IN3070/4070 – Logic – Autumn 2020

Lecture 5: Soundness & Completeness for 1st-order LK

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Today's Plan

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- Proof of Completeness
- Examples of Counter-model Construction

Outline

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- Soundness Proof
- Completeness: Preliminaries
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- Some rules require "fresh" constants, so we assume that the set of constant symbols $\mathcal{A}$ is (countably) infinite.
- A root sequent $\Gamma \implies \Delta$ consists of *closed* formulae.
- We show that if $\Gamma \implies \Delta$ is provable, then $\Gamma \implies \Delta$ is valid

Reminer: Semantics for Sequents

Definition 1.2 (Valid sequent).

A sequent $\Gamma \implies \Delta$ is valid if all interpretations that satisfy all formulae in Γ satisfy at least one formula in Δ.

Definition 1.3 (Countermodel/falsifiable sequent).

- *An interpretation $\mathcal{I}$ is a countermodel for the sequent $\Gamma \implies \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$*

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- *We say that a countermodel for a sequent falsifies the sequent.*
- *A sequent is falsifiable if it has a countermodel.*

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  - substitutions (= syntactic operations)
  - variable assignments (= semantic objects)
- This connection is given by the Substitution Lemma

Reminder: Substitution Lemma

Theorem 1.2 (Substitution Lemma for Formulae).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is capture-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

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Definition 1.4 (Capture-free substitution).

A substitution σ is capture-free for a formula A if for every free variable x in A, none of the variables in $\sigma(x)$ is bound in A.

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Note: if $t \in \mathcal{T}$ is a *closed* term, then $\{y \backslash t\}$ is capture-free for any A.

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Definition 2.1.

An LK*-rule* θ *preserves falsifiability (upwards) if whenever the conclusion* w *of an instance* $\frac{w_1 \cdots w_n}{w}$ *of* θ *is falsifiabile, then also at least one of the premises* w_i *is falsifiable*

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Lemma 2.1.

All LK*-rules preserve falsifiability.*

- We have shown that the rules for propositional connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$) have this property.
- It remains to show that also the $\forall$ and $\exists$ rules preserve falsifiability.

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$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left}$$

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- Since $\mathcal{I} \models \forall x\, A$, we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$ and any α.

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- And therefore: $\mathcal{I} \models A[x\backslash t]$.

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    - $v_{\mathcal{I}'}(\alpha\{x \leftarrow d\}, A) = v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$, and $d = v_{\mathcal{I}'}(\alpha, a)$, so $\mathcal{I}' \models A[x\backslash a]$, by the Substitution Lemma.

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Proof: ∃-right and ∀-right preserve satisfiability

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How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
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Finally, we use these lemmas to show the soundness theorem.

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Lemma 2.2.

If the root sequent $\mathcal{I}$ of an an LK*-derivation is falsifiable, then at least one of the leaf sequents is falsifiable.*

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- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.

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- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- So $\mathcal{P}$ cannot be an LK-proof.

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Outline

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- Proof of Completeness
- Examples of Counter-model Construction

Herbrand Universe

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- Intuitively, the Herbrand universe of T is the set of all *closed* terms that can be constructed from the constant and function symbols in T.

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- If all branches in a derivation can be closed, then the derivation is finite. I.e. proofs are finite.

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We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite. (Otherwise T would be finite.) Let u_1 be the root of such a sub-tree. If the branch $u_0, u_1, \ldots, u_n$ is defined, we find the next node u_{n+1} by the same kind of reasoning.

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Corollary 3.1.

If T is a finitley branching tree, where all branches are finitely long, then T is finite.

Outline

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- **Proof of Completeness**
- Examples of Counter-model Construction

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- Such an intepretation is often called a Herbrand model or a term model.

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- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.

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Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$.

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We only need to cover quantified formulae

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- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^{\perp}$
  - $v_{\mathcal{I}}(A[x \backslash t]) = F$ (induction hypothesis)

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
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  - $v_{\mathcal{I}}(\alpha\{x{\leftarrow}v_{\mathcal{I}}(t)\}, A) = F$ for any α (substitution lemma)

Proof of Completeness (∃ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^\perp$.

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- By fairness of the derivation, the ∃-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^\perp$
  - $v_\mathcal{I}(A[x \backslash t]) = F$ (induction hypothesis)
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Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

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- By fairness of the derivation, $\forall$-right was applied to $\exists x\, A$ on the branch.

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- There may be finite counter-models even when this method does not terminate. Finding finite counter-models is a topic of active research.
- The idea of the completeness proof is important: we construct an interpretation from something purely syntactic.

Outline

Counter-model Construction, Ex. 1

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

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$$\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx$$

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- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \to qa$.
- Since $qb \in \mathcal{B}^{\perp}$, define $b \notin q^{\iota}$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- Since $qb \in \mathcal{B}^{\perp}$, define $b \notin q^{\iota}$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.
- Since $pb \in \mathcal{B}^{\perp}$, define $b \notin p^{\iota}$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- Since $qb \in \mathcal{B}^\perp$, define $b \notin q^\iota$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.
- Since $pb \in \mathcal{B}^\perp$, define $b \notin p^\iota$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.
- Therefore also $\mathcal{I} \models \forall x\,(px \rightarrow qx)$.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\, (px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- Since $qb \in \mathcal{B}^{\perp}$, define $b \notin q^{\iota}$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.
- Since $pb \in \mathcal{B}^{\perp}$, define $b \notin p^{\iota}$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.
- Therefore also $\mathcal{I} \models \forall x\, (px \rightarrow qx)$.
- $\mathcal{I}$ makes all of $\mathcal{B}^{\top}$ true and all of $\mathcal{B}^{\perp}$ false.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab$$

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \to pxb)}_{A},\ paa \lor pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A},\; paa \vee pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\frac{A, paa \Rightarrow pab \qquad\qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{A, paa \Rightarrow pab \qquad\qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad A, {\color{red}pba} \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}}{}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad A, {\color{red}pba} \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad {\color{red}A}, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad \dfrac{A, paa \to pab, pba \Rightarrow pab}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad \dfrac{A, {\color{red} paa \to pab}, pba \Rightarrow pab}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, paa \qquad pab, A, pba \Rightarrow pab}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, paa \qquad \color{red}{pab}\color{black}{, A, pba \Rightarrow} \color{red}{pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\begin{array}{c}\text{(Both branches closeable)}\\ A, paa \to pab, paa \Rightarrow pab\end{array}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, paa \qquad \dfrac{}{{\color{red}pab}, A, pba \Rightarrow {\color{red}pab}}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, \color{red}{paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$
\dfrac{
  \dfrac{
    \dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}
  }{A, paa \Rightarrow pab}
  \qquad
  \dfrac{
    \dfrac{A, pba \Rightarrow pab, {\color{red}paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}
  }{A, pba \Rightarrow pab}
}{
  \underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab
}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\bot}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{\color{red}{A}, pba \Rightarrow pab, paa \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$
\dfrac{
  \dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
  \qquad
  \dfrac{\dfrac{\dfrac{\dfrac{A, pba \to pbb, pba \Rightarrow pab, paa}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{}
}{
  \underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab
}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b \rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a \rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a \rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{\dfrac{A, {\color{red}pba \to pbb}, pba \Rightarrow pab, paa}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\bot}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A, pba \Rightarrow pab, paa, pba \qquad pbb, A, pba \Rightarrow pab, paa}{A, pba \rightarrow pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \rightarrow pab$.

Counter-model Construction, Ex. 2

$$\dfrac{A, \textcolor{red}{pba} \Rightarrow pab, paa, \textcolor{red}{pba} \qquad pbb, A, pba \Rightarrow pab, paa}{A, pba \to pbb, pba \Rightarrow pab, paa}$$

$$\dfrac{A, pba \Rightarrow pab, paa \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{\dfrac{A, paa \to pab, pba \Rightarrow pab}{A, pba \Rightarrow pab}}$$

(Both branches closeable)

$$\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}$$

$$\dfrac{A, paa \Rightarrow pab \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\text{(Both branches closeable)}}{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, {\color{red}pba} \Rightarrow pab, paa, {\color{red}pba}} \quad pbb, A, pba \Rightarrow pab, paa}{A, pba \to pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa} \quad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$\dfrac{}{A, pba \Rightarrow pab, paa, pba} \qquad \dfrac{}{{\color{red}pbb}, A, pba \Rightarrow pab, paa}$$

$$A, pba \rightarrow pbb, pba \Rightarrow pab, paa$$

$$A, pba \Rightarrow pab, paa \qquad\qquad \dfrac{}{pab, A, pba \Rightarrow pab}$$

(Both branches closeable)

$$A, paa \rightarrow pab, paa \Rightarrow pab \qquad\qquad A, paa \rightarrow pab, pba \Rightarrow pab$$

$$A, paa \Rightarrow pab \qquad\qquad A, pba \Rightarrow pab$$

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \rightarrow pab$.

Counter-model Construction, Ex. 2

$$
\dfrac{
\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\overline{A, pba \Rightarrow pab, paa, pba} \qquad {\color{red}pbb}, A, pba \Rightarrow pab, paa}{A, pba \rightarrow pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa} \qquad \overline{pab, A, pba \Rightarrow pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}
}{
\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab
}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \rightarrow pab$.
- Since $pbb \in \mathcal{B}^{\top}$ vil $\langle b, b\rangle \in p^{\iota}$, so $\mathcal{I} \models pbb$ and $\mathcal{I} \models pba \rightarrow pbb$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, pba \Rightarrow pab, paa, pba} \quad pbb, A, pba \Rightarrow pab, paa}{A, pba \rightarrow pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa} \quad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
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- Since $pbb \in \mathcal{B}^{\top}$ vil $\langle b, b\rangle \in p^{\iota}$, so $\mathcal{I} \models pbb$ and $\mathcal{I} \models pba \rightarrow pbb$.
- We thus have $\mathcal{I} \models \forall x\,(pxa \rightarrow pxb)$.

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$$
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- We thus have $\mathcal{I} \models \forall x\,(pxa \to pxb)$.
- $\mathcal{I}$ makes all of $\mathcal{B}^{\top}$ true and all of $\mathcal{B}^{\perp}$ false.

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