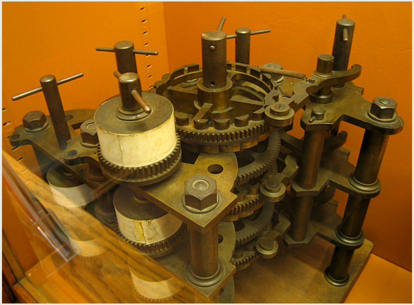

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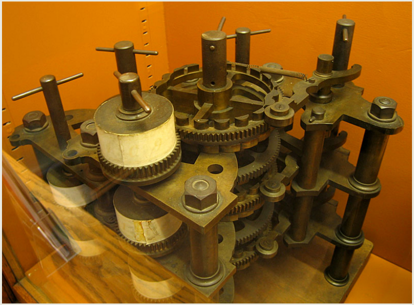

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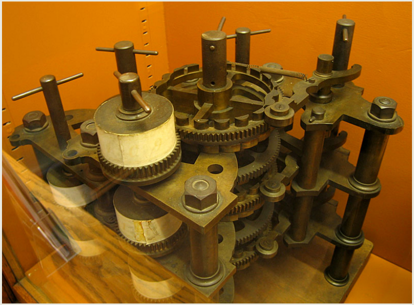

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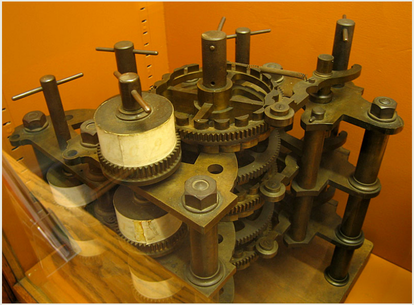

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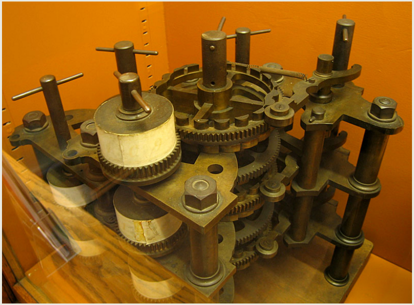

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This means that there is an interpretation that makes all formulae in Γ true and all formulae in Δ false.

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  - the leaf sequent of $\mathcal{B}$ is not an axiom.

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Assume $\Gamma \implies \Delta$ is not provable.

- Construct a derivation $\mathcal{D}$ from $\Gamma \implies \Delta$ such that no further rule applications are possible. "A maximal derivation."
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom. We then have:
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  $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on $\mathcal{B}$, and

  $\mathcal{B}^\bot$ be the set of formulae that occur in an succedent on $\mathcal{B}$, and

  $\mathcal{I}_\mathcal{B}$ be the interpretation that makes all atomic formulae in $\mathcal{B}^\top$ true and all other atomic formulae (in particular those in $\mathcal{B}^\bot$) false.

Example

$$
\dfrac{\dfrac{\dfrac{}{p \implies q, p} \qquad \dfrac{}{q, p \implies q}}{p \to q, p \implies q} \qquad \dfrac{\dfrac{}{r \implies q, p} \qquad \dfrac{}{q, r \implies q}}{p \to q, r \implies q}}{\dfrac{p \to q, p \vee r \implies q}{\color{red}{p \to q \implies (p \vee r) \to q}}}
$$

Example

$$\dfrac{\dfrac{\dfrac{}{p \Longrightarrow q, p} \quad \dfrac{}{q, p \Longrightarrow q}}{p \to q, p \Longrightarrow q} \qquad \dfrac{r \Longrightarrow q, p \quad \dfrac{}{q, r \Longrightarrow q}}{p \to q, r \Longrightarrow q}}{\dfrac{p \to q, p \lor r \Longrightarrow q}{p \to q \Longrightarrow \color{red}{(p \lor r) \to q}}}$$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{}{p \implies q, p} \qquad \dfrac{}{q, p \implies q}
  }{p \to q, p \implies q}
  \qquad
  \dfrac{
    r \implies q, p \qquad \dfrac{}{q, r \implies q}
  }{p \to q, r \implies q}
}{
  \dfrac{p \to q, {\color{red} p \lor r} \implies q}{p \to q \implies (p \lor r) \to q}
}
$$

Example

$$\dfrac{\dfrac{\dfrac{}{p \implies q, p} \qquad \dfrac{}{q, p \implies q}}{{\color{red}p \rightarrow q}, p \implies q} \qquad \dfrac{\dfrac{}{r \implies q, p} \qquad \dfrac{}{q, r \implies q}}{p \rightarrow q, r \implies q}}{\dfrac{p \rightarrow q, p \lor r \implies q}{p \rightarrow q \implies (p \lor r) \rightarrow q}}$$

Example

$$
\dfrac{
  \dfrac{
    \overline{{\color{red}p} \implies q, {\color{red}p}}
    \qquad
    \overline{q, p \implies q}
  }{
    p \to q, p \implies q
  }
  \qquad
  \dfrac{
    r \implies q, p
    \qquad
    \overline{q, r \implies q}
  }{
    p \to q, r \implies q
  }
}{
  \dfrac{
    p \to q, p \lor r \implies q
  }{
    p \to q \implies (p \lor r) \to q
  }
}
$$

Example

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\dfrac{
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    \dfrac{}{p \implies q, p}
    \qquad
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  }{p \to q, p \implies q}
  \qquad
  \dfrac{
    r \implies q, p
    \qquad
    \dfrac{}{\color{red}{q}, r \implies \color{red}{q}}
  }{p \to q, r \implies q}
}{
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}
$$

Example

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We see that the branch $\mathcal{B}$ with leaf sequent $r \implies q, p$ is not closed.

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$$\mathcal{B}^{\top} = \{r, p \to q, p \vee r\}$$

Example

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$$\mathcal{B}^{\bot} = \{q, p, (p \lor r) \to q\}$$

Example

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$$\mathcal{B}^{\top} = \{r, p \to q, p \vee r\}$$

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$$\mathcal{I}_{\mathcal{B}} = \text{interpretation with } \mathcal{I}_{\mathcal{B}}(r) = T \text{ og } \mathcal{I}_{\mathcal{B}}(q) = \mathcal{I}_{\mathcal{B}}(p) = F$$

Example

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$$\begin{aligned}
\mathcal{B}^{\top} &= \{r, p \to q, p \lor r\} \\
\mathcal{B}^{\bot} &= \{q, p, (p \lor r) \to q\} \\
\mathcal{I}_{\mathcal{B}} &= \text{interpretation with } \mathcal{I}_{\mathcal{B}}(r) = T \text{ og } \mathcal{I}_{\mathcal{B}}(q) = \mathcal{I}_{\mathcal{B}}(p) = F \\
& \text{To show: this interpretation falsifies the root sequent.}
\end{aligned}$$

Proof of Completeness, cont.

- We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_{\mathcal{B}}$ makes all formulae in $\mathcal{B}^{\top}$ true, and all formulae in $\mathcal{B}^{\perp}$ false.

Proof of Completeness, cont.

- We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_{\mathcal{B}}$ makes all formulae in $\mathcal{B}^{\top}$ true, and all formulae in $\mathcal{B}^{\perp}$ false.
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  If $A \in \mathcal{B}^{\top}$, then $\mathcal{I}_{\mathcal{B}} \models A$.

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<u>Induction base:</u> A is an atomic formula in $\mathcal{B}^{\top}/\mathcal{B}^{\perp}$.

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- Our statment holds for $A \in \mathcal{B}^{\top}$ because that is how we defined $\mathcal{I}_{\mathcal{B}}$.

Proof of Completeness, cont.

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- For $A \in \mathcal{B}^{\perp}$, $A \notin \mathcal{B}^{\top}$ because atoms do not disappear from a branch and $\mathcal{B}$ contains no axiom.

Proof of Completeness, cont.

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Proof of Completeness, cont.

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Induction step: From the assumption (IH) that the statement holds for A and B, we must show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$ og $(A \rightarrow B)$. These are four cases, of which we show three here.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

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- A appears in a succedent, so we have $A \in \mathcal{B}^{\perp}$.

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- A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.

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- A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

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- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\perp$.

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- Since the derivation is maximal, $\neg$-left *is* eventually applied
- A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\perp$.

- $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$-right is applied
- Since the derivation is maximal, $\neg$-right *is* eventually applied
- A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

- $\neg A$ appears in an antecedent, it can't 'go away' unless $\neg$-left is applied
- Since the derivation is maximal, $\neg$-left *is* eventually applied
- A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\perp$.

- $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$-right is applied
- Since the derivation is maximal, $\neg$-right *is* eventually applied
- A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

- $\neg A$ appears in an antecedent, it can't 'go away' unless $\neg$-left is applied
- Since the derivation is maximal, $\neg$-left *is* eventually applied
- A appears in a succedent, so we have $A \in \mathcal{B}^\bot$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\bot$.

- $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$-right is applied
- Since the derivation is maximal, $\neg$-right *is* eventually applied
- A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models \neg A$.

Case: Conjunction in antecedent/succedent

Assume that $(A \land B) \in \mathcal{B}^\top$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
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- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

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Assume that $(A \wedge B) \in \mathcal{B}^{\top}$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^{\top}$ and $B \in \mathcal{B}^{\top}$.
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- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^{\perp}$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\wedge$-right is eventually applied...

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^{\top}$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^{\top}$ and $B \in \mathcal{B}^{\top}$.
- By the IH, we have $\mathcal{I}_{\mathcal{B}} \models A$ and $\mathcal{I}_{\mathcal{B}} \models B$.
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^{\perp}$.

- Since the derivation is maximal, $\wedge$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.

Case: Conjunction in antecedent/succedent

Assume that $(A \land B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \land B)$.

Assume that $(A \land B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\land$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\perp$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\wedge$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \not\models B$

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^{\top}$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^{\top}$ and $B \in \mathcal{B}^{\top}$.
- By the IH, we have $\mathcal{I}_{\mathcal{B}} \models A$ and $\mathcal{I}_{\mathcal{B}} \models B$.
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^{\perp}$.

- Since the derivation is maximal, $\wedge$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^{\perp}$ or $B \in \mathcal{B}^{\perp}$.
- By the IH, we have $\mathcal{I}_{\mathcal{B}} \not\models A$ or $\mathcal{I}_{\mathcal{B}} \not\models B$
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \not\models (A \wedge B)$

Case: Implication in antecedent/succedent

Assume that $(A \rightarrow B) \in \mathcal{B}^\top$.

Case: Implication in antecedent/succedent

Assume that $(A \rightarrow B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, $\rightarrow$-left is eventually applied. . .

Case: Implication in antecedent/succedent

Assume that $(A \to B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, $\to$-left is eventually applied...
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- Structural induction on formulae, while soundness was by induction on derivations
- Not possible to prove completeness 'one rule at a time'

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- Can do the same with empty antecedents $\implies \Delta$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p, p, \neg q \implies} \qquad \dfrac{}{q, p, \neg q \implies}}{p \to q, p, \neg q \implies} \qquad \dfrac{\neg p, r, \neg q \implies \qquad \dfrac{}{q, r, \neg q \implies}}{p \to q, r, \neg q \implies}}{p \to q, p \lor r, \neg q \implies}}{\color{red}{p \to q, \neg((p \lor r) \to q) \implies}}$$

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\dfrac{
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    \quad
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  \qquad
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- Soundness and completeness very similar to two-sided LK.

Semantic Tableaux (Ben-Ari 2.6)

- Others call these 'block tableaux'
- Sequent arrow $\Longrightarrow$ not needed for one-sided calculus
- More handy to write top-down, like everybody else
- Mark 'closed' branches (with axioms) with $\times$

$$p \rightarrow q, \neg((p \vee r) \rightarrow q)$$

$$p \rightarrow q, p \vee r, \neg q$$

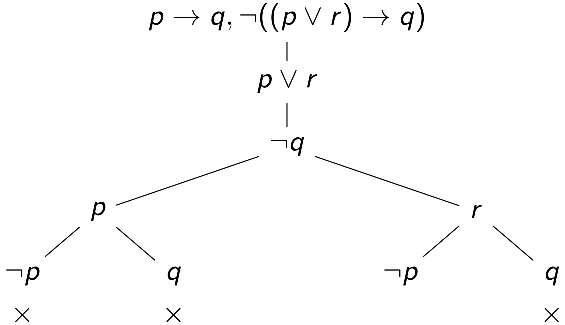
$$p \rightarrow q, p, \neg q \qquad\qquad p \rightarrow q, r, \neg q$$

$$\neg p, p, \neg q \quad q, p, \neg q \qquad\qquad \neg p, r, \neg q \quad q, r, \neg q$$

$$\times \qquad \times \qquad\qquad\qquad \times$$

Short Hand Notation for Tableaux

- Only write the new formula in every node.
- Even more handy to write
- Close branch using literals A and $\neg A$ anywhere on a branch.
- Have to make sure that all rules were used on every branch!



Summary and Outlook

Until now:

- Propositional logic and model semantics
- LK Calculus
- Soundness
- Completeness

Next three weeks:

- First-order logic and model semantics
- LK Calculus for first-order logic
- Soundness
- Completeness

After that: resolution, DPLL, Prolog,...