IN3070/4070 – Logic – Autumn 2020

Lecture 7: Resolution

Martin Giese

1st October 2019

Department of Informatics

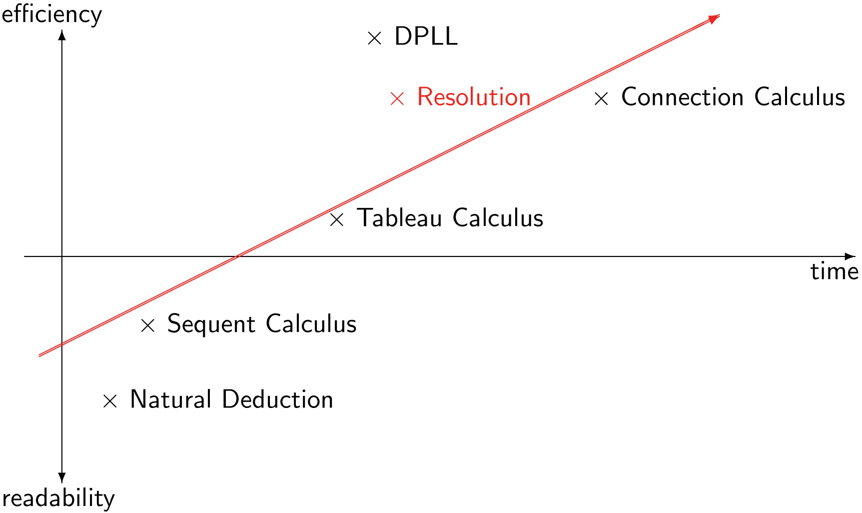
University of Oslo

Today's Plan

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

Outline

Proof Search Calculi



Robinson's Resolution Calculus

"A formulation of first-order logic which is specifically designed for use as the basis theoretical instrument of a computer theorem-proving program."

Robinson's Resolution Calculus

"A formulation of first-order logic which is specifically designed for use as the basis theoretical instrument of a computer theorem-proving program."

- the resolution calculus was published by Alan Robinson in 1965

Robinson's Resolution Calculus

"A formulation of first-order logic which is specifically designed for use as the basis theoretical instrument of a computer theorem-proving program."

- the resolution calculus was published by Alan Robinson in 1965
- works for first-order formulae in clausal form (e.g. conjunctive or disjunctive normal form)
- consists of one (two for first-order) inference rules and one axiom

Robinson's Resolution Calculus

"A formulation of first-order logic which is specifically designed for use as the basis theoretical instrument of a computer theorem-proving program."

- the resolution calculus was published by Alan Robinson in 1965
- works for first-order formulae in clausal form (e.g. conjunctive or disjunctive normal form)
- consists of one (two for first-order) inference rules and one axiom
- is one of the most popular proof search calculi
- has been implemented in many automated theorem provers

Outline

Negation Normal Form

Definition 2.1 (Negation Normal Form).

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Example.

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Example.

- $p \rightarrow q$ *is not in NNF*

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Example.

- $p \rightarrow q$ *is not in NNF*
- $\neg p \vee q$ *is in NNF*

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Example.

- $p \rightarrow q$ *is not in NNF*
- $\neg p \vee q$ *is in NNF*
- $\neg(p \vee \forall x\, \neg q(x))$ *is not in NNF*

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Example.

- $p \to q$ *is not in NNF*
- $\neg p \lor q$ *is in NNF*
- $\neg(p \lor \forall x \neg q(x))$ *is not in NNF*
- $\neg p \land \exists x\, q(x)$ *is in NNF*

Negation Normal Form

Definition 2.1 (Negation Normal Form).

A formula is in negation normal form (NNF) if it contains no implications, and all negations are in front of literals.

Example.

- $p \to q$ *is not in NNF*
- $\neg p \lor q$ *is in NNF*
- $\neg(p \lor \forall x \, \neg q(x))$ *is not in NNF*
- $\neg p \land \exists x \, q(x)$ *is in NNF*

Theorem 2.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$
\begin{aligned}
A \to B &\equiv \neg A \lor B \\
\neg(A \land B) &\equiv \neg A \lor \neg B \\
\neg(A \lor B) &\equiv \neg A \land \neg B \\
\neg(\forall x\, A) &\equiv \exists x\, \neg A \\
\neg(\exists x\, A) &\equiv \forall x\, \neg A \\
\neg(\neg A) &\equiv A
\end{aligned}
$$

$\square$

Outline

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Example.

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Example.

$(p \lor \neg q) \land (\neg p \lor q)$ *is in CNF.*

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Example.

$(p \vee \neg q) \wedge (\neg p \vee q)$ *is in CNF.*

$(p \vee \neg q) \wedge (\neg p \vee (q \wedge q))$ *is not in CNF.*

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Example.

$(p \lor \neg q) \land (\neg p \lor q)$ *is in CNF.*

$(p \lor \neg q) \land (\neg p \lor (q \land q))$ *is not in CNF.*

What about just p *or* $(p \lor q)$*?*

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Example.

$(p \vee \neg q) \wedge (\neg p \vee q)$ *is in CNF.*

$(p \vee \neg q) \wedge (\neg p \vee (q \wedge q))$ *is not in CNF.*

What about just p or $(p \vee q)$? Yes, if we consider a literal to be both a conjunction and a disjunction.

Conjunctive Normal Form

Definition 3.1 (Conjunctive Normal Form).

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Example.

$(p \lor \neg q) \land (\neg p \lor q)$ *is in CNF.*

$(p \lor \neg q) \land (\neg p \lor (q \land q))$ *is not in CNF.*

What about just p *or* $(p \lor q)$*? Yes, if we consider a literal to be both a conjunction and a disjunction.*

Theorem 3.1.

Every formula in propositional logic can be transformed into an equivalent formula in CNF.

Proof.

□

Proof.

To convert an arbitrary propositional formula to a formula in CNF perform the following steps, each of which preserves logical equivalence:

□

Proof.

To convert an arbitrary propositional formula to a formula in CNF perform the following steps, each of which preserves logical equivalence:

(1) Convert to negation normal form.

□

Proof.

To convert an arbitrary propositional formula to a formula in CNF perform the following steps, each of which preserves logical equivalence:

(1) Convert to negation normal form.

(2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

$\square$

Outline

Clausal Form

Definition 4.1 (Clausal Form).

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal. The empty set of literals is the empty clause, denoted by $\Box$.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal. The empty set of literals is the empty clause, denoted by $\square$. A formula in clausal form is a set of clauses.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal. The empty set of literals is the empty clause, denoted by $\square$. A formula in clausal form is a set of clauses. A formula is considered to be an implicit conjunction of its clauses.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal. The empty set of literals is the empty clause, denoted by $\square$. A formula in clausal form is a set of clauses. A formula is considered to be an implicit conjunction of its clauses. The formula that is the empty set of clauses is denoted by $\emptyset$.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal. The empty set of literals is the empty clause, denoted by $\square$. A formula in clausal form is a set of clauses. A formula is considered to be an implicit conjunction of its clauses. The formula that is the empty set of clauses is denoted by $\emptyset$.

The only significant difference between clausal form and the standard syntax is that clausal form is defined in terms of sets.

Clausal Form

Definition 4.1 (Clausal Form).

A clause is a set of literals. A clause is considered to be an implicit disjunction of its literals. A unit clause is a clause consisting of exactly one literal. The empty set of literals is the empty clause, denoted by $\square$. A formula in clausal form is a set of clauses. A formula is considered to be an implicit conjunction of its clauses. The formula that is the empty set of clauses is denoted by $\emptyset$.

The only significant difference between clausal form and the standard syntax is that clausal form is defined in terms of sets.

$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

Transformation to Clausal Form

Corollary 4.1.

Every formula ϕ in propositional logic can be transformed into an logically equivalent formula in clausal form.

Transformation to Clausal Form

Corollary 4.1.

Every formula ϕ in propositional logic can be transformed into an logically equivalent formula in clausal form.

Proof.

□

Transformation to Clausal Form

Corollary 4.1.

Every formula ϕ in propositional logic can be transformed into an logically equivalent formula in clausal form.

Proof.

This follows from the previous theorem, where we transformed a formula to CNF.

$\square$

Transformation to Clausal Form

Corollary 4.1.

Every formula ϕ in propositional logic can be transformed into an logically equivalent formula in clausal form.

Proof.

This follows from the previous theorem, where we transformed a formula to CNF. Each disjunction is then transformed to a clause (of literals), and the clausal form is the set of these clauses. □

Empty Clause and Empty Set of Clauses

Lemma 4.1.

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\Box$, *the empty clause, is unsatisfiable.*

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\square$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\Box$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Proof.

$\Box$

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\square$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Proof.

A clause is satisfiable iff there is some interpretation under which at least one literal in the clause is true.

$\square$

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\square$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Proof.

A clause is satisfiable iff there is some interpretation under which at least one literal in the clause is true. Let $\mathcal{I}$ be an arbitrary interpretation.

$\square$

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\Box$, the empty clause, is unsatisfiable.

$\emptyset$, the empty set of clauses, is valid.

Proof.

A clause is satisfiable iff there is some interpretation under which at least one literal in the clause is true. Let $\mathcal{I}$ be an arbitrary interpretation. Since there are no literals in $\Box$, there are no literals whose value is true under $\mathcal{I}$.

$\Box$

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\Box$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Proof.

A clause is satisfiable iff there is some interpretation under which at least one literal in the clause is true. Let $\mathcal{I}$ be an arbitrary interpretation. Since there are no literals in $\Box$, there are no literals whose value is true under $\mathcal{I}$. But $\mathcal{I}$ was an arbitrary interpretation, so $\Box$ is unsatisfiable.

$\Box$

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\square$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Proof.

A clause is satisfiable iff there is some interpretation under which at least one literal in the clause is true. Let $\mathcal{I}$ be an arbitrary interpretation. Since there are no literals in $\square$, there are no literals whose value is true under $\mathcal{I}$. But $\mathcal{I}$ was an arbitrary interpretation, so $\square$ is unsatisfiable.

A set of clauses is valid iff every clause in the set is true in every interpretation.

$\square$

Empty Clause and Empty Set of Clauses

Lemma 4.1.

$\square$, *the empty clause, is unsatisfiable.*

$\emptyset$, *the empty set of clauses, is valid.*

Proof.

A clause is satisfiable iff there is some interpretation under which at least one literal in the clause is true. Let $\mathcal{I}$ be an arbitrary interpretation. Since there are no literals in $\square$, there are no literals whose value is true under $\mathcal{I}$. But $\mathcal{I}$ was an arbitrary interpretation, so $\square$ is unsatisfiable.

A set of clauses is valid iff every clause in the set is true in every interpretation. But there are no clauses in $\emptyset$ that need be true, so $\emptyset$ is valid. $\square$

Outline

The Resolution Rule

The resolution calculus is a refutation procedure.

- in order to determine whether a formula F (in clausal form) is valid, we check whether $\neg F$ is unsatisfiable

The Resolution Rule

The resolution calculus is a refutation procedure.

- in order to determine whether a formula F (in clausal form) is valid, we check whether $\neg F$ is unsatisfiable

Definition 5.1 (Complementary Literal).

The complementary literal $\overline{L}$ of a literal L is A if L is of the form $\neg A$, otherwise it is $\neg L$.

The Resolution Rule

The resolution calculus is a refutation procedure.

- in order to determine whether a formula F (in clausal form) is valid, we check whether $\neg F$ is unsatisfiable

Definition 5.1 (Complementary Literal).

The complementary literal $\overline{L}$ of a literal L is A if L is of the form $\neg A$, otherwise it is $\neg L$.

Definition 5.2 (Resolution Rule).

Let C_1, C_2 be clauses with $L \in C_1$ and $\overline{L} \in C_2$. The resolvent C' of C_1 and C_2 is $(C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$. C_1 and C_2 are the parents of C'.

The Resolution Rule

The resolution calculus is a refutation procedure.

- in order to determine whether a formula F (in clausal form) is valid, we check whether $\neg F$ is unsatisfiable

Definition 5.1 (Complementary Literal).

The complementary literal $\overline{L}$ of a literal L is A if L is of the form $\neg A$, otherwise it is $\neg L$.

Definition 5.2 (Resolution Rule).

Let C_1, C_2 be clauses with $L \in C_1$ and $\overline{L} \in C_2$. The resolvent C' of C_1 and C_2 is $(C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$. C_1 and C_2 are the parents of C'.

- the resolution rule maintains satisfiability: If $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$ then $\mathcal{I} \models C'$

The Resolution Rule

The resolution calculus is a refutation procedure.

- in order to determine whether a formula F (in clausal form) is valid, we check whether $\neg F$ is unsatisfiable

Definition 5.1 (Complementary Literal).

The complementary literal $\overline{L}$ of a literal L is A if L is of the form $\neg A$, otherwise it is $\neg L$.

Definition 5.2 (Resolution Rule).

Let C_1, C_2 be clauses with $L \in C_1$ and $\overline{L} \in C_2$. The resolvent C' of C_1 and C_2 is $(C_1 \backslash \{L\}) \cup (C_2 \backslash \{\overline{L}\})$. C_1 and C_2 are the parents of C'.

- the resolution rule maintains satisfiability: If $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$ then $\mathcal{I} \models C'$
- if a set of clauses S is satisfiable and $C_1, C_2 \in S$, then $S \cup \{C'\}$ is satisfiable.

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

$$\{a, b, \neg c\} \qquad \{b, c, \neg e\}$$

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

$$\begin{array}{ccc} \{a, b, \neg c\} & & \{b, c, \neg e\} \\ & \searrow \quad \swarrow & \\ & \{a, b, \neg e\} & \end{array}$$

The resolvent of C_1 and C_2 is $\{a, b, \neg e\}$.

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

$$\begin{array}{ccc} \{a, b, \neg c\} & & \{b, c, \neg e\} \\ & \searrow \quad \swarrow & \\ & \{a, b, \neg e\} & \end{array}$$

The resolvent of C_1 and C_2 is $\{a, b, \neg e\}$.

Observations:

- if $\{a, b, \neg c\}$ and $\{b, c, \neg e\} \equiv (a \vee b \vee \neg c) \wedge (b \vee c \vee \neg e)$ are true in $\mathcal{I}$, then $(a \vee b)$ is true (if c is true) or $(b \vee \neg e)$ is true (if c is false); hence $(a \vee b \vee \neg e)$ is true

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

$$
\begin{array}{ccc}
\{a, b, \neg c\} & & \{b, c, \neg e\} \\
\searrow & & \swarrow \\
& \{a, b, \neg e\} &
\end{array}
$$

The resolvent of C_1 and C_2 is $\{a, b, \neg e\}$.

Observations:

- if $\{a, b, \neg c\}$ and $\{b, c, \neg e\} \equiv (a \vee b \vee \neg c) \wedge (b \vee c \vee \neg e)$ are true in $\mathcal{I}$, then $(a \vee b)$ is true (if c is true) or $(b \vee \neg e)$ is true (if c is false); hence $(a \vee b \vee \neg e)$ is true
- if resolvent is unsatisfiable, then conj. of parents is unsatisfiable
- the empty clause $\square$ is unsatisfiable

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

$$
\begin{array}{ccc}
\{a, b, \neg c\} & & \{b, c, \neg e\} \\
& \searrow \quad \swarrow & \\
& \{a, b, \neg e\} &
\end{array}
$$

The resolvent of C_1 and C_2 is $\{a, b, \neg e\}$.

Observations:

- if $\{a, b, \neg c\}$ and $\{b, c, \neg e\} \equiv (a \vee b \vee \neg c) \wedge (b \vee c \vee \neg e)$ are true in $\mathcal{I}$, then $(a \vee b)$ is true (if c is true) or $(b \vee \neg e)$ is true (if c is false); hence $(a \vee b \vee \neg e)$ is true
- if resolvent is unsatisfiable, then conj. of parents is unsatisfiable
- the empty clause $\square$ is unsatisfiable
- goal: derive empty clause $\square$

The Resolution Calculus

- a set of clauses is unsatisfiable iff the empty clause can be derived
- a clause C is true iff at least one of its literals is true; if there is no literal in C, then C is false and every set of clauses (in CNF) that contains C is false, i.e.unsatisfiable

The Resolution Calculus

- a set of clauses is unsatisfiable iff the empty clause can be derived
- a clause C is true iff at least one of its literals is true; if there is no literal in C, then C is false and every set of clauses (in CNF) that contains C is false, i.e.unsatisfiable

Definition 5.3 (Resolution Procedure).

Given a set of clauses S.

1. *apply the resolution rule to a pair of clauses $\{C_1, C_2\} \subseteq S$ that has not been chosen before; let C' be the resolvent*

The Resolution Calculus

- a set of clauses is unsatisfiable iff the empty clause can be derived
- a clause C is true iff at least one of its literals is true; if there is no literal in C, then C is false and every set of clauses (in CNF) that contains C is false, i.e.unsatisfiable

Definition 5.3 (Resolution Procedure).

Given a set of clauses S.

1. *apply the resolution rule to a pair of clauses $\{C_1, C_2\} \subseteq S$ that has not been chosen before; let C' be the resolvent*
2. $S' := S \cup \{C'\}$, $S := S'$

The Resolution Calculus

- a set of clauses is unsatisfiable iff the empty clause can be derived
- a clause C is true iff at least one of its literals is true; if there is no literal in C, then C is false and every set of clauses (in CNF) that contains C is false, i.e.unsatisfiable

Definition 5.3 (Resolution Procedure).

Given a set of clauses S.

1. *apply the resolution rule to a pair of clauses $\{C_1, C_2\} \subseteq S$ that has not been chosen before; let C' be the resolvent*
2. $S' := S \cup \{C'\}$, $S := S'$
3. *if $C' = \square$, then output "unsatisfiable";*

The Resolution Calculus

- a set of clauses is unsatisfiable iff the empty clause can be derived
- a clause C is true iff at least one of its literals is true; if there is no literal in C, then C is false and every set of clauses (in CNF) that contains C is false, i.e.unsatisfiable

Definition 5.3 (Resolution Procedure).

Given a set of clauses S.

1. *apply the resolution rule to a pair of clauses $\{C_1, C_2\} \subseteq S$ that has not been chosen before; let C' be the resolvent*
2. $S' := S \cup \{C'\}$, $S := S'$
3. *if $C' = \square$, then output "unsatisfiable";*
   if all possible resolvents have been considered, then output "satisfiable"; otherwise continue with 1.

Resolution Calculus – Example 1

- Prove validity of: $(p \wedge q) \rightarrow p$

Resolution Calculus – Example 1

- Prove validity of: $(p \wedge q) \to p$
- Show unsatisfiability of: $\neg((p \wedge q) \to p)$

Resolution Calculus – Example 1

- Prove validity of: $(p \land q) \rightarrow p$
- Show unsatisfiability of: $\neg((p \land q) \rightarrow p)$
- CNF: $p \land q \land \neg p$

Resolution Calculus – Example 1

- Prove validity of: $(p \land q) \to p$
- Show unsatisfiability of: $\neg((p \land q) \to p)$
- CNF: $p \land q \land \neg p$
- Clause set: $\{\{p\}, \{q\}, \{\neg p\}\}$

Resolution Calculus – Example 1

- Prove validity of: $(p \wedge q) \rightarrow p$
- Show unsatisfiability of: $\neg((p \wedge q) \rightarrow p)$
- CNF: $p \wedge q \wedge \neg p$
- Clause set: $\{\{p\}, \{q\}, \{\neg p\}\}$
- Resolve $\{p\}$ with $\{\neg p\}$

Resolution Calculus – Example 1

- Prove validity of: $(p \land q) \rightarrow p$
- Show unsatisfiability of: $\neg((p \land q) \rightarrow p)$
- CNF: $p \land q \land \neg p$
- Clause set: $\{\{p\}, \{q\}, \{\neg p\}\}$
- Resolve $\{p\}$ with $\{\neg p\}$
- Resolvent: $\square$

Resolution Calculus – Example 2

- Prove validity of: $p \land (p \to q) \to q$

Resolution Calculus – Example 2

- Prove validity of: $p \land (p \to q) \to q$
- Show unsatisfiability of: $\neg(p \land (p \to q) \to q)$

Resolution Calculus – Example 2

- Prove validity of: $p \land (p \to q) \to q$
- Show unsatisfiability of: $\neg(p \land (p \to q) \to q)$
- Equivalent to: $p \land (p \to q) \land \neg q$

Resolution Calculus – Example 2

- Prove validity of: $p \land (p \to q) \to q$
- Show unsatisfiability of: $\neg(p \land (p \to q) \to q)$
- Equivalent to: $p \land (p \to q) \land \neg q$
- CNF: $p \land (\neg p \lor q) \land \neg q$

Resolution Calculus – Example 2

- Prove validity of: $p \land (p \to q) \to q$
- Show unsatisfiability of: $\neg(p \land (p \to q) \to q)$
- Equivalent to: $p \land (p \to q) \land \neg q$
- CNF: $p \land (\neg p \lor q) \land \neg q$
- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
- CNF: $p \wedge (\neg p \vee q) \wedge \neg q$
- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
- Resolution step 1: between $\{p\}$ and $\{\neg p, q\}$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
- CNF: $p \wedge (\neg p \vee q) \wedge \neg q$
- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
- Resolution step 1: between $\{p\}$ and $\{\neg p, q\}$
- Resolvent: $\{q\}$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
- CNF: $p \wedge (\neg p \vee q) \wedge \neg q$
- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
- Resolution step 1: between $\{p\}$ and $\{\neg p, q\}$
- Resolvent: $\{q\}$
- New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$

Resolution Calculus – Example 2

- Prove validity of: $p \land (p \to q) \to q$
- Show unsatisfiability of: $\neg(p \land (p \to q) \to q)$
- Equivalent to: $p \land (p \to q) \land \neg q$
- CNF: $p \land (\neg p \lor q) \land \neg q$
- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
- Resolution step 1: between $\{p\}$ and $\{\neg p, q\}$
- Resolvent: $\{q\}$
- New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$
- Resolution step 2: between $\{\neg q\}$ and $\{q\}$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
- CNF: $p \wedge (\neg p \vee q) \wedge \neg q$
- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
- Resolution step 1: between $\{p\}$ and $\{\neg p, q\}$
- Resolvent: $\{q\}$
- New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$
- Resolution step 2: between $\{\neg q\}$ and $\{q\}$
- Resolvent: $\Box$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \wedge q \to r)$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \land q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \land q \to r))$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \wedge q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \wedge q \to r))$
- Equivalent to $(p \to (q \to r)) \wedge (p \wedge q) \wedge \neg r$

Resolution Calculus – Example 3

- Prove validity of: $(p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r)$
- Show unsatisfiability of: $\neg((p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r))$
- Equivalent to $(p \rightarrow (q \rightarrow r)) \wedge (p \wedge q) \wedge \neg r$
- Clauses:

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \land q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \land q \to r))$
- Equivalent to $(p \to (q \to r)) \land (p \land q) \land \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \land q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \land q \to r))$
- Equivalent to $(p \to (q \to r)) \land (p \land q) \land \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$
2. $\{p\}$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \wedge q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \wedge q \to r))$
- Equivalent to $(p \to (q \to r)) \wedge (p \wedge q) \wedge \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$
2. $\{p\}$
3. $\{q\}$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \wedge q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \wedge q \to r))$
- Equivalent to $(p \to (q \to r)) \wedge (p \wedge q) \wedge \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$
2. $\{p\}$
3. $\{q\}$
4. $\{\neg r\}$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \land q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \land q \to r))$
- Equivalent to $(p \to (q \to r)) \land (p \land q) \land \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$
2. $\{p\}$
3. $\{q\}$
4. $\{\neg r\}$
5. $\{\neg q, r\}$ — resolvent of 1. and 2.

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \land q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \land q \to r))$
- Equivalent to $(p \to (q \to r)) \land (p \land q) \land \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$
2. $\{p\}$
3. $\{q\}$
4. $\{\neg r\}$
5. $\{\neg q, r\}$ — resolvent of 1. and 2.
6. $\{r\}$ — resolvent of 3. and 5.

Resolution Calculus – Example 3

- Prove validity of: $(p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r)$
- Show unsatisfiability of: $\neg((p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r))$
- Equivalent to $(p \rightarrow (q \rightarrow r)) \wedge (p \wedge q) \wedge \neg r$
- Clauses:

1. $\{\neg p, \neg q, r\}$
2. $\{p\}$
3. $\{q\}$
4. $\{\neg r\}$
5. $\{\neg q, r\}$ — resolvent of 1. and 2.
6. $\{r\}$ — resolvent of 3. and 5.
7. $\Box$ — resolvent of 4. and 6.

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$

Resolution Calculus – Example 4

- Prove validity of: $(p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow (q \wedge r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$
5. $\{q\}$ — resolvent of 1. and 3.

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$
5. $\{q\}$ — resolvent of 1. and 3.
6. $\{r\}$ — resolvent of 2. and 3.

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$
5. $\{q\}$ — resolvent of 1. and 3.
6. $\{r\}$ — resolvent of 2. and 3.
7. $\{\neg r\}$ — resolvent of 4. and 5.

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$
5. $\{q\}$ — resolvent of 1. and 3.
6. $\{r\}$ — resolvent of 2. and 3.
7. $\{\neg r\}$ — resolvent of 4. and 5.
8. $\square$ — resolvent of 6. and 7.

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$
5. $\{q\}$ — resolvent of 1. and 3.
6. $\{r\}$ — resolvent of 2. and 3.
7. $\{\neg r\}$ — resolvent of 4. and 5.
8. $\Box$ — resolvent of 6. and 7.

- May have to use same clause several times

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
- Clauses:

1. $\{\neg p, q\}$
2. $\{\neg p, r\}$
3. $\{p\}$
4. $\{\neg q, \neg r\}$
5. $\{q\}$ — resolvent of 1. and 3.
6. $\{r\}$ — resolvent of 2. and 3.
7. $\{\neg r\}$ — resolvent of 4. and 5.
8. $\Box$ — resolvent of 6. and 7.

- May have to use same clause several times
- Order of resolution steps does not matter for completeness

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

$$\frac{}{C_1, \ldots, \square, \ldots C_n} \textit{ axiom}$$

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

$$\frac{}{C_1, ..., \Box, ... C_n}\ axiom$$

$$\frac{C_1, ..., C_i \cup \{L\}, ..., C_j \cup \{\overline{L}\}, ..., C_n, C_i \cup C_j}{C_1, ..., C_i \cup \{L\}, ..., C_j \cup \{\overline{L}\}, ..., C_n}\ resolution$$

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The *resolution calculus* *has one axiom and one (inference) rule.*

$$\frac{}{C_1, \dots, \square, \dots C_n}\ \textit{axiom}$$

$$\frac{C_1, \dots, C_i \cup \{L\}, \dots, C_j \cup \{\overline{L}\}, \dots, C_n, C_i \cup C_j}{C_1, \dots, C_i \cup \{L\}, \dots, C_j \cup \{\overline{L}\}, \dots, C_n}\ \textit{resolution}$$

A *resolution proof* *of a set of clauses S is a derivation of S in the resolution calculus.*

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

$$\frac{}{C_1, \ldots, \Box, \ldots C_n}\ \textit{axiom}$$

$$\frac{C_1, \ldots, C_i \cup \{L\}, \ldots, C_j \cup \{\overline{L}\}, \ldots, C_n, C_i \cup C_j}{C_1, \ldots, C_i \cup \{L\}, \ldots, C_j \cup \{\overline{L}\}, \ldots, C_n}\ \textit{resolution}$$

A resolution proof of a set of clauses S is a derivation of S in the resolution calculus.

- in contrast to natural deduction or the sequent calculus, the resolution calculus has no rule with more than one premise

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

$$\frac{}{C_1, \ldots, \Box, \ldots C_n}\ \textit{axiom}$$

$$\frac{C_1, \ldots, C_i \cup \{L\}, \ldots, C_j \cup \{\overline{L}\}, \ldots, C_n, C_i \cup C_j}{C_1, \ldots, C_i \cup \{L\}, \ldots, C_j \cup \{\overline{L}\}, \ldots, C_n}\ \textit{resolution}$$

A resolution proof of a set of clauses S is a derivation of S in the resolution calculus.

- in contrast to natural deduction or the sequent calculus, the resolution calculus has no rule with more than one premise
- hence, a derivation in the resolution calculus has only one branch

The Formal Resolution Calculus

Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

$$\frac{}{C_1, \ldots, \Box, \ldots C_n}\ \textit{axiom}$$

$$\frac{C_1, \ldots, C_i \cup \{L\}, \ldots, C_j \cup \{\overline{L}\}, \ldots, C_n, C_i \cup C_j}{C_1, \ldots, C_i \cup \{L\}, \ldots, C_j \cup \{\overline{L}\}, \ldots, C_n}\ \textit{resolution}$$

A resolution proof of a set of clauses S is a derivation of S in the resolution calculus.

- in contrast to natural deduction or the sequent calculus, the resolution calculus has no rule with more than one premise
- hence, a derivation in the resolution calculus has only one branch
- terminates, if all clauses $C_i \cup \{L\}, C_j \cup \{\overline{L}\}$ have been considered

Outline

Soundness of Resolution

- Recall: to prove A, we 'refute' $\neg A$

Soundness of Resolution

- Recall: to prove A, we 'refute' $\neg A$
- I.e. we derive a 'contradiction' (the empty clause) from $\neg A$...

Soundness of Resolution

- Recall: to prove A, we 'refute' $\neg A$
- I.e. we derive a 'contradiction' (the empty clause) from $\neg A$...
- ...meaning that $\neg A$ was unsatisfiable, and therefore A valid.

Soundness of Resolution

- Recall: to prove A, we 'refute' $\neg A$
- I.e. we derive a 'contradiction' (the empty clause) from $\neg A$...
- ... meaning that $\neg A$ was unsatisfiable, and therefore A valid.

We need to prove the following statements:

Soundness of Resolution

- Recall: to prove A, we 'refute' $\neg A$
- I.e. we derive a 'contradiction' (the empty clause) from $\neg A$...
- ...meaning that $\neg A$ was unsatisfiable, and therefore A valid.

We need to prove the following statements:

1. If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Soundness of Resolution

- Recall: to prove A, we 'refute' $\neg A$
- I.e. we derive a 'contradiction' (the empty clause) from $\neg A$...
- ...meaning that $\neg A$ was unsatisfiable, and therefore A valid.

We need to prove the following statements:

1. If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.
2. A set of clauses containing the empty clause is unsatisfiable

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

$\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals,

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.

A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.

Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

$\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2,

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

$\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

$\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.

A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.

Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

- $\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.
- $\mathcal{I} \models \overline{L}$ By the same reasoning $\mathcal{I} \models C_1 \setminus \{L\}$.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

- $\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.
- $\mathcal{I} \models \overline{L}$ By the same reasoning $\mathcal{I} \models C_1 \setminus \{L\}$.

So $\mathcal{I}$ satisfies at least one literal in either $C_1 \setminus \{L\}$ or $C_2 \setminus \{\overline{L}\}$.

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

- $\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.
- $\mathcal{I} \models \overline{L}$ By the same reasoning $\mathcal{I} \models C_1 \setminus \{L\}$.

So $\mathcal{I}$ satisfies at least one literal in either $C_1 \setminus \{L\}$ or $C_2 \setminus \{\overline{L}\}$.
I.e. $\mathcal{I} \models (C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$.
Now either $\mathcal{I} \models L$ or $\mathcal{I} \models \overline{L}$:

- $\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.
- $\mathcal{I} \models \overline{L}$ By the same reasoning $\mathcal{I} \models C_1 \setminus \{L\}$.

So $\mathcal{I}$ satisfies at least one literal in either $C_1 \setminus \{L\}$ or $C_2 \setminus \{\overline{L}\}$.
I.e. $\mathcal{I} \models (C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$, the resolvent of C_1 and C_2. □

The Empty Clause is unsatisfiable

Lemma 6.2.

A set of clauses containing the empty clause is unsatisfiable.

The Empty Clause is unsatisfiable

Lemma 6.2.

A set of clauses containing the empty clause is unsatisfiable.

Proof.

Let S be a set of clauses and $\square \in S$.

The Empty Clause is unsatisfiable

Lemma 6.2.

A set of clauses containing the empty clause is unsatisfiable.

Proof.

Let S be a set of clauses and $\Box \in S$.
Assume for the sake of contradiction that $\mathcal{I} \models S$.

The Empty Clause is unsatisfiable

Lemma 6.2.

A set of clauses containing the empty clause is unsatisfiable.

Proof.

Let S be a set of clauses and $\square \in S$.
Assume for the sake of contradiction that $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so in particular $\mathcal{I} \models \square$.

The Empty Clause is unsatisfiable

Lemma 6.2.

A set of clauses containing the empty clause is unsatisfiable.

Proof.

Let S be a set of clauses and $\square \in S$.
Assume for the sake of contradiction that $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so in particular $\mathcal{I} \models \square$.
Since clauses are *disjunctions*, to satisfy a clause C, an interpretation has to satisfy *at least one* of its literals $L \in C$.

The Empty Clause is unsatisfiable

Lemma 6.2.

A set of clauses containing the empty clause is unsatisfiable.

Proof.

Let S be a set of clauses and $\square \in S$.
Assume for the sake of contradiction that $\mathcal{I} \models S$.
A clause set is a *conjunction* of its clauses, so in particular $\mathcal{I} \models \square$.
Since clauses are *disjunctions*, to satisfy a clause C, an interpretation has to satisfy *at least one* of its literals $L \in C$.
But the empty clause $\mathcal{I}$ contains no literals, so that is a contradiction.

$\square$

Outline

Prove Completeness like for LK?

- Plan:

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\square \notin S'$.

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$
  - And we can't use structural induction either

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$
  - And we can't use structural induction either
- This can be fixed

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\square \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$
  - And we can't use structural induction either
- This can be fixed
  - $\mathcal{I}_{S'}$ is not defined on only the one-literal clauses

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S…
  - …build a fair limit derivation where all resolutions are applied…
  - …giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately…
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$
  - And we can't use structural induction either
- This can be fixed
  - $\mathcal{I}_{S'}$ is not defined on only the one-literal clauses
  - Argument doesn't use structural induction on clauses

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$
  - And we can't use structural induction either
- This can be fixed
  - $\mathcal{I}_{S'}$ is not defined on only the one-literal clauses
  - Argument doesn't use structural induction on clauses
  - The proof is rather advanced!

Prove Completeness like for LK?

- Plan:
  - Starting from a set of clauses S...
  - ...build a fair limit derivation where all resolutions are applied...
  - ...giving a set of clauses S' with $\Box \notin S'$.
  - Define an interpretation $\mathcal{I}_{S'}$ based on the "smallest" clauses (literals)
  - Show by structural induction that $\mathcal{I}_{S'}$ satsifies all clauses in S'
  - So in particular the ones in S.
- Nice plan, but unfortunately...
  - Resolution does not make clauses smaller (resolvent can be larger!)
  - So we don't always get lots of one-literal clauses in $\mathcal{I}_S$
  - And we can't use structural induction either
- This can be fixed
  - $\mathcal{I}_{S'}$ is not defined on only the one-literal clauses
  - Argument doesn't use structural induction on clauses
  - The proof is rather advanced!
- We will go through Robinson's original proof

Semantic Trees

The completeness proof uses the following concept:

Definition 7.1 (Semantic Trees).

A semantic tree is a binary tree where:

- *The root is labelled by the symbol* $\perp$*,*
- *Every node has either no children or two children,*
- *For every node that has children, there is some atom* A *such that one child is labeled with* A *and the other with* $\neg A$
- *There are not two complementary literals* A *and* $\neg A$ *on any path starting at the root.*

Semantic Trees

The completeness proof uses the following concept:

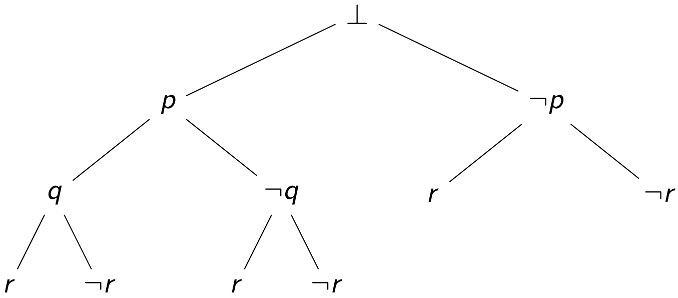
Definition 7.1 (Semantic Trees).

A semantic tree is a binary tree where:

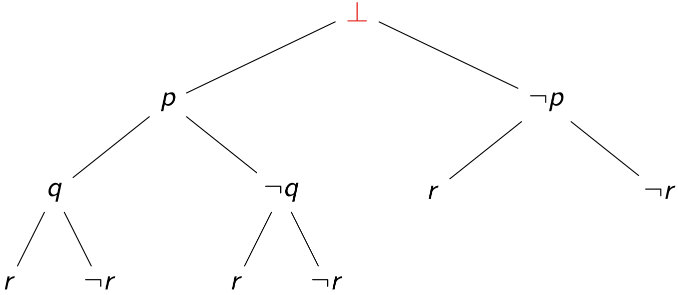
- *The root is labelled by the symbol $\perp$,*
- *Every node has either no children or two children,*
- *For every node that has children, there is some atom A such that one child is labeled with A and the other with $\neg A$*
- *There are not two complementary literals A and $\neg A$ on any path starting at the root.*

Not a data structure, just needed for the completeness proof

Semantic Trees — Example

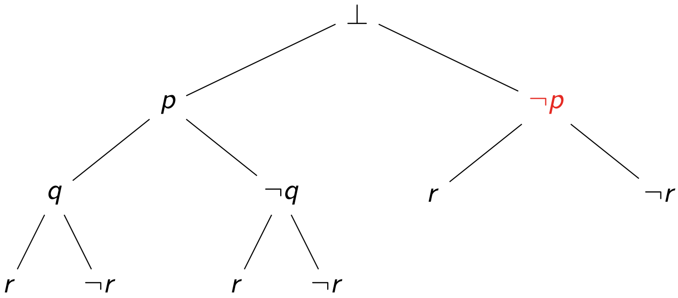


Semantic Trees — Example



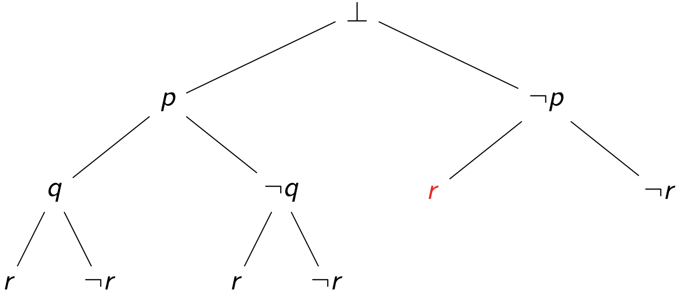
- Root labelled with $\perp$

Semantic Trees — Example



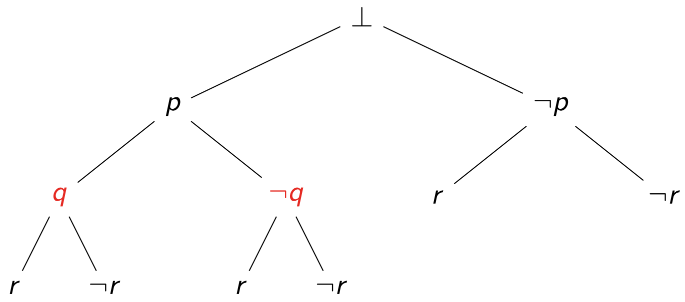
- Root labelled with $\perp$
- Either two children,

Semantic Trees — Example



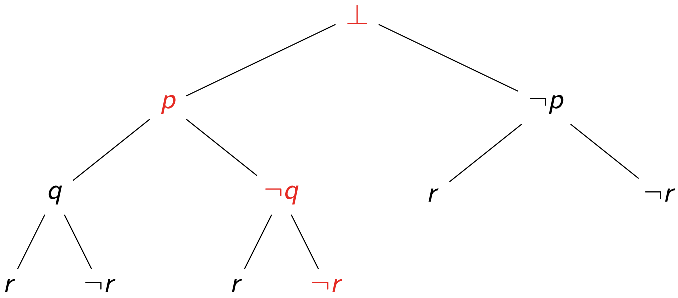
- Root labelled with $\perp$
- Either two children, or no children

Semantic Trees — Example



- Root labelled with $\bot$
- Either two children, or no children
- Complementary siblings

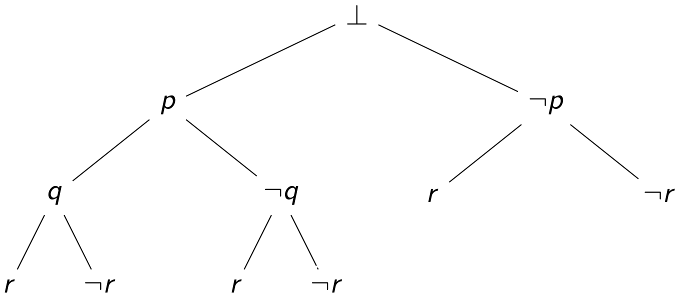
Semantic Trees — Example



- Root labelled with $\perp$
- Either two children, or no children
- Complementary siblings
- No complementary pairs on a path

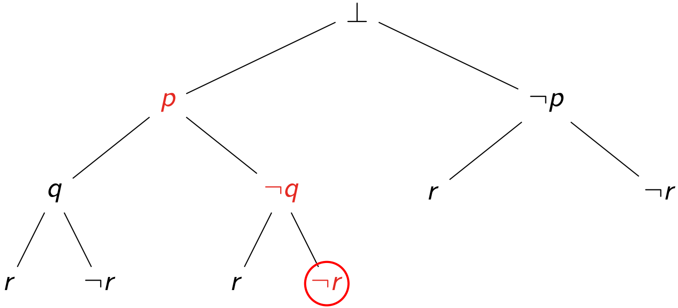
Partial Interpretations

The path to every node n in a semantic tree gives a 'partial interpretation' $\mathcal{I}_n$:



Partial Interpretations

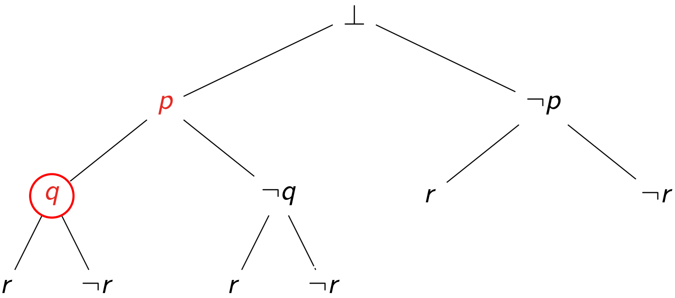
The path to every node n in a semantic tree gives a 'partial interpretation' $\mathcal{I}_n$:



$\mathcal{I}_n \models p$, $\mathcal{I}_n \models \neg q$, $\mathcal{I}_n \models \neg r$

Partial Interpretations

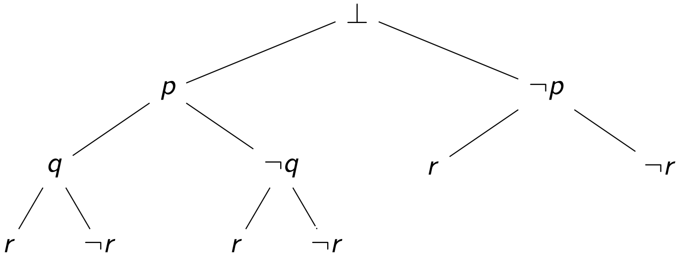
The path to every node n in a semantic tree gives a ‘partial interpretation’ $\mathcal{I}_n$:



$\mathcal{I}_n \models p$, $\mathcal{I}_n \models q$

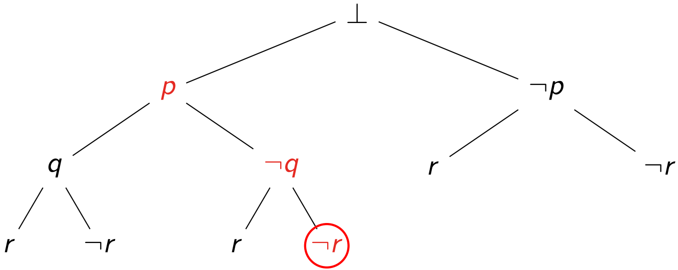
Failure Nodes – Motivation

Sometimes, such a ‘partial interpretation’ is enough to falsify a clause:



Failure Nodes – Motivation

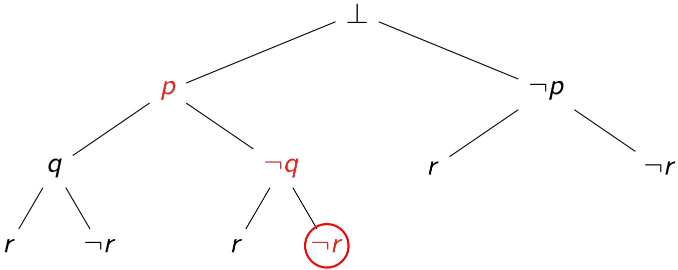
Sometimes, such a ‘partial interpretation’ is enough to falsify a clause:



- At the marked node, the clause $\neg p \vee q \vee r$ is false

Failure Nodes – Motivation

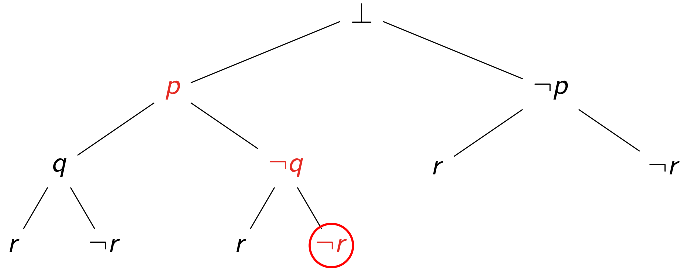
Sometimes, such a 'partial interpretation' is enough to falsify a clause:



- At the marked node, the clause $\neg p \vee q \vee r$ is false
- At the marked node, the clause $\neg p \vee r$ is false

Failure Nodes – Motivation

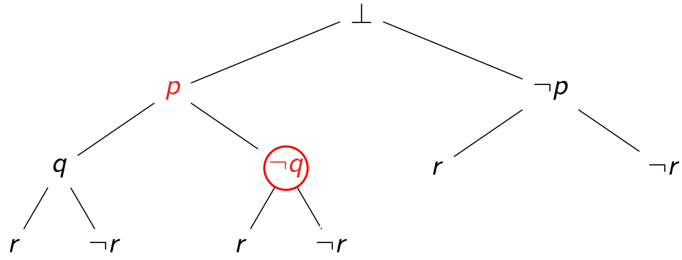
Sometimes, such a ‘partial interpretation’ is enough to falsify a clause:



- At the marked node, the clause $\neg p \vee q \vee r$ is false
- At the marked node, the clause $\neg p \vee r$ is false
- At the marked node, the clause $\neg p \vee q$ is false

Failure Nodes – Motivation

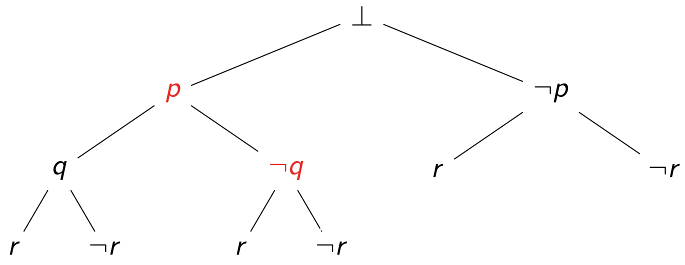
Sometimes, such a ‘partial interpretation’ is enough to falsify a clause:



- At the marked node, the clause $\neg p \vee q \vee r$ is false
- At the marked node, the clause $\neg p \vee r$ is false
- At the marked node, the clause $\neg p \vee q$ is false
- The clause $\neg p \vee q$ is already false at the parent node!

Failure Nodes – Motivation

Sometimes, such a 'partial interpretation' is enough to falsify a clause:



- At the marked node, the clause $\neg p \lor q \lor r$ is false
- At the marked node, the clause $\neg p \lor r$ is false
- At the marked node, the clause $\neg p \lor q$ is false
- The clause $\neg p \lor q$ is already false at the parent node!
- It remains false further down.

Failure Nodes – Definition

Definition 7.2.

A node n in a semantic tree is a falsifies a clause C if for every literal $L \in C$, the complement $\overline{L}$ is on the branch leading to n.

Failure Nodes – Definition

Definition 7.2.

A node n in a semantic tree is a falsifies a clause C if for every literal $L \in C$, the complement $\overline{L}$ is on the branch leading to n.

Definition 7.3.

A node n in a semantic tree is a failure node for a clause set S if it falsifies some clause $C \in S$, but the parent of n does not.

Failure Nodes – Definition

Definition 7.2.

A node n in a semantic tree is a falsifies a clause C if for every literal $L \in C$, the complement $\overline{L}$ is on the branch leading to n.

Definition 7.3.

A node n in a semantic tree is a failure node for a clause set S if it falsifies some clause $C \in S$, but the parent of n does not.

Failure nodes have just enough information to make sure some clause is falsified.

Failure Nodes – Definition

Definition 7.2.

A node n in a semantic tree is a falsifies a clause C if for every literal $L \in C$, the complement $\overline{L}$ is on the branch leading to n.

Definition 7.3.

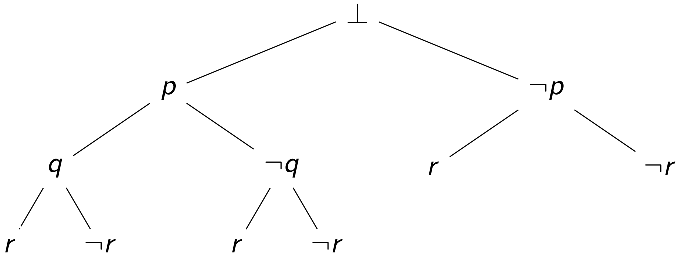
A node n in a semantic tree is a failure node for a clause set S if it falsifies some clause $C \in S$, but the parent of n does not.

Failure nodes have just enough information to make sure some clause is falsified.

Note: A has the root as a failure node iff $\Box \in S$.

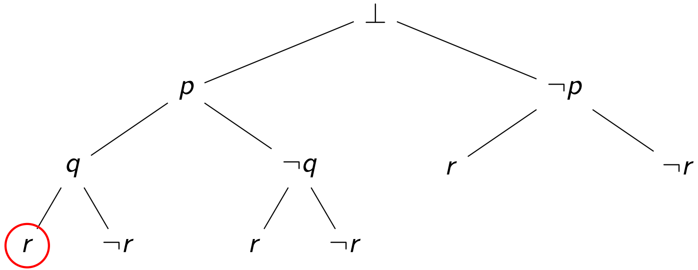
Failure Nodes – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



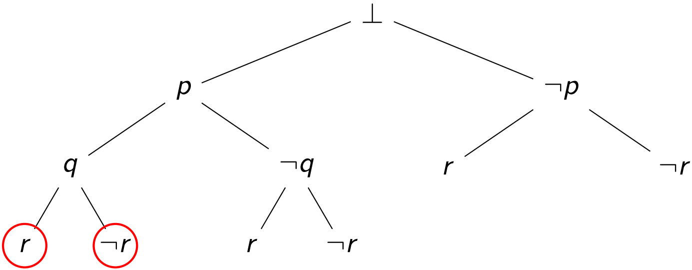
Failure Nodes – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



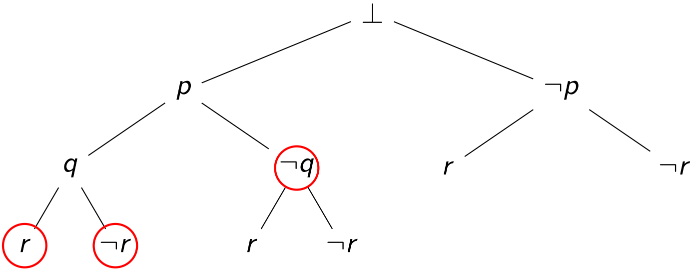
Failure Nodes – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



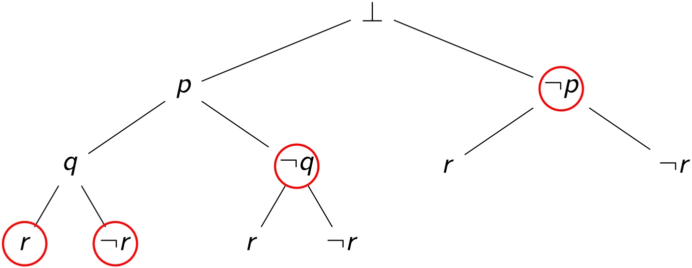
Failure Nodes – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



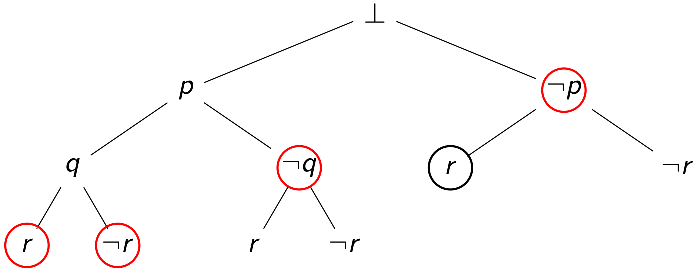
Failure Nodes – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



Failure Nodes – Example

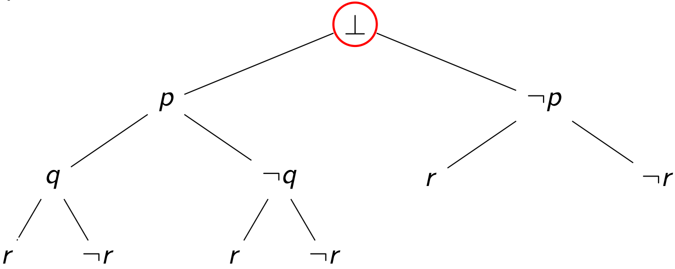
1. $\neg p \lor \neg q \lor \neg r$
2. $\neg q \lor r$
3. $\neg p \lor q$
4. p
5. $p \lor \neg r$



Not a failure node: parent node falsifies clause 4.

Failure Nodes – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$
6. $\square$



The empty clause is falsified by the root node

Closed Semantic Trees

Definition 7.4.

Given a semantic tree and a clause set S, a branch of the tree is closed if it contains a failure node.

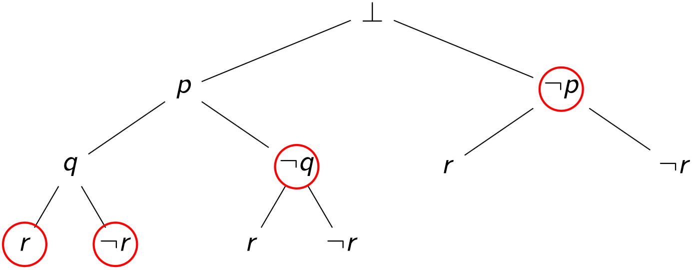
Closed Semantic Trees

Definition 7.4.

Given a semantic tree and a clause set S, a branch of the tree is closed if it contains a failure node.
The semantic tree is closed if all branches contain failure nodes.

Closed Semantic Tree – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$

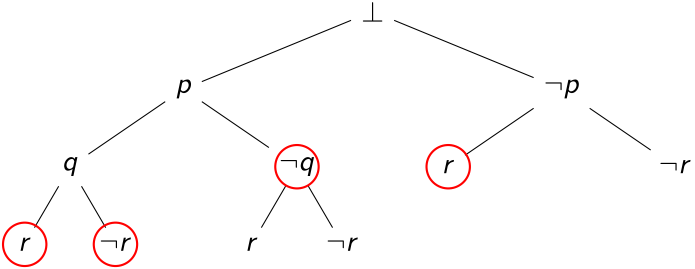


The semantic tree is closed for these 5 clauses.

Closed Semantic Tree – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$

5. $p \vee \neg r$



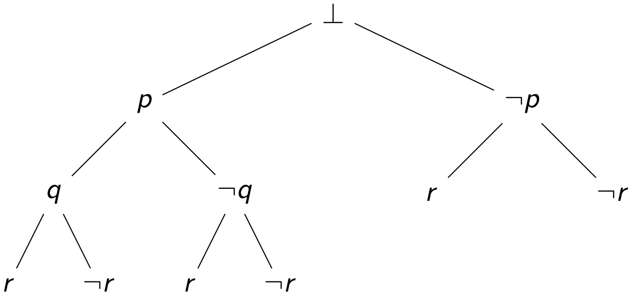
Without p, it is not closed.

Complete Semantic Trees

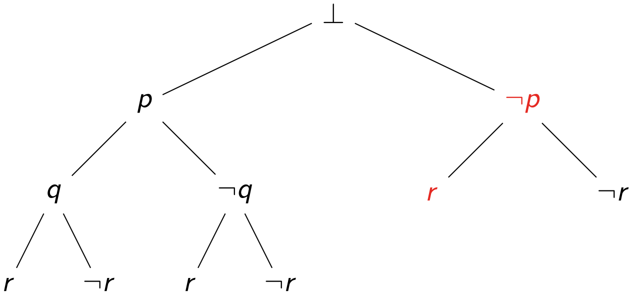
Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Example: Complete Semantic Tree

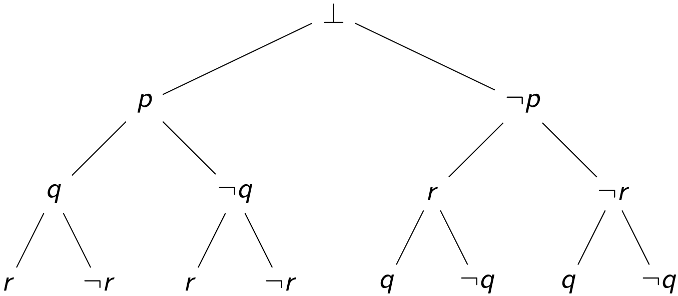


Example: Complete Semantic Tree



Not complete, since neither q nor $\neg q$ on branch

Example: Complete Semantic Tree



Complete for vocabulary $\{p, q, r\}$

Complete Semantic Trees

Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Complete Semantic Trees

Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Every branch $\mathcal{B}$ in a *complete* semantic tree corresponds to an interpretation $\mathcal{I}_{\mathcal{B}}$ with $\mathcal{I} \models A$ iff A is on the branch.

Complete Semantic Trees

Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Every branch $\mathcal{B}$ in a *complete* semantic tree corresponds to an interpretation $\mathcal{I}_{\mathcal{B}}$ with $\mathcal{I} \models A$ iff A is on the branch.

Lemma 7.1.

For every interpretation $\mathcal{I}$ there is a branch $\mathcal{B}$ in a complete semantic tree with $\mathcal{I} = \mathcal{I}_{\mathcal{B}}$.

Complete Semantic Trees

Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Every branch $\mathcal{B}$ in a *complete* semantic tree corresponds to an interpretation $\mathcal{I}_{\mathcal{B}}$ with $\mathcal{I} \models A$ iff A is on the branch.

Lemma 7.1.

For every interpretation $\mathcal{I}$ there is a branch $\mathcal{B}$ in a complete semantic tree with $\mathcal{I} = \mathcal{I}_{\mathcal{B}}$.

A complete semantic tree 'enumerates' all possible interpretations.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

$\Rightarrow$ Let S be an unsatisfiable clause set.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

$\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

$\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$,

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

$\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

$\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_\mathcal{B} \not\models S$, so $\mathcal{I}_\mathcal{B} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_\mathcal{B} \not\models S$, so $\mathcal{I}_\mathcal{B} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$,

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_\mathcal{B} \not\models S$, so $\mathcal{I}_\mathcal{B} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch,

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$,

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$, so $\mathcal{I} \not\models C$,

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$, so $\mathcal{I} \not\models C$, and thus $\mathcal{I} \not\models S$.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

Proof.

- $\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$, so there is a node on the branch that falsifies C.
  The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- $\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$, so $\mathcal{I} \not\models C$, and thus $\mathcal{I} \not\models S$.
  This holds for arbitrary interpretations $\mathcal{I}$, so S is unsatisfiable. $\square$

Resolution Steps from Closed Semantic Trees

Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree

Resolution Steps from Closed Semantic Trees

Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree, and $\Box \notin S$.

Resolution Steps from Closed Semantic Trees

Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree, and $\Box \notin S$. Then

- *a resolution step is possible from S,*

Resolution Steps from Closed Semantic Trees

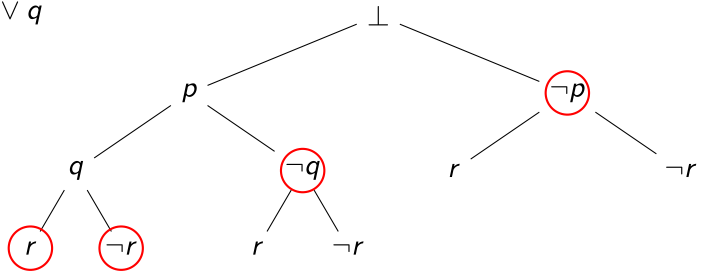
Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree, and $\square \notin S$. Then

- *a resolution step is possible from S,*
- *and the resulting clause set S' has a smaller closed semantic tree*

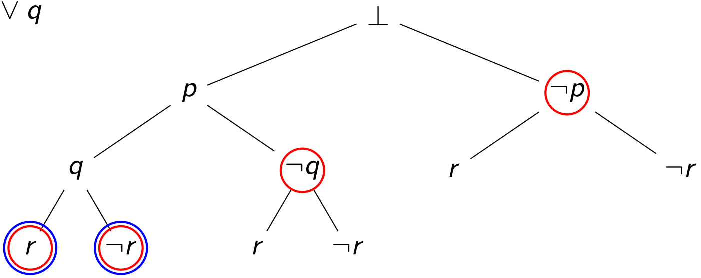
Idea of proof

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p



Idea of proof

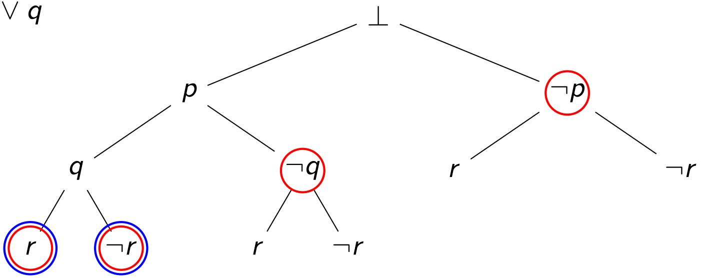
1. $\neg p \lor \neg q \lor \neg r$
2. $\neg q \lor r$
3. $\neg p \lor q$
4. p



- There are two sibling failure nodes

Idea of proof

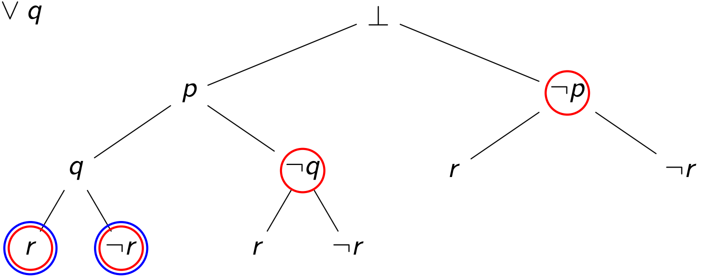
1. $\neg p \lor \neg q \lor \neg r$
2. $\neg q \lor r$
3. $\neg p \lor q$
4. p



- There are two sibling failure nodes
- They falsify two clauses

Idea of proof

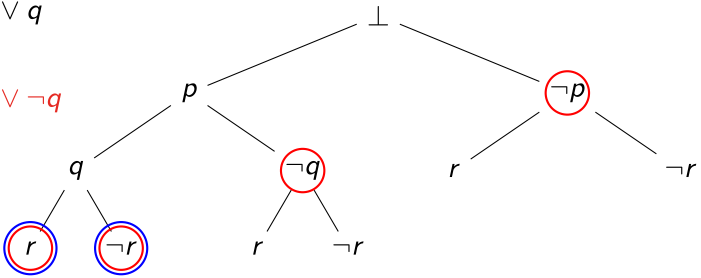
1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p



- There are two sibling failure nodes
- They falsify two clauses with complementary literals

Idea of proof

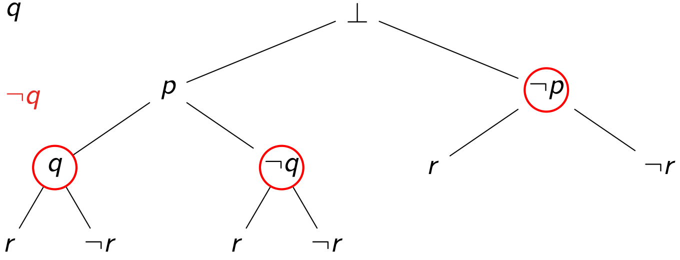
1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $\neg p \vee \neg q$



- There are two sibling failure nodes
- They falsify two clauses with complementary literals
- They can be resolved to a new clause $\neg p \vee \neg q$

Idea of proof

1. $\neg p \lor \neg q \lor \neg r$
2. $\neg q \lor r$
3. $\neg p \lor q$
4. p
5. $\neg p \lor \neg q$



- There are two sibling failure nodes
- They falsify two clauses with complementary literals
- They can be resolved to a new clause $\neg p \lor \neg q$
- Which is falsified by the parent node

There are two sibling failure nodes

- Let n_0 be the root.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\Box \notin S$, n_0 is not a failure node.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\Box \notin S$, n_0 is not a failure node.
- n_0 has two children.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\square \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\square \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.
- Otherwise, let n_1 be one of the siblings that is not a failure node.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\Box \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.
- Otherwise, let n_1 be one of the siblings that is not a failure node.
- n_1 has two children.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\square \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.
- Otherwise, let n_1 be one of the siblings that is not a failure node.
- n_1 has two children.
- If both are failure nodes, we are done.

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\square \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.
- Otherwise, let n_1 be one of the siblings that is not a failure node.
- n_1 has two children.
- If both are failure nodes, we are done.
- ...

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\square \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.
- Otherwise, let n_1 be one of the siblings that is not a failure node.
- n_1 has two children.
- If both are failure nodes, we are done.
- ...
- This either finds sibling failure nodes...

There are two sibling failure nodes

- Let n_0 be the root.
- Since $\Box \notin S$, n_0 is not a failure node.
- n_0 has two children.
- If both are failure nodes, we are done.
- Otherwise, let n_1 be one of the siblings that is not a failure node.
- n_1 has two children.
- If both are failure nodes, we are done.
- ...
- This either finds sibling failure nodes...
- or it constructs a path in the tree without a failure node, but that is not possible.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$
- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$
- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$
- Every literal in C has its negation in N

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$
- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$
- Every literal in C has its negation in N
- Adding C to the clause set will make n into a failure node.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$
- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$
- Every literal in C has its negation in N
- Adding C to the clause set will make n into a failure node.
- This gives a closed semantic tree with two nodes less than before.

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S

□

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S
- As long as S does not contain the empty clause,

□

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S
- As long as S does not contain the empty clause,
  - It is possible to apply a resolution step to S

□

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S
- As long as S does not contain the empty clause,
  - It is possible to apply a resolution step to S
  - Leading to a clause set with a smaller closed semantic tree

□

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S
- As long as S does not contain the empty clause,
  - It is possible to apply a resolution step to S
  - Leading to a clause set with a smaller closed semantic tree
- Since the tree is finite, this cannot go on forever.

□

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S
- As long as S does not contain the empty clause,
  - It is possible to apply a resolution step to S
  - Leading to a clause set with a smaller closed semantic tree
- Since the tree is finite, this cannot go on forever.
- Therefore, eventually the semantic tree must consist of only the root...

□

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

Proof.

- There exists a closed semantic tree for S
- As long as S does not contain the empty clause,
  - It is possible to apply a resolution step to S
  - Leading to a clause set with a smaller closed semantic tree
- Since the tree is finite, this cannot go on forever.
- Therefore, eventually the semantic tree must consist of only the root...
- ...and S contain the empty clause $\square$.

$\square$