IN3070/4070 – Logic – Autumn 2020

Lecture 4: First-order Logic

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Today's Plan

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- LK for First-order Logic
- Summary

Outline

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Limitations of Propositional Logic

Propositional logic: atomic formula (p, q, r), $\wedge$, $\vee$, $\neg$, $\rightarrow$, (,)

Problem: How do we represent the following statements?

- "all men are mortal" $\forall x(man(x) \rightarrow mortal(x))$
- "there exist prime numbers that are even" $\exists y(prime(y) \wedge even(y))$
- "1 is smaller than 3" $1 < 3$ or $<(1,3)$
- "transitivity of smaller" $\forall x\,\forall y\,\forall z\,(x<y \wedge y<z \rightarrow x<z)$
- $2 * 8 = 16$ $=(*(2,8),16)$
- "if x is even than $x+2$ is even" $\forall x\,(even(x) \rightarrow even(x+2))$
- "if x is prime than $x+2$ is prime" $\forall x\,(prime(x) \rightarrow prime(x+2))$

First-order logic: extension of propositional logic

First-Order Logic — Overview

Extending propositional logic by...

Syntax:

- constants (a, b, c), functions (f, g, h), variables (x, y, z)
- predicates (p, q, r)
- terms (t, u, v)
- quantifiers ($\forall$, $\exists$)
- scope of variables, free variables, variable assignment/substitution

Semantics:

- interpretation of constants, functions, variables
- interpretation of predicates
- value of terms
- truth value of (quantified) formulae
- satisfiability, validity, logical equivalence,...

Outline

Syntax — Terms

Terms are built up of constant (symbols), variable (symbols), and function (symbols).

Definition 2.1 (Terms).

Let $\mathcal{A} = \{a, b, \ldots\}$ be a countable set of constant symbols, $\mathcal{V} = \{x, y, z, \ldots\}$ be a countable set of variable symbols, and $\mathcal{F} = \{f, g, h, \ldots\}$ be a countable set of function symbols.

Terms, denoted t, u, v, are inductively defined as follows:

1. *Every variable $x \in \mathcal{V}$ is a term.*
2. *Every constant $a \in \mathcal{A}$ is a term.*
3. *If $f \in \mathcal{F}$ is an n-ary function (symbol) $n>0$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.*

Example: a, x, $f(a, x)$, $f(g(x), b)$, and $g(f(a, g(y)))$ are terms.

Syntax — First-Order Formulae

Formulae are built up of atomic formulae and the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\forall$ (universal quantifier), $\exists$ (existential quantifier).

Definition 2.2 (Atomic Formulae).

Let $\mathcal{P} = \{p, q, r, \ldots\}$ be a countable set of predicate symbols. If $p \in \mathcal{P}$ is an n-ary predicate (symbol) $n \geq 0$ and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$, $\top$, and $\bot$ are atomic formulae (or atoms).

Definition 2.3 ((First-Order) Formulae).

(First-order) formulae, denoted A, B, C, F, G, H, are inductively defined as follows:

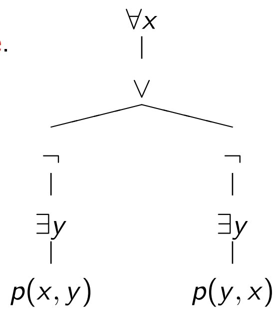
1. *Every atomic formula p is a formula.*
2. *If A and B are formulae and $x \in \mathcal{V}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$ are formulae.*

Formula Trees

A formula can be presented as formula tree.

Example:

$\forall x\, (\neg \exists y\, p(x, y) \lor \neg \exists y\, p(y, x))$



Definition 2.4 (Subformula, Main Operator).

Formula A is a (proper) subformula of formula B iff A is a (proper) subtree of B. If the root of a formula tree of A is a logical connective/quantifier, then it is called the main operator of A.

Outline

Free Variables

A free variable is a variable that is not in the scope of a quantifier.

Definition 3.1 (Free/Bound Variables, Closed Formula/Term).

Free variables in a formula A are inductively defined:

1. *If A is an atomic formula, then all variables in A are free.*
2. *If $A = \neg B$, then the free variables of A are exactly those of B.*
3. *If $A = B \land C$, $A = B \lor C$, or $A = B \to C$, then the free variables of A are those of B together with those of C.*
4. *If $A = \forall x\, B$ or $A = \exists x\, B$, then the free variables of A are those of B without the variable x.*

A bound variable in a formula C is a variable that appears in $\forall x$ or $\exists x$ in some subformula of C. A formula/term is closed iff it has no free variables.

Scope, Universal and Existential Closure

Definition 3.2 (Scope of Variables).

Let $\forall x\, A$ or $\exists x\, A$ be a universally or existentially quantified formula. Then x is the quantified variable and its scope is the formula A.

Remark: It is not required that x actually appears in the scope of its quantification, e.g. $\forall x\, \exists y\, p(y, y)$.

Definition 3.3 (Universal and Existential Closure).

If $\{x_1, \ldots, x_n\}$ are all the free variables of A, the universal closure of A is $\forall x_1 \ldots \forall x_n\, A$ and the existential closure of A is $\exists x_1 \ldots \exists x_n\, A$.

- $p(x, y)$ has the two free variables x and y. Its universal closure is $\forall x\, \forall y\, p(x, y)$ and its existential closure is $\exists x\, \exists y\, p(x, y)$; $\exists y\, p(x, y)$ has the only free variable x; $\forall x\, \exists y\, p(x, y)$ is closed
- In $\forall x\, p(x) \land q(x)$, the x occurs bound and free. The existential closure is $\exists x\, (\forall x\, p(x) \land q(x))$; renaming: $\exists y\, (\forall x\, p(x) \land q(y))$

Substitutions

Free variables in a first-order formula can be substituted by terms.

Definition 3.4 (Substitution).

Let $\mathcal{V}$ be a set of variables, $\mathcal{T}$ be the set of terms. A substitution $\sigma : \mathcal{V} \to \mathcal{T}$ assigns each variable a term.

Remark: The substitution σ is often represented as set $\{x \backslash t \mid \sigma(x) = t\}$.

Example: For the variable set $\{x, y\}$, $\sigma(x) = a$, $\sigma(y) = f(z, b)$ is a substitution and can also be represented as $\{x \backslash a, y \backslash f(z, b)\}$.

Ben-Ari: $\{x \leftarrow a, y \leftarrow f(z, b)\}$.

Others: $[a/x, f(z, b)/y]$

Application of substitutions

Definition 3.5 (Application of Substitutions, informally).

Let σ be a substitution. The application of σ to a term t or formula A, written $\sigma(t)$ or $\sigma(A)$, replaces every free variable in t or A according to its image under σ. Short hand: $A[x \backslash t] = \sigma(A)$ with $\sigma = \{x \backslash t\}$.

Example: Let $\sigma = \{x \backslash a, y \backslash f(z, b)\}$ be a substitution.
Then $\sigma(g(y)) = g(f(z, b))$
and $\sigma(p(x) \wedge \forall x\, q(x, g(y))) = p(a) \wedge \forall x\, q(x, g(f(z, b)))$

Problem: $\sigma(\forall z\, p(z, y)) = \forall z\, p(z, f(z, b))$
The free variable z in σ is captured by the quantifier.
This is bad because the effect depends on the choice of variable names

Definition 3.6 (Capture-free substitution).

A substitution σ is capture-free for a formula A if for every free variable x in A, none of the variables in $\sigma(x)$ is bound in A.

Application of Substitutions

Definition 3.7 (Application of Substitutions, formally).

The application of a subtitution σ to a term or formula is defined by structural induction:

- $\sigma(x) = \sigma(x)$ *for variables x in the range of σ*
- $\sigma(y) = y$ *for variables y not in the range of σ*
- $\sigma(a) = a$ *for constants $a \in \mathcal{A}$*
- $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$ *for a function symbol $f \in \mathcal{F}$*
- $\sigma(p(t_1, \ldots, t_n)) = p(\sigma(t_1), \ldots, \sigma(t_n))$ *for a predicate symbol $p \in \mathcal{P}$*
- $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$ *for formulae A, B*
- *. . . similarly for $\neg A$, $A \vee B$, $A \to B$. . .*
- $\sigma(\exists x\, A) = \exists x\, \sigma_x(A)$, $\quad \sigma(\forall x\, A) = \forall x\, \sigma_x(A)$

where we define σ_x by: $\sigma_x(x) = x$, and $\sigma_x(y) = \sigma(y)$ for all $y \neq x$

Outline

Semantics — Interpretation

An interpretation assigns concrete objects, functions and relations to constant symbols, function symbols, and predicate symbols.

Definition 4.1 (Interpretation/Structure).

An interpretation (or structure) $\mathcal{I} = (D, \iota)$ consists of the following elements:

1. *Domain D is a non-empty set*
2. *Interpretation of constant symbols assigns each constant $a \in \mathcal{A}$ an element $a^\iota \in D$*
3. *Interpretation of function symbols assigns each n-ary function symbol $f \in \mathcal{F}$ with $n{>}0$ a function $f^\iota : D^n \to D$*
4. *Interpretation of propositional variables assigns each 0-ary predicate symbol $p \in \mathcal{P}$ a truth value $p^\iota \in \{T, F\}$*
5. *Interpretation of predicate symbols assigns each n-ary predicate symbol $p \in \mathcal{P}$ with $n{>}0$ a relation $p^\iota \subseteq D^n$*

Semantics — Examples

Example: $\forall x\, p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c$

Remark: In Ben-Ari: $(\mathbb{N}, \{\leq\}, \{0\})$, $(\mathbb{N}, \{\leq\}, \{3\})$, $(\mathbb{Z}, \{\leq\}, \{0\})$

Example: $\forall x \forall y\, (p(x, y) \to p(f(x, a), f(y, a)))$ with interpretations

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$

Remark: In Ben-Ari: $(\mathbb{Z}, \{\leq\}, \{+\}, \{1\})$, $(\mathbb{Z}, \{>\}, \{*\}, \{-1\})$.

Semantics — Value of Closed Terms

Terms are evaluated according to the interpretation of their constant and function symbols.

Definition 4.2 (Term Value for Closed Terms).

Let $\mathcal{I} = (D, \iota)$ be an interpretation. The term value $v_\mathcal{I}(t)$ of a closed term $t \in \mathcal{T}$ under the interpretation $\mathcal{I}$ is inductively defined:

1. *For a constant symbol $a \in \mathcal{A}$ the term value is $v_\mathcal{I}(a) = a^\iota$;*
2. *Let $f \in \mathcal{F}$ be an n-ary function, $n{>}0$, and $t_1, \ldots, t_n$ be terms; the term value of $f(t_1, \ldots, t_n)$ is $v_\mathcal{I}(f(t_1, \ldots, t_n)) = f^\iota(v_\mathcal{I}(t_1), \ldots, v_\mathcal{I}(t_n))$*

Examples:

- $f(a, f(a, b))$ with $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 20$, $b^\iota = 2$; then $v_\mathcal{I}(f(a, f(a, b))) = 42$
- $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^\iota = *$ (multiplication), $*^\iota = -$ (subtraction), $1^\iota = -20$, $2^\iota = 0$, $4^\iota = 10$; then $v_\mathcal{I}(+(1, *(4, 2))) = -200$

Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
We need something additional.

Definition 4.3 (Variable Assignment).

Given the set of variables $\mathcal{V}$, and an interpretation $\mathcal{I} = (D, \iota)$, a variable assignment α for $\mathcal{I}$ is a function $\alpha : \mathcal{V} \to D$.

Ben-Ari (7.18) writes this $\sigma_{\mathcal{I}_A}$

Definition 4.4 (Term Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation, and α an variable assignment for $\mathcal{I}$. The term value $v_\mathcal{I}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and α is inductively defined:

1. $v_\mathcal{I}(\alpha, x) = \alpha(x)$ *for a variable $v \in \mathcal{V}$*
2. $v_\mathcal{I}(\alpha, a) = a^\iota$ *for a constant symbol $a \in \mathcal{A}$*
3. $v_\mathcal{I}(\alpha, f(t_1, \ldots, t_n)) = f^\iota(v_\mathcal{I}(\alpha, t_1), \ldots, v_\mathcal{I}(\alpha, t_n))$ *for an n-ary $f \in \mathcal{F}$*

Semantics — Term value Examples

- $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$
  - $\mathcal{V} = \{x, y\}$
  - $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
  - $v_\mathcal{I}(\alpha, f(a, f(a, x))) = 23$
- $\mathcal{I} = (\text{Strings}, \iota)$ with $g^\iota =$ concatenation, $a^\iota =$"Hello"
  - $\mathcal{V} = \{y\}$
  - $\alpha(y) =$"students"
  - $v_\mathcal{I}(\alpha, f(a, f(y, a))) =$ "HellostudentsHello"

Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$. The modified variable assignment $\alpha\{y{\leftarrow}d\}$ is defined as

$$\alpha\{y{\leftarrow}d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- $\mathcal{I} = (\mathbb{N}, \iota)$
- $\mathcal{V} = \{x, y\}$
- $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- $\alpha\{y{\leftarrow}7\}(x) =3$ and $\alpha\{y{\leftarrow}7\}(y) =7$

Semantics — Truth Value

Definition 4.6 (Truth Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The truth value $v_\mathcal{I}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

1. $v_\mathcal{I}(\alpha, p)=T$ for 0-ary $p \in \mathcal{P}$ iff $p^\iota=T$, otherwise $v_\mathcal{I}(\alpha, p)=F$
2. $v_\mathcal{I}(\alpha, p(t_1, \ldots, t_n))=T$ for $p \in \mathcal{P}$, $n>0$, iff $(v_\mathcal{I}(\alpha, t_1), \ldots, v_\mathcal{I}(\alpha, t_n)) \in p^\iota$, otherwise $v_\mathcal{I}(\alpha, p(t_1, \ldots, t_n))=F$
3. $v_\mathcal{I}(\alpha, \neg A)=T$ iff $v_\mathcal{I}(\alpha, A)=F$, otherwise $v_\mathcal{I}(\alpha, \neg A)=F$
4. $v_\mathcal{I}(\alpha, A \wedge B)=T$ iff $v_\mathcal{I}(\alpha, A)=T$ and $v_\mathcal{I}(\alpha, B)=T$, otherwise $v_\mathcal{I}(\alpha, A \wedge B)=F$
5. $v_\mathcal{I}(\alpha, A \vee B)=T$ iff $v_\mathcal{I}(\alpha, A)=T$ or $v_\mathcal{I}(\alpha, B)=T$, otherwise $v_\mathcal{I}(\alpha, A \vee B)=F$
6. $v_\mathcal{I}(\alpha, A \to B)=T$ iff $v_\mathcal{I}(\alpha, A)=F$ or $v_\mathcal{I}(\alpha, B)=T$, otherwise $v_\mathcal{I}(\alpha, A \to B)=F$
7. $v_\mathcal{I}(\alpha, \forall x A)=T$ iff $v_\mathcal{I}(\alpha\{x{\leftarrow}d\}, A)=T$ for all $d \in D$, otherwise $v_\mathcal{I}(\alpha, \forall x A)=F$
8. $v_\mathcal{I}(\alpha, \exists x A)=T$ iff $v_\mathcal{I}(\alpha\{x{\leftarrow}d\}, A)=T$ for some $d \in D$, otherwise $v_\mathcal{I}(\alpha, \exists x A)=F$
9. $v_\mathcal{I}(\alpha, \top)=T$ and $v_\mathcal{I}(\alpha, \bot)=F$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a closed term or formula, the assignment has no influence on the term value or truth value. We can write $v_\mathcal{I}(A)$ instead of $v_\mathcal{I}(\alpha, A)$.

Example: $A = \forall x\, p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$ $\rightsquigarrow v_\mathcal{I}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$ $\rightsquigarrow v_\mathcal{I}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$ $\rightsquigarrow v_\mathcal{I}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c$ $\rightsquigarrow v_\mathcal{I}(A) = T$

Example: $B = \forall x \forall y\, (p(x, y) \to p(f(x, a), f(y, a)))$ with interpretations

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$ $\rightsquigarrow v_\mathcal{I}(B) = T$
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$ $\rightsquigarrow v_\mathcal{I}(B) = F$

Outline

The Substitution Lemma for Terms

Theorem 5.1 (Substitution Lemma for Terms).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, and terms $t, s \in \mathcal{T}$

$$v_{\mathcal{I}}(\alpha, t[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

Proof.

By structural induction on t. We abbreviate: $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

For a constant a, $a[y \backslash s] = a$, so $v_{\mathcal{I}}(\alpha, a[y \backslash s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

For a variable $x \neq y$, $x[y \backslash s] = x$, so
$v_{\mathcal{I}}(\alpha, x[y \backslash s]) = v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \alpha'(x) = v_{\mathcal{I}}(\alpha', x)$ □

Proof of substitution lemma, continued

Proof.

For the variable y, $y[y \backslash s] = s$, so
$v_{\mathcal{I}}(\alpha, y[y \backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, y)$

For a complex term, $f(\ldots t_i \ldots)[y \backslash s] = f(\ldots t_i[y \backslash s] \ldots)$, so

$$\begin{aligned} & v_{\mathcal{I}}(\alpha, f(\ldots t_i \ldots)[y \backslash s]) \\ = \; & v_{\mathcal{I}}(\alpha, f(\ldots t_i[y \backslash s] \ldots)) && \text{by def. of substitution} \\ = \; & f^{\iota}(\ldots v_{\mathcal{I}}(\alpha, t_i[y \backslash s]) \ldots) && \text{by model semantics} \\ = \; & f^{\iota}(\ldots v_{\mathcal{I}}(\alpha', t_i) \ldots) && \text{by the induction hypothesis} \\ = \; & v_{\mathcal{I}}(\alpha', f(\ldots t_i \ldots)) && \text{by model semantics} \end{aligned}$$

□

The Substitution Lemma for Formulae

Theorem 5.2 (Substitution Lemma for Formulae).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is capture-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

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Satisfiability and Validity

Definition 6.1 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let A be a closed (first-order) formula and $U=\{A_1,\ldots\}$ be a set of closed (first-order) formulae A_i.

- *A is satisfiable iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.*
- *A satisfying interpretation $\mathcal{I}$ for A is called a model for A.*
- *$U=\{A_1,\ldots\}$ is satisfiable iff there is (common) model for all A_i.*
- *A (resp. U) is unsatisfiable iff A (resp. U) is not satisfiable.*
- *A is valid, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$.*
- *A is invalid/falsifiable iff A is not valid.*

Theorem 6.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is invalid.

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x\, p(a,x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \quad \leadsto v_{\mathcal{I}}(A) = F$
   $\leadsto A$ is invalid
2. $\mathcal{I} = (\{c,d,e,f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c \quad \leadsto v_{\mathcal{I}}(A) = T$
   $\leadsto A$ is satisfiable ($\mathcal{I}$ is a model)

Example: $B = \forall x\, \forall y\, (p(x,y) \rightarrow p(f(x,a), f(y,a)))$

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1 \leadsto v_{\mathcal{I}}(B) = T$
   $\leadsto$ satisfiable ($\mathcal{I}$ is a model)
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1 \leadsto v_{\mathcal{I}}(B) = F$
   $\leadsto$ invalid ($\mathcal{I}$ is a "counter-model")

Example: $\forall x\, \forall y\, (p(x,y) \rightarrow p(y,x))$
$\leadsto$ satisfiable (e.g. $p^\iota =$"$=$"), but invalid (e.g. $p^\iota =$"$<$")

Example: $\exists x\, \exists y\, (p(x) \wedge \neg p(y))$
$\leadsto$ only satisfiable for $|D| \geq 2$, invalid (e.g. $D = \mathbb{N}$, $p^\iota = even$)

Logical Equivalence

The concept of logical equivalence can be adapted to first-order logic, i.e. to closed first-order formulae.

Definition 6.2 (Logical Equivalence).

Let A_1, A_2 be two closed formulae. A_1 is logically equivalent to A_2, denoted $A_1 \equiv A_2$ iff $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$.

Theorem 6.2 (Relation $\equiv$ and $\leftrightarrow$).

Let A, B be two closed formulae. Then $A \equiv B$ iff $\models A \leftrightarrow B$.

Remark: $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$

Important: even though $\equiv$ and $\leftrightarrow$ are closely related, they are different relations. Whereas $\leftrightarrow$ is part of the object language (i.e. the definition of formulae), $\equiv$ is used in the meta-language to talk about or relate formulae.

Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

- $\models \forall x\, A(x) \leftrightarrow \neg\exists x\, \neg A(x)$
- $\models \exists x\, A(x) \leftrightarrow \neg\forall x\, \neg A(x)$

Commutativity:

- $\models \forall x \forall y\, A(x,y) \leftrightarrow \forall y \forall x\, A(x,y)$
- $\models \exists x \exists y\, A(x,y) \leftrightarrow \exists y \exists x\, A(x,y)$
- $\models \exists x \forall y\, A(x,y) \rightarrow \forall y \exists x\, A(x,y)$ (other direction is not valid!)

Distributivity:

- $\models \exists x\, (A(x) \vee B(x)) \leftrightarrow \exists x\, A(x) \vee \exists x\, B(x)$
- $\models \forall x\, (A(x) \wedge B(x)) \leftrightarrow \forall x\, A(x) \wedge \forall x\, B(x)$
- $\models \forall x\, A(x) \vee \forall x\, B(x) \rightarrow \forall x\, (A(x) \vee B(x))$ (other direction not valid!)
- $\models \exists x\, (A(x) \wedge B(x)) \rightarrow \exists x\, A(x) \wedge \exists x\, B(x)$ (other direction not valid!)

See [Ben-Ari 2012] for more equivalences involving quantifiers.

Logical Consequence

Definition 6.3 (Logical Consequence).

Let A be a closed formula and U be a set of closed formulae. A is a logical consequence of U, denoted $U \models A$, iff every model of U is a model of A, i.e. $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$ implies $v_{\mathcal{I}}(A) = T$.

Theorem 6.3 (Logical Consequence and Validity).

Let A be a closed formula and $U = \{A_1, \ldots, A_n\}$ be a set of closed formulae. Then $U \models A$ iff $\models (A_1 \wedge \cdots \wedge A_n) \rightarrow A$.

- again, we can reduce the problem of "logical consequence" to the problem of determining if a formula is valid
- hence, we need methods or proof search calculi that can deal with first-order formulae

Outline

LK — Axiom and Propositional Rules

- axiom

$$\frac{}{\Gamma, A \Longrightarrow A, \Delta}\ \text{axiom}$$

- rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \Longrightarrow \Delta}{\Gamma, A \wedge B \Longrightarrow \Delta}\ \wedge\text{-left} \qquad \frac{\Gamma \Longrightarrow A, \Delta \qquad \Gamma \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \wedge B, \Delta}\ \wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \Longrightarrow \Delta \qquad \Gamma, B \Longrightarrow \Delta}{\Gamma, A \vee B \Longrightarrow \Delta}\ \vee\text{-left} \qquad \frac{\Gamma \Longrightarrow A, B, \Delta}{\Gamma \Longrightarrow A \vee B, \Delta}\ \vee\text{-right}$$

- rules for $\rightarrow$ (implication)

$$\frac{\Gamma \Longrightarrow A, \Delta \qquad \Gamma, B \Longrightarrow \Delta}{\Gamma, A \rightarrow B \Longrightarrow \Delta}\ \rightarrow\text{-left} \qquad \frac{\Gamma, A \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \rightarrow B, \Delta}\ \rightarrow\text{-right}$$

- rules for $\neg$ (negation)

$$\frac{\Gamma \Longrightarrow A, \Delta}{\Gamma, \neg A \Longrightarrow \Delta}\ \neg\text{-left} \qquad \frac{\Gamma, A \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg A, \Delta}\ \neg\text{-right}$$

LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad \frac{\Gamma \implies A[x\backslash a], \Delta}{\Gamma \implies \forall x\, A, \Delta}\ \forall\text{-right}^*$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\forall$-right*: a must not occur in the conclusion, i.e. in Γ, Δ, or A
  - the formula $\forall x\, A$ is preserved in the premise of the rule $\forall$-left

- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}^* \qquad \frac{\Gamma \implies \exists x\, A, A[x\backslash t], \Delta}{\Gamma \implies \exists x\, A, \Delta}\ \exists\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\exists$-left*: a must not occur in the conclusion, i.e. in Γ, Δ, or A
  - the formula $\exists x\, A$ is preserved in the premise of the rule $\exists$-right

Soundness and Completeness

Theorem 7.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

- *if A is provable in LK, then A is valid (if $\vdash A$ then $\models A$)*
- *if A is valid, then A is provable in LK (if $\models A$ then $\vdash A$)*

Proof.

Next week. □

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x\, p(x)$

$$\cfrac{\cfrac{\cfrac{}{p(a) \implies p(a), \exists x\, p(x)}\ \text{axiom}}{p(a) \implies \exists x\, p(x)}\ \exists\text{-right}}{\implies p(a) \rightarrow \exists x\, p(x)}\ \rightarrow\text{-right}$$

Example: $p(a) \rightarrow p(b)$

$$\cfrac{\cfrac{}{p(a) \implies p(b)}\ (?)}{\implies p(a) \rightarrow p(b)}\ \rightarrow\text{-right}$$

Example: $\forall x\, p(x) \rightarrow \exists x\, p(x)$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{p(c), \forall x\, p(x) \implies p(c), \exists x\, p(x)}\ \text{axiom}}{p(c), \forall x\, p(x) \implies \exists x\, p(x)}\ \exists\text{-right}}{\forall x\, p(x) \implies \exists x\, p(x)}\ \forall\text{-left}}{\implies \forall x\, p(x) \rightarrow \exists x\, p(x)}\ \rightarrow\text{-right}$$

Example: $\exists x\, p(x) \rightarrow p(a)$

$$\cfrac{\cfrac{}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}^*}{\implies \exists x\, p(x) \rightarrow p(a)}\ \rightarrow\text{-right}$$

rule $\exists$-left*with $p(x)[x\backslash a]$ cannot be applied as a occurs in the premise (Eigenvariable condition!)

Outline

Summary

- first-order logic extends the syntax of propositional logic by: constants, functions, variables, predicates, and the quantifiers $\forall/\exists$
- the semantics consists of a domain D and an interpretation ι
- the interpretation ι relates constants to elements of the domain, function symbols to actual functions, and predicates to relations
- variables are interpreted by a variable assignment α
- the formula $\forall x\, p(x)/\exists x\, p(x)$ evaluates to T iff $p(x)$ evaluates to T for all/some element(s) in D
- the truth value of formulae is inductively evaluated, and takes the value of terms into account
- most concepts from propositional logic can be adapted
- four semantical concepts: satisfiable, valid, unsatisfiable, invalid
- logical consequence in first-order logic can be reduced to validity
- Next week: Soundness and completeness