IN3070/4070 – Logic – Autumn 2020

Lecture 4: First-order Logic

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Today's Plan

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- LK for First-order Logic
- Summary

Outline

Limitations of Propositional Logic

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First-order logic: extension of propositional logic

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Semantics:

- interpretation of constants, functions, variables
- interpretation of predicates
- value of terms
- truth value of (quantified) formulae
- satisfiability, validity, logical equivalence,...

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Example: a, x, $f(a, x)$, $f(g(x), b)$, and $g(f(a, g(y)))$ are terms.

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Let $\mathcal{P} = \{p, q, r, \ldots\}$ be a countable set of predicate symbols. If $p \in \mathcal{P}$ is an n-ary predicate (symbol) $n \geq 0$ and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$, $\top$, and $\bot$ are atomic formulae (or atoms).

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2. *If A and B are formulae and $x \in \mathcal{V}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$ are formulae.*

Formula Trees

A formula can be presented as formula tree.

Example:

$\forall x\, (\neg \exists y\, p(x, y) \lor \neg \exists y\, p(y, x))$

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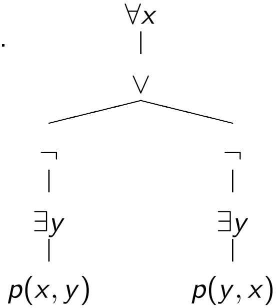

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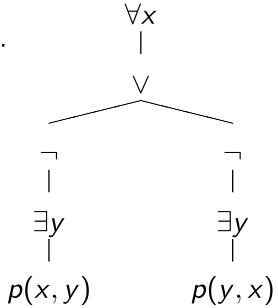

Definition 2.4 (Subformula, Main Operator).

Formula A is a (proper) subformula of formula B iff A is a (proper) subtree of B. If the root of a formula tree of A is a logical connective/quantifier, then it is called the main operator of A.

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A bound variable in a formula C is a variable that appears in $\forall x$ or $\exists x$ in some subformula of C. A formula/term is closed iff it has no free variables.

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If $\{x_1, \ldots, x_n\}$ are all the free variables of A, the universal closure of A is $\forall x_1 \ldots \forall x_n\, A$ and the existential closure of A is $\exists x_1 \ldots \exists x_n\, A$.

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- $p(x, y)$ has the two free variables x and y. Its universal closure is $\forall x\, \forall y\, p(x, y)$ and its existential closure is $\exists x\, \exists y\, p(x, y)$; $\exists y\, p(x, y)$ has the only free variable x; $\forall x\, \exists y\, p(x, y)$ is closed

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- In $\forall x\, p(x) \wedge q(x)$, the x occurs bound and free. The existential closure is $\exists x\, (\forall x\, p(x) \wedge q(x))$; renaming: $\exists y\, (\forall x\, p(x) \wedge q(y))$

Substitutions

Free variables in a first-order formula can be substituted by terms.

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Others: $[a/x, f(z, b)/y]$

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and $\sigma(p(x) \wedge \forall x\, q(x, g(y))) =$

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Definition 3.5 (Application of Substitutions, informally).

Let σ be a substitution. The application of σ to a term t or formula A, written $\sigma(t)$ or $\sigma(A)$, replaces every free variable in t or A according to its image under σ. Short hand: $A[x\backslash t]{=}\sigma(A)$ with $\sigma{=}\{x\backslash t\}$.

Example: Let $\sigma{=}\{x\backslash a, y\backslash f(z, b)\}$ be a substitution.
Then $\sigma(g(y)) = g(f(z, b))$
and $\sigma(p(x) \wedge \forall x\, q(x, g(y))) = p(a) \wedge \forall x\, q(x, g(f(z, b)))$

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Definition 3.6 (Capture-free substitution).

A substitution σ is capture-free for a formula A if for every free variable x in A, none of the variables in $\sigma(x)$ is bound in A.

Application of Substitutions

Definition 3.7 (Application of Substitutions, formally).

The application of a subtitution σ to a term or formula is defined by structural induction:

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- $\sigma(A \land B) = \sigma(A) \land \sigma(B)$ *for formulae* A, B
- *...similarly for* $\neg A$, $A \lor B$, $A \to B$...
- $\sigma(\exists x\, A) = \exists x\, \sigma_x(A), \quad \sigma(\forall x\, A) = \forall x\, \sigma_x(A)$

where we define σ_x *by:* $\sigma_x(x) = x$, *and* $\sigma_x(y) = \sigma(y)$ *for all* $y \neq x$

Outline

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4. *Interpretation of propositional variables assigns each* 0*-ary predicate symbol* $p \in \mathcal{P}$ *a truth value* $p^\iota \in \{T, F\}$

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4. *Interpretation of propositional variables assigns each 0-ary predicate symbol $p \in \mathcal{P}$ a truth value $p^\iota \in \{T, F\}$*
5. *Interpretation of predicate symbols assigns each n-ary predicate symbol $p \in \mathcal{P}$ with $n>0$ a relation $p^\iota \subseteq D^n$*

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1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$

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Examples:

- $f(a, f(a, b))$ with $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^{\iota} = +$, $a^{\iota} = 20$, $b^{\iota} = 2$;

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- $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$;

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1. *For a constant symbol* $a \in \mathcal{A}$ *the term value is* $v_{\mathcal{I}}(a) = a^{\iota}$*;*
2. *Let* $f \in \mathcal{F}$ *be an n-ary function,* $n{>}0$*, and* $t_1, \ldots, t_n$ *be terms; the term value of* $f(t_1, \ldots, t_n)$ *is* $v_{\mathcal{I}}(f(t_1, \ldots, t_n)) = f^{\iota}(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n))$

Examples:

- $f(a, f(a, b))$ with $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^{\iota} = +$, $a^{\iota} = 20$, $b^{\iota} = 2$; then $v_{\mathcal{I}}(f(a, f(a, b))) = 42$
- $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$; then $v_{\mathcal{I}}(+(1, *(4, 2))) = -200$

Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
We need something additional.

Definition 4.3 (Variable Assignment).

Given the set of variables $\mathcal{V}$*, and an interpretation* $\mathcal{I} = (D, \iota)$*, a variable assignment* α *for* $\mathcal{I}$ *is a function* $\alpha : \mathcal{V} \to D$*.*

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Definition 4.4 (Term Value).

Let $\mathcal{I} = (D, \iota)$ *be an interpretation, and* α *an variable assignment for* $\mathcal{I}$. *The term value* $v_{\mathcal{I}}(\alpha, t)$ *of a term* $t \in \mathcal{T}$ *under* $\mathcal{I}$ *and* α *is inductively defined:*

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Given the set of variables $\mathcal{V}$, and an interpretation $\mathcal{I} = (D, \iota)$, a variable assignment α for $\mathcal{I}$ is a function $\alpha : \mathcal{V} \to D$.

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Definition 4.4 (Term Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation, and α an variable assignment for $\mathcal{I}$. The term value $v_{\mathcal{I}}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and α is inductively defined:

1. $v_{\mathcal{I}}(\alpha, x) = \alpha(x)$ *for a variable $v \in \mathcal{V}$*

Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
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3. $v_{\mathcal{I}}(\alpha, f(t_1, \ldots, t_n)) = f^{\iota}(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n))$ *for an n-ary $f \in \mathcal{F}$*

Semantics — Term value Examples

- $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$

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  - $v_\mathcal{I}(\alpha, f(a, f(y, a))) =$ "HellostudentsHello"

Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$. The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

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The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

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- $\alpha\{y \leftarrow 7\}(x) = 3$

Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.
The modified variable assignment $\alpha\{y{\leftarrow}d\}$ is defined as

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Semantics — Modification of an assignment

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Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
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Semantics — Truth Value

Definition 4.6 (Truth Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The truth value $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

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1. $v_{\mathcal{I}}(\alpha, p) = T$ *for* 0*-ary* $p \in \mathcal{P}$ *iff* $p^{\iota} = T$*, otherwise* $v_{\mathcal{I}}(\alpha, p) = F$

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2. $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ *for* $p \in \mathcal{P}$*,* $n > 0$*, iff* $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$*, otherwise* $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$

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3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$*, otherwise* $v_{\mathcal{I}}(\alpha, \neg A) = F$

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2. $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ *for* $p \in \mathcal{P}$, $n > 0$, *iff* $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, *otherwise* $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$, *otherwise* $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = T$ *and* $v_{\mathcal{I}}(\alpha, B) = T$, *otherwise* $v_{\mathcal{I}}(\alpha, A \wedge B) = F$

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5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = T$ *or* $v_{\mathcal{I}}(\alpha, B) = T$*, otherwise* $v_{\mathcal{I}}(\alpha, A \vee B) = F$

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6. $v_{\mathcal{I}}(\alpha, A \to B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$ *or* $v_{\mathcal{I}}(\alpha, B) = T$, *otherwise* $v_{\mathcal{I}}(\alpha, A \to B) = F$

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Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The truth value $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

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2. $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7. $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$

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8. $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$

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1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7. $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$
8. $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$
9. $v_{\mathcal{I}}(\alpha, \top) = T$ and $v_{\mathcal{I}}(\alpha, \bot) = F$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a closed term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x\, p(a, x)$ with the interpretations

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Outline

- Motivation
- Syntax
- Variables
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The Substitution Lemma for Terms

Theorem 5.1 (Substitution Lemma for Terms).

Given an interpretation $\mathcal{I} = (D, \iota)$ *and a variable assignment* α *for* $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$, *and terms* $t, s \in \mathcal{T}$

$$v_{\mathcal{I}}(\alpha, t[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

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Proof.

By structural induction on t*.*

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By structural induction on t. We abbreviate: $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

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For a constant a, $a[y \backslash s] = a$, so $v_{\mathcal{I}}(\alpha, a[y \backslash s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

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For a constant a, $a[y \backslash s] = a$, so $v_{\mathcal{I}}(\alpha, a[y \backslash s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

For a variable $x \neq y$, $x[y \backslash s] = x$, so
$v_{\mathcal{I}}(\alpha, x[y \backslash s]) = v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \alpha'(x) = v_{\mathcal{I}}(\alpha', x)$

Proof of substitution lemma, continued

Proof.

For the variable y, $y[y\backslash s] = s$, so
$v_{\mathcal{I}}(\alpha, y[y\backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha\{y{\leftarrow}v_{\mathcal{I}}(\alpha, s)\}, y)$

Proof of substitution lemma, continued

Proof.

For the variable y, $y[y\backslash s] = s$, so
$v_{\mathcal{I}}(\alpha, y[y\backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha\{y{\leftarrow}v_{\mathcal{I}}(\alpha, s)\}, y)$

For a complex term, $f(\ldots t_i \ldots)[y\backslash s] = f(\ldots t_i[y\backslash s] \ldots)$, so

$$v_{\mathcal{I}}(\alpha, f(\ldots t_i \ldots)[y\backslash s])$$

Proof of substitution lemma, continued

Proof.

For the variable y, $y[y\backslash s] = s$, so
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Proof of substitution lemma, continued

Proof.

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$$
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= \; & f^{\iota}(\ldots v_{\mathcal{I}}(\alpha, t_i[y\backslash s]) \ldots) \quad \text{by model semantics} \\
= \; & f^{\iota}(\ldots v_{\mathcal{I}}(\alpha', t_i) \ldots) \quad \text{by the induction hypothesis}
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Proof of substitution lemma, continued

Proof.

For the variable y, $y[y\backslash s] = s$, so
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□

The Substitution Lemma for Formulae

Theorem 5.2 (Substitution Lemma for Formulae).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is capture-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

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Satisfiability and Validity

Definition 6.1 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let A be a closed (first-order) formula and $U = \{A_1, \ldots\}$ be a set of closed (first-order) formulae A_i.

- *A is satisfiable iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.*
- *A satisfying interpretation $\mathcal{I}$ for A is called a model for A.*

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- *A (resp. U) is unsatisfiable iff A (resp. U) is not satisfiable.*

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- *A (resp. U) is unsatisfiable iff A (resp. U) is not satisfiable.*
- *A is valid, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$.*
- *A is invalid/falsifiable iff A is not valid.*

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- *A is invalid/falsifiable iff A is not valid.*

Theorem 6.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is invalid.

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x\, p(a, x)$

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Example: $A = \forall x\, p(a, x)$

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   $\leadsto A$ is invalid

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x\, p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \quad \rightsquigarrow v_\mathcal{I}(A) = F$
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2. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c \quad \rightsquigarrow v_\mathcal{I}(A) = T$
   $\rightsquigarrow A$ is satisfiable ($\mathcal{I}$ is a model)

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x\, p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$ $\rightsquigarrow v_{\mathcal{I}}(A) = F$
   $\rightsquigarrow$ A is invalid
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   $\rightsquigarrow$ A is satisfiable ($\mathcal{I}$ is a model)

Example: $B = \forall x\, \forall y\, (p(x, y) \rightarrow p(f(x, a), f(y, a)))$

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Example: $\exists x\, \exists y\, (p(x) \land \neg p(y))$
$\rightsquigarrow$ only satisfiable for $|D| \geq 2$, invalid (e.g. $D = \mathbb{N}$, $p^\iota = even$)

Logical Equivalence

The concept of logical equivalence can be adapted to first-order logic, i.e. to closed first-order formulae.

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Definition 6.2 (Logical Equivalence).

Let A_1, A_2 be two closed formulae. A_1 is logically equivalent to A_2, denoted $A_1 \equiv A_2$ iff $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$.

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Theorem 6.2 (Relation $\equiv$ and $\leftrightarrow$).

Let A, B be two closed formulae. Then $A \equiv B$ iff $\models A \leftrightarrow B$.

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Important: even though $\equiv$ and $\leftrightarrow$ are closely related, they are different relations. Whereas $\leftrightarrow$ is part of the object language (i.e. the definition of formulae), $\equiv$ is used in the meta-language to talk about or relate formulae.

Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

- $\models \forall x \, A(x) \leftrightarrow \neg \exists x \, \neg A(x)$
- $\models \exists x \, A(x) \leftrightarrow \neg \forall x \, \neg A(x)$

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- $\models \exists x\, A(x) \leftrightarrow \neg\forall x\, \neg A(x)$

Commutativity:

- $\models \forall x\, \forall y\, A(x, y) \leftrightarrow \forall y\, \forall x\, A(x, y)$
- $\models \exists x\, \exists y\, A(x, y) \leftrightarrow \exists y\, \exists x\, A(x, y)$
- $\models \exists x\, \forall y\, A(x, y) \rightarrow \forall y\, \exists x\, A(x, y)$ (other direction is not valid!)

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- $\models \exists x\, \forall y\, A(x, y) \rightarrow \forall y\, \exists x\, A(x, y)$ (other direction is not valid!)

Distributivity:

- $\models \exists x\, (A(x) \vee B(x)) \leftrightarrow \exists x\, A(x) \vee \exists x\, B(x)$
- $\models \forall x\, (A(x) \wedge B(x)) \leftrightarrow \forall x\, A(x) \wedge \forall x\, B(x)$
- $\models \forall x\, A(x) \vee \forall x\, B(x) \rightarrow \forall x\, (A(x) \vee B(x))$ (other direction not valid!)
- $\models \exists x\, (A(x) \wedge B(x)) \rightarrow \exists x\, A(x) \wedge \exists x\, B(x)$ (other direction not valid!)

See [Ben-Ari 2012] for more equivalences involving quantifiers.

Logical Consequence

Definition 6.3 (Logical Consequence).

Let A be a closed formula and U be a set of closed formulae. A is a logical consequence of U, denoted $U \models A$, iff every model of U is a model of A, i.e. $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$ implies $v_{\mathcal{I}}(A) = T$.

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Theorem 6.3 (Logical Consequence and Validity).

Let A be a closed formula and $U = \{A_1, \ldots, A_n\}$ be a set of closed formulae. Then $U \models A$ iff $\models (A_1 \wedge \cdots \wedge A_n) \rightarrow A$.

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- again, we can reduce the problem of "logical consequence" to the problem of determining if a formula is valid
- hence, we need methods or proof search calculi that can deal with first-order formulae

Outline

LK — Axiom and Propositional Rules

- axiom

$$\frac{}{\Gamma, A \implies A, \Delta}\text{ axiom}$$

- rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \implies \Delta}{\Gamma, A \wedge B \implies \Delta}\wedge\text{-left} \qquad \frac{\Gamma \implies A, \Delta \qquad \Gamma \implies B, \Delta}{\Gamma \implies A \wedge B, \Delta}\wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \implies \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \vee B \implies \Delta}\vee\text{-left} \qquad \frac{\Gamma \implies A, B, \Delta}{\Gamma \implies A \vee B, \Delta}\vee\text{-right}$$

- rules for $\to$ (implication)

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta}\to\text{-left} \qquad \frac{\Gamma, A \implies B, \Delta}{\Gamma \implies A \to B, \Delta}\to\text{-right}$$

- rules for $\neg$ (negation)

$$\frac{\Gamma \implies A, \Delta}{\Gamma, \neg A \implies \Delta}\neg\text{-left} \qquad \frac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta}\neg\text{-right}$$

LK — Rules for Universal and Existential Quantifier

- rules for ∀ (universal quantifier)

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  - t is an arbitrary closed term
  - Eigenvariable condition for the rule ∀-right*: a must not occur in the conclusion, i.e. in Γ, Δ, or A
  - the formula $\forall x\, A$ is preserved in the premise of the rule ∀-left

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- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}^*$$

LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

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  - the formula $\exists x\, A$ is preserved in the premise of the rule $\exists$-right

Soundness and Completeness

Theorem 7.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

- *if A is provable in LK, then A is valid (if* $\vdash A$ *then* $\models A$*)*
- *if A is valid, then A is provable in LK (if* $\models A$ *then* $\vdash A$*)*

Proof.

Next week. □

Examples of LK Proofs

Example: $p(a) \to \exists x\, p(x)$

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Example: $p(a) \to \exists x\, p(x)$

$$\dfrac{p(a) \implies \exists x\, p(x)}{\implies p(a) \to \exists x\, p(x)}\ \to\text{-right}$$

Examples of LK Proofs

Example: $p(a) \to \exists x\, p(x)$

$$\cfrac{\cfrac{p(a) \implies p(a), \exists x\, p(x)}{p(a) \implies \exists x\, p(x)}\ \exists\text{-right}}{\implies p(a) \to \exists x\, p(x)}\ \to\text{-right}$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x\, p(x)$

$$
\dfrac{\dfrac{\dfrac{}{p(a) \;\Longrightarrow\; p(a), \exists x\, p(x)}\,\text{axiom}}{p(a) \;\Longrightarrow\; \exists x\, p(x)}\,\exists\text{-right}}{\Longrightarrow\; p(a) \rightarrow \exists x\, p(x)}\,\rightarrow\text{-right}
$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x\, p(x)$

$$\dfrac{\dfrac{\dfrac{}{p(a) \implies p(a), \exists x\, p(x)}\text{ axiom}}{p(a) \implies \exists x\, p(x)}\ \exists\text{-right}}{\implies p(a) \rightarrow \exists x\, p(x)}\ \rightarrow\text{-right}$$

Example: $p(a) \rightarrow p(b)$

Examples of LK Proofs

Example: $p(a) \to \exists x\, p(x)$

$$
\dfrac{\dfrac{\overline{\;p(a) \implies p(a), \exists x\, p(x)\;}\ \text{axiom}}{p(a) \implies \exists x\, p(x)}\ \exists\text{-right}}{\implies p(a) \to \exists x\, p(x)}\ \to\text{-right}
$$

Example: $p(a) \to p(b)$

$$\implies p(a) \to p(b)$$

Examples of LK Proofs

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Example: $p(a) \rightarrow p(b)$

$$\dfrac{p(a) \implies p(b)}{\implies p(a) \rightarrow p(b)}\ \rightarrow\text{-right}$$

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Example: $p(a) \to p(b)$

$$\dfrac{\dfrac{}{p(a) \implies p(b)}\ (?)}{\implies p(a) \to p(b)}\ \to\text{-right}$$

Examples of LK Proofs

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Example: $p(a) \to p(b)$

$$\dfrac{\dfrac{}{p(a) \implies p(b)}\ (?)}{\implies p(a) \to p(b)}\ \to\text{-right}$$

Example: $\forall x\, p(x) \to \exists x\, p(x)$

Examples of LK Proofs

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$$\dfrac{\dfrac{\overline{p(a) \implies p(a), \exists x\, p(x)}\ \text{axiom}}{p(a) \implies \exists x\, p(x)}\ \exists\text{-right}}{\implies p(a) \to \exists x\, p(x)}\ \to\text{-right}$$

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Example: $\forall x\, p(x) \to \exists x\, p(x)$

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rule $\exists$-left* with $p(x)[x \backslash a]$ cannot be applied as a occurs in the premise (Eigenvariable condition!)

Outline

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- Next week: Soundness and completeness