

# IN3070/4070 – Logic – Autumn 2020

## Lecture 4: First-order Logic

Martin Giese

10th September 2020



DEPARTMENT OF  
INFORMATICS



UNIVERSITY OF  
OSLO

# Today's Plan

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

# Limitations of Propositional Logic

**Propositional logic:** atomic formula  $(p, q, r), \wedge, \vee, \neg, \rightarrow, (, )$

**Problem:** How do we represent the following statements?

- ▶ “all men are mortal”  $\forall x(\text{man}(x) \rightarrow \text{mortal}(x))$
- ▶ “there exist prime numbers that are even”  $\exists y(\text{prime}(y) \wedge \text{even}(y))$
- ▶ “1 is smaller than 3”  $1 < 3$  or  $<(1, 3)$
- ▶ “transitivity of smaller”  $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$
- ▶  $2 * 8 = 16$   $= (*(2, 8), 16)$
- ▶ “if  $x$  is even then  $x + 2$  is even”  $\forall x (\text{even}(x) \rightarrow \text{even}(x + 2))$
- ▶ “if  $x$  is prime then  $x + 2$  is prime”  $\forall x (\text{prime}(x) \rightarrow \text{prime}(x + 2))$

**First-order logic:** extension of propositional logic

# First-Order Logic — Overview

Extending propositional logic by...

Syntax:

- ▶ constants  $(a, b, c)$ , functions  $(f, g, h)$ , variables  $(x, y, z)$
- ▶ predicates  $(p, q, r)$
- ▶ terms  $(t, u, v)$
- ▶ quantifiers  $(\forall, \exists)$
- ▶ scope of variables, free variables, variable assignment/substitution

Semantics:

- ▶ interpretation of constants, functions, variables
- ▶ interpretation of predicates
- ▶ value of terms
- ▶ truth value of (quantified) formulae
- ▶ satisfiability, validity, logical equivalence,...

# Outline

- ▶ Motivation
- ▶ **Syntax**
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

# Syntax — Terms

**Terms** are built up of constant (symbols), variable (symbols), and function (symbols).

## Definition 2.1 (Terms).

Let  $\mathcal{A} = \{a, b, \dots\}$  be a countable set of *constant symbols*,  
 $\mathcal{V} = \{x, y, z, \dots\}$  be a countable set of *variable symbols*, and  
 $\mathcal{F} = \{f, g, h, \dots\}$  be a countable set of *function symbols*.

*Terms*, denoted  $t, u, v$ , are inductively defined as follows:

1. Every variable  $x \in \mathcal{V}$  is a term.
2. Every constant  $a \in \mathcal{A}$  is a term.
3. If  $f \in \mathcal{F}$  is an  $n$ -ary function (symbol)  $n > 0$  and  $t_1, \dots, t_n$  are terms, then  $f(t_1, \dots, t_n)$  is a term.

**Example:**  $a, x, f(a, x), f(g(x), b)$ , and  $g(f(a, g(y)))$  are terms.

## Syntax — First-Order Formulae

Formulae are built up of **atomic formulae** and the **logical connectives**  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$ , and  $\forall$  (universal quantifier),  $\exists$  (existential quantifier).

### Definition 2.2 (Atomic Formulae).

Let  $\mathcal{P} = \{p, q, r, \dots\}$  be a countable set of **predicate symbols**. If  $p \in \mathcal{P}$  is an  $n$ -ary predicate (symbol)  $n \geq 0$  and  $t_1, \dots, t_n$  are terms, then  $p(t_1, \dots, t_n)$ ,  $\top$ , and  $\perp$  are **atomic formulae** (or **atoms**).

### Definition 2.3 ((First-Order) Formulae).

*(First-order) formulae*, denoted  $A, B, C, F, G, H$ , are inductively defined as follows:

1. Every atomic formula  $p$  is a formula.
2. If  $A$  and  $B$  are formulae and  $x \in \mathcal{V}$ , then  $(\neg A)$ ,  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ ,  $\forall x A$ , and  $\exists x A$  are formulae.

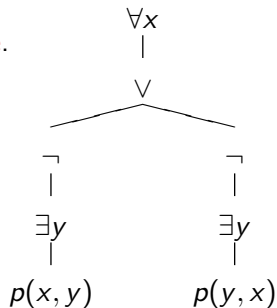


# Formula Trees

A formula can be presented as **formula tree**.

**Example:**

$$\forall x (\neg \exists y p(x, y) \vee \neg \exists y p(y, x))$$



## Definition 2.4 (Subformula, Main Operator).

Formula  $A$  is a (*proper*) **subformula** of formula  $B$  iff  $A$  is a (*proper*) subtree of  $B$ . If the root of a formula tree of  $A$  is a logical connective/quantifier, then it is called the **main operator** of  $A$ .

# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ **Variables**
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

# Free Variables

A **free variable** is a variable that is not in the **scope** of a quantifier.

## Definition 3.1 (Free/Bound Variables, Closed Formula/Term).

*Free variables* in a formula  $A$  are inductively defined:

1. If  $A$  is an atomic formula, then all variables in  $A$  are free.
2. If  $A = \neg B$ , then the free variables of  $A$  are exactly those of  $B$ .
3. If  $A = B \wedge C$ ,  $A = B \vee C$ , or  $A = B \rightarrow C$ , then the free variables of  $A$  are those of  $B$  together with those of  $C$ .
4. If  $A = \forall x B$  or  $A = \exists x B$ , then the free variables of  $A$  are those of  $B$  without the variable  $x$ .

A **bound variable** in a formula  $C$  is a variable that appears in  $\forall x$  or  $\exists x$  in some subformula of  $C$ . A formula/term is **closed** iff it has no free variables.

# Scope, Universal and Existential Closure

## Definition 3.2 (Scope of Variables).

Let  $\forall x A$  or  $\exists x A$  be a universally or existentially *quantified formula*. Then  $x$  is the *quantified variable* and its *scope* is the formula  $A$ .

**Remark:** It is not required that  $x$  actually appears in the scope of its quantification, e.g.  $\forall x \exists y p(y, y)$ .

## Definition 3.3 (Universal and Existential Closure).

If  $\{x_1, \dots, x_n\}$  are all the free variables of  $A$ , the *universal closure* of  $A$  is  $\forall x_1 \dots \forall x_n A$  and the *existential closure* of  $A$  is  $\exists x_1 \dots \exists x_n A$ .

- ▶  $p(x, y)$  has the two free variables  $x$  and  $y$ . Its universal closure is  $\forall x \forall y p(x, y)$  and its existential closure is  $\exists x \exists y p(x, y)$ ;  $\exists y p(x, y)$  has the only free variable  $x$ ;  $\forall x \exists y p(x, y)$  is closed
- ▶ In  $\forall x p(x) \wedge q(x)$ , the  $x$  occurs bound **and** free. The existential closure is  $\exists x (\forall x p(x) \wedge q(x))$ ; **renaming:**  $\exists y (\forall x p(x) \wedge q(y))$

# Substitutions

**Free variables** in a first-order formula can be substituted by terms.

## Definition 3.4 (Substitution).

Let  $\mathcal{V}$  be a set of variables,  $\mathcal{T}$  be the set of terms. A **substitution**  $\sigma : \mathcal{V} \rightarrow \mathcal{T}$  assigns each variable a term.

**Remark:** The substitution  $\sigma$  is often represented as set  $\{x \setminus t \mid \sigma(x) = t\}$ .

**Example:** For the variable set  $\{x, y\}$ ,  $\sigma(x) = a$ ,  $\sigma(y) = f(z, b)$  is a substitution and can also be represented as  $\{x \setminus a, y \setminus f(z, b)\}$ .

Ben-Ari:  $\{x \leftarrow a, y \leftarrow f(z, b)\}$ .

Others:  $[a/x, f(z, b)/y]$

# Application of substitutions

## Definition 3.5 (Application of Substitutions, informally).

Let  $\sigma$  be a substitution. The *application* of  $\sigma$  to a term  $t$  or formula  $A$ , written  $\sigma(t)$  or  $\sigma(A)$ , replaces every free variable in  $t$  or  $A$  according to its image under  $\sigma$ . Short hand:  $A[x \setminus t] = \sigma(A)$  with  $\sigma = \{x \setminus t\}$ .

**Example:** Let  $\sigma = \{x \setminus a, y \setminus f(z, b)\}$  be a substitution.

Then  $\sigma(g(y)) = g(f(z, b))$

and  $\sigma(p(x) \wedge \forall x q(x, g(y))) = p(a) \wedge \forall x q(x, g(f(z, b)))$

**Problem:**  $\sigma(\forall z p(z, y)) = \forall z p(z, f(z, b))$

The free variable  $z$  in  $\sigma$  is **captured** by the quantifier.

This is **bad** because the effect depends on the choice of variable names

## Definition 3.6 (Capture-free substitution).

A substitution  $\sigma$  is *capture-free* for a formula  $A$  if for every free variable  $x$  in  $A$ , none of the variables in  $\sigma(x)$  is bound in  $A$ .

# Application of Substitutions

## Definition 3.7 (Application of Substitutions, formally).

The application of a substitution  $\sigma$  to a term or formula is defined by structural induction:

- ▶  $\sigma(x) = \sigma(x)$  for variables  $x$  in the range of  $\sigma$
- ▶  $\sigma(y) = y$  for variables  $y$  not in the range of  $\sigma$
- ▶  $\sigma(a) = a$  for constants  $a \in \mathcal{A}$
- ▶  $\sigma(f(t_1, \dots, t_n)) = f(\sigma(t_1), \dots, \sigma(t_n))$  for a function symbol  $f \in \mathcal{F}$
- ▶  $\sigma(p(t_1, \dots, t_n)) = p(\sigma(t_1), \dots, \sigma(t_n))$  for a predicate symbol  $p \in \mathcal{P}$
- ▶  $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$  for formulae  $A, B$
- ▶ ... similarly for  $\neg A, A \vee B, A \rightarrow B$ ...
- ▶  $\sigma(\exists x A) = \exists x \sigma_x(A), \quad \sigma(\forall x A) = \forall x \sigma_x(A)$

where we define  $\sigma_x$  by:  $\sigma_x(x) = x$ , and  $\sigma_x(y) = \sigma(y)$  for all  $y \neq x$

# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ **Semantics**
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary



# Semantics — Interpretation

An **interpretation** assigns concrete objects, functions and relations to constant symbols, function symbols, and predicate symbols.

## Definition 4.1 (Interpretation/Structure).

An **interpretation** (or **structure**)  $\mathcal{I} = (D, \iota)$  consists of the following elements:

1. **Domain**  $D$  is a non-empty set
2. **Interpretation of constant symbols** assigns each constant  $a \in \mathcal{A}$  an element  $a^\iota \in D$
3. **Interpretation of function symbols** assigns each  $n$ -ary function symbol  $f \in \mathcal{F}$  with  $n > 0$  a function  $f^\iota : D^n \rightarrow D$
4. **Interpretation of propositional variables** assigns each 0-ary predicate symbol  $p \in \mathcal{P}$  a truth value  $p^\iota \in \{T, F\}$
5. **Interpretation of predicate symbols** assigns each  $n$ -ary predicate symbol  $p \in \mathcal{P}$  with  $n > 0$  a relation  $p^\iota \subseteq D^n$

## Semantics — Examples

**Example:**  $\forall x p(a, x)$  with the interpretations

1.  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $p^\iota = \leq$  and  $a^\iota = 0$
2.  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $p^\iota = \leq$  and  $a^\iota = 3$
3.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^\iota = \leq$  and  $a^\iota = 0$
4.  $\mathcal{I} = (\{c, d, e, f\}, \iota)$  with  $p^\iota = \leq_{lexi}$  and  $a^\iota = c$

**Remark:** In Ben-Ari:  $(\mathbb{N}, \{\leq\}, \{0\})$ ,  $(\mathbb{N}, \{\leq\}, \{3\})$ ,  $(\mathbb{Z}, \{\leq\}, \{0\})$

**Example:**  $\forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$  with interpretations

1.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^\iota = \leq$ ,  $f^\iota = +$ , and  $a^\iota = 1$
2.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^\iota = >$ ,  $f^\iota = *$ , and  $a^\iota = -1$

**Remark:** In Ben-Ari:  $(\mathbb{Z}, \{\leq\}, \{+\}, \{1\})$ ,  $(\mathbb{Z}, \{>\}, \{*\}, \{-1\})$ .

## Semantics — Value of Closed Terms

Terms are evaluated according to the interpretation of their constant and function symbols.

### Definition 4.2 (Term Value for Closed Terms).

Let  $\mathcal{I} = (D, \iota)$  be an interpretation. The *term value*  $v_{\mathcal{I}}(t)$  of a closed term  $t \in \mathcal{T}$  under the interpretation  $\mathcal{I}$  is inductively defined:

1. For a constant symbol  $a \in \mathcal{A}$  the term value is  $v_{\mathcal{I}}(a) = a^{\iota}$ ;
2. Let  $f \in \mathcal{F}$  be an  $n$ -ary function,  $n > 0$ , and  $t_1, \dots, t_n$  be terms; the term value of  $f(t_1, \dots, t_n)$  is  $v_{\mathcal{I}}(f(t_1, \dots, t_n)) = f^{\iota}(v_{\mathcal{I}}(t_1), \dots, v_{\mathcal{I}}(t_n))$

### Examples:

- ▶  $f(a, f(a, b))$  with  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $f^{\iota} = +$ ,  $a^{\iota} = 20$ ,  $b^{\iota} = 2$ ; then  $v_{\mathcal{I}}(f(a, f(a, b))) = 42$
- ▶  $+(1, *(4, 2))$  with  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $+^{\iota} = *$  (multiplication),  $*^{\iota} = -$  (subtraction),  $1^{\iota} = -20$ ,  $2^{\iota} = 0$ ,  $4^{\iota} = 10$ ; then  $v_{\mathcal{I}}(+(1, *(4, 2))) = -200$

# Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.  
We need something additional.

## Definition 4.3 (Variable Assignment).

Given the set of variables  $\mathcal{V}$ , and an interpretation  $\mathcal{I} = (D, \iota)$ , a variable assignment  $\alpha$  for  $\mathcal{I}$  is a function  $\alpha : \mathcal{V} \rightarrow D$ .

Ben-Ari (7.18) writes this  $\sigma_{\mathcal{I}_A}$

## Definition 4.4 (Term Value).

Let  $\mathcal{I} = (D, \iota)$  be an interpretation, and  $\alpha$  an variable assignment for  $\mathcal{I}$ . The **term value**  $v_{\mathcal{I}}(\alpha, t)$  of a term  $t \in \mathcal{T}$  under  $\mathcal{I}$  and  $\alpha$  is inductively defined:

1.  $v_{\mathcal{I}}(\alpha, x) = \alpha(x)$  for a variable  $x \in \mathcal{V}$
2.  $v_{\mathcal{I}}(\alpha, a) = a^{\iota}$  for a constant symbol  $a \in \mathcal{A}$
3.  $v_{\mathcal{I}}(\alpha, f(t_1, \dots, t_n)) = f^{\iota}(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n))$  for an  $n$ -ary  $f \in \mathcal{F}$

## Semantics — Term value Examples

- ▶  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $f^\iota = +$ ,  $a^\iota = 10$ 
  - ▶  $\mathcal{V} = \{x, y\}$
  - ▶  $\alpha(x) = 3 \in \mathbb{N}$  and  $\alpha(y) = 5 \in \mathbb{N}$  is an assignment for  $\mathcal{I}$
  - ▶  $v_{\mathcal{I}}(\alpha, f(a, f(a, x))) = 23$
- ▶  $\mathcal{I} = (\text{Strings}, \iota)$  with  $g^\iota = \text{concatenation}$ ,  $a^\iota = \text{"Hello"}$ 
  - ▶  $\mathcal{V} = \{y\}$
  - ▶  $\alpha(y) = \text{"students"}$
  - ▶  $v_{\mathcal{I}}(\alpha, f(a, f(y, a))) = \text{"HelloworldstudentsHello"}$

## Semantics — Modification of an assignment

**Definition 4.5 (Modification of a variable assignment).**

Given an interpretation  $\mathcal{I} = (D, \iota)$  and a variable assignment  $\alpha$  for  $\mathcal{I}$ .  
 Given also a variable  $y \in \mathcal{V}$  and a domain element  $d \in D$ .  
 The modified variable assignment  $\alpha\{y \leftarrow d\}$  is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- ▶  $\mathcal{I} = (\mathbb{N}, \iota)$
- ▶  $\mathcal{V} = \{x, y\}$
- ▶  $\alpha(x) = 3 \in \mathbb{N}$  and  $\alpha(y) = 5 \in \mathbb{N}$  is an assignment for  $\mathcal{I}$
- ▶  $\alpha\{y \leftarrow 7\}(x) = 3$  and  $\alpha\{y \leftarrow 7\}(y) = 7$

## Semantics — Truth Value

## Definition 4.6 (Truth Value).

Let  $\mathcal{I} = (D, \iota)$  be an interpretation and  $\alpha$  an assignment for  $\mathcal{I}$ . The **truth value**  $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$  of a formula  $A$  under  $\mathcal{I}$  and  $\alpha$  is defined inductively as follows:

1.  $v_{\mathcal{I}}(\alpha, p) = T$  for 0-ary  $p \in \mathcal{P}$  iff  $p^{\iota} = T$ , otherwise  $v_{\mathcal{I}}(\alpha, p) = F$
2.  $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$  for  $p \in \mathcal{P}$ ,  $n > 0$ , iff  $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$ , otherwise  $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3.  $v_{\mathcal{I}}(\alpha, \neg A) = T$  iff  $v_{\mathcal{I}}(\alpha, A) = F$ , otherwise  $v_{\mathcal{I}}(\alpha, \neg A) = F$
4.  $v_{\mathcal{I}}(\alpha, A \wedge B) = T$  iff  $v_{\mathcal{I}}(\alpha, A) = T$  and  $v_{\mathcal{I}}(\alpha, B) = T$ , otherwise  $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5.  $v_{\mathcal{I}}(\alpha, A \vee B) = T$  iff  $v_{\mathcal{I}}(\alpha, A) = T$  or  $v_{\mathcal{I}}(\alpha, B) = T$ , otherwise  $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6.  $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$  iff  $v_{\mathcal{I}}(\alpha, A) = F$  or  $v_{\mathcal{I}}(\alpha, B) = T$ , otherwise  $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7.  $v_{\mathcal{I}}(\alpha, \forall x A) = T$  iff  $v_{\mathcal{I}}(\alpha \{x \leftarrow d\}, A) = T$  for all  $d \in D$ , otherwise  $v_{\mathcal{I}}(\alpha, \forall x A) = F$
8.  $v_{\mathcal{I}}(\alpha, \exists x A) = T$  iff  $v_{\mathcal{I}}(\alpha \{x \leftarrow d\}, A) = T$  for some  $d \in D$ , otherwise  $v_{\mathcal{I}}(\alpha, \exists x A) = F$
9.  $v_{\mathcal{I}}(\alpha, \top) = T$  and  $v_{\mathcal{I}}(\alpha, \perp) = F$

## Semantics — Truth Value

**Theorem 4.1 (Value of closed formulae).**

For a *closed* term or formula, the assignment has no influence on the term value or truth value. We can write  $v_{\mathcal{I}}(A)$  instead of  $v_{\mathcal{I}}(\alpha, A)$ .

**Example:**  $A = \forall x p(a, x)$  with the interpretations

1.  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $p^{\iota} = \leq$  and  $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2.  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $p^{\iota} = \leq$  and  $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^{\iota} = \leq$  and  $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4.  $\mathcal{I} = (\{c, d, e, f\}, \iota)$  with  $p^{\iota} = \leq_{lex}$  and  $a^{\iota} = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

**Example:**  $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$  with interpretations

1.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^{\iota} = \leq$ ,  $f^{\iota} = +$ , and  $a^{\iota} = 1$   
 $\rightsquigarrow v_{\mathcal{I}}(B) = T$
2.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^{\iota} = >$ ,  $f^{\iota} = *$ , and  $a^{\iota} = -1$   
 $\rightsquigarrow v_{\mathcal{I}}(B) = F$



# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ **The Substitution Lemma**
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

# The Substitution Lemma for Terms

## Theorem 5.1 (Substitution Lemma for Terms).

Given an interpretation  $\mathcal{I} = (D, \iota)$  and a variable assignment  $\alpha$  for  $\mathcal{I}$ .  
 Given also a variable  $y \in \mathcal{V}$ , and terms  $t, s \in \mathcal{T}$

$$v_{\mathcal{I}}(\alpha, t[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

### Proof.

By structural induction on  $t$ . We abbreviate:  $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

For a constant  $a$ ,  $a[y \setminus s] = a$ , so  $v_{\mathcal{I}}(\alpha, a[y \setminus s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

For a variable  $x \neq y$ ,  $x[y \setminus s] = x$ , so

$$v_{\mathcal{I}}(\alpha, x[y \setminus s]) = v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \alpha'(x) = v_{\mathcal{I}}(\alpha', x) \quad \square$$

## Proof of substitution lemma, continued

## Proof.

For the variable  $y$ ,  $y[y \setminus s] = s$ , so

$$v_{\mathcal{I}}(\alpha, y[y \setminus s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha \{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, y)$$

For a complex term,  $f(\dots t_i \dots)[y \setminus s] = f(\dots t_i[y \setminus s] \dots)$ , so

$$\begin{aligned} & v_{\mathcal{I}}(\alpha, f(\dots t_i \dots)[y \setminus s]) \\ &= v_{\mathcal{I}}(\alpha, f(\dots t_i[y \setminus s] \dots)) \quad \text{by def. of substitution} \\ &= f^v(\dots v_{\mathcal{I}}(\alpha, t_i[y \setminus s]) \dots) \quad \text{by model semantics} \\ &= f^v(\dots v_{\mathcal{I}}(\alpha', t_i) \dots) \quad \text{by the induction hypothesis} \\ &= v_{\mathcal{I}}(\alpha', f(\dots t_i \dots)) \quad \text{by model semantics} \end{aligned}$$



# The Substitution Lemma for Formulae

## Theorem 5.2 (Substitution Lemma for Formulae).

Given an interpretation  $\mathcal{I} = (D, \iota)$  and a variable assignment  $\alpha$  for  $\mathcal{I}$ .  
Given also a variable  $y \in \mathcal{V}$ , a formula  $A$  and a term  $s \in \mathcal{T}$ , such that  
 $\{y \setminus s\}$  is capture-free for  $A$ .

$$v_{\mathcal{I}}(\alpha, A[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ **Satisfiability & Validity**
- ▶ LK for First-order Logic
- ▶ Summary

# Satisfiability and Validity

## Definition 6.1 (Satisfiable, Model, Unsatisfiable, Valid, Invalid).

Let  $A$  be a *closed* (first-order) formula and  $U = \{A_1, \dots\}$  be a set of *closed* (first-order) formulae  $A_i$ .

- ▶  $A$  is *satisfiable* iff  $v_{\mathcal{I}}(A) = T$  for some interpretation  $\mathcal{I}$ .
- ▶ A satisfying interpretation  $\mathcal{I}$  for  $A$  is called a *model* for  $A$ .
- ▶  $U = \{A_1, \dots\}$  is *satisfiable* iff there is (common) model for all  $A_i$ .
- ▶  $A$  (resp.  $U$ ) is *unsatisfiable* iff  $A$  (resp.  $U$ ) is *not* satisfiable.
- ▶  $A$  is *valid*, denoted  $\models A$ , iff  $v_{\mathcal{I}}(A) = T$  for all interpretations  $\mathcal{I}$ .
- ▶  $A$  is *invalid/falsifiable* iff  $A$  is *not* valid.

## Theorem 6.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

$A$  is *valid* iff  $\neg A$  is *unsatisfiable*.  $A$  is *satisfiable* iff  $\neg A$  is *invalid*.

# Examples for Satisfiable and Invalid Formulae

**Example:**  $A = \forall x p(a, x)$

1.  $\mathcal{I} = (\mathbb{N}, \iota)$  with  $p^\iota = \leq$  and  $a^\iota = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$   
 $\rightsquigarrow A$  is invalid
2.  $\mathcal{I} = (\{c, d, e, f\}, \iota)$  with  $p^\iota = \leq_{lex}$  and  $a^\iota = c \rightsquigarrow v_{\mathcal{I}}(A) = T$   
 $\rightsquigarrow A$  is satisfiable ( $\mathcal{I}$  is a model)

**Example:**  $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$

1.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^\iota = \leq$ ,  $f^\iota = +$ , and  $a^\iota = 1 \rightsquigarrow v_{\mathcal{I}}(B) = T$   
 $\rightsquigarrow$  satisfiable ( $\mathcal{I}$  is a model)
2.  $\mathcal{I} = (\mathbb{Z}, \iota)$  with  $p^\iota = >$ ,  $f^\iota = *$ , and  $a^\iota = -1 \rightsquigarrow v_{\mathcal{I}}(B) = F$   
 $\rightsquigarrow$  invalid ( $\mathcal{I}$  is a “counter-model”)

**Example:**  $\forall x \forall y (p(x, y) \rightarrow p(y, x))$

$\rightsquigarrow$  satisfiable (e.g.  $p^\iota = “=”$ ), but invalid (e.g.  $p^\iota = “<”$ )

**Example:**  $\exists x \exists y (p(x) \wedge \neg p(y))$

$\rightsquigarrow$  only satisfiable for  $|D| \geq 2$ , invalid (e.g.  $D = \mathbb{N}$ ,  $p^\iota = \text{even}$ )

# Logical Equivalence

The concept of **logical equivalence** can be adapted to first-order logic, i.e. to closed first-order formulae.

## Definition 6.2 (Logical Equivalence).

Let  $A_1, A_2$  be two closed formulae.  $A_1$  is **logically equivalent** to  $A_2$ , denoted  $A_1 \equiv A_2$  iff  $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$  for all interpretations  $\mathcal{I}$ .

## Theorem 6.2 (Relation $\equiv$ and $\leftrightarrow$ ).

Let  $A, B$  be two closed formulae. Then  $A \equiv B$  iff  $\models A \leftrightarrow B$ .

**Remark:**  $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$

**Important:** even though  $\equiv$  and  $\leftrightarrow$  are closely related, they are different relations. Whereas  $\leftrightarrow$  is part of the object language (i.e. the definition of formulae),  $\equiv$  is used in the meta-language to talk about or relate formulae.



# Logically Equivalent Formulae

**Duality:**  $\forall$  can be expressed with  $\exists$ , and vice versa

$$\blacktriangleright \models \forall x A(x) \leftrightarrow \neg \exists x \neg A(x)$$

$$\blacktriangleright \models \exists x A(x) \leftrightarrow \neg \forall x \neg A(x)$$

**Commutativity:**

$$\blacktriangleright \models \forall x \forall y A(x, y) \leftrightarrow \forall y \forall x A(x, y)$$

$$\blacktriangleright \models \exists x \exists y A(x, y) \leftrightarrow \exists y \exists x A(x, y)$$

$$\blacktriangleright \models \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y) \quad (\text{other direction is not valid!})$$

**Distributivity:**

$$\blacktriangleright \models \exists x (A(x) \vee B(x)) \leftrightarrow \exists x A(x) \vee \exists x B(x)$$

$$\blacktriangleright \models \forall x (A(x) \wedge B(x)) \leftrightarrow \forall x A(x) \wedge \forall x B(x)$$

$$\blacktriangleright \models \forall x A(x) \vee \forall x B(x) \rightarrow \forall x (A(x) \vee B(x)) \quad (\text{other direction not valid!})$$

$$\blacktriangleright \models \exists x (A(x) \wedge B(x)) \rightarrow \exists x A(x) \wedge \exists x B(x) \quad (\text{other direction not valid!})$$

See [Ben-Ari 2012] for **more equivalences** involving quantifiers.

# Logical Consequence

## Definition 6.3 (Logical Consequence).

Let  $A$  be a closed formula and  $U$  be a set of closed formulae.  $A$  is a **logical consequence** of  $U$ , denoted  $U \models A$ , iff every model of  $U$  is a model of  $A$ , i.e.  $v_{\mathcal{I}}(A_i) = T$  for all  $A_i \in U$  implies  $v_{\mathcal{I}}(A) = T$ .

## Theorem 6.3 (Logical Consequence and Validity).

Let  $A$  be a closed formula and  $U = \{A_1, \dots, A_n\}$  be a set of closed formulae. Then  $U \models A$  iff  $\models (A_1 \wedge \dots \wedge A_n) \rightarrow A$ .

- ▶ again, we can **reduce** the problem of “logical consequence” to the problem of determining if a formula is **valid**
- ▶ hence, we need methods or **proof search calculi** that can deal with **first-order formulae**

# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ **LK for First-order Logic**
- ▶ Summary

## LK — Axiom and Propositional Rules

## ► axiom

$$\frac{}{\Gamma, A \Rightarrow A, \Delta} \text{ axiom}$$

► rules for  $\wedge$  (conjunction)

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge\text{-left} \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \wedge\text{-right}$$

► rules for  $\vee$  (disjunction)

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee\text{-left} \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \vee\text{-right}$$

► rules for  $\rightarrow$  (implication)

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow\text{-left} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \rightarrow\text{-right}$$

► rules for  $\neg$  (negation)

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \neg\text{-left} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \neg\text{-right}$$

## LK — Rules for Universal and Existential Quantifier

► rules for  $\forall$  (universal quantifier)

$$\frac{\Gamma, A[x \setminus t], \forall x A \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \qquad \frac{\Gamma \implies A[x \setminus a], \Delta}{\Gamma \implies \forall x A, \Delta} \forall\text{-right}^*$$

- $t$  is an arbitrary closed term
- **Eigenvariable condition** for the rule  $\forall\text{-right}^*$ :  $a$  must not occur in the conclusion, i.e. in  $\Gamma$ ,  $\Delta$ , or  $A$
- the formula  $\forall x A$  is preserved in the premise of the rule  $\forall\text{-left}$

► rules for  $\exists$  (existential quantifier)

$$\frac{\Gamma, A[x \setminus a] \implies \Delta}{\Gamma, \exists x A \implies \Delta} \exists\text{-left}^* \qquad \frac{\Gamma \implies \exists x A, A[x \setminus t], \Delta}{\Gamma \implies \exists x A, \Delta} \exists\text{-right}$$

- $t$  is an arbitrary closed term
- **Eigenvariable condition** for the rule  $\exists\text{-left}^*$ :  $a$  must not occur in the conclusion, i.e. in  $\Gamma$ ,  $\Delta$ , or  $A$
- the formula  $\exists x A$  is preserved in the premise of the rule  $\exists\text{-right}$

# Soundness and Completeness

## Theorem 7.1 (Soundness and Completeness of LK).

*The calculus of natural deduction LK is sound and complete, i.e.*

- ▶ *if  $A$  is provable in LK, then  $A$  is valid (if  $\vdash A$  then  $\models A$ )*
- ▶ *if  $A$  is valid, then  $A$  is provable in LK (if  $\models A$  then  $\vdash A$ )*

Proof.

Next week. □

## Examples of LK Proofs

Example:  $p(a) \rightarrow \exists x p(x)$

$$\frac{\frac{p(a) \Rightarrow p(a), \exists x p(x)}{\quad} \text{axiom}}{p(a) \Rightarrow \exists x p(x)} \exists\text{-right}}{\Rightarrow p(a) \rightarrow \exists x p(x)} \rightarrow\text{-right}$$

Example:  $\forall x p(x) \rightarrow \exists x p(x)$

$$\frac{\frac{\frac{p(c), \forall x p(x) \Rightarrow p(c), \exists x p(x)}{\quad} \text{axiom}}{p(c), \forall x p(x) \Rightarrow \exists x p(x)} \exists\text{-right}}{\forall x p(x) \Rightarrow \exists x p(x)} \forall\text{-left}}{\Rightarrow \forall x p(x) \rightarrow \exists x p(x)} \rightarrow\text{-right}$$

Example:  $p(a) \rightarrow p(b)$

$$\frac{\frac{\quad}{p(a) \Rightarrow p(b)} (?)}}{\Rightarrow p(a) \rightarrow p(b)} \rightarrow\text{-right}$$

Example:  $\exists x p(x) \rightarrow p(a)$

$$\frac{\frac{\quad}{\exists x p(x) \Rightarrow p(a)} \exists\text{-left}^*}}{\Rightarrow \exists x p(x) \rightarrow p(a)} \rightarrow\text{-right}$$

rule  $\exists\text{-left}^*$  with  $p(x)[x \setminus a]$  **cannot** be applied as  $a$  occurs in the premise (Eigenvariable condition!)

# Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ **Summary**



# Summary

- ▶ **first-order logic** extends the **syntax** of propositional logic by: **constants**, **functions**, **variables**, **predicates**, and the **quantifiers**  $\forall/\exists$
- ▶ the **semantics** consists of a **domain**  $D$  and an **interpretation**  $\iota$
- ▶ the **interpretation**  $\iota$  relates constants to elements of the domain, function symbols to actual functions, and predicates to relations
- ▶ variables are interpreted by a **variable assignment**  $\alpha$
- ▶ the formula  $\forall x p(x)/\exists x p(x)$  evaluates to  $T$  iff  $p(x)$  evaluates to  $T$  **for all/some** element(s) in  $D$
- ▶ the **truth value** of formulae is inductively evaluated, and takes the **value of terms** into account
- ▶ most concepts from propositional logic can be adapted
- ▶ four **semantical concepts**: satisfiable, valid, unsatisfiable, invalid
- ▶ **logical consequence** in first-order logic can be reduced to **validity**
- ▶ **Next week**: Soundness and completeness